Expected topology of random real algebraic submanifolds

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Abstract

Let X be a smooth complex projective manifold of dimension n equipped with an ample line bundle L and a rank k holomorphic vector bundle E. We assume that $1 \leqslant k \leqslant n$, that X, E and L are defined over the reals and denote by $\mathbb{R}X$ the real locus of X. Then, we estimate from above and below the expected Betti numbers of the vanishing loci in $\mathbb{R}X$ of holomorphic real sections of $E \otimes L^d$, where d is a large enough integer. Moreover, given any closed connected codimension k submanifold Σ of $\mathbb{R}^n$ with trivial normal bundle, we prove that a real section of $E \otimes L^d$ has a positive probability, independent of d, to contain around $\sqrt{d}^n$ connected components diffeomorphic to Σ in its vanishing locus.

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Contents

1 Introduction

Let X be a smooth complex projective manifold of positive dimension n equipped with an ample line bundle L and let E be a holomorphic vector bundle of rank k over X. From the vanishing theorem of Kodaira and Serre, we know that the dimension N_d of the complex vector space $H^0(X, E \otimes L^d)$ of global holomorphic sections of $E \otimes L^d$ grows as a polynomial of degree n in d. We will assume throughout this paper that $1 \leqslant k \leqslant n$ and that X, E and L are defined over the reals. We denote by $\mathbb{R}X$ the real locus of X and by $\mathbb{R}H^0(X, E \otimes L^d)$ the real vector space of real holomorphic sections of $E \otimes L^d$, see (5). Its dimension equals N_d. The discriminant locus $\mathbb{R}\Delta_d \subset \mathbb{R}H^0(X, E \otimes L^d)$ of sections which do not vanish transversally is a codimension one submanifold for d large enough and for every σ in its complement, the real vanishing locus $\mathbb{R}C_\sigma$ of σ is a smooth codimension k submanifold of $\mathbb{R}X$. The topology of $\mathbb{R}C_\sigma$ drastically depends on the choice of $\sigma \in \mathbb{R}H^0(X, E \otimes L^d) \backslash \mathbb{R}\Delta_d$. When $n = k = 1$, $X = \mathbb{C}P^1$, $L = \mathcal{O}_{\mathbb{C}P^1}(1)$ and $E = \mathcal{O}_{\mathbb{C}P^1}$ for example, σ is a real polynomial of degree d in one variable and $\mathbb{R}C_\sigma$ the set of its real roots.

The space $\mathbb{R}H^0(X, E \otimes L^d)$ inherits classical probability measures. Indeed, let h_E be a Hermitian metric on E and h_L be a Hermitian metric of positive curvature on L, both h_E and h_L being real, that is invariant under the $\mathbb{Z}/2\mathbb{Z}$-Galois action of E and L. We denote by $h_{E,d} = h_E \otimes h_L^d$ the induced metric on $E \otimes L^d$. Then, the vector space $\mathbb{R}H^0(X, E \otimes L^d)$ becomes Euclidean, with the L^2-scalar product defined by

$$\forall \sigma, \tau \in \mathbb{R}H^0(X, E \otimes L^d),\ \langle \sigma, \tau \rangle = \int_X h_{E,d}(\sigma, \tau) dx,$$

where dx denotes any chosen volume form on X (our results being asymptotic in d, they turn out not to depend on the choice of dx). It thus inherits a Gaussian probability measure $\mu_{\mathbb{R}}$ whose density at $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$ with respect to the Lebesgue measure is $\frac{1}{\sqrt{\pi}^{N_d}} e^{-\|\sigma\|^2}$.

What is the typical topology of $\mathbb{R}C_\sigma$ for $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$ chosen at random for $d\mu_{\mathbb{R}}$? We do not know, but can estimate its average Betti numbers. To formulate our results, let us denote, for every $i \in \{0, \cdots, n-k\}$, by $b_i(\mathbb{R}C_\sigma, \mathbb{R}) = \dim H_i(\mathbb{R}C_\sigma, \mathbb{R})$ the i-th Betti number of $\mathbb{R}C_\sigma$ and by

$$\mathbb{E}(b_i) = \int_{\mathbb{R}H^0(X, E \otimes L^d) \backslash \mathbb{R}\Delta_d} b_i(\mathbb{R}C_\sigma, \mathbb{R}) d\mu_{\mathbb{R}}(\sigma)$$

its expected value.

1.1 Upper estimates

As in [14], for every $i \in \{0, \cdots, n-k\}$, we denote by $Sym_{\mathbb{R}}(i, n-k-i)$ the open cone of real symmetric matrices of size $n-k$ and signature $(i, n-k-i)$, by $\mu_{\mathbb{R}}$ the classical Gaussian measure on the space of real symmetric matrices and by $e_{\mathbb{R}}(i, n-k-i)$ the numbers

$$e_{\mathbb{R}}(i, n-k-i) = \int_{Sym_{\mathbb{R}}(i,n-k-i)} |\det A| d\mu_{\mathbb{R}}(A), \tag{1}$$

see §3.1. We then denote by $Vol_{h_L}(\mathbb{R}X)$ the volume of $\mathbb{R}X$ for the Riemannian metric induced by the Kähler metric g_{h_L} defined by the curvature form of h_L, see (3) and (4).

Theorem 1.1.1 *Let X be a smooth real projective manifold of dimension n, (L, h_L) be a real holomorphic Hermitian line bundle of positive curvature over X and (E, h_E) be a rank k real holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n$, $k \neq n$. Then, for every $0 \leqslant i \leqslant n-k$,*

$$\limsup_{d\to\infty} \frac{1}{\sqrt{d}^n} \mathbb{E}(b_i) \quad \leqslant \quad \binom{n-1}{k-1} e_{\mathbb{R}}(i, n-k-i) \frac{Vol_{h_L}(\mathbb{R}X)}{Vol_{FS}(\mathbb{R}P^k)}.$$

Moreover, when $k = n$, $\frac{1}{\sqrt{d}^n}\mathbb{E}(b_0)$ converges to $\frac{Vol_{h_L}(\mathbb{R}X)}{Vol_{FS}(\mathbb{R}P^n)}$ as d grows to infinity.

In fact, the right hand side of the inequality given by Theorem 1.1.1 also involves the determinant of random matrices of size $k-1$ and the volume of the Grassmann manifold of $(k-1)$ linear subspaces of $\mathbb{R}^{n-1}$, see Theorem 3.1.2, but these can be computed explicitly. Note that when E is the trivial line bundle, Theorem 1.1.1 reduces to Theorem 1.1 of [14].

Theorem 1.1.1 relies on Theorem 3.1.3, which establishes the asymptotic equidistribution of clouds of critical points, see §3.1. We obtain a similar result in a complex projective setting, for critical points of Lefschetz pencils, see Theorem 3.5.1.

1.2 Lower estimates and topology

Let Σ be a closed submanifold of codimension k of $\mathbb{R}^n$, $1 \leqslant k \leqslant n$, which we do not assume to be connected. For every $\sigma \in \mathbb{R}H^0(X, E \otimes L^d) \backslash \mathbb{R}\Delta_d$, we denote by $N_\Sigma(\sigma)$ the maximal number of disjoint open subsets of $\mathbb{R}X$ having the property that each such open subset U' contains a codimension k submanifold Σ' such that $\Sigma' \subset \mathbb{R}C_\sigma$ and (U', Σ') is diffeomorphic to $(\mathbb{R}^n, \Sigma)$. We then set

$$\mathbb{E}(N_\Sigma) = \int_{\mathbb{R}H^0(X,E\otimes L^d)\backslash\mathbb{R}\Delta_d} N_\Sigma(\sigma) d\mu_{\mathbb{R}}(\sigma) \tag{2}$$

and we associate to Σ, in fact to its isotopy class in $\mathbb{R}^n$, a constant c_Σ which is positive if and only if Σ has trivial normal bundle in $\mathbb{R}^n$, see (14) for its definition and Lemma 2.2.3. The latter measures à la Donaldson the amount of transversality that a polynomial map $\mathbb{R}^n \to \mathbb{R}^k$ vanishing along a submanifold isotopic to Σ may have.

Theorem 1.2.1 *Let X be a smooth real projective manifold of dimension n, (L, h_L) be a real holomorphic Hermitian line bundle of positive curvature over X and (E, h_E) be a rank k real holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n$. Let Σ be a closed submanifold of codimension k of $\mathbb{R}^n$ with trivial normal bundle, which does not need to be connected. Then,*

$$\liminf_{d\to\infty} \frac{1}{\sqrt{d}^n} \mathbb{E}(N_\Sigma) \geqslant c_\Sigma Vol_{h_L}(\mathbb{R}X).$$

In particular, when Σ is connected, Theorem 1.2.1 bounds from below the expected number of connected components diffeomorphic to Σ in the real vanishing locus of a random section $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$. The constant c_Σ does not depend on the choice of the triple $(X, (L, h_L), (E, h_E))$, it only depends on Σ. When $k = 1$ and $E = \mathcal{O}_X$, Theorem 1.2.1 coincides with Theorem 1.2 of [16]. Computing c_Σ for explicit submanifolds Σ yields the following lower bounds for the Betti numbers.

Corollary 1.2.2 *Under the hypotheses of Theorem 1.2.1, for every $i \in \{0, \cdots, n-k\}$,*

$$\liminf_{d\to\infty} \frac{1}{\sqrt{d}^n} \mathbb{E}(b_i) \geqslant \exp(-e^{84+6n}) Vol_{h_L}(\mathbb{R}X).$$

1.3 Some related results

The case $X = \mathbb{C}P^1$, $E = \mathcal{O}_{\mathbb{C}P^1}$ and $L = \mathcal{O}_{\mathbb{C}P^1}(1)$ was first considered by M. Kac in [18] for a different measure. In this case and with our measure, Kostlan [19] and Shub and Smale [34] gave an exact formula for the mean number of real roots of a polynomial, as well as the mean number of intersection points of n hypersurfaces in $\mathbb{R}P^n$. Still in $\mathbb{R}P^n$, Podkorytov [27] computed the mean Euler characteristics of random algebraic hypersufaces, and Bürgisser [4] extented this result to complete intersections. In [13], we proved the exponential rarefaction of real curves with a maximal number of components in real algebraic surfaces. In [15] and [14], we bounded from above the mean Betti numbers of random real hypersurfaces in real projective manifolds and in [16], we gave a lower bound for them.

A similar probabilistic study of complex projective manifolds has been performed by Shiffman and Zelditch, see [30], [33], [2] for example, or also [3], [36]. In particular, the asymptotic equidistribution of critical points of random sections over a fixed projective manifold has been studied in [8], [9] and [22], or also [11], [1], [5], while we studied critical points of the restriction of a fixed Morse function on random real hypersurfaces, see [15], [14].

A similar question concerns the mean number of components of the vanishing locus of eigenfunctions of the Laplacian. It has been studied on the round sphere by Nazarov and Sodin [25] (see also [35]), Lerario and Lundberg [20] or Sarnak and Wigman [28]. In a general Riemannian setting, Zelditch proved in [38] the equidistribution of the vanishing locus, whereas critical points of random eigenfunctions of the Laplacian have been addressed by Nicolaescu in [26].

Section 2 is devoted to lower estimates and the proof of Theorem 1.2.1. In this proof, the L^2-estimates of Hörmander play a crucial rôle, see §2.3, and we follow the same approach as in [16] (see also [12] for a similar construction). Section 3 is

devoted to upper estimates and the proof of Theorem 1.1.1.

