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# Expected topology of random real algebraic submanifolds

Damien Gayet, Jean-Yves Welschinger

April 14, 2014

## Abstract

Let  $X$  be a smooth complex projective manifold of dimension  $n$  equipped with an ample line bundle  $L$  and a rank  $k$  holomorphic vector bundle  $E$ . We assume that  $1 \leq k \leq n$ , that  $X$ ,  $E$  and  $L$  are defined over the reals and denote by  $\mathbb{R}X$  the real locus of  $X$ . Then, we estimate from above and below the expected Betti numbers of the vanishing loci in  $\mathbb{R}X$  of holomorphic real sections of  $E \otimes L^d$ , where  $d$  is a large enough integer. Moreover, given any closed connected codimension  $k$  submanifold  $\Sigma$  of  $\mathbb{R}^n$  with trivial normal bundle, we prove that a real section of  $E \otimes L^d$  has a positive probability, independent of  $d$ , to contain around  $\sqrt{d}^n$  connected components diffeomorphic to  $\Sigma$  in its vanishing locus.

MATHEMATICS SUBJECT CLASSIFICATION 2010: 14P25, 32Q15, 60D05

KEYWORDS: Real projective manifold, ample line bundle, random polynomial, Betti numbers

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## 1 Introduction

Let  $X$  be a smooth complex projective manifold of positive dimension  $n$  equipped with an ample line bundle  $L$  and let  $E$  be a holomorphic vector bundle of rank  $k$  over  $X$ . From the vanishing theorem of Kodaira and Serre, we know that the dimension  $N_d$  of the complex vector space  $H^0(X, E \otimes L^d)$  of global holomorphic sections of  $E \otimes L^d$  grows as a polynomial of degree  $n$  in  $d$ . We will assume throughout this paper that  $1 \leq k \leq n$  and that  $X$ ,  $E$  and  $L$  are defined over the reals. We denote by  $\mathbb{R}X$  the real locus of  $X$  and by  $\mathbb{R}H^0(X, E \otimes L^d)$  the real vector space of real holomorphic sections of  $E \otimes L^d$ , see (5). Its dimension equals  $N_d$ . The discriminant locus  $\mathbb{R}\Delta_d \subset \mathbb{R}H^0(X, E \otimes L^d)$  of sections which do not vanish transversally is a codimension one submanifold for  $d$  large enough and for every  $\sigma$  in its complement, the real vanishing locus  $\mathbb{R}C_\sigma$  of  $\sigma$  is a smooth codimension  $k$  submanifold of  $\mathbb{R}X$ . The topology of  $\mathbb{R}C_\sigma$  drastically depends on the choice of  $\sigma \in \mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_d$ . When  $n = k = 1$ ,  $X = \mathbb{C}P^1$ ,  $L = \mathcal{O}_{\mathbb{C}P^1}(1)$  and  $E = \mathcal{O}_{\mathbb{C}P^1}$  for example,  $\sigma$  is a real polynomial of degree  $d$  in one variable and  $\mathbb{R}C_\sigma$  the set of its real roots.

The space  $\mathbb{R}H^0(X, E \otimes L^d)$  inherits classical probability measures. Indeed, let  $h_E$  be a Hermitian metric on  $E$  and  $h_L$  be a Hermitian metric of positive curvature on  $L$ , both  $h_E$  and  $h_L$  being real, that is invariant under the  $\mathbb{Z}/2\mathbb{Z}$ -Galois action of  $E$  and  $L$ . We denote by  $h_{E,d} = h_E \otimes h_L^d$  the induced metric on  $E \otimes L^d$ . Then, the vector space  $\mathbb{R}H^0(X, E \otimes L^d)$  becomes Euclidean, with the  $L^2$ -scalar product defined by

$$\forall \sigma, \tau \in \mathbb{R}H^0(X, E \otimes L^d), \langle \sigma, \tau \rangle = \int_X h_{E,d}(\sigma, \tau) dx,$$

where  $dx$  denotes any chosen volume form on  $X$  (our results being asymptotic in  $d$ , they turn out not to depend on the choice of  $dx$ ). It thus inherits a Gaussian probability measure  $\mu_{\mathbb{R}}$  whose density at  $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$  with respect to the Lebesgue measure is  $\frac{1}{\sqrt{\pi}^{N_d}} e^{-\|\sigma\|^2}$ .

What is the typical topology of  $\mathbb{R}C_\sigma$  for  $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$  chosen at random for  $d\mu_{\mathbb{R}}$ ? We do not know, but can estimate its average Betti numbers. To formulate our results, let us denote, for every  $i \in \{0, \dots, n - k\}$ , by  $b_i(\mathbb{R}C_\sigma, \mathbb{R}) = \dim H_i(\mathbb{R}C_\sigma, \mathbb{R})$  the  $i$ -th Betti number of  $\mathbb{R}C_\sigma$  and by

$$\mathbb{E}(b_i) = \int_{\mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_d} b_i(\mathbb{R}C_\sigma, \mathbb{R}) d\mu_{\mathbb{R}}(\sigma)$$

its expected value.

## 1.1 Upper estimates

As in [14], for every  $i \in \{0, \dots, n-k\}$ , we denote by  $Sym_{\mathbb{R}}(i, n-k-i)$  the open cone of real symmetric matrices of size  $n-k$  and signature  $(i, n-k-i)$ , by  $\mu_{\mathbb{R}}$  the classical Gaussian measure on the space of real symmetric matrices and by  $e_{\mathbb{R}}(i, n-k-i)$  the numbers

$$e_{\mathbb{R}}(i, n-k-i) = \int_{Sym_{\mathbb{R}}(i, n-k-i)} |\det A| d\mu_{\mathbb{R}}(A), \quad (1)$$

see §3.1. We then denote by  $Vol_{h_L}(\mathbb{R}X)$  the volume of  $\mathbb{R}X$  for the Riemannian metric induced by the Kähler metric  $g_{h_L}$  defined by the curvature form of  $h_L$ , see (3) and (4).

**Theorem 1.1.1** *Let  $X$  be a smooth real projective manifold of dimension  $n$ ,  $(L, h_L)$  be a real holomorphic Hermitian line bundle of positive curvature over  $X$  and  $(E, h_E)$  be a rank  $k$  real holomorphic Hermitian vector bundle, with  $1 \leq k \leq n$ ,  $k \neq n$ . Then, for every  $0 \leq i \leq n-k$ ,*

$$\limsup_{d \rightarrow \infty} \frac{1}{\sqrt{d}^n} \mathbb{E}(b_i) \leq \binom{n-1}{k-1} e_{\mathbb{R}}(i, n-k-i) \frac{Vol_{h_L}(\mathbb{R}X)}{Vol_{FS}(\mathbb{R}P^k)}.$$

Moreover, when  $k = n$ ,  $\frac{1}{\sqrt{d}^n} \mathbb{E}(b_0)$  converges to  $\frac{Vol_{h_L}(\mathbb{R}X)}{Vol_{FS}(\mathbb{R}P^n)}$  as  $d$  grows to infinity.

In fact, the right hand side of the inequality given by Theorem 1.1.1 also involves the determinant of random matrices of size  $k-1$  and the volume of the Grassmann manifold of  $(k-1)$  linear subspaces of  $\mathbb{R}^{n-1}$ , see Theorem 3.1.2, but these can be computed explicitly. Note that when  $E$  is the trivial line bundle, Theorem 1.1.1 reduces to Theorem 1.1 of [14].

Theorem 1.1.1 relies on Theorem 3.1.3, which establishes the asymptotic equidistribution of clouds of critical points, see §3.1. We obtain a similar result in a complex projective setting, for critical points of Lefschetz pencils, see Theorem 3.5.1.

## 1.2 Lower estimates and topology

Let  $\Sigma$  be a closed submanifold of codimension  $k$  of  $\mathbb{R}^n$ ,  $1 \leq k \leq n$ , which we do not assume to be connected. For every  $\sigma \in \mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_d$ , we denote by  $N_{\Sigma}(\sigma)$  the maximal number of disjoint open subsets of  $\mathbb{R}X$  having the property that each such open subset  $U'$  contains a codimension  $k$  submanifold  $\Sigma'$  such that  $\Sigma' \subset \mathbb{R}C_{\sigma}$  and  $(U', \Sigma')$  is diffeomorphic to  $(\mathbb{R}^n, \Sigma)$ . We then set

$$\mathbb{E}(N_{\Sigma}) = \int_{\mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_d} N_{\Sigma}(\sigma) d\mu_{\mathbb{R}}(\sigma) \quad (2)$$

and we associate to  $\Sigma$ , in fact to its isotopy class in  $\mathbb{R}^n$ , a constant  $c_{\Sigma}$  which is positive if and only if  $\Sigma$  has trivial normal bundle in  $\mathbb{R}^n$ , see (14) for its definition and Lemma 2.2.3. The latter measures à la Donaldson the amount of transversality that a polynomial map  $\mathbb{R}^n \rightarrow \mathbb{R}^k$  vanishing along a submanifold isotopic to  $\Sigma$  may have.

**Theorem 1.2.1** *Let  $X$  be a smooth real projective manifold of dimension  $n$ ,  $(L, h_L)$  be a real holomorphic Hermitian line bundle of positive curvature over  $X$  and  $(E, h_E)$  be a rank  $k$  real holomorphic Hermitian vector bundle, with  $1 \leq k \leq n$ . Let  $\Sigma$  be a closed submanifold of codimension  $k$  of  $\mathbb{R}^n$  with trivial normal bundle, which does not need to be connected. Then,*

$$\liminf_{d \rightarrow \infty} \frac{1}{\sqrt{d}^n} \mathbb{E}(N_\Sigma) \geq c_\Sigma \text{Vol}_{h_L}(\mathbb{R}X).$$

In particular, when  $\Sigma$  is connected, Theorem 1.2.1 bounds from below the expected number of connected components diffeomorphic to  $\Sigma$  in the real vanishing locus of a random section  $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$ . The constant  $c_\Sigma$  does not depend on the choice of the triple  $(X, (L, h_L), (E, h_E))$ , it only depends on  $\Sigma$ . When  $k = 1$  and  $E = \mathcal{O}_X$ , Theorem 1.2.1 coincides with Theorem 1.2 of [16]. Computing  $c_\Sigma$  for explicit submanifolds  $\Sigma$  yields the following lower bounds for the Betti numbers.

**Corollary 1.2.2** *Under the hypotheses of Theorem 1.2.1, for every  $i \in \{0, \dots, n-k\}$ ,*

$$\liminf_{d \rightarrow \infty} \frac{1}{\sqrt{d}^n} \mathbb{E}(b_i) \geq \exp(-e^{84+6n}) \text{Vol}_{h_L}(\mathbb{R}X).$$

### 1.3 Some related results

The case  $X = \mathbb{C}P^1$ ,  $E = \mathcal{O}_{\mathbb{C}P^1}$  and  $L = \mathcal{O}_{\mathbb{C}P^1}(1)$  was first considered by M. Kac in [18] for a different measure. In this case and with our measure, Kostlan [19] and Shub and Smale [34] gave an exact formula for the mean number of real roots of a polynomial, as well as the mean number of intersection points of  $n$  hypersurfaces in  $\mathbb{R}P^n$ . Still in  $\mathbb{R}P^n$ , Podkorytov [27] computed the mean Euler characteristics of random algebraic hypersurfaces, and Bürgisser [4] extended this result to complete intersections. In [13], we proved the exponential rarefaction of real curves with a maximal number of components in real algebraic surfaces. In [15] and [14], we bounded from above the mean Betti numbers of random real hypersurfaces in real projective manifolds and in [16], we gave a lower bound for them.

A similar probabilistic study of complex projective manifolds has been performed by Shiffman and Zelditch, see [30], [33], [2] for example, or also [3], [36]. In particular, the asymptotic equidistribution of critical points of random sections over a fixed projective manifold has been studied in [8], [9] and [22], or also [11], [1], [5], while we studied critical points of the restriction of a fixed Morse function on random real hypersurfaces, see [15], [14].

A similar question concerns the mean number of components of the vanishing locus of eigenfunctions of the Laplacian. It has been studied on the round sphere by Nazarov and Sodin [25] (see also [35]), Lerario and Lundberg [20] or Sarnak and Wigman [28]. In a general Riemannian setting, Zelditch proved in [38] the equidistribution of the vanishing locus, whereas critical points of random eigenfunctions of the Laplacian have been addressed by Nicolaescu in [26].

Section 2 is devoted to lower estimates and the proof of Theorem 1.2.1. In this proof, the  $L^2$ -estimates of Hörmander play a crucial rôle, see §2.3, and we follow the same approach as in [16] (see also [12] for a similar construction). Section 3 is

devoted to upper estimates and the proof of Theorem 1.1.1.

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## 2 Lower estimates for the expected Betti numbers

### 2.1 Statement of the results

#### 2.1.1 Framework

Let us first recall our framework. We denote by  $X$  a smooth complex projective manifold of dimension  $n$  defined over the reals, by  $c_X : X \rightarrow X$  the induced Galois antiholomorphic involution and by  $\mathbb{R}X = \text{Fix}(c_X)$  the real locus of  $X$  which we implicitly assume to be non-empty. We then consider an ample line bundle  $L$  over  $X$ , also defined over the reals. It comes thus equipped with an antiholomorphic involution  $c_L : L \rightarrow L$  which turns the bundle projection map  $\pi : L \rightarrow X$  into a  $\mathbb{Z}/2\mathbb{Z}$ -equivariant one, so that  $c_X \circ \pi = \pi \circ c_L$ . We equip  $L$  in addition with a real Hermitian metric  $h_L$ , thus invariant under  $c_L$ , which has a positive curvature form  $\omega$  locally defined by

$$\omega = \frac{1}{2i\pi} \partial \bar{\partial} \log h_L(e, e) \quad (3)$$

for any non-vanishing local holomorphic section  $e$  of  $L$ . This metric induces a Kähler metric

$$g_{h_L} = \omega(\cdot, i \cdot) \quad (4)$$

on  $X$ , which reduces to a Riemannian metric  $g_{h_L}$  on  $\mathbb{R}X$ . Let finally  $E$  be a holomorphic vector bundle of rank  $k$ ,  $1 \leq k \leq n$ , defined over the reals and equipped with an antiholomorphic involution  $c_E$  and a real Hermitian metric  $h_E$ . For every  $d > 0$ , we denote by

$$\mathbb{R}H^0(X, E \otimes L^d) = \{\sigma \in H^0(X, E \otimes L^d) \mid (c_E \otimes c_{L^d}) \circ \sigma = \sigma \circ c_X\} \quad (5)$$

the space of global real holomorphic sections of  $E \otimes L^d$ . It is equipped with the  $L^2$ -scalar product defined by the formula

$$\forall (\sigma, \tau) \in \mathbb{R}H^0(X, E \otimes L^d), \langle \sigma, \tau \rangle = \int_X h_{E,d}(\sigma, \tau)(x) dx, \quad (6)$$

where  $h_{E,d} = h_E \otimes h_L^d$ . Here,  $dx$  denotes any volume form of  $X$ . For instance,  $dx$  can be chosen to be the normalized volume form  $dV_{h_L} = \frac{\omega^n}{\int_X \omega^n}$ . This  $L^2$ -scalar product finally induces a Gaussian probability measure  $\mu_{\mathbb{R}}$  on  $\mathbb{R}H^0(X, E \otimes L^d)$  whose density with respect to the Lebesgue one at  $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$  writes  $\frac{1}{\sqrt{\pi^{N_d}}} e^{-\|\sigma\|^2}$ , where  $N_d = \dim H^0(X, E \otimes L^d)$ . It is with respect to this probability measure that we consider random real codimension  $k$  submanifolds (as in the works [19] and [34], [15], [14] and [16]).

### 2.1.2 The lower estimates

The aim of Section 2 is to prove Theorem 1.2.1. In addition to Theorem 1.2.1, we also get the following Theorem 2.1.1, which is a consequence of Proposition 2.4.2 below.