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2 Lower estimates for the expected Betti numbers

2.1 Statement of the results

2.1.1 Framework

Let us first recall our framework. We denote by X a smooth complex projective manifold of dimension n defined over the reals, by $c_X : X \to X$ the induced Galois antiholomorphic involution and by $\mathbb{R}X = \mathrm{Fix}(c_X)$ the real locus of X which we implicitly assume to be non-empty. We then consider an ample line bundle L over X, also defined over the reals. It comes thus equipped with an antiholomorphic involution $c_L : L \to L$ which turns the bundle projection map $\pi : L \to X$ into a $\mathbb{Z}/2\mathbb{Z}$-equivariant one, so that $c_X \circ \pi = \pi \circ c_L$. We equip L in addition with a real Hermitian metric h_L, thus invariant under c_L, which has a positive curvature form ω locally defined by

$$\omega = \frac{1}{2i\pi}\partial\bar{\partial}\log h_L(e,e) \tag{3}$$

for any non-vanishing local holomorphic section e of L. This metric induces a Kähler metric

$$g_{h_L} = \omega(.\,,i.\,) \tag{4}$$

on X, which reduces to a Riemannian metric g_{h_L} on $\mathbb{R}X$. Let finally E be a holomorphic vector bundle of rank k, $1 \leqslant k \leqslant n$, defined over the reals and equipped with a antiholomorphic involution c_E and a real Hermitian metric h_E. For every $d > 0$, we denote by

$$\mathbb{R}H^0(X, E\otimes L^d) = \{\sigma \in H^0(X, E\otimes L^d) \mid (c_E \otimes c_{L^d})\circ\sigma = \sigma\circ c_X\} \tag{5}$$

the space of global real holomorphic sections of $E \otimes L^d$. It is equipped with the L^2-scalar product defined by the formula

$$\forall(\sigma,\tau)\in\mathbb{R}H^0(X,E\otimes L^d),\ \langle\sigma,\tau\rangle = \int_X h_{E,d}(\sigma,\tau)(x)dx, \tag{6}$$

where $h_{E,d} = h_E \otimes h_L^d$. Here, dx denotes any volume form of X. For instance, dx can be chosen to be the normalized volume form $dV_{h_L} = \frac{\omega^n}{\int_X \omega^n}$. This L^2-scalar product finally induces a Gaussian probability measure $\mu_{\mathbb{R}}$ on $\mathbb{R}H^0(X, E\otimes L^d)$ whose density with respect to the Lebesgue one at $\sigma \in \mathbb{R}H^0(X, E\otimes L^d)$ writes $\frac{1}{\sqrt{\pi}^{N_d}}e^{-\|\sigma\|^2}$, where $N_d = \dim H^0(X, E\otimes L^d)$. It is with respect to this probability measure that we consider random real codimension k submanifolds (as in the works [19] and [34], [15], [14] and [16]).

2.1.2 The lower estimates

The aim of Section 2 is to prove Theorem 1.2.1. In addition to Theorem 1.2.1, we also get the following Theorem 2.1.1, which is a consequence of Proposition 2.4.2 below.

Theorem 2.1.1 *Under the hypotheses of Theorem 1.2.1, for every* $0 \leqslant \epsilon < 1$,

$$\liminf_{d\to\infty} \mu_{\mathbb{R}} \left\{ \sigma \in \mathbb{R}H^0(X, E\otimes L^d) \mid N_\Sigma(\sigma) \geqslant \epsilon c_\Sigma Vol_{h_L}(\mathbb{R}X)\sqrt{d}^n \right\} > 0.$$

In fact, the positive lower bound given by Theorem 2.1.1 can be made explicit, see (30).

Let us now denote, for every $1 \leqslant k \leqslant n$, by $\mathcal{H}_{n,k}$ the set of diffeormophism classes of smooth closed connected codimension k submanifolds of $\mathbb{R}^n$. For every $i \in \{0, \cdots, n-k\}$ and every $[\Sigma] \in \mathcal{H}_{n,k}$, we denote by $b_i(\Sigma) = \dim H_i(\Sigma; \mathbb{R})$ its i-th Betti number with real coefficients and by $m_i(\Sigma)$ its i-th Morse number. This is the infimum over all Morse functions f on Σ of the number of critical points of index i of f. Then, we set $c_{[\Sigma]} = \sup_{\Sigma\in[\Sigma]} c_\Sigma$ and

$$\mathbb{E}(m_i) = \int_{\mathbb{R}H^0(X,E\otimes L^d)\setminus\mathbb{R}\Delta_d} m_i(\mathbb{R}C_\sigma) d\mu_{\mathbb{R}}(\sigma).$$

Corollary 2.1.2 *Let X be a smooth real projective manifold of dimension n, (L, h_L) be a real holomorphic Hermitian line bundle of positive curvature over X and (E, h_E) be a rank k real holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n$. Then, for every $i \in \{0, \cdots, n-k\}$,*

$$\liminf_{d\to\infty} \frac{1}{\sqrt{d}^n}\mathbb{E}(b_i) \;\geqslant\; \Big(\sum_{[\Sigma]\in\mathcal{H}_{n,k}} c_{[\Sigma]} b_i(\Sigma)\Big) Vol_{h_L}(\mathbb{R}X) \text{ and likewise} \tag{7}$$

$$\liminf_{d\to\infty} \frac{1}{\sqrt{d}^n}\mathbb{E}(m_i) \;\geqslant\; \Big(\sum_{[\Sigma]\in\mathcal{H}_{n,k}} c_{[\Sigma]} m_i(\Sigma)\Big) Vol_{h_L}(\mathbb{R}X). \tag{8}$$

Note that in Corollary 2.1.2, we could have chosen one representative Σ in each diffeomorphism class $[\Sigma] \in \mathcal{H}_{n,k}$ and obtained the lower estimates (7), (8) with constants c_Σ instead of $c_{[\Sigma]}$. But it turns out that in the proof of Corollary 2.1.2 we are free to choose the representative we wish in every diffeomorphism class and that the higher c_Σ is, the better the estimates (7), (8) are. This is why we introduce the constant $c_{[\Sigma]}$, which is positive if and only if $[\Sigma]$ has a representative Σ with trivial normal bundle in $\mathbb{R}^n$, see (14) and Lemma 2.2.3.

2.2 Closed affine real algebraic submanifolds

We introduce here the notion of regular pair, see Definition 2.2.1, and the constant c_Σ associated to any isotopy class of smooth closed codimension k submanifold Σ of $\mathbb{R}^n$, see (14).

Definition 2.2.1 *Let U be a bounded open subset of $\mathbb{R}^n$ and $P \in \mathbb{R}[x_1, \cdots x_n]^k$, $1 \leqslant k \leqslant n$. The pair (U, P) is said to be regular if and only if*

1. *zero is a regular value of the restriction of P to U,*

2. *the vanishing locus of P in U is compact.*

Hence, for every regular pair (U, P), the vanishing locus of P does not intersect the boundary of U and it meets U in a smooth compact codimension k submanifold.

In the sequel, for every integer p and every vector $v \in \mathbb{R}^p$, we denote by $|v|$ its Euclidian norm, and for every integers p and q, and every linear map $F : \mathbb{R}^p \to \mathbb{R}^q$, we denote by F^* the adjoint of F, defined by the property

$$\forall v \in \mathbb{R}^p, \forall w \in \mathbb{R}^q,\ \langle F(v), w\rangle = \langle v, F^*(w)\rangle,$$

and denote by $\|F\|$ its operator norm, that is

$$\|F\| = \sup_{v\in\mathbb{R}^p\setminus\{0\}} |F(v)|/|v|.$$

We will also use the norm

$$\|F\|_2 = \sqrt{\mathrm{Tr}\ FF^*}.$$

These norms satisfy $\|F\| \leqslant \|F\|_2$. Finally, if $P = (P_1, \cdots, P_k) \in \mathbb{R}[x_1, \cdots, x_n]^k$, we denote by $\|P\|_{L^2}$ its L^2-norm defined by

$$\|P\|_{L^2}^2 = \int_{\mathbb{C}^n} |P(z)|^2 e^{-\pi|z|^2} dz = \sum_{i=1}^k \int_{\mathbb{C}^n} |P_i(z)|^2 e^{-\pi|z|^2} dz = \sum_{i=1}^k \|P_i\|_{L^2}^2. \tag{9}$$

Definition 2.2.2 *For every regular pair (U, P) given by Definition 2.2.1, we denote by $\mathcal{T}_{(U,P)}$ the set of $(\delta, \epsilon) \in (\mathbb{R}_+^*)^2$ such that*

1. *there exists a compact subset K of U satisfying $\inf_{x\in U\setminus K} |P(x)| > \delta$,*

2. *for every $y \in U$, $|P(y)| < \delta \Rightarrow \forall w \in \mathbb{R}^k,\ |(d_{|y}P)^*(w)| \geqslant \epsilon|w|$.*

Hence, for every regular pair (U, P) given by Definition 2.2.1, (δ, ϵ) belongs to $\mathcal{T}_{(U,P)}$ provided the δ-sublevel of P does not intersect the boundary of U while inside this δ-sublevel, P is in a sense ϵ-far from having a critical point. This quantifies how much transversally P vanishes in a way similar to the one used by Donaldson in [7].

Then, for every regular pair (U, P), we set $R_{(U,P)} = \max(1, \sup_{y\in U} |y|)$, so that U is contained in the ball centered at the origin and of radius $R_{(U,P)}$. Finally, we set

$$\tau_{(U,P)} = 24k\rho_{R_{(U,P)}}\|P\|_{L^2}^2 \inf_{(\delta,\epsilon)\in\mathcal{T}_{(U,P)}} (\frac{1}{\delta^2} + \frac{\pi n}{\epsilon^2}) \in \mathbb{R}_+^*, \tag{10}$$

where, for every $R > 0$,

$$\rho_R = \inf_{\mathbb{R}^+} g_R, \tag{11}$$

$$g_R : s \in \mathbb{R}_+^* \mapsto \frac{(R+s)^{2n}}{s^{2n}} e^{\pi(R+s)^2}, \tag{12}$$

so that

$$e^{\pi R^2} \leqslant \rho_R \leqslant 4^n e^{4\pi R^2}. \tag{13}$$

This constant $\tau_{(U,P)}$ is the main ingredient in the definition of c_Σ, see (14). The lower $\tau_{(U,P)}$ is, the larger c_Σ is and the better the estimates given by Theorem 1.2.1 are. Note that $\tau_{(U,P)}$ remains small whenever δ, ϵ are not too small, that is when P vanishes quite transversally in U.

Now, let Σ be a closed submanifold of codimension k of $\mathbb{R}^n$, not necessarily connected. We denote by $\mathcal{I}_\Sigma$ the set of regular pairs (U, P) given by Definition 2.2.1, such that the vanishing locus of P in U contains a subset isotopic to Σ in $\mathbb{R}^n$.

Lemma 2.2.3 *Let Σ be a closed submanifold of codimension $k > 0$ of $\mathbb{R}^n$, not necessarily connected. Then, $\mathcal{I}_\Sigma$ is non empty if and only if the normal bundle of Σ in $\mathbb{R}^n$ is trivial.*

Proof. If $(U, P) \in \mathcal{I}_\Sigma$, then $P : \mathbb{R}^n \to \mathbb{R}^k$ contains in its vanishing locus a codimension k submanifold $\widehat{\Sigma}$ which is isotopic to Σ in $\mathbb{R}^n$. The normal bundle of Σ in $\mathbb{R}^n$ is thus trivial if and only if the normal bundle of $\widehat{\Sigma}$ in $\mathbb{R}^n$ is trivial. But the differential of P at every point of $\widehat{\Sigma}$ provides an isomorphism between the normal bundle of $\widehat{\Sigma}$ in $\mathbb{R}^n$ and the product $\widehat{\Sigma} \times \mathbb{R}^k$.