**Theorem 2.1.1** *Under the hypotheses of Theorem 1.2.1, for every  $0 \leq \epsilon < 1$ ,*

$$\liminf_{d \rightarrow \infty} \mu_{\mathbb{R}} \left\{ \sigma \in \mathbb{R}H^0(X, E \otimes L^d) \mid N_{\Sigma}(\sigma) \geq \epsilon c_{\Sigma} \text{Vol}_{h_L}(\mathbb{R}X) \sqrt{d}^n \right\} > 0.$$

In fact, the positive lower bound given by Theorem 2.1.1 can be made explicit, see (30).

Let us now denote, for every  $1 \leq k \leq n$ , by  $\mathcal{H}_{n,k}$  the set of diffeomorphism classes of smooth closed connected codimension  $k$  submanifolds of  $\mathbb{R}^n$ . For every  $i \in \{0, \dots, n-k\}$  and every  $[\Sigma] \in \mathcal{H}_{n,k}$ , we denote by  $b_i(\Sigma) = \dim H_i(\Sigma; \mathbb{R})$  its  $i$ -th Betti number with real coefficients and by  $m_i(\Sigma)$  its  $i$ -th Morse number. This is the infimum over all Morse functions  $f$  on  $\Sigma$  of the number of critical points of index  $i$  of  $f$ . Then, we set  $c_{[\Sigma]} = \sup_{\Sigma \in [\Sigma]} c_{\Sigma}$  and

$$\mathbb{E}(m_i) = \int_{\mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_d} m_i(\mathbb{R}C_{\sigma}) d\mu_{\mathbb{R}}(\sigma).$$

**Corollary 2.1.2** *Let  $X$  be a smooth real projective manifold of dimension  $n$ ,  $(L, h_L)$  be a real holomorphic Hermitian line bundle of positive curvature over  $X$  and  $(E, h_E)$  be a rank  $k$  real holomorphic Hermitian vector bundle, with  $1 \leq k \leq n$ . Then, for every  $i \in \{0, \dots, n-k\}$ ,*

$$\liminf_{d \rightarrow \infty} \frac{1}{\sqrt{d}^n} \mathbb{E}(b_i) \geq \left( \sum_{[\Sigma] \in \mathcal{H}_{n,k}} c_{[\Sigma]} b_i(\Sigma) \right) \text{Vol}_{h_L}(\mathbb{R}X) \text{ and likewise} \quad (7)$$

$$\liminf_{d \rightarrow \infty} \frac{1}{\sqrt{d}^n} \mathbb{E}(m_i) \geq \left( \sum_{[\Sigma] \in \mathcal{H}_{n,k}} c_{[\Sigma]} m_i(\Sigma) \right) \text{Vol}_{h_L}(\mathbb{R}X). \quad (8)$$

Note that in Corollary 2.1.2, we could have chosen one representative  $\Sigma$  in each diffeomorphism class  $[\Sigma] \in \mathcal{H}_{n,k}$  and obtained the lower estimates (7), (8) with constants  $c_{\Sigma}$  instead of  $c_{[\Sigma]}$ . But it turns out that in the proof of Corollary 2.1.2 we are free to choose the representative we wish in every diffeomorphism class and that the higher  $c_{\Sigma}$  is, the better the estimates (7), (8) are. This is why we introduce the constant  $c_{[\Sigma]}$ , which is positive if and only if  $[\Sigma]$  has a representative  $\Sigma$  with trivial normal bundle in  $\mathbb{R}^n$ , see (14) and Lemma 2.2.3.

## 2.2 Closed affine real algebraic submanifolds

We introduce here the notion of regular pair, see Definition 2.2.1, and the constant  $c_{\Sigma}$  associated to any isotopy class of smooth closed codimension  $k$  submanifold  $\Sigma$  of  $\mathbb{R}^n$ , see (14).

**Definition 2.2.1** *Let  $U$  be a bounded open subset of  $\mathbb{R}^n$  and  $P \in \mathbb{R}[x_1, \dots, x_n]^k$ ,  $1 \leq k \leq n$ . The pair  $(U, P)$  is said to be regular if and only if*

1. zero is a regular value of the restriction of  $P$  to  $U$ ,
2. the vanishing locus of  $P$  in  $U$  is compact.

Hence, for every regular pair  $(U, P)$ , the vanishing locus of  $P$  does not intersect the boundary of  $U$  and it meets  $U$  in a smooth compact codimension  $k$  submanifold.

In the sequel, for every integer  $p$  and every vector  $v \in \mathbb{R}^p$ , we denote by  $|v|$  its Euclidian norm, and for every integers  $p$  and  $q$ , and every linear map  $F : \mathbb{R}^p \rightarrow \mathbb{R}^q$ , we denote by  $F^*$  the adjoint of  $F$ , defined by the property

$$\forall v \in \mathbb{R}^p, \forall w \in \mathbb{R}^q, \langle F(v), w \rangle = \langle v, F^*(w) \rangle,$$

and denote by  $\|F\|$  its operator norm, that is

$$\|F\| = \sup_{v \in \mathbb{R}^p \setminus \{0\}} |F(v)|/|v|.$$

We will also use the norm

$$\|F\|_2 = \sqrt{\text{Tr } FF^*}.$$

These norms satisfy  $\|F\| \leq \|F\|_2$ . Finally, if  $P = (P_1, \dots, P_k) \in \mathbb{R}[x_1, \dots, x_n]^k$ , we denote by  $\|P\|_{L^2}$  its  $L^2$ -norm defined by

$$\|P\|_{L^2}^2 = \int_{\mathbb{C}^n} |P(z)|^2 e^{-\pi|z|^2} dz = \sum_{i=1}^k \int_{\mathbb{C}^n} |P_i(z)|^2 e^{-\pi|z|^2} dz = \sum_{i=1}^k \|P_i\|_{L^2}^2. \quad (9)$$

**Definition 2.2.2** For every regular pair  $(U, P)$  given by Definition 2.2.1, we denote by  $\mathcal{T}_{(U,P)}$  the set of  $(\delta, \epsilon) \in (\mathbb{R}_+^*)^2$  such that

1. there exists a compact subset  $K$  of  $U$  satisfying  $\inf_{x \in U \setminus K} |P(x)| > \delta$ ,
2. for every  $y \in U$ ,  $|P(y)| < \delta \Rightarrow \forall w \in \mathbb{R}^k, |(d|_y P)^*(w)| \geq \epsilon|w|$ .

Hence, for every regular pair  $(U, P)$  given by Definition 2.2.1,  $(\delta, \epsilon)$  belongs to  $\mathcal{T}_{(U,P)}$  provided the  $\delta$ -sublevel of  $P$  does not intersect the boundary of  $U$  while inside this  $\delta$ -sublevel,  $P$  is in a sense  $\epsilon$ -far from having a critical point. This quantifies how much transversally  $P$  vanishes in a way similar to the one used by Donaldson in [7].

Then, for every regular pair  $(U, P)$ , we set  $R_{(U,P)} = \max(1, \sup_{y \in U} |y|)$ , so that  $U$  is contained in the ball centered at the origin and of radius  $R_{(U,P)}$ . Finally, we set

$$\tau_{(U,P)} = 24k\rho_{R_{(U,P)}}\|P\|_{L^2}^2 \inf_{(\delta,\epsilon) \in \mathcal{T}_{(U,P)}} \left( \frac{1}{\delta^2} + \frac{\pi n}{\epsilon^2} \right) \in \mathbb{R}_+^*, \quad (10)$$

where, for every  $R > 0$ ,

$$\rho_R = \inf_{\mathbb{R}_+} g_R, \quad (11)$$

$$g_R : s \in \mathbb{R}_+^* \mapsto \frac{(R+s)^{2n}}{s^{2n}} e^{\pi(R+s)^2}, \quad (12)$$

so that

$$e^{\pi R^2} \leq \rho_R \leq 4^n e^{4\pi R^2}. \quad (13)$$



This constant  $\tau_{(U,P)}$  is the main ingredient in the definition of  $c_\Sigma$ , see (14). The lower  $\tau_{(U,P)}$  is, the larger  $c_\Sigma$  is and the better the estimates given by Theorem 1.2.1 are. Note that  $\tau_{(U,P)}$  remains small whenever  $\delta, \epsilon$  are not too small, that is when  $P$  vanishes quite transversally in  $U$ .

Now, let  $\Sigma$  be a closed submanifold of codimension  $k$  of  $\mathbb{R}^n$ , not necessarily connected. We denote by  $\mathcal{I}_\Sigma$  the set of regular pairs  $(U, P)$  given by Definition 2.2.1, such that the vanishing locus of  $P$  in  $U$  contains a subset isotopic to  $\Sigma$  in  $\mathbb{R}^n$ .

**Lemma 2.2.3** *Let  $\Sigma$  be a closed submanifold of codimension  $k > 0$  of  $\mathbb{R}^n$ , not necessarily connected. Then,  $\mathcal{I}_\Sigma$  is non empty if and only if the normal bundle of  $\Sigma$  in  $\mathbb{R}^n$  is trivial.*

**Proof.** If  $(U, P) \in \mathcal{I}_\Sigma$ , then  $P : \mathbb{R}^n \rightarrow \mathbb{R}^k$  contains in its vanishing locus a codimension  $k$  submanifold  $\widehat{\Sigma}$  which is isotopic to  $\Sigma$  in  $\mathbb{R}^n$ . The normal bundle of  $\Sigma$  in  $\mathbb{R}^n$  is thus trivial if and only if the normal bundle of  $\widehat{\Sigma}$  in  $\mathbb{R}^n$  is trivial. But the differential of  $P$  at every point of  $\widehat{\Sigma}$  provides an isomorphism between the normal bundle of  $\widehat{\Sigma}$  in  $\mathbb{R}^n$  and the product  $\widehat{\Sigma} \times \mathbb{R}^k$ .

Conversely, if  $\Sigma$  has a trivial normal bundle in  $\mathbb{R}^n$ , it has been proved by Seifert [29] (see also [24]) that there exist a polynomial map  $P : \mathbb{R}^n \rightarrow \mathbb{R}^k$  and a tubular neighbourhood  $U$  of  $\Sigma$  in  $\mathbb{R}^n$  such that  $P^{-1}(0) \cap U$  is isotopic to  $\Sigma$  in  $U$ . The strategy of the proof is to first find a smooth function  $U \rightarrow \mathbb{R}^k$  in a neighborhood of  $\Sigma$  which vanishes transversally along  $\Sigma$  and then to suitably approximate the coordinates of this function by some polynomial, see [29], [24]. The pair  $(U, P)$  then belongs to  $\mathcal{I}_\Sigma$  by Definition 2.2.1.  $\square$

We then set  $c_\Sigma = 0$  if  $\Sigma$  does not have a trivial normal bundle in  $\mathbb{R}^n$  and

$$c_\Sigma = \sup_{(U,P) \in \mathcal{I}_\Sigma} \left( \frac{m_{\tau_{(U,P)}}}{2^n \text{Vol}(B(R_{(U,P)}))} \right) \text{ otherwise,} \quad (14)$$

where  $\text{Vol}(B(R_{(U,P)}))$  denotes the volume of the Euclidean ball of radius  $R_{(U,P)}$  in  $\mathbb{R}^n$ , and where, for every  $\tau > 0$ ,

$$m_\tau = \sup_{[\sqrt{\tau}, +\infty[} f_\tau, \quad (15)$$

with  $f_\tau : a \in [\sqrt{\tau}, +\infty[ \mapsto \frac{1}{\sqrt{\pi}} (1 - \frac{\tau}{a^2}) \int_a^{+\infty} e^{-t^2} dt$ . For large values of  $m_\tau$ , as the ones which appear in §2.6, the estimate

$$c_\Sigma \geq e^{-2\tau_{(U,P)}} \quad (16)$$

holds, compare (2.8) of [16].

### 2.3 Hörmander sections

Our key tool to prove Theorems 1.1.1 and 1.2.1 has been developed by L. Hörmander. We introduce in this §2.3 the material we need. For every positive  $d$  and every  $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$ , we set

$$\|\sigma\|_{L^2(h_L)}^2 = \int_X \|\sigma\|_{h_{E,d}}^2 dV_{h_L},$$

where  $dV_{h_L} = \omega^n / \int_X \omega^n$ , compare (6). Let us choose a field of  $h_L$ -trivializations of  $L$  on  $\mathbb{R}X$  given by Definition 4 of [16]. It provides in particular, for every  $x \in \mathbb{R}X$ , a local holomorphic chart  $\psi_x : (W_x, x) \subset X \rightarrow (V_x, 0) \subset \mathbb{C}^n$  isometric at  $x$ , and a non-vanishing holomorphic section  $e$  of  $L$  defined over  $W_x$  such that  $\phi = -\log h_L(e, e)$  vanishes at  $x$  and is positive elsewhere. Moreover, there exist a positive constant  $\alpha_1$  such that

$$\forall y \in V_x, |\phi \circ \psi_x^{-1}(y) - \pi|y|^2| \leq \alpha_1|y|^3. \quad (17)$$

Restricting  $W_x$  if necessary, we choose a holomorphic trivialization  $(e_1, \dots, e_k)$  of  $E|_{W_x}$  which is orthonormal at  $x$ . This provides a trivialization  $(e_1 \otimes e^d, \dots, e_k \otimes e^d)$  of  $E \otimes L|_{W_x}^d$ . In this trivialization, the restriction of  $\sigma$  to  $W_x$  writes

$$\sigma = \sum_{j=1}^k f_\sigma^j e_j \otimes e^d \quad (18)$$

for some holomorphic functions  $f_\sigma^j : W_x \rightarrow \mathbb{C}$ . We write  $f_\sigma = (f_\sigma^1, \dots, f_\sigma^k)$  and we set

$$|\sigma| = |f_\sigma|, \quad (19)$$

so that on  $W_x$ ,  $\|\sigma\|_{h_{E,d}}^2 = \left\| \sum_{j=1}^k f_\sigma^j e_j \right\|_{h_E}^2 e^{-d\phi}$  and  $\|\sigma(x)\|_{h_{E,d}}^2 = |\sigma(x)|^2$  since the frames  $(e_1, \dots, e_k)$  and  $e$  are orthonormal at the point  $x$  so that in particular  $\phi(x) = 0$ . For every  $z \in W_x$ , we define

$$\|d|_z \sigma\|_2 = \|d|_y (f_\sigma \circ \psi_x^{-1})\|_2, \quad (20)$$

$$\|d|_z \sigma\| = \|d|_y (f_\sigma \circ \psi_x^{-1})\|, \quad (21)$$

and

$$(d|_z \sigma)^* = (d|_y (f_\sigma \circ \psi_x^{-1}))^*, \quad (22)$$

where  $y = \psi_x(z)$ . Finally, we denote, for every small enough  $r > 0$ , by  $B(x, r) \subset W_x$  the ball centered at  $x$  and of radius  $r$  for the flat metric of  $V_x$  pulled back by  $\psi_x$ , so that

$$B(x, r) = \psi_x^{-1}(B(0, r)). \quad (23)$$

**Proposition 2.3.1** *Let  $X$  be a smooth real projective manifold of dimension  $n$ ,  $(L, h_L)$  be a real holomorphic Hermitian line bundle of positive curvature over  $X$  and  $(E, h_E)$  be a rank  $k$  real holomorphic Hermitian vector bundle, with  $1 \leq k \leq n$ . We choose a field of  $h_L$ -trivializations on  $\mathbb{R}X$ . Then, for every regular pair  $(U, P)$ , every large enough integer  $d$ , every  $x$  in  $\mathbb{R}X$  and every local trivialization of  $E$  orthonormal at  $x$ , there exist  $\sigma_{(U,P)} \in \mathbb{R}H^0(X, E \otimes L^d)$  and an open subset  $U_d$  of  $B(x, \frac{R_{(U,P)}}{\sqrt{d}}) \cap \mathbb{R}X$  such that*

1.  $\|\sigma_{(U,P)}\|_{L^2(h_L)}$  be equivalent to  $\frac{\|P\|_{L^2}}{\sqrt{\delta_L}}$  as  $d$  grows to infinity, where  $\|P\|_{L^2}$  is defined by (9) and  $\delta_L = \int_X \omega^n$ ,
2.  $(U_d, \sigma_{(U,P)}^{-1}(0) \cap U_d)$  be diffeomorphic to  $(U, P^{-1}(0) \cap U) \subset \mathbb{R}^n$ ,

3. for every  $(\delta, \epsilon) \in \mathcal{T}_{(U,P)}$  given by Definition 2.2.2, there exists a compact subset  $K_d \subset U_d$  such that

$$\inf_{U_d \setminus K_d} |\sigma_{(U,P)}| > \frac{\delta}{2} \sqrt{d}^n,$$

while for every  $y$  in  $U_d$ ,

$$|\sigma_{(U,P)}(y)| < \frac{\delta}{2} \sqrt{d}^n \Rightarrow \forall w \in \mathbb{R}^k, |(d|_y \sigma_{(U,P)})^*(w)| \geq \frac{\epsilon}{2} \sqrt{d}^{n+1} |w|. \quad (24)$$

**Proof.** We proceed as in the proof of Proposition 3.2 of [16]. Let  $(U, P)$  be a regular pair,  $x \in \mathbb{R}X$  and  $d$  large enough. We set  $U_d = \psi_x^{-1}(\frac{1}{\sqrt{d}}U) \subset B(x, \frac{R_{(U,P)}}{\sqrt{d}})$  and  $K_d = \psi_x^{-1}(\frac{1}{\sqrt{d}}K)$ . Let  $\chi : \mathbb{C}^n \rightarrow [0, 1]$  be a smooth function with compact support in  $B(0, R_{(U,P)})$ , which equals one in a neighbourhood of the origin. Then, let  $\sigma$  be the global smooth section of  $E \otimes L^d$  defined by  $\sigma|_{X \setminus W_x} = 0$  and

$$\sigma|_{W_x} = (\chi \circ \psi_x) \left( \sum_{j=1}^k P_j(\sqrt{d}\psi_x) e_j \otimes e^d \right),$$

where  $P = (P_1, \dots, P_k)$  is now considered as a function  $\mathbb{C}^n \rightarrow \mathbb{C}^k$ . From the  $L^2$ -estimates of Hörmander, see [17] or [21], there exists a global section  $\tau$  of  $E \otimes L^d$  such that  $\bar{\partial}\tau = \bar{\partial}\sigma$  and  $\|\tau\|_{L^2(h_{E,d})} \leq \|\bar{\partial}\sigma\|_{L^2(h_{E,d})}$  for  $d$  large enough. This section  $\tau$  can be chosen orthogonal to holomorphic sections and is then unique, in particular real. Moreover, there exist positive constants  $c_1$  and  $c_2$ , which do not depend on  $x$ , such that  $\|\tau\|_{L^2(h_{E,d})} \leq c_1 e^{-c_2 d}$  and  $\sup_{U_d} (|\tau| + \|\tau\|_2) \leq c_2 e^{-c_2 d}$ , see Lemma 3.3 of [16]. We then set  $\sigma_{(U,P)} = \sqrt{d}^n (\sigma - \tau)$ . It has the desired properties as can be checked along the same lines as in the proof of Proposition 3.2 of [16] and thanks to Lemma 2.3.2.  $\square$

**Lemma 2.3.2** *Let  $U$  be an open subset of  $\mathbb{R}^n$ ,  $1 \leq k \leq n$ ,  $f : U \rightarrow \mathbb{R}^k$  be a function of class  $C^1$  and  $(\delta, \epsilon) \in (\mathbb{R}_+^*)^2$  be such that*

1. *there exists a compact subset  $K$  of  $U$  such that  $\inf_{U \setminus K} |f| > \delta$ ,*
2. *for every  $y$  in  $U$ ,  $|f(y)| < \delta \Rightarrow \forall w \in \mathbb{R}^k, |(d|_y f)^*(w)| \geq \epsilon |w|$ .*

*Then, for every function  $g : U \rightarrow \mathbb{R}^k$  of class  $C^1$  such that  $\sup_U |g| < \delta$  and  $\sup_U \|dg\| < \epsilon$ , zero is a regular value of  $f + g$  and  $(f + g)^{-1}(0)$  is compact and isotopic to  $f^{-1}(0)$  in  $U$ .*

**Proof.** The proof is analogous to the one of Lemma 3.4 of [16], since  $\|(dg)^*\| = \|dg\|$ .  $\square$

The following Lemma 2.3.3 establishes the existence of peak sections for higher rank vector bundles.

**Lemma 2.3.3 (compare Lemma 1.2 of [37])** *Let  $X$  be a smooth real projective manifold of dimension  $n$ ,  $(L, h_L)$  be a real holomorphic Hermitian line bundle of positive curvature over  $X$  and  $(E, h_E)$  be a rank  $k$  real holomorphic Hermitian vector bundle, with  $1 \leq k \leq n$ . Let  $x \in \mathbb{R}X$ ,  $(p_1, \dots, p_n) \in \mathbb{N}^n$ ,  $i \in \{1, \dots, k\}$  and  $p' >$*

$p_1 + \dots + p_n$ . There exists  $d_0 \in \mathbb{N}$  independent of  $x$  such that for every  $d > d_0$ , there exists  $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$  with the property that  $\|\sigma\|_{L^2(h_L)} = 1$  and if  $(y_1, \dots, y_n)$  are local real holomorphic coordinates in the neighbourhood of  $x$  and  $(e_1, \dots, e_k)$  is a local real holomorphic trivialization of  $E$  orthonormal at  $x$ , we can assume that in a neighbourhood of  $x$ ,

$$\sigma(y_1, \dots, y_n) = \lambda y_1^{p_1} \dots y_n^{p_n} e_i \otimes e^d (1 + O(d^{-2p'})) + O(\lambda |y|^{2p'}), \quad (25)$$

where  $\lambda^{-2} = \int_{B(x, \frac{\log d}{\sqrt{d}})} |y_1^{p_1} \dots y_n^{p_n}|^2 \|e^d\|_{h_L}^2 dV_{h_L}$ , with  $dV_{h_L} = \omega^n / \int_X \omega^n$  and where  $e$  is a local trivialization of  $L$  whose potential  $-\log h_L(e, e)$  reaches a local minimum at  $x$  with Hessian  $\pi\omega(\cdot, \cdot)$ .

**Proof.** The proof goes along the same lines as the one of Lemma 1.2 of [37]. Let  $\eta$  be a cut-off function on  $\mathbb{R}$  with  $\eta = 1$  in a neighbourhood of 0, and

$$\psi = (n + 2p')\eta\left(\frac{d\|z\|^2}{\log^2 d}\right) \log\left(\frac{d\|z\|^2}{\log^2 d}\right)$$

in the coordinates  $z$  on  $X$ . Then,  $i\partial\bar{\partial}\psi$  is bounded from below by  $-C\omega$ , where  $C$  is some uniform constant independent of  $d$  and  $x$ . Let  $s \in C^\infty(X, E \otimes L^d)$  be the real section defined by

$$s = \eta\left(\frac{d\|z\|^2}{\log^2 d}\right) y_1^{p_1} \dots y_n^{p_n} e_i \otimes e^d.$$

Then, from Theorem 5.1 of [6], for  $d$  large enough not depending on  $x$ , there exists a real section  $u \in C^\infty(X, E \otimes L^d)$  such that  $\bar{\partial}u = \bar{\partial}s$  and satisfying the Hörmander  $L^2$ -estimates

$$\int_X \|u\|_{h_{E,d}}^2 e^{-\psi} dV_{h_L} \leq \int_X \|\bar{\partial}s\|_{h_{E,d}}^2 e^{-\psi} dV_{h_L}.$$

The presence of the singular weight  $e^{-\psi}$  forces the jets of  $u$  to vanish up to order  $2p'$  at  $x$ . As in Lemma 1.2 of [37], we conclude that the real holomorphic section  $\sigma = (s - u) / \|s - u\|_{L^2(h_{E,d})}$  satisfies the required properties.  $\square$

In this first section we will only need peak sections given by Lemma 2.3.3 with  $\sum_{i=1}^n p_i = 0$ , whereas in the second one we will need those given with  $\sum_{i=1}^n p_i \leq 2$ .

**Definition 2.3.4** For  $i \in \{1, \dots, k\}$ , let  $\sigma_0^i$  be the section given by Lemma 2.3.3 with  $p' = 3$  and  $p_1 = \dots = p_n = 0$ . Likewise, for every  $j \in \{1, \dots, n\}$ , let  $\sigma_j^i$  be a section given by (25) with  $p' = 3$ ,  $p_j = 1$  and  $p_l = 0$  for  $l \in \{1, \dots, n\} \setminus \{j\}$ . Finally, for every  $1 \leq l \leq m \leq n$ , let  $\sigma_{lm}^i$  be a section given by (25) with  $p' = 3$ ,  $p_j = 0$  for every  $j \in \{1, \dots, n\} \setminus \{l, m\}$  and  $p_l = p_m = 1$  if  $l \neq m$ , while  $p_l = 2$  otherwise.

The asymptotic values of the constants  $\lambda$  in (25) are given by Lemma 2.3.5 (compare Lemma 2.1 of [37]).

**Lemma 2.3.5** For every  $i \in \{1, \dots, k\}$ , the sections given by Definition 2.3.4 satisfy

$$\sigma_0^i / \sqrt{\delta_L d^n} \underset{d \rightarrow \infty}{\sim} e_i \otimes e^d + O(\|y\|^6), \quad (26)$$

$$\forall j \in \{1, \dots, n\}, \sigma_j^i / \sqrt{\pi \delta_L d^{n+1}} \underset{d \rightarrow \infty}{\sim} y_j e_i \otimes e^d + O(\|y\|^6), \quad (27)$$

$$\forall l, m \in \{1, \dots, n\}, l \neq m, \sigma_{lm}^i / (\pi \sqrt{\delta_L d^{n+2}}) \underset{d \rightarrow \infty}{\sim} y_l y_m e_i \otimes e^d + O(\|y\|^6), \quad (28)$$

$$\text{and } \forall l \in \{1, \dots, n\}, \sigma_{ll}^i / (\pi \sqrt{\delta_L d^{n+2}}) \underset{d \rightarrow \infty}{\sim} \frac{1}{\sqrt{2}} y_l^2 e_i \otimes e^d + O(\|y\|^6). \quad (29)$$

Moreover, these sections are asymptotically orthonormal as  $d$  grows to infinity, as follows from Lemma 2.3.6.

**Lemma 2.3.6 (compare Lemma 3.1 of [37])** *For every  $x \in \mathbb{R}X$ , the sections  $(\sigma_j^i)_{\substack{1 \leq i \leq k \\ 0 \leq j \leq n}}$  and  $(\sigma_{lm}^i)_{\substack{1 \leq i \leq k \\ 1 \leq l \leq m \leq n}}$  given by Definition 2.3.4 have  $L^2$ -norm equal to one and their pairwise scalar product are dominated by a  $O(d^{-1})$  which does not depend on  $x$ . Likewise, their scalar products with every section of  $\mathbb{R}H^0(X, E \otimes L^d)$  of  $L^2$ -norm equal to one and whose 2-jet at  $x$  vanishes is dominated by a  $O(d^{-3/2})$  which does not depend on  $x$ .*

**Proof.** The proof goes along the same lines as the one of Lemma 3.1 of [37].  $\square$

**Lemma 2.3.7** *Denote by  $v$  the density of  $dV_{h_L} = \omega^n / \int_X \omega^n$  with respect to the volume form  $dx$  chosen in (6), so that  $dV_{h_L} = v(x)dx$ . Then the sections given by Definition 2.3.4 times  $\sqrt{v(x)}$  are still asymptotically orthonormal for (6).*

**Proof.** This is a direct consequence of Lemmas 2.3.3 and 2.3.6 and the asymptotic concentration of the support of the peak sections near  $x$ .  $\square$

**Remark 2.3.8** *The complex analogues of Lemmas 2.3.3, 2.3.5 and 2.3.6 hold, compare [37].*

## 2.4 Proof of Theorem 1.2.1

We first compute the expected local  $C^1$ -norm of sections.

**Proposition 2.4.1** *Let  $X$  be a smooth real projective manifold of dimension  $n$ ,  $(L, h_L)$  be a real holomorphic Hermitian line bundle of positive curvature over  $X$  and  $(E, h_E)$  be a rank  $k$  real holomorphic Hermitian vector bundle, with  $1 \leq k \leq n$ . We equip  $\mathbb{R}X$  with a field of  $h_L$ -trivializations, see §2.3. Then, for every positive  $R$ ,*

$$\limsup_{d \rightarrow \infty} \sup_{x \in \mathbb{R}X} \frac{1}{d^n} E \left( \sup_{B(x, \frac{R}{\sqrt{d}})} \frac{|\sigma|^2}{v(x)} \right) \leq 6k\delta_L \rho_R \text{ and}$$

$$\limsup_{d \rightarrow \infty} \sup_{x \in \mathbb{R}X} \frac{1}{d^{n+1}} E \left( \sup_{B(x, \frac{R}{\sqrt{d}})} \frac{\|d\sigma\|_2^2}{v(x)} \right) \leq 6\pi n k \delta_L \rho_R,$$

where  $v$  is given by Lemma 2.3.7 and  $\rho_R$  is given by (11), see (19) and (20) for the definitions of  $|\sigma|$  and  $\|d\sigma\|_2$ .

Note that a global estimate on the sup norm of  $L^2$  random holomorphic sections is given by Theorem 1.1 of [32].

**Proof.** The proof goes along the same lines as the proof of Proposition 3.5 of [16]. We first establish from the mean value inequality that for every  $x \in \mathbb{R}X$ ,  $R > 0$  and  $s > 0$ ,

$$E \left( \sup_{B(x, \frac{R}{\sqrt{d}})} |\sigma|^2 \right) \leq \frac{1}{\text{Vol}(B(\frac{s}{\sqrt{d}}))} \int_{B(x, \frac{R+s}{\sqrt{d}})} \mathbb{E}(|\sigma|^2) \psi_x^* dy$$

for  $d$  large enough not depending on  $x$ . Then, for every  $z \in B(x, \frac{R+s}{\sqrt{d}}) \cap \mathbb{R}X$ , we write  $\sigma = \sum_{i=1}^k a_i \sigma_0^i + \tau$ , where  $\tau \in \mathbb{R}H^0(X, E \otimes L^d)$  vanishes at  $z$  and  $(\sigma_0^i)_{i=1, \dots, k}$  are the peak sections at  $z$  given by Definition 2.3.4. In particular, by Lemma 2.3.5, at the point  $z$ , for every  $i = 1, \dots, k$ ,  $\|\sigma_0^i\|_{h_{E,d}} \underset{d \rightarrow \infty}{\sim} \sqrt{\delta_L d^n}$ . Moreover, since  $(e_1, \dots, e_n)$  is orthonormal at  $x$ ,

$$\begin{aligned} |\sigma_0^i(z)|^2 &= \|\sigma_0^i(z)\|_{h_{E,d}}^2 (1 + O(|z-x|)) e^{d\phi(z)} \\ &\leq \delta_L d^n e^{\pi(R+s)^2} (1 + o(1)) \end{aligned}$$

from the inequalities (17), where the  $o(d^n)$  can be chosen not to depend on  $x \in \mathbb{R}X$ . Suppose that  $dy = dV_{h_L}$ . Then, by Lemma 2.3.6, the peak sections are asymptotically orthogonal to each other for the scalar product defined by (6), and asymptotically orthogonal to the space of sections  $\tau$  vanishing at  $x$ . We deduce that

$$\begin{aligned} \mathbb{E}(|\sigma(z)|^2) &= \mathbb{E}\left(\left|\sum_{i=1}^k a_i \sigma_0^i\right|^2\right) (1 + o(1)) \\ &= \left(\sum_{i=1}^k |\sigma_0^i(z)|^2\right) \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} a^2 e^{-a^2} da (1 + o(1)) \\ &\leq \frac{1}{2} k \delta_L d^n e^{\pi(R+s)^2} (1 + o(1)). \end{aligned}$$

When  $z \notin B(x, \frac{R+s}{\sqrt{d}}) \cap \mathbb{R}X$ , the space of real sections vanishing at  $z$  gets of real codimension  $2k$  in  $\mathbb{R}H^0(X, E \otimes L^d)$ . Let  $\langle \theta_1^i, \theta_2^i, i \in \{1, \dots, k\} \rangle$  be an orthonormal basis of its orthogonal complement. From Remark 2.3.8, for every  $i \in \{1, \dots, k\}$ ,  $j \in \{1, 2\}$ ,

$$\limsup_{d \rightarrow \infty} \frac{1}{d^n} |\theta_j^i(z)|^2 \leq 2\delta_L e^{\pi(R+s)^2},$$

an upper bound which does not depend on  $z$ . We deduce that

$$\begin{aligned} \mathbb{E}(|\sigma(z)|^2) &= \int_{\mathbb{R}^{2k}} \left| \sum_{i=1}^k (a_{01}^i \theta_1^i(z) + a_{02}^i \theta_2^i(z)) \right|^2 e^{-\sum_{i=1}^k (a_{01}^i)^2 + (a_{02}^i)^2} \frac{1}{\pi^k} \prod_{i=1}^k da_{01}^i da_{02}^i \\ &\leq 2\delta_L d^n e^{\pi(R+s)^2} (1 + o(1)) \sum_{i=1}^k \int_{\mathbb{R}^2} ((a_{01}^i)^2 + (a_{02}^i)^2 + 2|a_{01}^i| |a_{02}^i|) \dots \\ &\quad \dots \frac{1}{\pi} e^{-(a_{01}^i)^2 - (a_{02}^i)^2} da_{01}^i da_{02}^i \\ &\leq 6\delta_L d^n e^{\pi(R+s)^2} (1 + o(1)). \end{aligned}$$

We deduce the first part of Proposition 2.4.1 by taking the supremum over  $\mathbb{R}X$ , choosing  $s$  which minimize  $g_{R(U,P)}$  and taking the lim sup as  $d$  grows to infinity.