Conversely, if Σ has a trivial normal bundle in $\mathbb{R}^n$, it has been proved by Seifert [29] (see also [24]) that there exist a polynomial map $P : \mathbb{R}^n \to \mathbb{R}^k$ and a tubular neighbourhood U of Σ in $\mathbb{R}^n$ such that $P^{-1}(0) \cap U$ is isotopic to Σ in U. The strategy of the proof is to first find a smooth function $U \to \mathbb{R}^k$ in a neighborhood of Σ which vanishes transversally along Σ and then to suitably approximate the coordinates of this function by some polynomial, see [29], [24]. The pair (U, P) then belongs to $\mathcal{I}_\Sigma$ by Definition 2.2.1. □

We then set $c_\Sigma = 0$ if Σ does not have a trivial normal bundle in $\mathbb{R}^n$ and

$$c_\Sigma = \sup_{(U,P)\in\mathcal{I}_\Sigma} \left(\frac{m_{\tau_{(U,P)}}}{2^n Vol(B(R_{(U,P)}))} \right) \text{ otherwise,} \tag{14}$$

where $Vol(B(R_{(U,P)}))$ denotes the volume of the Euclidean ball of radius $R_{(U,P)}$ in $\mathbb{R}^n$, and where, for every $\tau > 0$,

$$m_\tau = \sup_{[\sqrt{\tau},+\infty[} f_\tau, \tag{15}$$

with $f_\tau : a \in [\sqrt{\tau}, +\infty[\mapsto \frac{1}{\sqrt{\pi}}(1 - \frac{\tau}{a^2}) \int_a^{+\infty} e^{-t^2} dt$. For large values of m_τ, as the ones which appear in §2.6, the estimate

$$c_\Sigma \geqslant e^{-2\tau_{(U,P)}} \tag{16}$$

holds, compare (2.8) of [16].

2.3 Hörmander sections

Our key tool to prove Theorems 1.1.1 and 1.2.1 has been developped by L. Hörmander. We introduce in this §2.3 the material we need. For every positive d and every $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$, we set

$$\|\sigma\|^2_{L^2(h_L)} = \int_X \|\sigma\|^2_{h_{E,d}} dV_{h_L},$$

where $dV_{h_L} = \omega^n / \int_X \omega^n$, compare (6). Let us choose a field of h_L-trivializations of L on $\mathbb{R}X$ given by Definition 4 of [16]. It provides in particular, for every $x \in \mathbb{R}X$, a local holomorphic chart $\psi_x : (W_x, x) \subset X \to (V_x, 0) \subset \mathbb{C}^n$ isometric at x, and a non-vanishing holomorphic section e of L defined over W_x such that $\phi = -\log h_L(e,e)$ vanishes at x and is positive elsewhere. Moreover, there exist a positive constant α_1 such that

$$\forall y \in V_x, |\phi \circ \psi_x^{-1}(y) - \pi|y|^2| \leqslant \alpha_1 |y|^3. \tag{17}$$

Restricting W_x if necessary, we choose a holomophic trivialization $(e_1, \cdots, e_k)$ of $E_{|W_x}$ which is orthonormal at x. This provides a trivialization $(e_1 \otimes e^d, \cdots, e_k \otimes e^d)$ of $E \otimes L^d_{|W_x}$. In this trivialization, the restriction of σ to W_x writes

$$\sigma = \sum_{j=1}^{k} f_\sigma^j e_j \otimes e^d \tag{18}$$

for some holomorphic functions $f_\sigma^j : W_x \to \mathbb{C}$, We write $f_\sigma = (f_\sigma^1, \cdots, f_\sigma^k)$ and we set

$$|\sigma| = |f_\sigma|, \tag{19}$$

so that on W_x, $\|\sigma\|^2_{h_{E,d}} = \left\| \sum_{j=1}^k f_\sigma^k e_j \right\|^2_{h_E} e^{-d\phi}$ and $\|\sigma(x)\|^2_{h_{E,d}} = |\sigma(x)|^2$ since the frames $(e_1, \cdots, e_k)$ and e are orthonormal at the point x so that in particular $\phi(x) = 0$. For every $z \in W_x$, we define

$$\|d_{|z}\sigma\|_2 = \|d_{|y}(f_\sigma \circ \psi_x^{-1})\|_2, \tag{20}$$

$$\|d_{|z}\sigma\| = \|d_{|y}(f_\sigma \circ \psi_x^{-1})\|, \tag{21}$$

and

$$(d_{|z}\sigma)^* = (d_{|y}(f_\sigma \circ \psi_x^{-1}))^*, \tag{22}$$

where $y = \psi_x(z)$. Finally, we denote, for every small enough $r > 0$, by $B(x,r) \subset W_x$ the ball centered at x and of radius r for the flat metric of V_x pulled back by ψ_x, so that

$$B(x,r) = \psi_x^{-1}(B(0,r)). \tag{23}$$

Proposition 2.3.1 *Let X be a smooth real projective manifold of dimension n, (L, h_L) be a real holomorphic Hermitian line bundle of positive curvature over X and (E, h_E) be a rank k real holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n$. We choose a field of h_L-trivializations on $\mathbb{R}X$. Then, for every regular pair (U, P), every large enough integer d, every x in $\mathbb{R}X$ and every local trivialization of E orthonormal at x, there exist $\sigma_{(U,P)} \in \mathbb{R}H^0(X, E \otimes L^d)$ and an open subset U_d of $B(x, \frac{R_{(U,P)}}{\sqrt{d}}) \cap \mathbb{R}X$ such that*

1. *$\|\sigma_{(U,P)}\|_{L^2(h_L)}$ be equivalent to $\frac{\|P\|_{L^2}}{\sqrt{\delta_L}}$ as d grows to infinity, where $\|P\|_{L^2}$ is defined by (9) and $\delta_L = \int_X \omega^n$,*
2. *$(U_d, \sigma^{-1}_{(U,P)}(0) \cap U_d)$ be diffeomorphic to $(U, P^{-1}(0) \cap U) \subset \mathbb{R}^n$,*

3. *for every* $(\delta,\epsilon)\in\mathcal{T}_{(U,P)}$ *given by Definition 2.2.2, there exists a compact subset* $K_d\subset U_d$ *such that*

$$\inf_{U_d\setminus K_d}|\sigma_{(U,P)}|>\frac{\delta}{2}\sqrt{d}^n,$$

while for every y *in* U_d,

$$|\sigma_{(U,P)}(y)|<\frac{\delta}{2}\sqrt{d}^n\Rightarrow\forall w\in\mathbb{R}^k,\ |(d_{|y}\sigma_{(U,P)})^*(w)|\geqslant\frac{\epsilon}{2}\sqrt{d}^{n+1}|w|. \qquad (24)$$

Proof. We proceed as in the proof of Proposition 3.2 of [16]. Let (U,P) be a regular pair, $x\in\mathbb{R}X$ and d large enough. We set $U_d=\psi_x^{-1}(\frac{1}{\sqrt{d}}U)\subset B(x,\frac{R_{(U,P)}}{\sqrt{d}})$ and $K_d=\psi_x^{-1}(\frac{1}{\sqrt{d}}K)$. Let $\chi:\mathbb{C}^n\to[0,1]$ be a smooth function with compact support in $B(0,R_{(U,P)})$, which equals one in a neighbourhood of the origin. Then, let σ be the global smooth section of $E\otimes L^d$ defined by $\sigma_{|X\setminus W_x}=0$ and

$$\sigma_{|W_x}=(\chi\circ\psi_x)\big(\sum_{j=1}^k P_j(\sqrt{d}\psi_x)e_j\otimes e^d\big),$$

where $P=(P_1,\cdots,P_k)$ is now considered as a function $\mathbb{C}^n\to\mathbb{C}^k$. From the L^2-estimates of Hörmander, see [17] or [21], there exists a global section τ of $E\otimes L^d$ such that $\bar\partial\tau=\bar\partial\sigma$ and $\|\tau\|_{L^2(h_{E,d})}\leqslant\|\bar\partial\sigma\|_{L^2(h_{E,d})}$ for d large enough. This section τ can be chosen orthogonal to holomorphic sections and is then unique, in particular real. Moreover, there exist positive constants c_1 and c_2, which do not depend on x, such that $\|\tau\|_{L^2(h_{E,d})}\leqslant c_1e^{-c_2d}$ and $\sup_{U_d}(|\tau|+\|\tau\|_2)\leqslant c_2e^{-c_2d}$, see Lemma 3.3 of [16]. We then set $\sigma_{(U,P)}=\sqrt{d}^n(\sigma-\tau)$. It has the desired properties as can be checked along the same lines as in the proof of Proposition 3.2 of [16] and thanks to Lemma 2.3.2.□

Lemma 2.3.2 *Let* U *be an open subset of* $\mathbb{R}^n$, $1\leqslant k\leqslant n$, $f:U\to\mathbb{R}^k$ *be a function of class* C^1 *and* $(\delta,\epsilon)\in(\mathbb{R}_+^*)^2$ *be such that*

1. *there exists a compact subset* K *of* U *such that* $\inf_{U\setminus K}|f|>\delta$,
2. *for every* y *in* U, $|f(y)|<\delta\Rightarrow\forall w\in\mathbb{R}^k,\ |(d_{|y}f)^*(w)|\geqslant\epsilon|w|$.

Then, for every function $g:U\to\mathbb{R}^k$ *of class* C^1 *such that* $\sup_U|g|<\delta$ *and* $\sup_U\|dg\|<\epsilon$, *zero is a regular value of* $f+g$ *and* $(f+g)^{-1}(0)$ *is compact and isotopic to* $f^{-1}(0)$ *in* U.

Proof. The proof is analogous to the one of Lemma 3.4 of [16], since $\|(dg)^*\|=\|dg\|$. □

The following Lemma 2.3.3 establishes the existence of peak sections for higher rank vector bundles.

Lemma 2.3.3 (compare Lemma 1.2 of [37]) *Let* X *be a smooth real projective manifold of dimension* n, (L,h_L) *be a real holomorphic Hermitian line bundle of positive curvature over* X *and* (E,h_E) *be a rank* k *real holomorphic Hermitian vector bundle, with* $1\leqslant k\leqslant n$. *Let* $x\in\mathbb{R}X$, $(p_1,\cdots,p_n)\in\mathbb{N}^n$, $i\in\{1,\cdots,k\}$ *and* $p'>$

$p_1 + \cdots + p_n$. *There exists* $d_0 \in \mathbb{N}$ *independent of* x *such that for every* $d > d_0$, *there exists* $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$ *with the property that* $\|\sigma\|_{L^2(h_L)} = 1$ *and if* $(y_1, \cdots, y_n)$ *are local real holomorphic coordinates in the neighbourhood of* x *and* $(e_1, \cdots e_k)$ *is a local real holomorphic trivialization of* E *orthonormal at* x, *we can assume that in a neighbourhood of* x,

$$\sigma(y_1, \cdots, y_n) = \lambda y_1^{p_1} \cdots y_n^{p_n} e_i \otimes e^d (1 + O(d^{-2p'})) + O(\lambda |y|^{2p'}), \tag{25}$$

where $\lambda^{-2} = \int_{B(x, \frac{\log d}{\sqrt{d}})} |y_1^{p_1} \cdots y_n^{p_n}|^2 \|e^d\|^2_{h_L^d} dV_{h_L}$, *with* $dV_{h_L} = \omega^n / \int_X \omega^n$ *and where* e *is a local trivialization of* L *whose potential* $-\log h_L(e, e)$ *reaches a local minimum at* x *with Hessian* $\pi\omega(., i.)$.

Proof. The proof goes along the same lines as the one of Lemma 1.2 of [37]. Let η be a cut-off function on $\mathbb{R}$ with $\eta = 1$ in a neighbourhood of 0, and

$$\psi = (n + 2p')\eta\big(\frac{d\|z\|^2}{\log^2 d}\big) \log\big(\frac{d\|z\|^2}{\log^2 d}\big)$$

in the coordinates z on X. Then, $i\partial\bar{\partial}\psi$ is bounded from below by $-C\omega$, where C is some uniform constant independent of d and x. Let $s \in C^\infty(X, E \otimes L^d)$ be the real section defined by

$$s = \eta\big(\frac{d\|z\|^2}{\log^2 d}\big) y_1^{p_1} \cdots y_n^{p_n} e_i \otimes e^d.$$

Then, from Theorem 5.1 of [6], for d large enough not depending on x, there exists a real section $u \in C^\infty(X, E \otimes L^d)$ such that $\bar{\partial} u = \bar{\partial} s$ and satisfying the Hörmander L^2-estimates

$$\int_X \|u\|^2_{h_{E,d}} e^{-\psi} dV_{h_L} \leqslant \int_X \|\bar{\partial} s\|^2_{h_{E,d}} e^{-\psi} dV_{h_L}.$$

The presence of the singular weight $e^{-\psi}$ forces the jets of u to vanish up to order $2p'$ at x. As in Lemma 1.2 of [37], we conclude that the real holomorphic section $\sigma = (s - u)/\|s - u\|_{L^2(h_{E,d})}$ satisfies the required properties. □

In this first section we will only need peak sections given by Lemma 2.3.3 with $\sum_{i=1}^n p_i = 0$, whereas in the second one we will need those given with $\sum_{i=1}^n p_i \leqslant 2$.