In general, the Bergman section at  $x$  for the  $L^2$ -product (6) associated to the volume form  $dx$  is equivalent to the Bergman section  $\sigma_0$  at  $x$  for  $dV_h$  times  $\sqrt{v(x)}$ , see Lemma 2.3.7. The same holds true for the  $\sigma_j$ 's, and the result follows by replacing  $\delta_L$  with  $v(x)\delta_L$ .

The proof of the second assertion goes along the same lines, see the proof of Proposition 3.5 of [16] (and [31] for similar results).  $\square$

As in [16], we then compute the probability of presence of closed affine real algebraic submanifolds, inspired by an approach of Nazarov and Sodin [25], see also [20]. Let  $(U, P)$  be a regular pair given by Definition 2.2.1 and  $\Sigma = P^{-1}(0) \subset U$ . Then, for every  $x \in \mathbb{R}X$ , we set  $B_d = B(x, \frac{R_{(U,P)}}{\sqrt{d}}) \cap \mathbb{R}X$ , see (23), and denote by  $Prob_{x,\Sigma}(E \otimes L^d)$  the probability that  $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$  has the property that  $\sigma^{-1}(0) \cap B_d$  contains a closed submanifold  $\Sigma'$  such that the pair  $(B_d, \Sigma')$  be diffeomorphic to  $(\mathbb{R}^n, \Sigma)$ . That is,

$$Prob_{x,\Sigma}(E \otimes L^d) = \mu_{\mathbb{R}} \left\{ \sigma \in \mathbb{R}H^0(X, E \otimes L^d) \mid (\sigma^{-1}(0) \cap B_d) \supset \Sigma', (B_d, \Sigma') \sim (\mathbb{R}^n, \Sigma) \right\}.$$

We then set  $Prob_{\Sigma}(E \otimes L^d) = \inf_{x \in \mathbb{R}X} Prob_{x,\Sigma}(E \otimes L^d)$ .

**Proposition 2.4.2** *Let  $X$  be a smooth real projective manifold of dimension  $n$ ,  $(L, h_L)$  be a real holomorphic Hermitian line bundle of positive curvature over  $X$  and  $(E, h_E)$  be a rank  $k$  real holomorphic Hermitian vector bundle, with  $1 \leq k \leq n$ . Let  $(U, P)$  be a regular pair given by Definition 2.2.1 and  $\Sigma = P^{-1}(0) \subset U$ . Then,*

$$\liminf_{d \rightarrow \infty} Prob_{\Sigma}(E \otimes L^d) \geq m_{\tau_{(U,P)}},$$

see (15).

**Proof.** The proof is the same as the one of Proposition 3.6 of [16] and is not reproduced here.  $\square$

The proof of Theorem 1.2.1 (resp. Corollary 2.1.2) then just goes along the same lines as the one of Theorem 1.2 (resp. Corollary 1.3) of [16].

## 2.5 Proof of Theorem 2.1.1

Let  $(U, P)$  be a regular pair given by Definition 2.2.1. For every  $d > 0$ , let  $\Lambda_d$  be a maximal subset of  $\mathbb{R}X$  with the property that two distinct points of  $\Lambda_d$  are at distance greater than  $\frac{2R_{(U,P)}}{\sqrt{d}}$ . The balls centered at points of  $\Lambda_d$  and of radius  $\frac{R_{(U,P)}}{\sqrt{d}}$  are disjoint, whereas the ones of radius  $\frac{2R_{(U,P)}}{\sqrt{d}}$  cover  $\mathbb{R}X$ . Note that if we use the local flat metric given by a trivial  $h_L$ -trivialization, then the associated lattice has asymptotically the same number of balls than  $\Lambda_d$  as  $d$  grows to infinity, so we can suppose from now on that the balls are defined for this local metric. For every  $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$ , denote by  $N_{\Sigma}(\Lambda_d, \sigma)$  the number of  $x \in \Lambda_d$  such that the ball  $B_d = B(x, \frac{R_{(U,P)}}{\sqrt{d}}) \cap \mathbb{R}X$  contains a codimension  $k$  submanifold  $\Sigma'$  with  $\Sigma' \subset \sigma^{-1}(0)$  and  $(B_d, \Sigma')$  diffeomorphic to  $(\mathbb{R}^n, \Sigma)$ . By definition of  $N_{\Sigma}(\sigma)$ ,  $N_{\Sigma}(\Lambda_d, \sigma) \leq N_{\Sigma}(\sigma)$ , see §1.2, while from Proposition 2.4.2, for every  $0 < \epsilon < 1$ ,

$$\begin{aligned} |\Lambda_d| m_{\tau_{(U,P)}} &\leq \sum_{x \in \Lambda_d} Prob_{x,\Sigma}(E \otimes L^d) \\ &\leq \sum_{j=1}^{|\Lambda_d|} j \mu_{\mathbb{R}} \{ \sigma \mid N_{\Sigma}(\Lambda_d, \sigma) = j \} \\ &\leq \epsilon m_{\tau_{(U,P)}} |\Lambda_d| \mu_{\mathbb{R}} \{ \sigma \mid N_{\Sigma}(\Lambda_d, \sigma) \leq \epsilon m_{\tau_{(U,P)}} |\Lambda_d| \} \\ &\quad + |\Lambda_d| \mu_{\mathbb{R}} \{ \sigma \mid N_{\Sigma}(\Lambda_d, \sigma) \geq \epsilon m_{\tau_{(U,P)}} |\Lambda_d| \}. \end{aligned}$$



We deduce that

$$(1 - \epsilon)m_{\tau(U,P)} \leq \mu_{\mathbb{R}} \{ \sigma \mid N_{\Sigma}(\sigma) \geq \epsilon m_{\tau(U,P)} |\Lambda_d| \} \quad (30)$$

and the result follows by choosing a sequence  $(U_p, P_p)_p \in \mathcal{I}_{\Sigma}$  such that

$$\lim_{p \rightarrow \infty} m_{\tau(U_p, P_p)} |\Lambda_d| = c_{\Sigma} \text{Vol}_{h_L}(\mathbb{R}X) \sqrt{d}^n,$$

see (14).  $\square$

## 2.6 Proof of Corollary 1.2.2

In this paragraph, for every positive integer  $p$ ,  $S^p$  denotes the unit sphere in  $\mathbb{R}^{p+1}$ . Corollary 1.2.2 is a consequence of Theorem 1.2.1 and the following Propositions 2.6.1 and 2.6.3.

**Proposition 2.6.1** *For every  $1 \leq k \leq n$ ,  $c_{S^{n-k}} \geq \exp(-e^{54+5n})$ .*

Recall the following.

**Lemma 2.6.2 (Lemma 2.2 of [16])** *If  $P = \sum_{(i_1, \dots, i_n) \in \mathbb{N}^n} a_{i_1, \dots, i_n} z_1^{i_1} \cdots z_n^{i_n} \in \mathbb{R}[z_1, \dots, z_n]$ , then*

$$\|P\|_{L^2}^2 = \int_{\mathbb{C}^n} |P(z)|^2 e^{-\pi|z|^2} dz = \sum_{(i_1, \dots, i_n) \in \mathbb{N}^n} |a_{i_1, \dots, i_n}|^2 \frac{i_1! \cdots i_n!}{\pi^{i_1 + \dots + i_n}}.$$

**Proof of Proposition 2.6.1.** For every  $n > 0$ , we set  $P_k(x_1, \dots, x_n) = \sum_{j=k}^n x_j^2 - 1$ . For every  $x \in \mathbb{R}^n$  and  $\delta > 0$ ,

$$|P_k(x)| < \delta \Leftrightarrow 1 - \delta < \sum_{i=k}^n x_i^2 < 1 + \delta \Rightarrow \|d_x P_k\|_2^2 = 4 \sum_{i=k}^n x_i^2 > 4(1 - \delta).$$

Moreover from Lemma 2.6.2,

$$\|P_k\|_{L^2}^2 = 1 + \frac{2(n-k+1)}{\pi^2} \leq n - k + 2.$$

Now set  $P_S = (P_1, \dots, P_k)$  with  $P_j(x) = x_j$  for  $1 \leq j \leq k-1$ , so that

$$\|P_S\|_{L^2}^2 \leq (k-1)/\pi + (n-k+2) \leq n+1 \leq 2n.$$

Since for every  $w = (w_1, \dots, w_k) \in \mathbb{R}^k$  and every  $x \in \mathbb{R}^n$ ,

$$|d_x P_S^*(w)|^2 = \sum_{i=1}^{k-1} w_i^2 + w_k^2 \|d_x P_k\|_2^2,$$

we get that  $\|d_x P_S^*\|^2 \geq \min(1, 4(1-\delta))$  if  $|P_k(x)| < \delta$ . Choose

$$U_S = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{j=1}^n x_j^2 < 4\}.$$



Then if  $0 < \delta < 1$ ,

$$K_\delta = \left\{ x \in U_S \mid 1 - \delta \leq \sum_{i=k}^n x_i^2 \leq 1 + \delta \text{ and } \sum_{i=1}^{k-1} x_k^2 \leq 1 - \frac{1}{2}(1 + \delta)^2 \right\}$$

is compact in  $U_S$  and taking  $R_{(U_S, P_S)}^2 = 4$ , we see that the pair  $(U_S, P_S)$  is regular in the sense of Definition 2.2.1. The submanifold  $P_S^{-1}(0) \subset U_S$  is isotopic in  $\mathbb{R}^n$  to the unit sphere  $S^{n-k}$ . We deduce that  $(3/4, 1) \in \mathcal{T}_{(U_S, P_S)}$ . From (10) and (13) we deduce

$$\tau_{(U_S, P_S)} \leq 24k4^n e^{16\pi} 2n(2 + \pi n) \leq e^{53+5n}.$$

The estimate  $c_{S^{n-1}} \geq \exp(-e^{54+5n})$  follows then from (16).  $\square$

**Proposition 2.6.3** *For every  $1 \leq k \leq n$  and every  $0 \leq i \leq n - k$ ,  $c_{S^i \times S^{n-i-k}} \geq \exp(-e^{82+6n})$ .*

**Proof.** For every  $1 \leq k \leq n$  and every  $0 \leq i \leq n - k$ , we set

$$Q_k((x_1, \dots, x_{i+1}), (y_1, \dots, y_{n-i-1})) = (|x|^2 - 2)^2 + \sum_{j=1}^{n-k-i} y_j^2 - 1.$$

For every  $(x, y) \in \mathbb{R}^{i+1} \times \mathbb{R}^{n-i-1}$  and  $0 < \delta < 1/2$ ,

$$\begin{aligned} |Q_k(x, y)| < \delta &\Leftrightarrow 1 - \delta < (|x|^2 - 2)^2 + \sum_{j=1}^{n-k-i} y_j^2 < 1 + \delta \\ &\Rightarrow \|d_{|(x,y)} Q_k\|_2^2 = 4 \sum_{j=1}^{n-k-i} y_j^2 + 16|x|^2(|x|^2 - 2)^2, \end{aligned}$$

with  $|x|^2 > 2 - \sqrt{1 + \delta} > 1/2$ . Thus  $\|d_{|(x,y)} Q_k\|_2^2 > 4(1 - \delta)$ , compare Lemma 2.6 of [16]. Moreover from Lemma 2.6.2,  $\|Q_k\|_{L^2}^2 \leq 13n^2$ , compare §2.3.2 of [16]. Now set  $Q = (Q_1, \dots, Q_k)$  with  $Q_j(x, y) = y_{n-i-j}$  for  $1 \leq j \leq k - 1$ , so that

$$\|Q\|_{L^2}^2 \leq (k - 1)/\pi + 13n^2 \leq 13(n + 1)^2.$$

For every  $w = (w_1, \dots, w_k) \in \mathbb{R}^k$  and every  $(x, y) \in \mathbb{R}^{i+1} \times \mathbb{R}^{n-i-1}$ ,

$$\begin{aligned} |d_{|(x,y)} Q^*(w)|^2 &= \sum_{i=1}^{k-1} w_i^2 + w_k^2 \|d_{|(x,y)} Q_k\|_2^2 \\ &> \min(1, 4(1 - \delta)) |w|^2 \end{aligned}$$

if  $|Q_k(x, y)| \leq \delta < 1/2$ . We choose

$$U = \{(x, y) \in \mathbb{R}^{i+1} \times \mathbb{R}^{n-i-1} \mid |x|^2 + |y|^2 < 6\},$$

$$K_\delta = \left\{ (x, y) \in U \mid 1 - \delta \leq (|x|^2 - 2)^2 + \sum_{j=1}^{n-k-i} y_j^2 \leq 1 + \delta \text{ and } \sum_{j=1}^{k-1} y_{n-i-j}^2 \leq 1 - \delta \right\},$$

and  $R_{(U,Q)}^2 = 6$ . The pair  $(U, Q)$  is regular in the sense of Definition 2.2.1 and  $Q^{-1}(0) \subset U$  is isotopic in  $\mathbb{R}^n$  to the product  $S^i \times S^{n-i-k}$  of unit spheres in  $\mathbb{R}^{i+1}$  and  $\mathbb{R}^{n-i-k+1}$ . We deduce that for every positive  $\epsilon$ ,  $(1/2 - \epsilon, 1) \in \mathcal{T}_{(U,Q)}$  and from (10) and (13) that

$$\tau_{(U,Q)} \leq 24k4^n e^{24\pi} 13(n+1)^2(4+\pi n) \leq e^{81+6n}.$$

The estimate  $c_{S^i \times S^{n-i-k}} \geq \exp(-e^{82+6n})$  follows then from (16).  $\square$

### 3 Upper estimates for the expected Betti numbers

#### 3.1 Statement of the results

For every  $1 \leq k \leq n$ , we denote by  $\text{Gr}(k-1, n-1)$  the Grassmann manifold of  $(k-1)$ -dimensional linear subspaces of  $\mathbb{R}^{n-1}$ . The tangent space of  $\text{Gr}(k-1, n-1)$  at every  $H \in \text{Gr}(k-1, n-1)$  is canonically isomorphic to the space of linear maps  $L(H, H^\perp)$  from  $H$  to its orthogonal  $H^\perp$  and we equip it with the norm

$$A \in L(H, H^\perp) \mapsto \|A\|_2 = \sqrt{\text{Tr}(A^*A)} \in \mathbb{R}^+.$$

The total volume of  $\text{Gr}(k-1, n-1)$  for this Riemannian metric is denoted by  $\text{Vol}(\text{Gr}(k-1, n-1))$  and we set

$$V_{k-1, n-1} = \frac{1}{\sqrt{\pi}^{(k-1)(n-k)}} \text{Vol}(\text{Gr}(k-1, n-1))$$

its volume for the rescaled metric  $A \in L(H, H^\perp) \mapsto \frac{1}{\sqrt{\pi}}\|A\|_2$ . Likewise, we equip  $M_{k-1}(\mathbb{R})$  with the Euclidean norm  $A \in M_{k-1}(\mathbb{R}) \mapsto \|A\|_2 = \sqrt{\text{Tr}(A^*A)}$  and set  $d\mu(A) = \frac{1}{\pi^{k-1}} e^{-\|A\|_2^2} dA$  the associated Gaussian measure on  $M_{k-1}(\mathbb{R})$ . Then, we set

$$E_{k-1}(|\det|^{n-k+2}) = \int_{M_{k-1}(\mathbb{R})} |\det A|^{n-k+2} d\mu(A).$$

**Remark 3.1.1** 1. The orthogonal group  $O_{n-1}(\mathbb{R})$  acts transitively on the Grassmannian  $\text{Gr}(k-1, n-1)$  with fixators isomorphic to  $O_{k-1}(\mathbb{R}) \times O_{n-k}(\mathbb{R})$ . We deduce that

$$\begin{aligned} \text{Vol}(\text{Gr}(k-1, n-1)) &= \text{Vol}(O_{n-1}(\mathbb{R})) / (\text{Vol}(O_{k-1}(\mathbb{R})) \times \text{Vol}(O_{n-k}(\mathbb{R}))) \\ &= \binom{n-1}{k-1} \sqrt{\pi}^{(k-1)(n-k)} \frac{\prod_{j=1}^{k-1} \Gamma(1+j/2)}{\prod_{j=n-k+1}^{n-1} \Gamma(1+j/2)}, \end{aligned}$$

where  $\Gamma$  denotes the Gamma function of Euler, see for example Lemma 3.4 of [14].