Definition 2.3.4 *For* $i \in \{1, \cdots, k\}$, *let* σ_0^i *be the section given by Lemma 2.3.3 with* $p' = 3$ *and* $p_1 = \cdots = p_n = 0$. *Likewise, for every* $j \in \{1, \cdots, n\}$, *let* σ_j^i *be a section given by (25) with* $p' = 3$, $p_j = 1$ *and* $p_l = 0$ *for* $l \in \{1, \cdots, n\} \backslash \{j\}$. *Finally, for every* $1 \leqslant l \leqslant m \leqslant n$, *let* σ_{lm}^i *be a section given by (25) with* $p' = 3$, $p_j = 0$ *for every* $j \in \{1, \cdots, n\} \backslash \{l, m\}$ *and* $p_l = p_m = 1$ *if* $l \neq m$, *while* $p_l = 2$ *otherwise.*

The asymptotic values of the constants λ in (25) are given by Lemma 2.3.5 (compare Lemma 2.1 of [37]).

Lemma 2.3.5 *For every* $i \in \{1, \cdots, k\}$, *the sections given by Definition 2.3.4 satisfy*

$$\sigma_0^i / \sqrt{\delta_L d^n} \underset{d \to \infty}{\sim} e_i \otimes e^d + O(\|y\|^6), \tag{26}$$

$$\forall j \in \{1, \cdots, n\}, \sigma_j^i / \sqrt{\pi \delta_L d^{n+1}} \underset{d \to \infty}{\sim} y_j e_i \otimes e^d + O(\|y\|^6), \tag{27}$$

$$\forall l, m \in \{1, \cdots, n\}, l \neq m, \sigma_{lm}^i / \big(\pi \sqrt{\delta_L d^{n+2}}\big) \underset{d \to \infty}{\sim} y_l y_m e_i \otimes e^d + O(\|y\|^6), \tag{28}$$

$$\text{and } \forall l \in \{1, \cdots, n\}, \sigma_{ll}^i / \big(\pi \sqrt{\delta_L d^{n+2}}\big) \underset{d \to \infty}{\sim} \frac{1}{\sqrt{2}} y_l^2 e_i \otimes e^d + O(\|y\|^6). \tag{29}$$

Moreover, these sections are asymptotically orthonormal as d grows to infinity, as follows from Lemma 2.3.6.

Lemma 2.3.6 (compare Lemma 3.1 of [37]) *For every $x \in \mathbb{R}X$, the sections $(\sigma_j^i)_{\substack{1\leqslant i\leqslant k\\ 0\leqslant j\leqslant n}}$ and $(\sigma_{lm}^i)_{\substack{1\leqslant i\leqslant k\\ 1\leqslant l\leqslant m\leqslant n}}$ given by Definition 2.3.4 have L^2-norm equal to one and their pairwise scalar product are dominated by a $O(d^{-1})$ which does not depend on x. Likewise, their scalar products with every section of $\mathbb{R}H^0(X, E\otimes L^d)$ of L^2-norm equal to one and whose 2-jet at x vanishes is dominated by a $O(d^{-3/2})$ which does not depend on x.*

Proof. The proof goes along the same lines as the one of Lemma 3.1 of [37]. □

Lemma 2.3.7 *Denote by v the density of $dV_{h_L} = \omega^n/\int_X \omega^n$ with respect to the volume form dx chosen in (6), so that $dV_{h_L} = v(x)dx$. Then the sections given by Defintion 2.3.4 times $\sqrt{v(x)}$ are still asymptotically orthonormal for (6).*

Proof. This is a direct consequence of Lemmas 2.3.3 and 2.3.6 and the asymptotic concentration of the support of the peak sections near x. □

Remark 2.3.8 *The complex analogues of Lemmas 2.3.3, 2.3.5 and 2.3.6 hold, compare [37].*

2.4 Proof of Theorem 1.2.1

We first compute the expected local C^1-norm of sections.

Proposition 2.4.1 *Let X be a smooth real projective manifold of dimension n, (L, h_L) be a real holomorphic Hermitian line bundle of positive curvature over X and (E, h_E) be a rank k real holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n$. We equip $\mathbb{R}X$ with a field of h_L-trivializations, see §2.3. Then, for every positive R,*

$$\limsup_{d\to\infty} \sup_{x\in\mathbb{R}X} \frac{1}{d^n} E\Big(\sup_{B(x,\frac{R}{\sqrt{d}})} \frac{|\sigma|^2}{v(x)}\Big) \leqslant 6k\delta_L\rho_R \text{ and}$$

$$\limsup_{d\to\infty} \sup_{x\in\mathbb{R}X} \frac{1}{d^{n+1}} E\Big(\sup_{B(x,\frac{R}{\sqrt{d}})} \frac{\|d\sigma\|_2^2}{v(x)}\Big) \leqslant 6\pi nk\delta_L\rho_R,$$

where v is given by Lemma 2.3.7 and ρ_R is given by (11), see (19) and (20) for the definitions of $|\sigma|$ and $\|d\sigma\|_2$.

Note that a global estimate on the sup norm of L^2 random holomorphic sections is given by Theorem 1.1 of [32].

Proof. The proof goes along the same lines as the proof of Proposition 3.5 of [16]. We first establish from the mean value inequality that for every $x \in \mathbb{R}X$, $R > 0$ and $s > 0$,

$$E\Big(\sup_{B(x,\frac{R}{\sqrt{d}})} |\sigma|^2\Big) \leqslant \frac{1}{Vol(B(\frac{s}{\sqrt{d}}))} \int_{B(x,\frac{R+s}{\sqrt{d}})} \mathbb{E}(|\sigma|^2)\psi_x^* dy$$

for d large enough not depending on x. Then, for every $z \in B(x, \frac{R+s}{\sqrt{d}}) \cap \mathbb{R}X$, we write $\sigma = \sum_{i=1}^k a_i \sigma_0^i + \tau$, where $\tau \in \mathbb{R}H^0(X, E \otimes L^d)$ vanishes at z and $(\sigma_0^i)_{i=1,\cdots k}$ are the peak sections at z given by Definition 2.3.4. In particular, by Lemma 2.3.5, at the point z, for every $i = 1, \cdots, k$, $\|\sigma_0^i\|_{h_{E,d}} \underset{d\to\infty}{\sim} \sqrt{\delta_L d^n}$. Moreover, since $(e_1, \cdots, e_n)$ is orthonormal at x,

$$\begin{aligned} |\sigma_0^i(z)|^2 &= \|\sigma_0^i(z)\|^2_{h_{E,d}}(1 + O(|z-x|)e^{d\phi(z)} \\ &\leqslant \delta_L d^n e^{\pi(R+s)^2}(1 + o(1)) \end{aligned}$$

from the inequalities (17), where the $o(d^n)$ can be chosen not to depend on $x \in \mathbb{R}X$. Suppose that $dy = dV_{h_L}$. Then, by Lemma 2.3.6, the peak sections are asymptotically orthogonal to each other for the scalar product defined by (6), and asymptotically orthogonal to the space of sections τ vanishing at x. We deduce that

$$\begin{aligned} \mathbb{E}(|\sigma(z)|^2) &= \mathbb{E}(\Big|\sum_{i=1}^k a_i \sigma_0^i\Big|^2)\,(1 + o(1)) \\ &= (\sum_{i=1}^k |\sigma_0^i(z)|^2)\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} a^2 e^{-a^2} da\,(1 + o(1)) \\ &\leqslant \frac{1}{2} k \delta_L d^n e^{\pi(R+s)^2}(1 + o(1)). \end{aligned}$$

When $z \notin B(x, \frac{R+s}{\sqrt{d}}) \cap \mathbb{R}X$, the space of real sections vanishing at z gets of real codimension $2k$ in $\mathbb{R}H^0(X, E \otimes L^d)$. Let $\langle \theta_1^i, \theta_2^i,\ i \in \{1, \cdots, k\}\rangle$ be an orthonormal basis of its orthogonal complement. From Remark 2.3.8, for every $i \in \{1, \cdots, k\}$, $j \in \{1, 2\}$,

$$\limsup_{d\to\infty} \frac{1}{d^n}|\theta_j^i(z)|^2 \leqslant 2\delta_L e^{\pi(R+s)^2},$$

an upper bound which does not depend on z. We deduce that

$$\begin{aligned} \mathbb{E}(|\sigma(z)|^2) &= \int_{\mathbb{R}^{2k}} |\sum_{i=1}^k (a_{01}^i \theta_1^i(z) + a_{02}^i \theta_2^i(z))|^2 e^{-\sum_{i=1}^k (a_{01}^i)^2 + (a_{02}^i)^2} \frac{1}{\pi^k} \Pi_{i=1}^k da_{01}^i da_{02}^i \\ &\leqslant 2\delta_L d^n e^{\pi(R+s)^2}(1 + o(1)) \sum_{i=1}^k \int_{\mathbb{R}^2} \left((a_{01}^i)^2 + (a_{02}^i)^2 + 2|a_{01}^i||a_{02}^i|\right) \dots \\ &\quad \dots \frac{1}{\pi} e^{-(a_{01}^i)^2 - (a_{02}^i)^2} da_{01}^i da_{02}^i \\ &\leqslant 6\delta_L d^n e^{\pi(R+s)^2}(1 + o(1)). \end{aligned}$$

We deduce the first part of Proposition 2.4.1 by taking the supremum over $\mathbb{R}X$, choosing s which minimize $g_{R_{(U,P)}}$ and taking the lim sup as d grows to infinity.

In general, the Bergman section at x for the L^2-product (6) associated to the volume form dx is equivalent to the Bergman section σ_0 at x for dV_h times $\sqrt{v(x)}$, see Lemma 2.3.7. The same holds true for the σ_j's, and the result follows by replacing δ_L with $v(x)\delta_L$.

The proof of the second assertion goes along the same lines, see the proof of Proposition 3.5 of [16] (and [31] for similar results). □

As in [16], we then compute the probability of presence of closed affine real algebraic submanifolds, inspired by an approach of Nazarov and Sodin [25], see also [20]. Let (U,P) be a regular pair given by Definition 2.2.1 and $\Sigma = P^{-1}(0) \subset U$. Then, for every $x \in \mathbb{R}X$, we set $B_d = B(x, \frac{R_{(U,P)}}{\sqrt{d}}) \cap \mathbb{R}X$, see (23), and denote by $Prob_{x,\Sigma}(E \otimes L^d)$ the probability that $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$ has the property that $\sigma^{-1}(0) \cap B_d$ contains a closed submanifold Σ' such that the pair (B_d, Σ') be diffeomorphic to $(\mathbb{R}^n, \Sigma)$. That is,

$$Prob_{x,\Sigma}(E \otimes L^d) = \mu_{\mathbb{R}} \left\{ \sigma \in \mathbb{R}H^0(X, E \otimes L^d) \,|\, (\sigma^{-1}(0) \cap B_d) \supset \Sigma',\ (B_d, \Sigma') \sim (\mathbb{R}^n, \Sigma) \right\}.$$

We then set $Prob_{\Sigma}(E \otimes L^d) = \inf_{x \in \mathbb{R}X} Prob_{x,\Sigma}(E \otimes L^d)$.

Proposition 2.4.2 *Let X be a smooth real projective manifold of dimension n, (L, h_L) be a real holomorphic Hermitian line bundle of positive curvature over X and (E, h_E) be a rank k real holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n$. Let (U,P) be a regular pair given by Definition 2.2.1 and $\Sigma = P^{-1}(0) \subset U$. Then,*

$$\liminf_{d \to \infty} Prob_{\Sigma}(E \otimes L^d) \geqslant m_{\tau_{(U,P)}},$$

see (15).

Proof. The proof is the same as the one of Proposition 3.6 of [16] and is not reproduced here. □

The proof of Theorem 1.2.1 (resp. Corollary 2.1.2) then just goes along the same lines as the one of Theorem 1.2 (resp. Corollary 1.3) of [16].

2.5 Proof of Theorem 2.1.1

Let (U,P) be a regular pair given by Definition 2.2.1. For every $d > 0$, let Λ_d be a maximal subset of $\mathbb{R}X$ with the property that two distinct points of Λ_d are at distance greater than $\frac{2R_{(U,P)}}{\sqrt{d}}$. The balls centered at points of Λ_d and of radius $\frac{R_{(U,P)}}{\sqrt{d}}$ are disjoints, whereas the ones of radius $\frac{2R_{(U,P)}}{\sqrt{d}}$ cover $\mathbb{R}X$. Note that if we use the local flat metric given by a trivial h_L-trivialization, then the associated lattice has asymptotically the same number of balls than Λ_d as d grows to infinity, so we can suppose from now on that the balls are defined for this local metric. For every $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$, denote by $N_{\Sigma}(\Lambda_d, \sigma)$ the number of $x \in \Lambda_d$ such that the ball $B_d = B(x, \frac{R_{(U,P)}}{\sqrt{d}}) \cap \mathbb{R}X$ contains a codimension k submanifold Σ' with $\Sigma' \subset \sigma^{-1}(0)$ and (B_d, Σ') diffeomorphic to $(\mathbb{R}^n, \Sigma)$. By definition of $N_{\Sigma}(\sigma)$, $N_{\Sigma}(\Lambda_d, \sigma) \leqslant N_{\Sigma}(\sigma)$, see §1.2, while from Proposition 2.4.2, for every $0 < \epsilon < 1$,

$$\begin{aligned}
|\Lambda_d| m_{\tau_{(U,P)}} &\leqslant \sum_{x \in \Lambda_d} Prob_{x,\Sigma}(E \otimes L^d) \\
&\leqslant \sum_{j=1}^{|\Lambda_d|} j \mu_{\mathbb{R}} \{\sigma | N_{\Sigma}(\Lambda_d, \sigma) = j\} \\
&\leqslant \epsilon m_{\tau_{(U,P)}} |\Lambda_d| \mu_{\mathbb{R}} \left\{\sigma | N_{\Sigma}(\Lambda_d, \sigma) \leqslant \epsilon m_{\tau_{(U,P)}} |\Lambda_d|\right\} \\
&\quad + |\Lambda_d| \mu_{\mathbb{R}} \left\{\sigma | N_{\Sigma}(\Lambda_d, \sigma) \geqslant \epsilon m_{\tau_{(U,P)}} |\Lambda_d|\right\}.
\end{aligned}$$

We deduce that

$$(1-\epsilon)m_{\tau_{(U,P)}} \leqslant \mu_{\mathbb{R}}\left\{\sigma |\, N_\Sigma(\sigma) \geqslant \epsilon m_{\tau_{(U,P)}}|\Lambda_d|\right\} \tag{30}$$

and the result follows by choosing a sequence $(U_p, P_p)_p \in \mathcal{I}_\Sigma$ such that

$$\lim_{p\to\infty} m_{\tau_{(U_p,P_p)}}|\Lambda_d| = c_\Sigma Vol_{h_L}(\mathbb{R}X)\sqrt{d}^n,$$

see (14). □

2.6 Proof of Corollary 1.2.2

In this paragraph, for every positive integer p, S^p denotes the unit sphere in $\mathbb{R}^{p+1}$. Corollary 1.2.2 is a consequence of Theorem 1.2.1 and the following Propositions 2.6.1 and 2.6.3.

Proposition 2.6.1 *For every* $1 \leqslant k \leqslant n$, $c_{S^{n-k}} \geqslant \exp(-e^{54+5n})$.

Recall the following.

Lemma 2.6.2 (Lemma 2.2 of [16]) *If* $P = \sum_{(i_1,\cdots,i_n)\in\mathbb{N}^n} a_{i_1,\cdots,i_n} z_1^{i_1}\cdots z_n^{i_n} \in \mathbb{R}[z_1,\cdots,z_n]$, *then*

$$\|P\|_{L^2}^2 = \int_{\mathbb{C}^n} |P(z)|^2 e^{-\pi|z|^2} dz = \sum_{(i_1,\cdots,i_n)\in\mathbb{N}^n} |a_{i_1,\cdots,i_n}|^2 \frac{i_1!\cdots i_n!}{\pi^{i_1+\cdots+i_n}}.$$

Proof of Proposition 2.6.1. For every $n > 0$, we set $P_k(x_1,\cdots,x_n) = \sum_{j=k}^n x_j^2 - 1$. For every $x \in \mathbb{R}^n$ and $\delta > 0$,

$$|P_k(x)| < \delta \Leftrightarrow 1-\delta < \sum_{i=k}^n x_i^2 < 1+\delta \Rightarrow \|d_{|x}P_k\|_2^2 = 4\sum_{i=k}^n x_i^2 > 4(1-\delta).$$

Moreover from Lemma 2.6.2,

$$\|P_k\|_{L^2}^2 = 1 + \frac{2(n-k+1)}{\pi^2} \leqslant n-k+2.$$

Now set $P_S = (P_1,\cdots,P_k)$ with $P_j(x) = x_j$ for $1 \leqslant j \leqslant k-1$, so that

$$\|P_S\|_{L_2}^2 \leqslant (k-1)/\pi + (n-k+2) \leqslant n+1 \leqslant 2n.$$

Since for every $w = (w_1,\cdots,w_k) \in \mathbb{R}^k$ and every $x \in \mathbb{R}^n$,

$$|d_{|x}P_S^*(w)|^2 = \sum_{i=1}^{k-1} w_i^2 + w_k^2 \|d_{|x}P_k\|_2^2,$$

we get that $\|d_{|x}P_S^*\|^2 \geqslant \min\left(1, 4(1-\delta)\right)$ if $|P_k(x)| < \delta$. Choose

$$U_S = \{(x_1,\cdots,x_n) \in \mathbb{R}^n \mid \sum_{j=1}^n x_j^2 < 4\}.$$

Then if $0 < \delta < 1$,

$$K_\delta = \left\{ x \in U_S \mid 1 - \delta \leqslant \sum_{i=k}^{n} x_i^2 \leqslant 1 + \delta \text{ and } \sum_{i=1}^{k-1} x_k^2 \leqslant 1 - \frac{1}{2}(1+\delta)^2 \right\}$$

is compact in U_S and taking $R^2_{(U_S,P_S)} = 4$, we see that the pair (U_S, P_S) is regular in the sense of Definition 2.2.1. The submanifold $P_S^{-1}(0) \subset U_S$ is isotopic in $\mathbb{R}^n$ to the unit sphere S^{n-k}. We deduce that $(3/4, 1) \in \mathcal{T}_{(U_S,P_S)}$. From (10) and (13) we deduce

$$\tau_{(U_S,P_S)} \leqslant 24k4^n e^{16\pi} 2n(2 + \pi n) \leqslant e^{53+5n}.$$

The estimate $c_{S^{n-1}} \geqslant \exp(-e^{54+5n})$ follows then from (16). □

Proposition 2.6.3 *For every $1 \leqslant k \leqslant n$ and every $0 \leqslant i \leqslant n-k$, $c_{S^i \times S^{n-i-k}} \geqslant \exp(-e^{82+6n})$.*

Proof. For every $1 \leqslant k \leqslant n$ and every $0 \leqslant i \leqslant n-k$, we set

$$Q_k((x_1, \cdots, x_{i+1}), (y_1, \cdots, y_{n-i-1})) = \left(|x|^2 - 2\right)^2 + \sum_{j=1}^{n-k-i} y_j^2 - 1.$$

For every $(x, y) \in \mathbb{R}^{i+1} \times \mathbb{R}^{n-i-1}$ and $0 < \delta < 1/2$,

$$\begin{aligned} |Q_k(x,y)| < \delta \;\; &\Leftrightarrow \;\; 1 - \delta < (|x|^2 - 2)^2 + \sum_{j=1}^{n-k-i} y_j^2 < 1 + \delta \\ &\Rightarrow \;\; \|d_{|(x,y)} Q_k\|_2^2 = 4 \sum_{j=1}^{n-k-i} y_j^2 + 16|x|^2(|x|^2 - 2)^2, \end{aligned}$$

with $|x|^2 > 2 - \sqrt{1+\delta} > 1/2$. Thus $\|d_{|(x,y)} Q_k\|_2^2 > 4(1-\delta)$, compare Lemma 2.6 of [16]. Moreover from Lemma 2.6.2, $\|Q_k\|_{L^2}^2 \leqslant 13n^2$, compare §2.3.2 of [16]. Now set $Q = (Q_1, \cdots, Q_k)$ with $Q_j(x, y) = y_{n-i-j}$ for $1 \leqslant j \leqslant k-1$, so that

$$\|Q\|_{L_2}^2 \leqslant (k-1)/\pi + 13n^2 \leqslant 13(n+1)^2.$$

For every $w = (w_1, \cdots, w_k) \in \mathbb{R}^k$ and every $(x, y) \in \mathbb{R}^{i+1} \times \mathbb{R}^{n-i-1}$,

$$\begin{aligned} |d_{|(x,y)} Q^*(w)|^2 &= \sum_{i=1}^{k-1} w_i^2 + w_k^2 \|d_{|(x,y)} Q_k\|_2^2 \\ &> \min(1, 4(1-\delta))|w|^2 \end{aligned}$$

if $|Q_k(x, y)| \leqslant \delta < 1/2$. We choose

$$U = \{(x, y) \in \mathbb{R}^{i+1} \times \mathbb{R}^{n-i-1} \mid |x|^2 + |y|^2 < 6\},$$

$$K_\delta = \left\{ (x,y) \in U \mid 1 - \delta \leqslant (|x|^2 - 2)^2 + \sum_{j=1}^{n-k-i} y_j^2 \leqslant 1 + \delta \text{ and } \sum_{j=1}^{k-1} y_{n-i-j}^2 \leqslant 1 - \delta \right\},$$

and $R^2_{(U,Q)} = 6$. The pair (U,Q) is regular in the sense of Definition 2.2.1 and $Q^{-1}(0) \subset U$ is isotopic in $\mathbb{R}^n$ to the product $S^i \times S^{n-i-k}$ of unit spheres in $\mathbb{R}^{i+1}$ and $\mathbb{R}^{n-i-k+1}$. We deduce that for every positive ϵ, $(1/2-\epsilon, 1) \in \mathcal{T}_{(U,Q)}$ and from (10) and (13) that

$$\tau_{(U,Q)} \leqslant 24k4^n e^{24\pi} 13(n+1)^2(4+\pi n) \leqslant e^{81+6n}.$$

The estimate $c_{S^i \times S^{n-i-k}} \geqslant \exp(-e^{82+6n})$ follows then from (16). □

3 Upper estimates for the expected Betti numbers

3.1 Statement of the results

For every $1 \leqslant k \leqslant n$, we denote by $\mathrm{Gr}(k-1, n-1)$ the Grassmann manifold of $(k-1)$-dimensional linear subspaces of $\mathbb{R}^{n-1}$. The tangent space of $\mathrm{Gr}(k-1,n-1)$ at every $H \in \mathrm{Gr}(k-1,n-1)$ is canonically isomorphic to the space of linear maps $L(H, H^\perp)$ from H to its orthogonal $H^\perp$ and we equip it with the norm

$$A \in L(H, H^\perp) \mapsto \|A\|_2 = \sqrt{Tr(A^*A)} \in \mathbb{R}^+.$$

The total volume of $\mathrm{Gr}(k-1,n-1)$ for this Riemannian metric is denoted by $Vol(\mathrm{Gr}(k-1,n-1))$ and we set

$$V_{k-1,n-1} = \frac{1}{\sqrt{\pi}^{(k-1)(n-k)}} Vol(\mathrm{Gr}(k-1,n-1))$$

its volume for the rescaled metric $A \in L(H,H^\perp) \mapsto \frac{1}{\sqrt{\pi}}\|A\|_2$. Likewise, we equip $M_{k-1}(\mathbb{R})$ with the Euclidean norm $A \in M_{k-1}(\mathbb{R}) \mapsto \|A\|_2 = \sqrt{Tr(A^*A)}$ and set $d\mu(A) = \frac{1}{\pi^{k-1}} e^{-\|A\|_2^2} dA$ the associated Gaussian measure on $M_{k-1}(\mathbb{R})$. Then, we set

$$E_{k-1}(|\det|^{n-k+2}) = \int_{M_{k-1(\mathbb{R})}} |\det A|^{n-k+2} d\mu(A).$$

Remark 3.1.1 *1. The orthogonal group $O_{n-1}(\mathbb{R})$ acts transitively on the Grassmannian $Gr(k-1,n-1)$ with fixators isomorphic to $O_{k-1}(\mathbb{R}) \times O_{n-k}(\mathbb{R})$. We deduce that*

$$\begin{aligned} Vol\big(Gr(k-1,n-1)\big) &= Vol(O_{n-1}(\mathbb{R}))/\big(Vol(O_{k-1}(\mathbb{R})) \times Vol\big(O_{n-k}(\mathbb{R})\big)\big) \\ &= \binom{n-1}{k-1} \sqrt{\pi}^{(k-1)(n-k)} \frac{\prod_{j=1}^{k-1} \Gamma(1+j/2)}{\prod_{j=n-k+1}^{n-1} \Gamma(1+j/2)}, \end{aligned}$$

where Γ denotes the Gamma function of Euler, see for example Lemma 3.4 of [14].