2. From formula (15.4.12) of [23] follows that

$$E_{k-1}(|\det|^{n-k+2}) = \prod_{j=1}^{k-1} \frac{\Gamma(\frac{n-k+2+j}{2})}{\Gamma(\frac{j}{2})},$$

so that  $V_{k-1, n-1} E_{k-1}(|\det|^{n-k+2}) = \frac{(n-1)!}{(n-k)! 2^{k-1}}$ .

We now keep the framework of §2.1. Let us denote, for every  $i \in \{0, \dots, n-k\}$ , by  $b_i(\mathbb{R}C_\sigma, \mathbb{R}) = \dim H_i(\mathbb{R}C_\sigma, \mathbb{R})$  the  $i$ -th Betti number of  $\mathbb{R}C_\sigma$  and by

$$m_i(\mathbb{R}C_\sigma) = \inf_{f \text{ Morse on } \mathbb{R}C_\sigma} |\text{Crit}_i(f)|$$

its  $i$ -th Morse number, where  $|\text{Crit}_i(f)|$  denotes the number of critical points of index  $i$  of  $f$ . We then denote by

$$\mathbb{E}(b_i) = \int_{\mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_d} b_i(\mathbb{R}C_\sigma, \mathbb{R}) d\mu_{\mathbb{R}}(\sigma)$$

and

$$\mathbb{E}(m_i) = \int_{\mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_d} m_i(\mathbb{R}C_\sigma) d\mu_{\mathbb{R}}(\sigma)$$

their expected values. The aim of §3 is to prove the following Theorem 3.1.2, see (1) for the definition of  $e_{\mathbb{R}}(i, n-k-i)$ .

**Theorem 3.1.2** *Let  $X$  be a smooth real projective manifold of dimension  $n$ ,  $(L, h_L)$  be a real holomorphic Hermitian line bundle of positive curvature over  $X$  and  $(E, h_E)$  be a rank  $k$  real holomorphic Hermitian vector bundle, with  $1 \leq k \leq n-1$ . Then, for every  $0 \leq i \leq n-k$ ,*

$$\limsup_{d \rightarrow \infty} \frac{1}{\sqrt{d}^n} \mathbb{E}(m_i) \leq \frac{1}{\Gamma(\frac{k}{2})} V_{k-1, n-1} E_{k-1} (|\det|^{n-k+2}) e_{\mathbb{R}}(i, n-k-i) \text{Vol}_{h_L}(\mathbb{R}X).$$

Note that the case  $k = n$  is covered by Theorems 1.1.1 and 3.1.3. When  $k = 1$  and  $E = \mathcal{O}_X$ ,  $\text{Vol}_{FS}(\mathbb{R}P^k) = \sqrt{\pi}$ , see Remark 2.14 of [14], so that Theorem 3.1.2 reduces to Theorem 1.0.1 of [14] in this case. The proof of Theorem 3.1.2 actually goes along the same lines as the one of Theorem 1.1 of [14]. The strategy goes as follows. We fix a Morse function  $p : \mathbb{R}X \rightarrow \mathbb{R}$ . Then, almost surely on  $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$ , the restriction of  $p$  to  $\mathbb{R}C_\sigma$  is itself a Morse function. For  $i \in \{0, \dots, n-k\}$ , we denote by  $\text{Crit}_i(p|_{\mathbb{R}C_\sigma})$  the set of critical points of index  $i$  of this restriction and set

$$\nu_i(\mathbb{R}C_\sigma) = \frac{1}{\sqrt{d}^n} \sum_{x \in \text{Crit}_i(p|_{\mathbb{R}C_\sigma})} \delta_x$$

if  $n > k$  and  $\nu_0(\mathbb{R}C_\sigma) = \frac{1}{\sqrt{d}^n} \sum_{x \in \mathbb{R}C_\sigma} \delta_x$  if  $k = n$ . We then set

$$\mathbb{E}(\nu_i) = \int_{\mathbb{R}H^0(X, E \otimes L^d)} \nu_i(\mathbb{R}C_\sigma) d\mu_{\mathbb{R}}(\sigma)$$

and prove the following equidistribution result (compare Theorem 1.2 of [14]).

**Theorem 3.1.3** *Under the hypotheses of Theorem 3.1.2, let  $p : \mathbb{R}X \rightarrow \mathbb{R}$  be a Morse function. Then, for every  $i \in \{0, \dots, n-k\}$ , the measure  $\mathbb{E}(\nu_i)$  weakly converges to*

$$\frac{1}{\Gamma(\frac{k}{2})} V_{k-1, n-1} E_{k-1} (|\det|^{n-k+2}) e_{\mathbb{R}}(i, n-k-i) d\text{vol}_{h_L}$$

as  $d$  grows to infinity. When  $k = n$ ,  $\mathbb{E}(\nu_0)$  converges weakly to  $\frac{1}{\sqrt{\pi}} \Gamma(\frac{n+1}{2}) d\text{vol}_{h_L}$ .

In Theorem 3.1.3  $dvol_{h_L}$  denotes the Lebesgue measure of  $\mathbb{R}X$  induced by the Kähler metric. Theorem 3.1.2 is deduced from Theorem 3.1.3 by integration of 1 over  $\mathbb{R}X$ . The next paragraphs are devoted to the proof of Theorem 3.1.3.

**Proof of Theorem 1.1.1.** It follows from Theorem 3.1.2, the Morse inequalities, Remark 3.1.1 and the computation  $Vol_{FS}\mathbb{R}P^n = \sqrt{\pi}/\Gamma(\frac{n+1}{2})$  (see Remark 2.14 of [14]) when  $k \leq n-1$  and from Theorem 3.1.3 when  $k = n$ .  $\square$

### 3.2 Incidence varieties

Under the hypotheses of Theorem 3.1.3, we set

$$\mathbb{R}\Delta_p^d = \{\sigma \in \mathbb{R}H^0(X, E \otimes L^d) \mid \sigma \in \mathbb{R}\Delta_d \text{ or } p|_{\mathbb{R}C_\sigma} \text{ is not Morse}\}$$

and

$$\mathcal{I}_i = \{(\sigma, x) \in (\mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_p^d) \times (\mathbb{R}X \setminus \text{Crit}(p)) \mid x \in \text{Crit}_i(p|_{\mathbb{R}C_\sigma})\}.$$

We set

$$\pi_1 : (\sigma, x) \in \mathcal{I}_i \mapsto \sigma \in \mathbb{R}H^0(X, E \otimes L^d) \text{ and} \quad (31)$$

$$\pi_2 : (\sigma, x) \in \mathcal{I}_i \mapsto x \in \mathbb{R}X. \quad (32)$$

Then, for every  $(\sigma_0, x_0) \in ((\mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_p^d) \times (\mathbb{R}X \setminus \text{Crit}(p)))$ ,  $\pi_1$  is invertible in a neighbourhood  $\mathbb{R}U$  of  $\sigma_0$ , defining an evaluation map at the critical point

$$ev_{(\sigma_0, x_0)} : \sigma \in \mathbb{R}U \mapsto \pi_2 \circ \pi_1^{-1}(\sigma) = x \in \text{Crit}_i(p|_{\mathbb{R}C_\sigma}) \cap \mathbb{R}V,$$

where  $\mathbb{R}V$  denotes a neighbourhood of  $x_0$  in  $\mathbb{R}X$ , compare §2.4.2 of [14]. We denote by  $d|_{\sigma_0} ev_{(\sigma_0, x_0)}^\perp$  the restriction of its differential map  $d|_{\sigma_0} ev_{(\sigma_0, x_0)}$  at  $\sigma_0$  to the orthogonal complement of  $\pi_1(\pi_2^{-1}(x_0))$  in  $\mathbb{R}H^0(X, E \otimes L^d)$ .

**Proposition 3.2.1** *Under the hypotheses of Theorem 3.1.3,*

$$\mathbb{E}(\nu_i) = \frac{1}{\sqrt{d}^n} (\pi_2)_* (\pi_1^* d\mu_{\mathbb{R}}).$$

Moreover, at every point  $x \in \mathbb{R}X \setminus \text{Crit}(p)$ ,

$$(\pi_2)_* (\pi_1^* d\mu_{\mathbb{R}})|_x = \frac{1}{\sqrt{\pi}^n} \int_{\pi_1(\pi_2^{-1}(x))} |\det d|_{\sigma} ev_{(\sigma, x)}^\perp|^{-1} d\mu_{\mathbb{R}}(\sigma) dvol_{h_L}.$$

**Proof.** The proof is the same as in the one of Proposition 2.10 of [14] and is not reproduced here.  $\square$

Fix  $x \in \mathbb{R}X \setminus \text{Crit}(p)$ . Then  $\pi_1(\pi_2^{-1}(x))$  is open in a subspace of  $\mathbb{R}H^0(X, E \otimes L^d)$ . Namely,

$$\pi_1(\pi_2^{-1}(x)) = \{\sigma \in \mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_p^d \mid \sigma(x) = 0 \text{ and} \quad (33)$$

$$\exists \lambda \in \mathbb{R}(E \otimes L^d)_{|x}^*, \lambda \circ \nabla|_x \sigma = d|_x p\}, \quad (34)$$

where  $\mathbb{R}((E \otimes L^d)_x^*)$  is the real part of the fibre  $(E \otimes L^d)_x^*$ . We deduce a well-defined map

$$\rho_x : \pi_1(\pi_2^{-1}(x)) \rightarrow \text{Gr}(n-k, \ker d_{|x}p) \times (\mathbb{R}(E \otimes L^d)_x^* \setminus \{0\}) \quad (35)$$

$$\sigma \mapsto (\ker \nabla_{|x}\sigma, \lambda). \quad (36)$$

For every  $\sigma \in \mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_p^d$ , the tangent space of  $\pi_1(\pi_2^{-1}(x))$  at  $\sigma$  reads

$$\begin{aligned} T_\sigma \pi_1(\pi_2^{-1}(x)) &= \{ \dot{\sigma} \in \mathbb{R}H^0(X, E \otimes L^d) \mid \dot{\sigma}(x) = 0 \text{ and} \\ &\quad \exists \dot{\lambda} \in \mathbb{R}(E \otimes L^d)_x^* \mid \dot{\lambda} \circ \nabla_{|x}\sigma + \lambda \circ \nabla_{|x}\dot{\sigma} = 0 \}. \end{aligned}$$

Likewise, for every  $\lambda \in \mathbb{R}(E \otimes L^d)_x^* \setminus \{0\}$ , the tangent space of  $\rho_x^{-1}(\text{Gr}(n-k, \ker d_{|x}p) \times \{\lambda\})$  at  $\sigma$  reads

$$T_\sigma \rho_x^{-1}(\text{Gr}(n-k, \ker d_{|x}p) \times \{\lambda\}) = \{ \dot{\sigma} \in \mathbb{R}H^0(X, E \otimes L^d) \mid \dot{\sigma}(x) = 0 \text{ and } \lambda \circ \nabla_{|x}\dot{\sigma} = 0 \}.$$

Finally, for every  $K \in \text{Gr}(n-k, \ker d_{|x}p)$ , the tangent space of  $\rho_x^{-1}(K, \lambda)$  at  $\sigma$  reads

$$T_\sigma \rho_x^{-1}(K, \lambda) = \{ \dot{\sigma} \in \mathbb{R}H^0(X, E \otimes L^d) \mid \dot{\sigma}(x) = 0, \nabla_{|x}\dot{\sigma}|_K = 0 \text{ and } \lambda \circ \nabla_{|x}\dot{\sigma} = 0 \}.$$

Let us choose local real holomorphic coordinates  $(x_1, \dots, x_n)$  of  $X$  near  $x$  such that  $(\partial/\partial x_1, \dots, \partial/\partial x_n)$  be orthonormal at  $x$ , with  $d_{|x}p$  being colinear to  $dx_1$  and such that  $K = \ker \nabla_{|x}\sigma = \langle \partial/\partial x_{k+1}, \dots, \partial/\partial x_n \rangle$ . Let us choose a local real holomorphic trivialization  $(e_1, \dots, e_k)$  of  $E$  near  $x$  that is orthonormal at  $x$  and be such that  $\ker \lambda_{|x} = \langle e_2 \otimes e^d, \dots, e_k \otimes e^d \rangle_{|x}$ . For  $d$  large enough, we define the following subspaces of  $\mathbb{R}H^0(X, E \otimes L^d)$ :

$$H_x = \langle (\sigma_0^i)_{1 \leq i \leq k}, (\sigma_j^1)_{k+1 \leq j \leq n} \rangle \quad (37)$$

$$H_\lambda = \langle (\sigma_j^1)_{1 \leq j \leq k} \rangle \quad (38)$$

$$H_K = \langle (\sigma_j^i)_{\substack{2 \leq i \leq k \\ k+1 \leq j \leq n}} \rangle, \quad (39)$$

where the sections  $(\sigma_0^i)_{1 \leq i \leq k}$  and  $(\sigma_j^i)_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}}$  of  $\mathbb{R}H^0(X, E \otimes L^d)$  are given by Lemma 2.3.3 and Definition 2.3.4.

$H_K$  is a complement of  $T_\sigma \rho_x^{-1}(K, \lambda)$  in  $T_\sigma \rho_x^{-1}(\text{Gr}(n-k, \ker d_{|x}p) \times \{\lambda\})$ ,  $H_\lambda$  is a complement of  $T_\sigma \rho_x^{-1}(\text{Gr}(n-k, \ker d_{|x}p) \times \{\lambda\})$  in  $T_\sigma \pi_1(\pi_2^{-1}(x))$  and  $H_x$  is a complement of  $T_\sigma \pi_1(\pi_2^{-1}(x))$  in  $\mathbb{R}H^0(X, E \otimes L^d)$ . Then, from Lemmas 2.3.6 and 2.3.7, up to a uniform rescaling by  $\sqrt{v(x)}$ , these complements are asymptotically orthogonal and their given basis orthonormal. Hence, we can assume from now on that  $v = 1$ .