2. From formula (15.4.12) of [23] follows that

$$E_{k-1}(|\det|^{n-k+2}) = \prod_{j=1}^{k-1} \frac{\Gamma(\frac{n-k+2+j}{2})}{\Gamma(\frac{j}{2})},$$

so that $V_{k-1,n-1} E_{k-1}(|\det|^{n-k+2}) = \frac{(n-1)!}{(n-k)! 2^{k-1}}$.

We now keep the framework of §2.1. Let us denote, for every $i \in \{0, \cdots, n-k\}$, by $b_i(\mathbb{R}C_\sigma, \mathbb{R}) = \dim H_i(\mathbb{R}C_\sigma, \mathbb{R})$ the i-th Betti number of $\mathbb{R}C_\sigma$ and by

$$m_i(\mathbb{R}C_\sigma) = \inf_{f \text{ Morse on } \mathbb{R}C_\sigma} |\mathrm{Crit}_i(f)|$$

its i-th Morse number, where $|\mathrm{Crit}_i(f)|$ denotes the number of critical points of index i of f. We then denote by

$$\mathbb{E}(b_i) = \int_{\mathbb{R}H^0(X, E\otimes L^d)\setminus \mathbb{R}\Delta_d} b_i(\mathbb{R}C_\sigma, \mathbb{R}) d\mu_{\mathbb{R}}(\sigma)$$

and

$$\mathbb{E}(m_i) = \int_{\mathbb{R}H^0(X, E\otimes L^d)\setminus \mathbb{R}\Delta_d} m_i(\mathbb{R}C_\sigma) d\mu_{\mathbb{R}}(\sigma)$$

their expected values. The aim of §3 is to prove the following Theorem 3.1.2, see (1) for the definition of $e_{\mathbb{R}}(i, n-k-i)$.

Theorem 3.1.2 *Let X be a smooth real projective manifold of dimension n, (L, h_L) be a real holomorphic Hermitian line bundle of positive curvature over X and (E, h_E) be a rank k real holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n-1$. Then, for every $0 \leqslant i \leqslant n-k$,*

$$\limsup_{d\to\infty} \frac{1}{\sqrt{d}^n} \mathbb{E}(m_i) \quad \leqslant \quad \frac{1}{\Gamma(\frac{k}{2})} V_{k-1,n-1} E_{k-1}(|\det|^{n-k+2}) e_{\mathbb{R}}(i, n-k-i) Vol_{h_L}(\mathbb{R}X).$$

Note that the case $k = n$ is covered by Theorems 1.1.1 and 3.1.3. When $k = 1$ and $E = \mathcal{O}_X$, $Vol_{FS}(\mathbb{R}P^k) = \sqrt{\pi}$, see Remark 2.14 of [14], so that Theorem 3.1.2 reduces to Theorem 1.0.1 of [14] in this case. The proof of Theorem 3.1.2 actually goes along the same lines as the one of Theorem 1.1 of [14]. The strategy goes as follows. We fix a Morse function $p : \mathbb{R}X \to \mathbb{R}$. Then, almost surely on $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$, the restriction of p to $\mathbb{R}C_\sigma$ is itself a Morse function. For $i \in \{0, \cdots, n-k\}$, we denote by $\mathrm{Crit}_i(p_{|\mathbb{R}C_\sigma})$ the set of critical points of index i of this restriction and set

$$\nu_i(\mathbb{R}C_\sigma) = \frac{1}{\sqrt{d}^n} \sum_{x \in \mathrm{Crit}_i(p_{|\mathbb{R}C_\sigma})} \delta_x$$

if $n > k$ and $\nu_0(\mathbb{R}C_\sigma) = \frac{1}{\sqrt{d}^n} \sum_{x\in\mathbb{R}C_\sigma} \delta_x$ if $k = n$. We then set

$$\mathbb{E}(\nu_i) = \int_{\mathbb{R}H^0(X, E\otimes L^d)} \nu_i(\mathbb{R}C_\sigma) d\mu_{\mathbb{R}}(\sigma)$$

and prove the following equidistribution result (compare Theorem 1.2 of [14]).

Theorem 3.1.3 *Under the hypotheses of Theorem 3.1.2, let $p : \mathbb{R}X \to \mathbb{R}$ be a Morse function. Then, for every $i \in \{0, \cdots, n-k\}$, the measure $\mathbb{E}(\nu_i)$ weakly converges to*

$$\frac{1}{\Gamma(\frac{k}{2})} V_{k-1,n-1} E_{k-1}(|\det|^{n-k+2}) e_{\mathbb{R}}(i, n-k-i) dvol_{h_L}$$

as d grows to infinity. When $k = n$, $\mathbb{E}(\nu_0)$ converges weakly to $\frac{1}{\sqrt{\pi}} \Gamma(\frac{n+1}{2}) dvol_{h_L}$.

In Theorem 3.1.3 $dvol_{h_L}$ denotes the Lebesgue measure of $\mathbb{R}X$ induced by the Kähler metric. Theorem 3.1.2 is deduced from Theorem 3.1.3 by integration of 1 over $\mathbb{R}X$. The next paragraphs are devoted to the proof of Theorem 3.1.3.

Proof of Theorem 1.1.1. It follows from Theorem 3.1.2, the Morse inequalities, Remark 3.1.1 and the computation $Vol_{FS}\mathbb{R}P^n = \sqrt{\pi}/\Gamma(\frac{n+1}{2})$ (see Remark 2.14 of [14]) when $k \leqslant n-1$ and from Theorem 3.1.3 when $k = n$. □

3.2 Incidence varieties

Under the hypotheses of Theorem 3.1.3, we set

$$\mathbb{R}\Delta_p^d = \{\sigma \in \mathbb{R}H^0(X, E \otimes L^d) |\, \sigma \in \mathbb{R}\Delta_d \text{ or } p_{|\mathbb{R}C_\sigma} \text{ is not Morse}\}$$

and

$$\mathcal{I}_i = \{(\sigma, x) \in (\mathbb{R}H^0(X, E \otimes L^d)\backslash\mathbb{R}\Delta_p^d) \times (\mathbb{R}X\backslash Crit(p)) \,|\, x \in \mathrm{Crit}_i(p_{|\mathbb{R}C_\sigma})\}.$$

We set

$$\pi_1 : (\sigma, x) \in \mathcal{I}_i \;\mapsto\; \sigma \in \mathbb{R}H^0(X, E \otimes L^d) \text{ and} \tag{31}$$
$$\pi_2 : (\sigma, x) \in \mathcal{I}_i \;\mapsto\; x \in \mathbb{R}X. \tag{32}$$

Then, for every $(\sigma_0, x_0) \in ((\mathbb{R}H^0(X, E \otimes L^d)\backslash\mathbb{R}\Delta_p^d) \times (\mathbb{R}X\backslash Crit(p)))$, π_1 is invertible in a neighbourhood $\mathbb{R}U$ of σ_0, defining an evaluation map at the critical point

$$ev_{(\sigma_0,x_0)} : \sigma \in \mathbb{R}U \mapsto \pi_2 \circ \pi_1^{-1}(\sigma) = x \in \mathrm{Crit}_i(p_{|\mathbb{R}C_\sigma}) \cap \mathbb{R}V,$$

where $\mathbb{R}V$ denotes a neighbourhood of x_0 in $\mathbb{R}X$, compare §2.4.2 of [14]. We denote by $d_{|\sigma_0}ev^{\perp}_{(\sigma_0,x_0)}$ the restriction of its differential map $d_{|\sigma_0}ev_{(\sigma_0,x_0)}$ at σ_0 to the orthogonal complement of $\pi_1(\pi_2^{-1}(x_0))$ in $\mathbb{R}H^0(X, E \otimes L^d)$.

Proposition 3.2.1 *Under the hypotheses of Theorem 3.1.3,*

$$\mathbb{E}(\nu_i) = \frac{1}{\sqrt{d}^n}(\pi_2)_*(\pi_1^* d\mu_{\mathbb{R}}).$$

Moreover, at every point $x \in \mathbb{R}X\backslash Crit(p)$,

$$(\pi_2)_*(\pi_1^* d\mu_{\mathbb{R}})_{|x} = \frac{1}{\sqrt{\pi}^n} \int_{\pi_1(\pi_2^{-1}(x))} |\det d_{|\sigma} ev^{\perp}_{(\sigma,x)}|^{-1} d\mu_{\mathbb{R}}(\sigma) dvol_{h_L}.$$

Proof. The proof is the same as in the one of Proposition 2.10 of [14] and is not reproduced here. □

Fix $x \in \mathbb{R}X\backslash Crit(p)$. Then $\pi_1(\pi_2^{-1}(x))$ is open in a subspace of $\mathbb{R}H^0(X, E \otimes L^d)$. Namely,

$$\pi_1(\pi_2^{-1}(x)) \;=\; \big\{\sigma \in \mathbb{R}H^0(X, E \otimes L^d)\backslash\mathbb{R}\Delta_p^d \mid \sigma(x) = 0 \text{ and} \tag{33}$$
$$\exists \lambda \in \mathbb{R}(E \otimes L^d)^*_{|x},\ \lambda \circ \nabla_{|x}\sigma = d_{|x}p\big\}, \tag{34}$$

where $\mathbb{R}((E\otimes L^d)^*_{|x})$ is the real part of the fibre $(E\otimes L^d)^*_{|x}$. We deduce a well-defined map

$$\begin{aligned} \rho_x : \pi_1(\pi_2^{-1}(x)) &\to \mathrm{Gr}(n-k, \ker d_{|x}p) \times (\mathbb{R}(E\otimes L^d)^*_{|x}\backslash\{0\}) \quad (35)\\ \sigma &\mapsto (\ker \nabla_{|x}\sigma, \lambda). \quad (36)\end{aligned}$$

For every $\sigma \in \mathbb{R}H^0(X, E\otimes L^d)\backslash\mathbb{R}\Delta^d_p$, the tangent space of $\pi_1\big(\pi_2^{-1}(x)\big)$ at σ reads

$$\begin{aligned} T_\sigma\pi_1(\pi_2^{-1}(x)) &= \big\{\dot\sigma \in \mathbb{R}H^0(X, E\otimes L^d) \mid \dot\sigma(x) = 0 \text{ and}\\ &\quad \exists \dot\lambda \in \mathbb{R}(E\otimes L^d)^*_{|x} \mid \dot\lambda\circ\nabla_{|x}\sigma + \lambda\circ\nabla\dot\sigma_{|x} = 0\big\}.\end{aligned}$$

Likewise, for every $\lambda \in \mathbb{R}(E\otimes L^d)^*_{|x}\backslash\{0\}$, the tangent space of $\rho_x^{-1}(\mathrm{Gr}(n-k, \ker d_{|x}p)\times\{\lambda\})$ at σ reads

$$T_\sigma\rho_x^{-1}(\mathrm{Gr}(n-k, \ker d_{|x}p)\times\{\lambda\}) = \{\dot\sigma \in \mathbb{R}H^0(X, E\otimes L^d) \mid \dot\sigma(x) = 0 \text{ and } \lambda\circ\nabla_{|x}\dot\sigma = 0\}.$$

Finally, for every $K \in \mathrm{Gr}(n-k, \ker d_{|x}p)$, the tangent space of $\rho_x^{-1}(K, \lambda)$ at σ reads

$$T_\sigma\rho_x^{-1}(K, \lambda) = \{\dot\sigma \in \mathbb{R}H^0(X, E\otimes L^d) | \dot\sigma(x) = 0, \nabla_{|x}\dot\sigma_{|K} = 0 \text{ and } \lambda\circ\nabla_{|x}\dot\sigma = 0\}.$$

Let us choose local real holomorphic coordinates $(x_1, \cdots, x_n)$ of X near x such that $(\partial/\partial x_1, \cdots, \partial/\partial x_n)$ be orthonormal at x, with $d_{|x}p$ being colinear to dx_1 and such that $K = \ker\nabla_{|x}\sigma = \langle\partial/\partial x_{k+1}, \cdots, \partial/\partial x_n\rangle$. Let us choose a local real holomorphic trivialization $(e_1, \cdots, e_k)$ of E near x that is orthonormal at x and be such that $\ker\lambda_{|x} = \langle e_2\otimes e^d, \cdots, e_k\otimes e^d\rangle_{|x}$. For d large enough, we define the following subspaces of $\mathbb{R}H^0(X, E\otimes L^d)$:

$$\begin{aligned} H_x &= \big\langle(\sigma_0^i)_{1\leqslant i\leqslant k}, (\sigma_j^1)_{k+1\leqslant j\leqslant n}\big\rangle \quad (37)\\ H_\lambda &= \big\langle(\sigma_j^1)_{1\leqslant j\leqslant k}\big\rangle \quad (38)\\ H_K &= \big\langle(\sigma_j^i)_{\substack{2\leqslant i\leqslant k\\ k+1\leqslant j\leqslant n}}\big\rangle, \quad (39)\end{aligned}$$

where the sections $(\sigma_0^i)_{1\leqslant i\leqslant k}$ and $(\sigma_j^i)_{\substack{1\leqslant i\leqslant k\\ 1\leqslant j\leqslant n}}$ of $\mathbb{R}H^0(X, E\otimes L^d)$ are given by Lemma 2.3.3 and Definition 2.3.4.

H_K is a complement of $T_\sigma\rho_x^{-1}(K, \lambda)$ in $T_\sigma\rho_x^{-1}(\mathrm{Gr}(n-k, \ker d_{|x}p)\times\{\lambda\})$, H_λ is a complement of $T_\sigma\rho_x^{-1}(\mathrm{Gr}(n-k, \ker d_{|x}p)\times\{\lambda\})$ in $T_\sigma\pi_1(\pi_2^{-1}(x))$ and H_x is a complement of $T_\sigma\pi_1(\pi_2^{-1}(x))$ in $\mathbb{R}H^0(X, E\otimes L^d)$. Then, from Lemmas 2.3.6 and 2.3.7, up to a uniform rescaling by $\sqrt{v(x)}$, these complements are asymptotically orthogonal and their given basis orthonormal. Hence, we can assume from now on that $v = 1$.

Lemma 3.2.2 *Under the hypotheses of Theorem 3.1.3, let $(\sigma, x) \in \mathcal{I}_i$ and $\lambda \in \mathbb{R}(E\otimes L^d)^*_{|x}\backslash\{0\}$ such that $\lambda\circ\nabla_{|x}\sigma = d_{|x}p$. Then, $\lambda\circ\nabla^2\sigma_{|K_x} = \nabla^2(p_{|\mathbb{R}C_\sigma})_{|x}$, so that the quadratic form $\lambda\circ\nabla^2\sigma_{|K_x}$ is non-degenerated of index i.*

Proof. The proof is similar to the one of Lemma 2.9 of [14]. □

3.3 Computation of the Jacobian determinants

3.3.1 Jacobian determinant of ρ_x

Under the hypotheses of Theorem 3.1.3, let $(\sigma, x) \in \mathcal{I}_i$. We set $(K, \lambda) = \rho_x(\sigma)$ and denote by $d_{|\sigma}\rho_x^H$ the restriction of $d_{|\sigma}\rho_x$ to $H_K \oplus H_\lambda$. We then denote by $\det(d_{|\sigma}\rho_x^H)$ the Jacobian determinant of $d_{|\sigma}\rho_x^H$ computed in the given basis of H_λ and H_K, see (38), (39) and in orthonormal basis of $T_K \mathrm{Gr}(n-k, \ker d_{|x}p) \times \mathbb{R}(E \otimes L^d)^*_{|x}$. By assumption, the operator $\nabla_{|x}\sigma$ does not depend on the choice of a connection ∇ on $E \otimes L^d$ and is onto. We denote by $\nabla_{|x}\sigma^\perp$ its restriction to the orthogonal $K^\perp$ of $K = \ker \nabla_{|x}\sigma$,

$$\nabla_{|x}\sigma^\perp : K^\perp \to \mathbb{R}(E \otimes L^d)_{|x}.$$

Likewise, for every $(\dot\sigma_K, \dot\sigma_\lambda) \in H_K \oplus H_\lambda$, the operators $\nabla_{|x}\dot\sigma_K$ and $\nabla_{|x}\dot\sigma_\lambda$ do not depend on the choice of a connection ∇ on $E \otimes L^d$. Finally, we write at a point $y \in \mathbb{R}X$ near x

$$\sigma(y) = \sum_{i=1}^{k} \big(a_0^i \sigma_0^i + \sum_{j=1}^{n} a_j^i \sigma_j^i + \sum_{1 \leqslant l \leqslant m \leqslant n} a_{lm}^i \sigma_{lm}^i\big)(y) + o(|y|^2),$$

where (a_0^i), (a_j^i) and (a_{lm}^i) are real numbers and (σ_0^i), (σ_j^i) and (σ_{lm}^i) are given by Definition 2.3.4. From Lemma 2.3.5 and (33), we deduce that $a_0^i = 0 = a_j^1$ for $1 \leqslant i \leqslant k$ and $k+1 \leqslant j \leqslant n$, and that

$$\|\lambda\| \sqrt{\pi \delta_L} \sqrt{d}^{n+1} |a_1^1| = \|d_{|x}p\| + o(1), \tag{40}$$

where the $o(1)$ term is uniformly bounded over $\mathbb{R}X$.

Lemma 3.3.1 *Under the hypotheses of Theorem 3.1.3, let $(\sigma, x) \in \mathcal{I}_i$ and $(K, \lambda) = \rho_x(\sigma)$. Then, $d_{|x}\rho_x^H$ writes*

$$\begin{array}{rcl} H_K \oplus H_\lambda & \to & T_K Gr(n-k, \ker d_{|x}p) \times \mathbb{R}(E \otimes L^d)^*_{|x} \\ (\dot\sigma_K, \dot\sigma_\lambda) & \mapsto & \big(-(\nabla_{|x}\sigma^\perp)^{-1}_{|\ker\lambda} \circ \nabla_{|x}\dot\sigma_{K|K} \ , \ -\lambda \circ \nabla_{|x}\dot\sigma_\lambda \circ (\nabla_{|x}\sigma^\perp)^{-1} \big). \end{array}$$

Moreover, $|\det\ d_{|\sigma}\rho_x^H|^{-1} = \frac{|a_1^1|}{\|\lambda\|^k} |\det(a_j^i)_{2 \leqslant i,j \leqslant k}|^{n-k+1}(1+o(1))$, where the $o(1)$ term is uniformly bounded over $\mathbb{R}X$.

Proof. Let $(\dot\sigma_K, \dot\sigma_\lambda) \in H_K \oplus H_\lambda$ and $(\sigma_s)_{s \in]-\epsilon, \epsilon[}$ be a path of $\pi_1(\pi_2^{-1}(x))$ such that $\sigma_0 = \sigma$ and $\dot\sigma_0 = \dot\sigma_K + \dot\sigma_\lambda$. Then, for every $s \in]-\epsilon, \epsilon[$ and every $v_s \in \ker \nabla_{|x}\sigma_s$, there exists $\lambda_s \in \mathbb{R}(E \otimes L^d)^*_{|x}$ such that

$$\left\{ \begin{array}{rcll} \nabla_{|x}\sigma_s(v_s) & = & 0 & \text{and} \\ \lambda_s \circ \nabla_{|x}\sigma_s & = & d_{|x}p. & \end{array} \right.$$

By derivation, we deduce

$$\left\{ \begin{array}{rcll} \nabla_{|x}\dot\sigma_0(v_0) + \nabla_{|x}\sigma(\dot v_0) & = & 0 & \text{and} \\ \dot\lambda_0 \circ \nabla_{|x}\sigma + \lambda \circ \nabla_{|x}\dot\sigma_0 & = & 0. & \end{array} \right.$$

By setting $\dot v$ the orthogonal projection of $\dot v_0$ onto $K^\perp$, we deduce that

$$\left\{ \begin{array}{rcll} \dot v & = & -(\nabla_{|x}\sigma^\perp)^{-1} \circ \nabla_{|x}\dot\sigma_K(v_0) & \text{and} \\ \dot\lambda_0 & = & -\lambda \circ \nabla_{|x}\dot\sigma_\lambda \circ (\nabla_{|x}\sigma^\perp)^{-1}. & \end{array} \right.$$

The first part of Lemma 3.3.1 follows. Now, recall that $d_{|x}p$ is colinear to dx_1, that K is equipped with the orthonormal basis $(\partial/\partial x_{k+1}, \cdots, \partial/\partial x_n)$, $K^\perp$ with the orthonormal basis $(\partial/\partial x_1, \cdots, \partial/\partial x_k)$, and that $\ker \lambda_{|x}$ is spanned by the orthonormal basis $(e_2, \cdots, e_k)_{|x}$. From Lemma 2.3.3, the map

$$\dot\sigma_K \in H_K \mapsto \nabla_{|x}\dot\sigma_{K|K} \in L(K, \ker\lambda)$$

just dilates the norm by the factor $\sqrt{\pi\delta_L d^{n+1}}(1+o(1))$, where the $o(1)$ term is uniformly bounded over $\mathbb{R}X$. Now, since the matrix of the restriction of $\nabla_{|x}\sigma^\perp$ to $K^\perp \cap \ker d_{|x}p$ in the given basis of $K^\perp \cap \ker d_{|x}p$ and $\ker\lambda$ equals

$$\sqrt{\pi\delta_L d^{n+1}}(a^i_j)_{2\leqslant i,j\leqslant k} + o(\sqrt{d}^{n+1}),$$

where the $o(\sqrt{d}^{n+1})$ term is uniformly bounded over $\mathbb{R}X$. We deduce that the Jacobian determinant of the map

$$M \in L(K, \ker\lambda) \mapsto (\nabla_{|x}\sigma^\perp_{|\ker\lambda})^{-1} \circ M \in L(K, K^\perp \cap \ker d_{|x}p))$$

equals

$$\big((\sqrt{\pi\delta_L d^{n+1}})^{k-1} |\det(a^i_j)_{2\leqslant i,j\leqslant k}|(1+o(1))\big)^{k-n}.$$