**Lemma 3.2.2** *Under the hypotheses of Theorem 3.1.3, let  $(\sigma, x) \in \mathcal{I}_i$  and  $\lambda \in \mathbb{R}(E \otimes L^d)_x^* \setminus \{0\}$  such that  $\lambda \circ \nabla_{|x}\sigma = d_{|x}p$ . Then,  $\lambda \circ \nabla^2 \sigma|_{K_x} = \nabla^2(p|_{\mathbb{R}C_\sigma})_{|x}$ , so that the quadratic form  $\lambda \circ \nabla^2 \sigma|_{K_x}$  is non-degenerated of index  $i$ .*

**Proof.** The proof is similar to the one of Lemma 2.9 of [14].  $\square$

### 3.3 Computation of the Jacobian determinants

#### 3.3.1 Jacobian determinant of $\rho_x$

Under the hypotheses of Theorem 3.1.3, let  $(\sigma, x) \in \mathcal{I}_i$ . We set  $(K, \lambda) = \rho_x(\sigma)$  and denote by  $d_{|\sigma\rho_x^H}$  the restriction of  $d_{|\sigma\rho_x}$  to  $H_K \oplus H_\lambda$ . We then denote by  $\det(d_{|\sigma\rho_x^H})$  the Jacobian determinant of  $d_{|\sigma\rho_x^H}$  computed in the given basis of  $H_\lambda$  and  $H_K$ , see (38), (39) and in orthonormal basis of  $T_K \text{Gr}(n-k, \ker d_{|x}p) \times \mathbb{R}(E \otimes L^d)_{|x}^*$ . By assumption, the operator  $\nabla_{|x}\sigma$  does not depend on the choice of a connection  $\nabla$  on  $E \otimes L^d$  and is onto. We denote by  $\nabla_{|x}\sigma^\perp$  its restriction to the orthogonal  $K^\perp$  of  $K = \ker \nabla_{|x}\sigma$ ,

$$\nabla_{|x}\sigma^\perp : K^\perp \rightarrow \mathbb{R}(E \otimes L^d)_{|x}.$$

Likewise, for every  $(\dot{\sigma}_K, \dot{\sigma}_\lambda) \in H_K \oplus H_\lambda$ , the operators  $\nabla_{|x}\dot{\sigma}_K$  and  $\nabla_{|x}\dot{\sigma}_\lambda$  do not depend on the choice of a connection  $\nabla$  on  $E \otimes L^d$ . Finally, we write at a point  $y \in \mathbb{R}X$  near  $x$

$$\sigma(y) = \sum_{i=1}^k (a_0^i \sigma_0^i + \sum_{j=1}^n a_j^i \sigma_j^i + \sum_{1 \leq l \leq m \leq n} a_{lm}^i \sigma_{lm}^i)(y) + o(|y|^2),$$

where  $(a_0^i)$ ,  $(a_j^i)$  and  $(a_{lm}^i)$  are real numbers and  $(\sigma_0^i)$ ,  $(\sigma_j^i)$  and  $(\sigma_{lm}^i)$  are given by Definition 2.3.4. From Lemma 2.3.5 and (33), we deduce that  $a_0^i = 0 = a_j^i$  for  $1 \leq i \leq k$  and  $k+1 \leq j \leq n$ , and that

$$\|\lambda\| \sqrt{\pi \delta_L} \sqrt{d}^{n+1} |a_1^1| = \|d_{|x}p\| + o(1), \quad (40)$$

where the  $o(1)$  term is uniformly bounded over  $\mathbb{R}X$ .

**Lemma 3.3.1** *Under the hypotheses of Theorem 3.1.3, let  $(\sigma, x) \in \mathcal{I}_i$  and  $(K, \lambda) = \rho_x(\sigma)$ . Then,  $d_{|x}\rho_x^H$  writes*

$$\begin{aligned} H_K \oplus H_\lambda &\rightarrow T_K \text{Gr}(n-k, \ker d_{|x}p) \times \mathbb{R}(E \otimes L^d)_{|x}^* \\ (\dot{\sigma}_K, \dot{\sigma}_\lambda) &\mapsto \left( -(\nabla_{|x}\sigma^\perp)_{|\ker \lambda}^{-1} \circ \nabla_{|x}\dot{\sigma}_K|_K, -\lambda \circ \nabla_{|x}\dot{\sigma}_\lambda \circ (\nabla_{|x}\sigma^\perp)^{-1} \right). \end{aligned}$$

Moreover,  $|\det d_{|\sigma\rho_x^H}|^{-1} = \frac{|a_1^1|}{\|\lambda\|^k} |\det(a_j^i)_{2 \leq i, j \leq k}|^{n-k+1} (1 + o(1))$ , where the  $o(1)$  term is uniformly bounded over  $\mathbb{R}X$ .

**Proof.** Let  $(\dot{\sigma}_K, \dot{\sigma}_\lambda) \in H_K \oplus H_\lambda$  and  $(\sigma_s)_{s \in ]-\epsilon, \epsilon[}$  be a path of  $\pi_1(\pi_2^{-1}(x))$  such that  $\sigma_0 = \sigma$  and  $\dot{\sigma}_0 = \dot{\sigma}_K + \dot{\sigma}_\lambda$ . Then, for every  $s \in ]-\epsilon, \epsilon[$  and every  $v_s \in \ker \nabla_{|x}\sigma_s$ , there exists  $\lambda_s \in \mathbb{R}(E \otimes L^d)_{|x}^*$  such that

$$\begin{cases} \nabla_{|x}\sigma_s(v_s) &= 0 & \text{and} \\ \lambda_s \circ \nabla_{|x}\sigma_s &= d_{|x}p. \end{cases}$$

By derivation, we deduce

$$\begin{cases} \nabla_{|x}\dot{\sigma}_0(v_0) + \nabla_{|x}\sigma(\dot{v}_0) &= 0 & \text{and} \\ \dot{\lambda}_0 \circ \nabla_{|x}\sigma + \lambda_0 \circ \nabla_{|x}\dot{\sigma}_0 &= 0. \end{cases}$$

By setting  $\dot{v}$  the orthogonal projection of  $\dot{v}_0$  onto  $K^\perp$ , we deduce that

$$\begin{cases} \dot{v} &= -(\nabla_{|x}\sigma^\perp)^{-1} \circ \nabla_{|x}\dot{\sigma}_K(v_0) & \text{and} \\ \dot{\lambda}_0 &= -\lambda_0 \circ \nabla_{|x}\dot{\sigma}_\lambda \circ (\nabla_{|x}\sigma^\perp)^{-1}. \end{cases}$$

The first part of Lemma 3.3.1 follows. Now, recall that  $d_{|x}p$  is colinear to  $dx_1$ , that  $K$  is equipped with the orthonormal basis  $(\partial/\partial x_{k+1}, \dots, \partial/\partial x_n)$ ,  $K^\perp$  with the orthonormal basis  $(\partial/\partial x_1, \dots, \partial/\partial x_k)$ , and that  $\ker \lambda|_x$  is spanned by the orthonormal basis  $(e_2, \dots, e_k)|_x$ . From Lemma 2.3.3, the map

$$\dot{\sigma}_K \in H_K \mapsto \nabla_{|x} \dot{\sigma}_{K|K} \in L(K, \ker \lambda)$$

just dilates the norm by the factor  $\sqrt{\pi \delta_L d^{n+1}}(1 + o(1))$ , where the  $o(1)$  term is uniformly bounded over  $\mathbb{R}X$ . Now, since the matrix of the restriction of  $\nabla_{|x} \sigma^\perp$  to  $K^\perp \cap \ker d_{|x}p$  in the given basis of  $K^\perp \cap \ker d_{|x}p$  and  $\ker \lambda$  equals

$$\sqrt{\pi \delta_L d^{n+1}}(a_j^i)_{2 \leq i, j \leq k} + o(\sqrt{d^{n+1}}),$$

where the  $o(\sqrt{d^{n+1}})$  term is uniformly bounded over  $\mathbb{R}X$ . We deduce that the Jacobian determinant of the map

$$M \in L(K, \ker \lambda) \mapsto (\nabla_{|x} \sigma^\perp|_{\ker \lambda})^{-1} \circ M \in L(K, K^\perp \cap \ker d_{|x}p)$$

equals

$$((\sqrt{\pi \delta_L d^{n+1}})^{k-1} |\det(a_j^i)_{2 \leq i, j \leq k}| (1 + o(1)))^{k-n}.$$

The Jacobian determinant of the map

$$\dot{\sigma}_K \in H_K \mapsto (\nabla_{|x} \sigma^\perp|_{\ker \lambda})^{-1} \circ \nabla_{|x} \dot{\sigma}_{K|K} \in T_K \text{Gr}(n - k, \ker d_{|x}p)$$

thus equals  $|\det(a_j^i)_{2 \leq i, j \leq k}|^{k-n} + o(1)$ , where the  $o(1)$  is uniformly bounded over  $\mathbb{R}X$ . Likewise, from Lemma 2.3.3, the map

$$\dot{\sigma}_\lambda \in H_\lambda \mapsto \lambda \circ \nabla_{|x} \dot{\sigma}_\lambda \in (K^\perp)^*$$

just dilates the norm by a factor  $\sqrt{\pi \delta_L d^{n+1}} \|\lambda\| + o(\sqrt{d^{n+1}})$ , where the  $o(\sqrt{d^{n+1}})$  is uniformly bounded over  $\mathbb{R}X$ , while the Jacobian determinant of the map

$$M \in (K^\perp)^* \mapsto M \circ (\nabla_{|x} \sigma^\perp)^{-1} \in \mathbb{R}(E \otimes L^d)|_x^*$$

equals  $(\sqrt{\pi \delta_L} \sqrt{d^{n+1}})^{-k} |\det(a_j^i)_{1 \leq i, j \leq k}|^{-1} (1 + o(1))$  so that the Jacobian determinant of the map

$$\dot{\sigma}_\lambda \in H_\lambda \mapsto \lambda \circ \nabla_{|x} \dot{\sigma}_\lambda \circ (\nabla_{|x} \sigma^\perp)^{-1} \in \mathbb{R}(E \otimes L^d)|_x^*$$

equals  $\|\lambda\|^k |\det(a_j^i)_{1 \leq i, j \leq k}|^{-1} + o(1)$ , with a  $o(1)$  uniformly bounded over  $\mathbb{R}X$ . As a consequence,

$$|\det d_{|\sigma} \rho_x^H|^{-1} = \|\lambda\|^{-k} |\det(a_j^i)_{2 \leq i, j \leq k}|^{n-k+1} |a_1^1| (1 + o(1)),$$

with a  $o(1)$  uniformly bounded over  $\mathbb{R}X$ , since the relation  $\lambda \circ \nabla_{|x} \sigma = d_{|x}p$  implies that  $a_j^1$  vanishes for  $2 \leq j \leq n$ .  $\square$

### 3.3.2 Jacobian determinant of the evaluation map

Again, under the hypotheses of Theorem 3.1.3 and for  $(\sigma, x) \in \mathcal{I}_i$ , we set for every  $y$  in a neighbourhood of  $x$ ,

$$\sigma(y) = \sum_{i=1}^k (a_0^i \sigma_0^i + \sum_{j=1}^n a_j^i \sigma_j^i + \sum_{1 \leq l \leq m \leq n} a_{lm}^i \sigma_{lm}^i)(y) + o(|y|^2), \quad (41)$$

where  $a_0^i$ ,  $a_j^i$  and  $a_{lm}^i$  are real numbers. We then set, for  $1 \leq l, m \leq n$ ,  $\tilde{a}_{ll}^1 = \sqrt{2}a_{ll}^1$ ,  $\tilde{a}_{lm}^1 = a_{lm}^1$  if  $l < m$  and  $\tilde{a}_{lm}^1 = a_{ml}^1$  if  $l > m$ . We denote by  $d_{|\sigma} ev_{(\sigma, x)}^H$  the restriction of  $d_{|\sigma} ev_{(\sigma, x)}$  to  $H_x$ , see (37) and by  $\det d_{|\sigma} ev_{(\sigma, x)}^H$  its Jacobian determinant computed in the given basis of  $H_x$  and orthonormal basis of  $T_x \mathbb{R}X$ .

**Lemma 3.3.2** *Under the hypotheses of Theorem 3.1.3, let  $(\sigma, x) \in \mathcal{I}_i$ . Then,*

$$|\det d_{|\sigma} ev_{(\sigma, x)}^H|^{-1} = \sqrt{\pi^n d^n} |a_1^1| |\det(a_j^i)_{2 \leq i, j \leq k}| |\det(\tilde{a}_{lm}^1)_{k+1 \leq l, m \leq n}| (1 + o(1)),$$

where the  $o(1)$  term has poles of order at most  $n - k$  near the critical points of  $p$ .

**Remark 3.3.3** *In Lemma 3.3.2, a function  $f$  is said to have a pole of order at most  $n - k$  near a point  $x$  if  $r^{n-k} f$  is bounded near  $x$ , where  $r$  denotes the distance function to  $x$ . Such a function thus belongs to  $L^1(\mathbb{R}X, dvol_h)$ .*

**Proof.** We choose a torsion free connection  $\nabla^{TX}$  (resp. a connection  $\nabla^{E \otimes L^d}$ ) on  $\mathbb{R}X \setminus \text{Crit}(p)$  (resp. on  $E \otimes L^d$ ) such that  $\nabla^{TX} dp = 0$ . They induce a connection on  $T^*X \otimes E \otimes L^d$  which makes it possible to differentiate twice the elements of  $\mathbb{R}H^0(X, E \otimes L^d)$ . The tangent space of  $\mathcal{I}_i$  then reads

$$T_{(\sigma, x)} \mathcal{I}_i = \{(\dot{\sigma}, \dot{x}) \in \mathbb{R}H^0(X, E \otimes L^d) \times T_x \mathbb{R}X \mid \dot{\sigma}(x) + \nabla_{\dot{x}} \sigma = 0 \text{ and} \quad (42)$$

$$\exists \dot{\lambda} \in \mathbb{R}(E \otimes L^d)_{|x}^*, \dot{\lambda} \circ \nabla_{|x} \sigma + \lambda \circ \nabla_{|x} \dot{\sigma} + \lambda \circ \nabla_{\dot{x}, \cdot}^2 \sigma = 0\}. \quad (43)$$

Recall that  $T_x \mathbb{R}X$  is the direct sum  $K \oplus K^\perp$ , where  $K = \ker \nabla_{|x} \sigma$ . We write  $\dot{x} = (\dot{x}_K, \dot{x}_{K^\perp})$  the coordinates of  $\dot{x}$  in this decomposition. From the first equation we deduce, keeping the notations of §3.3.1, that  $\dot{x}_{K^\perp} = -(\nabla_{|x} \sigma^\perp)^{-1}(\dot{\sigma}(x))$ . From Lemma 2.3.3, the evaluation map at  $x$

$$\dot{\sigma} \in \langle (\sigma_0^i)_{1 \leq i \leq k} \rangle \mapsto \dot{\sigma}(x) \in E \otimes L_{|x}^d$$

just dilates the norm by a factor  $\sqrt{\delta_L d^n} (1 + o(1))$ , where the  $o(1)$  term is uniformly bounded over  $\mathbb{R}X$ , while

$$|\det(\nabla_{|x} \sigma^\perp)| = (\sqrt{\pi \delta_L d^{n+1}})^k |\det(a_j^i)_{1 \leq i, j \leq k}| (1 + o(1)).$$

We deduce by composition that the Jacobian of the map

$$\dot{\sigma} \in \langle (\sigma_0^i)_{1 \leq i \leq k} \rangle \mapsto \dot{x}_{K^\perp} = -(\nabla_{|x} \sigma^\perp)^{-1}(\dot{\sigma}(x))$$

equals  $(\sqrt{\pi^k d^k} |\det(a_j^i)_{2 \leq i, j \leq k}| |a_1^1|)^{-1} (1 + o(1))$ , where the  $o(1)$  term is uniformly bounded over  $\mathbb{R}X$ . Now, equation (43) restricted to  $K$  reads

$$\lambda \circ \nabla_{\dot{x}_K, \cdot}^2 \sigma|_K = -\lambda \circ \nabla_{|x} \dot{\sigma}|_K.$$



From Lemma 2.3.3, the map

$$\dot{\sigma} \in \langle (\sigma_j^1)_{k+1 \leq j \leq n} \rangle \mapsto -\lambda \circ \nabla_{|x} \dot{\sigma}|_K \in K^*$$

just dilates the norm by a factor  $\|\lambda\| \sqrt{\pi \delta_L d^{n+1}} (1 + o(1))$ , with  $o(1)$  term is uniformly bounded over  $\mathbb{R}X$ . Likewise, from Lemma 2.3.3, the Jacobian of the map  $\lambda \circ \nabla^2 \sigma|_K : K \rightarrow K^*$  equals

$$(\|\lambda\| \pi \sqrt{\delta_L d^{n+2}})^{n-k} |\det(\tilde{a}_{lm}^1)_{k+1 \leq l, m \leq n}| (1 + o(1)). \quad (44)$$

Here, the  $o(1)$  term is no more uniformly bounded over  $\mathbb{R}X$  though. Indeed, from Lemma 2.3.5 and (41),

$$\lambda \circ \nabla^2 \sigma|_K = a_1^1 (\|\lambda\| \sqrt{\pi \delta_L d^{n+1}}) (\nabla^{TX} dx_1) + \sum_{1 \leq l \leq m \leq n} \tilde{a}_{lm}^1 (\|\lambda\| \sqrt{\pi \delta_L d^{n+2}}) dx_l \otimes dx_m,$$

since the relation  $\lambda \circ \nabla_{|x} \sigma = d_{|x} p$  imposes that  $a_j^1$  vanishes for  $j > 1$ . Moreover, since  $dp = \sum_{i=1}^n \alpha_i dx_i$ , with  $\alpha_2(x) = \dots = \alpha_n(x) = 0$  and  $|\alpha_1(x)| = \|d_{|x} p\|$ , we get that

$$0 = \nabla^{TX} (dp)|_K = \alpha_1 (\nabla^{TX} dx_1)|_K + \sum_{i=1}^n (d\alpha_i \otimes dx_i)|_K,$$

so that  $\|\nabla^{TX} dx_1|_K\| = \frac{1}{\|d_{|x} p\|} \|\sum_{i=1}^n d\alpha_i \otimes dx_i\|$  has a pole of order one at  $x$ . In formula (44), the  $o(1)$  term has thus a pole of order at most  $n - k$  near the critical points of  $p$ .

We deduce that the Jacobian determinant of the map

$$\dot{\sigma} \in \langle (\sigma_j^1)_{k+1 \leq j \leq n} \rangle \mapsto \dot{x}_K = -(\lambda \circ \nabla^2 \sigma|_K)^{-1} \circ (\lambda \circ \nabla_{|x} \dot{\sigma}|_K) \in K$$

equals  $(\sqrt{\pi^{n-k} d^{n-k}} |\det(\tilde{a}_{lm}^1)_{k+1 \leq l, m \leq n}|)^{-1} (1 + o(1))$ , up to sign, where  $o(1)$  term has a pole of order at most  $n - k$  near the critical points of  $p$ . The result follows.  $\square$

### 3.4 Proof of Theorem 3.1.3

#### 3.4.1 The case $k < n$

From Proposition 2.4.1 we know that

$$\mathbb{E}(\nu_i) = \frac{1}{\sqrt{\pi^n d^n}} \left( \int_{\pi_1(\pi_2^{-1}(x))} |\det d_{|\sigma} ev_{(\sigma, x)}^\perp|^{-1} d\mu_{\mathbb{R}}(\sigma) \right) dvol_{h_L}.$$

From the coarea formula (see [10]), we likewise deduce that

$$\begin{aligned} \mathbb{E}(\nu_i) &= \frac{1}{\sqrt{\pi^n d^n}} \left( \int_{\text{Gr}(n-k, \ker d_{|x} p) \times \mathbb{R}(E \otimes L^d)_x^* \setminus \{0\}} e^{-(a_1^i)^2} \frac{dK \wedge d\lambda}{\sqrt{\pi}^{(n-k)(k-1)+k}} \dots \right. \\ &\quad \left. \dots \int_{\rho_x^{-1}(K, \lambda)} |\det d_{|\sigma} ev_{(\sigma, x)}^\perp|^{-1} |\det d_{|\sigma} \rho_x^\perp|^{-1} d\mu_{\mathbb{R}}(\sigma) \right) dvol_{h_L}, \end{aligned}$$

since with the notations (41),  $\sigma \in \rho_x^{-1}(K, \lambda)$  if and only if  $\forall i \in \{1, \dots, k\}$  and  $\forall j \in \{k+1, \dots, n\}$ ,  $a_0^i = 0 = a_j^i$  while  $\forall j \geq 2$ ,  $a_j^1 = 0$  and  $|a_1^1| = \frac{\|d_{|x} p\|}{\|\lambda\| \sqrt{\pi \delta_L} \sqrt{d^{n+1}}}$ . From

Lemma 2.3.6 and the relation (40), we deduce that for every  $x \in \mathbb{R}X \setminus \text{Crit}(p)$  and every  $(K, \lambda) \in \text{Gr}(n-k, \ker d|_x p) \times \mathbb{R}(E \otimes L^d)|_x^* \setminus \{0\}$ ,

$$\begin{aligned} & \int_{\rho_x^{-1}(K, \lambda)} |\det d|_\sigma ev_{(\sigma, x)}^\perp|^{-1} |\det d|_\sigma \rho_x^\perp|^{-1} d\mu_{\mathbb{R}}(\sigma) \\ & \stackrel{\sim}{d \rightarrow \infty} \int_{\rho_x^{-1}(K, \lambda)} |\det d|_\sigma ev_{(\sigma, x)}^H|^{-1} |\det d|_\sigma \rho_x^H|^{-1} d\mu_{\mathbb{R}}(\sigma). \end{aligned}$$

Thus, from Lemmas 3.3.1, 3.3.2 and 3.2.2,  $\mathbb{E}(\nu_i)$  converges to

$$\begin{aligned} & \int_{M_{k-1}(\mathbb{R})} |\det(a_j^i)_{2 \leq i, j \leq k}|^{n-k+2} d\mu(a_j^i) \int_{\text{Sym}_{\mathbb{R}}(i, n-k-i)} |\det(\tilde{a}_{lm}^1)_{k+1 \leq l, m \leq n}| d\mu(\tilde{a}_{lm}^1) \dots \\ & \dots \int_{\text{Gr}(n-k, \ker d|_x p) \times \mathbb{R}(E \otimes L^d)|_x^* \setminus \{0\}} \frac{(a_1^1)^2 e^{-(a_1^1)^2}}{\|\lambda\|^k} \frac{dK \wedge d\lambda}{\sqrt{\pi}^{(n-k)(k-1)+k}}, \end{aligned}$$

where the convergence is dominated by a function in  $L^1(\mathbb{R}X, d\text{vol}_{h_L})$ , see Remark 3.3.3. We deduce that  $\mathbb{E}(\nu_i)$  gets equivalent to

$$\frac{\|d|_x p\|^2}{\delta_L d^{n+1} \sqrt{\pi}^{k+2}} V_{k-1, n-1} E_{k-1} (|\det|^{n-k+2}) e_{\mathbb{R}}(i, n-k-i) \left( \int_{\mathbb{R}(E \otimes L^d)|_x^* \setminus \{0\}} \frac{e^{-(a_1^1)^2}}{\|\lambda\|^{k+2}} d\lambda \right) d\text{vol}_{h_L}.$$

Now,

$$\begin{aligned} \frac{\|d|_x p\|^2}{\pi \delta_L d^{n+1}} \int_{\mathbb{R}(E \otimes L^d)|_x^* \setminus \{0\}} \frac{e^{-(a_1^1)^2}}{\|\lambda\|^{k+2}} d\lambda &= \frac{\text{Vol}(S^{k-1}) \|d|_x p\|^2}{\pi \delta_L d^{n+1}} \int_0^{+\infty} \frac{e^{-r^2}}{\| \lambda \|^3} d\| \lambda \| \\ &= \text{Vol}(S^{k-1}) \int_0^{+\infty} e^{-r^2} r dr = \frac{1}{2} \text{Vol}(S^{k-1}). \end{aligned}$$

Since  $\text{Vol}(S^{k-1}) = \frac{2\sqrt{\pi}^k}{\Gamma(k/2)}$ , we finally deduce that  $\mathbb{E}(\nu_i)$  weakly converges to

$$\frac{1}{\Gamma(k/2)} V_{k-1, n-1} E_{k-1} (|\det|^{n-k+2}) e_{\mathbb{R}}(i, n-k-i) d\text{vol}_{h_L},$$

where the convergence is dominated by a function in  $L^1(\mathbb{R}X, d\text{vol}_{h_L})$ .  $\square$

### 3.4.2 The case $k = n$

When the rank of  $E$  equals the dimension of  $X$ , the vanishing locus of a generic section  $\sigma$  of  $\mathbb{R}H^0(X, E \otimes L^d)$  is a finite set of points. We set  $\nu = \frac{1}{\sqrt{d}^n} \sum_{x \in \mathbb{R}C_\sigma} \delta_x$ , and define the incidence variety as

$$\mathcal{I} = \{(\sigma, x) \in (\mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_d) \times \mathbb{R}X \mid \sigma(x) = 0\}.$$

The projections  $\pi_1$  and  $\pi_2$  are defined by (31) and (32). As before, for every  $(\sigma_0, x_0) \in (\mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_d) \times \mathbb{R}X$ ,  $\pi_1$  is invertible in a neighbourhood  $\mathbb{R}U$  of  $\sigma_0$ , defining an evaluation map at the critical point

$$ev_{(\sigma_0, x_0)} : \sigma \in \mathbb{R}U \mapsto \pi_2 \circ \pi_1^{-1}(\sigma) = x \in \mathbb{R}C_\sigma \cap \mathbb{R}V,$$

where  $\mathbb{R}V$  denotes a neighbourhood of  $x_0$  in  $\mathbb{R}X$ , compare §2.4.2 of [14]. We denote by  $d_{|\sigma_0} ev_{(\sigma_0, x_0)}^\perp$  the restriction of its differential map  $d_{|\sigma_0} ev_{(\sigma_0, x_0)}$  at  $\sigma_0$  to the orthogonal complement of  $\pi_1(\pi_2^{-1}(x_0))$  in  $\mathbb{R}H^0(X, E \otimes L^d)$ . Then, from Proposition 3.2.1,

$$\mathbb{E}(\nu) = \frac{1}{\sqrt{d^n}} (\pi_2)_* (\pi_1^* d\mu_{\mathbb{R}})|_x = \frac{1}{\sqrt{\pi d^n}} \int_{\pi_1(\pi_2^{-1}(x))} |\det d_{|\sigma} ev_{(\sigma, x)}^\perp|^{-1} d\mu_{\mathbb{R}}(\sigma) dvol_{h_L}.$$

The space  $H_x = \langle (\sigma_0^i)_{1 \leq i \leq k} \rangle$  is a complement to  $T_\sigma \pi_1(\pi_2^{-1}(x))$  in  $\mathbb{R}H^0(X, E \otimes L^d)$  and in the decomposition (41),  $a_0^i = 0$  for every  $i = 1, \dots, k$ . The tangent space of  $\mathcal{I}$  at  $(\sigma, x)$  reads

$$T_{(\sigma, x)} \mathcal{I} = \{(\dot{\sigma}, \dot{x}) \in \mathbb{R}H^0(X, E \otimes L^d) \times T_x \mathbb{R}X \mid \dot{\sigma}(x) + \nabla_{|x} \sigma(\dot{x}) = 0\}.$$

As in the proof of Lemma 3.3.2, we deduce that the Jacobian determinant of the map

$$\dot{\sigma} \in H_x \mapsto \dot{x} = -(\nabla_{|x} \sigma^\perp)^{-1}(\dot{\sigma}(x)) \in T_x \mathbb{R}X$$

equals  $\sqrt{\pi^n d^n} |\det(a_j^i)_{1 \leq i, j \leq n}|(1 + o(1))$ , so that

$$|\det d_{|\sigma} ev_{(\sigma, x)}^H|^{-1} = \sqrt{\pi d^n} |\det(a_j^i)_{1 \leq i, j \leq n}|(1 + o(1)),$$

where the  $o(1)$  term is uniformly bounded over  $\mathbb{R}X$ . From lemma 2.3.6 we deduce that  $\mathbb{E}(\nu)$  gets equivalent to

$$\left( \int_{M_n(\mathbb{R})} |\det(a_j^i)_{1 \leq i, j \leq n}| d\mu(a_j^i) \right) dvol_{h_L} = E_n(|\det|) dvol_{h_L}.$$

Formula (15.4.12) of [23], see Remark 3.1.1, now gives

$$E_n(|\det|) = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(1/2)} = \frac{1}{Vol_{FS}(\mathbb{R}P^n)},$$

see Remark 2.14 of [14], hence the result.  $\square$

### 3.5 Equidistribution of critical points in the complex case

Let  $X$  be a smooth complex projective manifold of dimension  $n$ ,  $(L, h_L)$  be a holomorphic Hermitian line bundle of positive curvature  $\omega$  over  $X$  and  $(E, h_E)$  be a rank  $k$  holomorphic Hermitian vector bundle, with  $1 \leq k \leq n$ . For every  $d > 0$ , we denote by  $L^d$  the  $d$ th tensor power of  $L$  and by  $h^d$  the induced Hermitian metric on  $L^d$ . We denote by  $H^0(X, L^d)$  its complex vector space of global holomorphic sections and by  $N_d$  the dimension of  $H^0(X, L^d)$ . We denote then by  $\langle \cdot, \cdot \rangle$  the  $L^2$ -Hermitian product on this vector space, defined by the relation

$$\forall \sigma, \tau \in H^0(X, L^d), \langle \sigma, \tau \rangle = \int_X h^d(\sigma, \tau) dx. \quad (45)$$

The associated Gaussian measure is denoted by  $\mu_{\mathbb{C}}$ . It is defined, for every open subset  $U$  of  $H^0(X, L^d)$ , by

$$\mu_{\mathbb{C}}(U) = \frac{1}{\pi^{N_d}} \int_U e^{-\|\sigma\|^2} d\sigma, \quad (46)$$

where  $d\sigma$  denotes the Lebesgue measure of  $H^0(X, L^d)$ . For every  $d > 0$ , we denote by  $\Delta^d$  the discriminant hypersurface of  $H^0(X, E \otimes L^d)$ , that is the set of sections  $\sigma \in H^0(X, E \otimes L^d)$  which do not vanish transversally. For every  $\sigma \in H^0(X, E \otimes L^d) \setminus \{0\}$ , we denote by  $C_\sigma$  the vanishing locus of  $\sigma$  in  $X$ . For every  $\sigma \in H^0(X, E \otimes L^d) \setminus \Delta^d$ ,  $C_\sigma$  is then a smooth codimension  $k$  complex submanifold of  $X$ . We equip  $X$  with a Lefschetz pencil  $p : X \dashrightarrow \mathbb{C}P^1$ . We then denote, for every  $d > 0$ , by  $\Delta_p^d$  the set of sections  $\sigma \in H^0(X, E \otimes L^d)$  such that  $\sigma \in \Delta^d$ , or  $C_\sigma$  intersects the critical locus of  $p$ , or the restriction of  $p$  to  $C_\sigma$  is not a Lefschetz pencil. For  $d$  large enough, this extended discriminant locus is of measure 0 for the measure  $\mu_{\mathbb{C}}$ .

For every  $\sigma \in H^0(X, E \otimes L^d) \setminus \Delta_p^d$ , we denote by  $\text{Crit}(p|_{C_\sigma})$  the set of critical points of the restriction of  $p$  to  $C_\sigma$  and set, for  $1 \leq k \leq n-1$ ,

$$\nu(C_\sigma) = \frac{1}{d^n} \sum_{x \in \text{Crit}(p|_{C_\sigma})} \delta_x, \quad (47)$$

where  $\delta_x$  denotes the Dirac measure of  $X$  at the point  $x$ . When  $k = n$ ,  $\nu(C_\sigma) = \frac{1}{d^n} \sum_{x \in C_\sigma} \delta_x$ .

**Theorem 3.5.1** *Let  $X$  be a smooth complex projective manifold of dimension  $n$ ,  $(L, h_L)$  be a holomorphic Hermitian line bundle of positive curvature  $\omega$  over  $X$  and  $(E, h_E)$  be a rank  $k$  holomorphic Hermitian vector bundle, with  $1 \leq k \leq n$ . Let  $p : X \dashrightarrow \mathbb{C}P^1$  be a Lefschetz pencil. Then, the measure  $\mathbb{E}(\nu)$  defined by (47) weakly converges to  $\binom{n-1}{k-1} \omega^n$  as  $d$  grows to infinity.*

When  $k = 1$ , Theorem 3.5.1 reduces to Theorem 3 of [15], see also Theorem 1.3 of [14].

**Proof.** The proof goes along the same lines as the one of Theorem 3.1.2, so we only give a sketch of it. Firstly, the analogue of Proposition 3.2.1 provides

$$\mathbb{E}(\nu) = \frac{1}{d^n} (\pi_2)_* (\pi_1^* d\mu_{\mathbb{C}}),$$

and at every point  $x \in X \setminus (\text{Crit}(p) \cup \text{Base}(p))$ , where  $\text{Base}(p)$  denotes the base locus of  $p$ ,

$$(\pi_2)_* (\pi_1^* d\mu_{\mathbb{C}})|_x = \frac{1}{\pi^n} \int_{\pi_1^{-1}(\pi_2^{-1}(x))} |\det d|_\sigma ev_{(\sigma,x)}^\perp|^{-2} d\mu_{\mathbb{R}}(\sigma) \frac{\omega^n}{n!},$$

see Proposition 2.10 of [14]. Choosing complex coefficients in decomposition (41), Lemmas 3.3.1 and 3.3.2 remain valid in the complex setting, see Remark 2.3.8. We deduce that

$$\begin{aligned} \mathbb{E}(\nu) &= \frac{1}{\pi^n d^n} \left( \int_{\pi_1^{-1}(\pi_2^{-1}(x))} |\det d|_\sigma ev_{(\sigma,x)}^\perp|^{-2} d\mu_{\mathbb{C}}(\sigma) \right) \frac{\omega^n}{n!} \\ &\stackrel{d \rightarrow \infty}{\sim} \frac{1}{\pi^n d^n} \left( \int_{\text{Gr}_{\mathbb{C}}(n-k, \ker d|_{xp}) \times (E \otimes L^d)|_x \setminus \{0\}} e^{-|a_1^1|^2} \frac{dK \wedge d\lambda}{\pi^{(n-k)(k-1)+k}} \cdots \right. \\ &\quad \left. \cdots \int_{\rho_x^{-1}(K, \lambda)} |\det d|_\sigma ev_{(\sigma,x)}^\perp|^{-2} |\det d|_\sigma \rho_x^\perp|^{-2} d\mu_{\mathbb{C}}(\sigma) \right) \frac{\omega^n}{n!}, \end{aligned}$$

with  $|a_1^1|$  given by (40), see Lemma 2.3.6 as before. Here,  $\text{Gr}_{\mathbb{C}}(n-k, \ker d|_{xp})$  denotes the Grassmann manifold of  $n-k$ -dimensional complex linear subspaces of  $\ker d|_{xp}$ .

From the complex versions of Lemma 2.3.5 and 2.3.6, see Remark 2.3.8 and the relation (40), we deduce that for every  $x \in X \setminus (\text{Crit}(p) \cup \text{Base}(p))$  and every  $(K, \lambda) \in \text{Gr}(n-k, \ker d_{|x}p) \times (E \otimes L^d)_{|x}^* \setminus \{0\}$ ,

$$\begin{aligned} & \int_{\rho_x^{-1}(K, \lambda)} |\det d_{|\sigma} ev_{(\sigma, x)}^\perp|^{-2} |\det d_{|\sigma} \rho_x^\perp|^{-2} d\mu_{\mathbb{C}}(\sigma) \\ \stackrel{d \rightarrow \infty}{\sim} & \frac{|a_1^1|^4 \pi^n d^n}{\|\lambda\|^{2k}} \int_{M_{k-1}(\mathbb{C})} |\det(a_j^i)_{2 \leq i, j \leq k}|^{2(n-k+2)} d\mu(a_j^i) \dots \\ & \dots \int_{\text{Sym}_{\mathbb{C}}(n-k)} |\det(\tilde{a}_{lm}^1)_{k+1 \leq l, m \leq n}|^2 d\mu(\tilde{a}_{lm}^1). \end{aligned}$$

We deduce that  $\mathbb{E}(\nu)$  is equivalent to

$$\begin{aligned} & \frac{\|d_{|x}p\|^4}{(\pi \delta_L d^{n+1})^2} \frac{1}{\pi^{(n-k)(k-1)+k}} \text{Vol}(\text{Gr}_{\mathbb{C}}(k-1, n-1)) \dots \\ & \dots E_{k-1}^{\mathbb{C}}(|\det|^{2(n-k+2)}) e_{\mathbb{C}}(n-k) \left( \int_{(E \otimes L^d)_{|x}^* \setminus \{0\}} \frac{e^{-|a_1^1|^2}}{\|\lambda\|^{2(k+2)}} d\lambda \right) \frac{\omega^n}{n!}, \end{aligned}$$

where  $e_{\mathbb{C}}(n-k) = \int_{\text{Sym}_{\mathbb{C}}(n-k)} |\det A|^2 d\mu_{\mathbb{C}}(A)$  and

$$E_{k-1}^{\mathbb{C}}(|\det|^{2(n-k+2)}) = \int_{M_{k-1}(\mathbb{C})} |\det A|^{2(n-k+2)} d\mu_{\mathbb{C}}(A).$$

Now,

$$\begin{aligned} \frac{\|d_{|x}p\|^4}{(\pi \delta_L d^{n+1})^2} \int_{(E \otimes L^d)_{|x}^* \setminus \{0\}} \frac{e^{-|a_1^1|^2}}{\|\lambda\|^{2k+4}} d\lambda &= \text{Vol}(S^{2k-1}) \frac{\|d_{|x}p\|^4}{(\pi \delta_L d^{n+1})^2} \int_0^{+\infty} \frac{e^{-|a_1^1|^2}}{\|\lambda\|^5} d\|\lambda\| \\ &= \text{Vol}(S^{2k-1}) \int_0^{+\infty} e^{-r^2} r^3 dr = \frac{1}{2} \text{Vol}(S^{2k-1}). \end{aligned}$$

Hence,  $\mathbb{E}(\nu)$  is equivalent to

$$\frac{1}{2\pi^{(n-k)(k-1)+k}} \text{Vol}(\text{Gr}_{\mathbb{C}}(k-1, n-1)) \text{Vol}(S^{2k-1}) E_{k-1}^{\mathbb{C}}(|\det|^{2(n-k+2)}) e_{\mathbb{C}}(n-k) \frac{\omega^n}{n!},$$

where  $e_{\mathbb{C}}(n-k) = (n-k+1)!$  by Proposition 3.8 of [14],  $\text{Vol}(S^{2k-1}) = 2\pi^k/(k-1)!$ ,

$$E_{k-1}^{\mathbb{C}}(|\det|^{2(n-k+2)}) = \frac{\prod_{j=1}^{k-1} \Gamma((n-k+2)+j)}{\prod_{j=1}^{k-1} \Gamma(j)} = \frac{\prod_{j=n-k+3}^{n+1} \Gamma(j)}{\prod_{j=1}^{k-1} \Gamma(j)}$$

by formula 15.4.12 of [23] and

$$\text{Vol}(\text{Gr}_{\mathbb{C}}(k-1, n-1)) = \frac{\prod_{j=1}^{k-1} \Gamma(j)}{\prod_{j=n-k+1}^{n-1} \Gamma(j)} \pi^{(k-1)(n-k)}$$

by a computation analogous to the one given in the real case by Remark 3.1.1. We conclude that  $\mathbb{E}(\nu)$  weakly converges to  $\binom{k-1}{n-1} \omega^n$ , where the convergence is dominated by a function in  $L^1(X, \frac{\omega^n}{n!})$ , for it has poles of order at most  $2(n-k)$  near the critical points of  $p$  and at most 2 near the base points, see [15].  $\square$

**Corollary 3.5.2** *Under the hypotheses of Theorem 3.5.1, for every generic  $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$ , let  $|\text{Crit } p|_{C_\sigma}|$  be the number of critical points of  $p|_{C_\sigma}$ . Then,*

$$\frac{1}{d^n} \mathbb{E}(|\text{Crit } p|_{C_\sigma}|) \underset{d \rightarrow \infty}{\sim} \binom{k-1}{n-1} \int_X c_1(L)^n.$$

**Proof.** Corollary 3.5.2 follows from Theorem 3.5.1 by integration of 1 over  $X$ . A direct proof can be given though. The modulus of  $p$  is a Morse function on  $C_\sigma \setminus (\text{Base}(p) \cup F_0 \cup F_\infty)$ , where  $F_0$  (resp.  $F_\infty$ ) is the fibre of 0 (resp. of  $\infty$ ) of  $p : X \dashrightarrow \mathbb{C}P^1$ . Moreover, the index of every critical point of  $|p|$  is  $n - k$ . As in the proofs of Propositions 1 and 2 in [15], we deduce that  $\mathbb{E}(|\text{Crit } p|_{C_\sigma}|)$  is equivalent to  $|\chi(C_\sigma)|$  as  $d$  grows to infinity. Now,

$$\chi(C_\sigma) = \int_{C_\sigma} c_{n-k}(C_\sigma) = \int_X c_{n-k}(C_\sigma) \wedge c_k(E \otimes L^d),$$

while from the adjunction formula,  $c(C_\sigma) \wedge c(E \otimes L^d)|_{C_\sigma} = c(X)$ . Moreover, for  $0 \leq i \leq k$ ,  $c_i(E \otimes L^d) = \binom{k}{i} d^i c_1(L)^i + o(d^i)$ , so that

$$c(E \otimes L^d) = (1 + dc_1(L))^k + o((1 + dc_1(L))^k).$$

From the formula  $(1 + x)^{-k} = \sum_{j=0}^{\infty} (-1)^j \frac{(k-1+j)!}{j!(k-1)!} x^j$ , we then deduce that  $c_{n-k}(C_\sigma) = (-1)^{n-k} \binom{n-1}{k-1} d^{n-k} c_1(L)^{n-k} + o(d^{n-k})$  and finally that

$$\chi(C_\sigma) = (-1)^{n-k} \binom{n-1}{k-1} d^n \int_X c_1(L)^n + o(d^n).$$

Hence the result.  $\square$

## References

- [1] A. Auffinger, G. Ben Arous, and J. Černý. Random matrices and complexity of spin glasses. *Commun. Pure Appl. Math.*, 66(2):165–201, 2013.
- [2] P. Bleher, B. Shiffman, and S. Zelditch. Universality and scaling of correlations between zeros on complex manifolds. *Invent. Math.*, 142(2):351–395, 2000.
- [3] E. Bogomolny, O. Bohigas, and P. Leboeuf. Quantum chaotic dynamics and random polynomials. *J. Statist. Phys.*, 85(5-6):639–679, 1996.
- [4] P. Bürgisser. Average Euler characteristic of random real algebraic varieties. *C. R. Math. Acad. Sci. Paris*, 345(9):507–512, 2007.
- [5] J.-P. Dedieu and G. Malajovich. On the number of minima of a random polynomial. *J. Complexity*, 24(2):89–108, 2008.
- [6] J.-P. Demailly. Estimations  $L^2$  pour l'opérateur  $\bar{\partial}$  d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète. *Ann. Sci. École Norm. Sup. (4)*, 15(3):457–511, 1982.

- [7] S. K. Donaldson. Symplectic submanifolds and almost-complex geometry. *J. Differential Geom.*, 44(4):666–705, 1996.
- [8] M. R. Douglas, B. Shiffman, and S. Zelditch. Critical points and supersymmetric vacua. I. *Commun. Math. Phys.*, 252(1-3):325–358, 2004.
- [9] M. R. Douglas, B. Shiffman, and S. Zelditch. Critical points and supersymmetric vacua. II: Asymptotics and extremal metrics. *J. Differ. Geom.*, 72(3):381–427, 2006.
- [10] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.
- [11] Y. V. Fyodorov. Complexity of random energy landscapes, glass transition, and absolute value of the spectral determinant of random matrices. *Phys. Rev. Lett.*, 92(24):240601, 4, 2004.
- [12] D. Gayet. Hypersurfaces symplectiques réelles et pinceaux de Lefschetz réels. *J. Symplectic Geom.*, 6(3):247–266, 2008.
- [13] D. Gayet and J.-Y. Welschinger. Exponential rarefaction of real curves with many components. *Publ. Math. Inst. Hautes Études Sci.*, (113):69–96, 2011.
- [14] D. Gayet and J.-Y. Welschinger. Betti numbers of random real hypersurfaces and determinants of random symmetric matrices. *To appear in J. Eur. Math. Soc.*, *arXiv:1207.1579*, 2012.
- [15] D. Gayet and J.-Y. Welschinger. What is the total Betti number of a random real hypersurface? *J. Reine Angew. Math.*, *published online, 10.1515/crelle-2012-0062*, 2012.
- [16] D. Gayet and J.-Y. Welschinger. Lower estimates for the expected betti numbers of random real hypersurfaces. *To appear in J. London Math. Soc.*, *arXiv:1303.3035*, 2013.
- [17] L. Hörmander. *An introduction to complex analysis in several variables*. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1966.
- [18] M. Kac. On the average number of real roots of a random algebraic equation. *Bull. Amer. Math. Soc.*, 49:314–320, 1943.
- [19] E. Kostlan. On the distribution of roots of random polynomials. In *From Topology to Computation: Proceedings of the Smalefest (Berkeley, CA, 1990)*, pages 419–431. Springer, New York, 1993.
- [20] A. Lerario and E. Lundberg. Statistics on Hilbert’s sixteenth problem. *arXiv:212.3823*, 2013.
- [21] X. Ma and G. Marinescu. *Holomorphic Morse inequalities and Bergman kernels*, volume 254 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.
- [22] B. Macdonald. Density of complex critical points of a real random  $SO(m + 1)$  polynomial. *J. Stat. Phys.*, 141(3):517–531, 2010.



- [23] M. L. Mehta. *Random matrices*, volume 142 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, third edition, 2004.
- [24] J. Nash. Real algebraic manifolds. *Ann. of Math. (2)*, 56:405–421, 1952.
- [25] F. Nazarov and M. Sodin. On the number of nodal domains of random spherical harmonics. *Amer. J. Math.*, 131(5):1337–1357, 2009.
- [26] L. I. Nicolaescu. Critical sets of random smooth functions on compact manifolds. *To appear in Asian J. Math.*, *arXiv:1101.5990*, 2011.
- [27] S. S. Podkorytov. The mean value of the Euler characteristic of an algebraic hypersurface. *Algebra i Analiz*, 11(5):185–193, 1999.
- [28] P. Sarnak and I. Wigman. Topologies of nodal sets of random band limited functions. *arXiv:1312.7858*, 2013.
- [29] H. Seifert. Algebraische Approximation von Mannigfaltigkeiten. *Math. Z.*, 41(1):1–17, 1936.
- [30] B. Shiffman and S. Zelditch. Distribution of zeros of random and quantum chaotic sections of positive line bundles. *Comm. Math. Phys.*, 200(3):661–683, 1999.
- [31] B. Shiffman and S. Zelditch. Addendum: “Asymptotics of almost holomorphic sections of ample line bundles on symplectic manifolds” [*J. Reine Angew. Math.* **544** (2002), 181–222; MR1887895 (2002m:58043)]. *Proc. Amer. Math. Soc.*, 131(1):291–302, 2003.
- [32] B. Shiffman and S. Zelditch. Random polynomials of high degree and Levy concentration of measure. *Asian J. Math.*, 7(4):627–646, 2003.
- [33] B. Shiffman and S. Zelditch. Number variance of random zeros on complex manifolds. *Geom. Funct. Anal.*, 18(4):1422–1475, 2008.
- [34] M. Shub and S. Smale. Complexity of Bezout’s theorem. II. Volumes and probabilities. In *Computational algebraic geometry (Nice, 1992)*, volume 109 of *Progr. Math.*, pages 267–285. Birkhäuser Boston, 1993.
- [35] M. Sodin. Lectures on random nodal portraits. *lecture notes for a mini-course given at the St. Petersburg Summer School in Probability and Statistical Physics (June, 2012)*, 2012.
- [36] M. Sodin and B. Tsirelson. Random complex zeroes. I. Asymptotic normality. *Israel J. Math.*, 144:125–149, 2004.
- [37] G. Tian. On a set of polarized Kähler metrics on algebraic manifolds. *J. Differential Geom.*, 32(1):99–130, 1990.
- [38] S. Zelditch. Real and complex zeros of Riemannian random waves. In *Spectral analysis in geometry and number theory*, volume 484 of *Contemp. Math.*, pages 321–342. Amer. Math. Soc., Providence, RI, 2009.



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