The Jacobian determinant of the map

$$\dot\sigma_K \in H_K \mapsto (\nabla_{|x}\sigma^\perp)^{-1}_{|\ker\lambda} \circ \nabla_{|x}\dot\sigma_{K|K} \in T_K\mathrm{Gr}(n-k, \ker d_{|x}p)$$

thus equals $|\det(a^i_j)_{2\leqslant i,j\leqslant k}|^{k-n} + o(1)$, where the $o(1)$ is uniformly bounded over $\mathbb{R}X$. Likewise, from Lemma 2.3.3, the map

$$\dot\sigma_\lambda \in H_\lambda \mapsto \lambda \circ \nabla_{|x}\dot\sigma_\lambda \in (K^\perp)^*$$

just dilates the norm by a factor $\sqrt{\pi\delta_L d^{n+1}}\|\lambda\| + o(\sqrt{d}^{n+1})$, where the $o(\sqrt{d}^{n+1})$ is uniformly bounded over $\mathbb{R}X$, while the Jacobian determinant of the map

$$M \in (K^\perp)^* \mapsto M \circ (\nabla_{|x}\sigma^\perp)^{-1} \in \mathbb{R}(E\otimes L^d)^*_{|x}$$

equals $(\sqrt{\pi\delta_L}\sqrt{d}^{n+1})^{-k}|\det(a^i_j)_{1\leqslant i,j\leqslant k}|^{-1}(1+o(1))$ so that the Jacobian determinant of the map

$$\dot\sigma_\lambda \in H_\lambda \mapsto \lambda \circ \nabla_{|x}\dot\sigma_\lambda \circ (\nabla_{|x}\sigma^\perp)^{-1} \in \mathbb{R}(E\otimes L^d)^*_{|x}$$

equals $\|\lambda\|^k|\det(a^i_j)_{1\leqslant i,j\leqslant k}|^{-1} + o(1)$, with a $o(1)$ uniformly bounded over $\mathbb{R}X$. As a consequence,

$$|\det d_{|\sigma}\rho^H_x|^{-1} = \|\lambda\|^{-k}|\det(a^i_j)_{2\leqslant i,j\leqslant k}|^{n-k+1}|a^1_1|(1+o(1)),$$

with a $o(1)$ uniformly bounded over $\mathbb{R}X$, since the relation $\lambda\circ\nabla_{|x}\sigma = d_{|x}p$ implies that a^1_j vanishes for $2\leqslant j\leqslant n$. □

3.3.2 Jacobian determinant of the evaluation map

Again, under the hypotheses of Theorem 3.1.3 and for $(\sigma, x) \in \mathcal{I}_i$, we set for every y in a neighbourhood of x,

$$\sigma(y) = \sum_{i=1}^{k} \big(a_0^i \sigma_0^i + \sum_{j=1}^{n} a_j^i \sigma_j^i + \sum_{1 \leqslant l \leqslant m \leqslant n} a_{lm}^i \sigma_{lm}^i\big)(y) + o(|y|^2), \tag{41}$$

where a_0^i, a_j^i and a_{lm}^i are real numbers. We then set, for $1 \leqslant l, m \leqslant n$, $\tilde{a}_{ll}^1 = \sqrt{2} a_{ll}^1$, $\tilde{a}_{lm}^1 = a_{lm}^1$ if $l < m$ and $\tilde{a}_{lm}^1 = a_{ml}^1$ if $l > m$. We denote by $d_{|\sigma} ev_{(\sigma,x)}^H$ the restriction of $d_{|\sigma} ev_{(\sigma,x)}$ to H_x, see (37) and by $\det d_{|\sigma} ev_{(\sigma,x)}^H$ its Jacobian determinant computed in the given basis of H_x and orthonormal basis of $T_x \mathbb{R}X$.

Lemma 3.3.2 *Under the hypotheses of Theorem 3.1.3, let $(\sigma, x) \in \mathcal{I}_i$. Then,*

$$|\det d_{|\sigma} ev_{(\sigma,x)}^H|^{-1} = \sqrt{\pi^n d^n} |a_1^1| |\det(a_j^i)_{2 \leqslant i,j \leqslant k}| |\det(\tilde{a}_{lm}^1)_{k+1 \leqslant l,m \leqslant n}| (1 + o(1)),$$

where the $o(1)$ term has poles of order at most $n-k$ near the critical points of p.

Remark 3.3.3 *In Lemma 3.3.2, a function f is said to have a pole of order at most $n-k$ near a point x if $r^{n-k} f$ is bounded near x, where r denotes the distance function to x. Such a function thus belongs to $L^1(\mathbb{R}X, dvol_h)$.*

Proof. We choose a torsion free connection ∇^{TX} (resp. a connection $\nabla^{E \otimes L^d}$) on $\mathbb{R}X \backslash Crit(p)$ (resp. on $E \otimes L^d$) such that $\nabla^{TX} dp = 0$. They induce a connection on $T^*X \otimes E \otimes L^d$ which makes it possible to differentiate twice the elements of $\mathbb{R}H^0(X, E \otimes L^d)$. The tangent space of $\mathcal{I}_i$ then reads

$$\begin{aligned} T_{(\sigma,x)} \mathcal{I}_i \;=\; & \big\{(\dot{\sigma}, \dot{x}) \in \mathbb{R}H^0(X, E \otimes L^d) \times T_x \mathbb{R}X \mid \dot{\sigma}(x) + \nabla_{\dot{x}} \sigma = 0 \text{ and} & (42) \\ & \exists \dot{\lambda} \in \mathbb{R}(E \otimes L^d)_{|x}^*,\ \dot{\lambda} \circ \nabla_{|x} \sigma + \lambda \circ \nabla_{|x} \dot{\sigma} + \lambda \circ \nabla_{\dot{x},\cdot}^2 \sigma = 0\big\}. & (43) \end{aligned}$$

Recall that $T_x \mathbb{R}X$ is the direct sum $K \oplus K^\perp$, where $K = \ker \nabla_{|x} \sigma$. We write $\dot{x} = (\dot{x}_K, \dot{x}_{K^\perp})$ the coordinates of $\dot{x}$ in this decomposition. From the first equation we deduce, keeping the notations of §3.3.1, that $\dot{x}_{K^\perp} = -(\nabla_{|x} \sigma^\perp)^{-1}(\dot{\sigma}(x))$. From Lemma 2.3.3, the evaluation map at x

$$\dot{\sigma} \in \langle (\sigma_0^i)_{1 \leqslant i \leqslant k} \rangle \mapsto \dot{\sigma}(x) \in E \otimes L_{|x}^d$$

just dilates the norm by a factor $\sqrt{\delta_L d^n}(1 + o(1))$, , where the $o(1)$ term is uniformly bounded over $\mathbb{R}X$, while

$$|\det(\nabla_{|x} \sigma^\perp)| = (\sqrt{\pi \delta_L d^{n+1}})^k |\det(a_j^i)_{1 \leqslant i,j \leqslant k}| (1 + o(1)).$$

We deduce by composition that the Jacobian of the map

$$\dot{\sigma} \in \langle (\sigma_0^i)_{1 \leqslant i \leqslant k} \rangle \mapsto \dot{x}_{K^\perp} = -(\nabla_{|x} \sigma^\perp)^{-1}(\dot{\sigma}(x))$$

equals $\big(\sqrt{\pi^k d^k} |\det(a_j^i)_{2 \leqslant i,j \leqslant k}| |a_1^1|\big)^{-1} (1 + o(1))$, where the $o(1)$ term is uniformly bounded over $\mathbb{R}X$. Now, equation (43) restricted to K reads

$$\lambda \circ \nabla_{\dot{x}_K,\cdot}^2 \sigma_{|K} = -\lambda \circ \nabla_{|x} \dot{\sigma}_{|K}.$$

From Lemma 2.3.3, the map

$$\dot\sigma \in \langle (\sigma_j^1)_{k+1\leqslant j\leqslant n}\rangle \mapsto -\lambda\circ\nabla_{|x}\dot\sigma_{|K}\in K^*$$

just dilates the norm by a factor $\|\lambda\|\sqrt{\pi\delta_L d^{n+1}}(1+o(1))$, with $o(1)$ term is uniformly bounded over $\mathbb{R}X$. Likewise, from Lemma 2.3.3, the Jacobian of the map $\lambda\circ\nabla^2\sigma_{|K}: K\to K^*$ equals

$$(\|\lambda\|\pi\sqrt{\delta_L d^{n+2}})^{n-k}|\det(\tilde a^1_{lm})_{k+1\leqslant l,m\leqslant n}|(1+o(1)). \tag{44}$$

Here, the $o(1)$ term is no more uniformly bounded over $\mathbb{R}X$ though. Indeed, from Lemma 2.3.5 and (41),

$$\lambda\circ\nabla^2\sigma_{|K}=a_1^1(\|\lambda\|\sqrt{\pi\delta_L d^{n+1}})(\nabla^{TX}dx_1)+\sum_{1\leqslant l\leqslant m\leqslant n}\tilde a^1_{lm}(\|\lambda\|\sqrt{\pi\delta_L d^{n+2}})dx_l\otimes dx_m,$$

since the relation $\lambda\circ\nabla_{|x}\sigma=d_{|x}p$ imposes that a_j^1 vanishes for $j>1$. Moreover, since $dp=\sum_{i=1}^n\alpha_i dx_i$, with $\alpha_2(x)=\cdots=\alpha_n(x)=0$ and $|\alpha_1(x)|=\|d_{|x}p\|$, we get that

$$0=\nabla^{TX}(dp)_{|K}=\alpha_1(\nabla^{TX}dx_1)_{|K}+\sum_{i=1}^n(d\alpha_i\otimes dx_i)_{|K},$$

so that $\|\nabla^{TX}dx_{1|K}\|=\frac{1}{\|d_{|x}p\|}\|\sum_{i=1}^n d\alpha_i\otimes dx_i\|$ has a pole of order one at x. In formula (44), the $o(1)$ term has thus a pole of order at most $n-k$ near the critical points of p.

We deduce that the Jacobian determinant of the map

$$\dot\sigma\in\langle(\sigma_j^1)_{k+1\leqslant j\leqslant n}\rangle\mapsto\dot x_K=-(\lambda\circ\nabla^2\sigma_{|K})^{-1}\circ(\lambda\circ\nabla_{|x}\dot\sigma_{|K})\in K$$

equals $(\sqrt{\pi^{n-k}d^{n-k}}|\det(\tilde a^1_{lm})_{k+1\leqslant l,m\leqslant n}|)^{-1}(1+o(1))$, up to sign, where $o(1)$ term has a pole of order at most $n-k$ near the critical points of p. The result follows. □

3.4 Proof of Theorem 3.1.3

3.4.1 The case $k<n$

From Proposition 2.4.1 we know that

$$\mathbb{E}(\nu_i)=\frac{1}{\sqrt{\pi^n d^n}}\Big(\int_{\pi_1(\pi_2^{-1}(x))}|\det d_{|\sigma}ev^\perp_{(\sigma,x)}|^{-1}d\mu_{\mathbb{R}}(\sigma)\Big)dvol_{h_L}.$$

From the coarea formula (see [10]), we likewise deduce that

$$\begin{aligned}\mathbb{E}(\nu_i) &= \frac{1}{\sqrt{\pi^n d^n}}\Big(\int_{\mathrm{Gr}(n-k,\ker d_{|x}p)\times\mathbb{R}(E\otimes L^d)^*_{|x}\setminus\{0\}}e^{-(a_1^1)^2}\frac{dK\wedge d\lambda}{\sqrt{\pi}^{(n-k)(k-1)+k}}\cdots\\ &\quad\cdots\int_{\rho_x^{-1}(K,\lambda)}|\det d_{|\sigma}ev^\perp_{(\sigma,x)}|^{-1}|\det d_{|\sigma}\rho_x^\perp|^{-1}d\mu_{\mathbb{R}}(\sigma)\Big)dvol_{h_L},\end{aligned}$$

since with the notations (41), $\sigma\in\rho_x^{-1}(K,\lambda)$ if and only if $\forall i\in\{1,\cdots,k\}$ and $\forall j\in\{k+1,\cdots,n\}$, $a_0^i=0=a_j^i$ while $\forall j\geqslant 2$, $a_j^1=0$ and $|a_1^1|=\frac{\|d_{|x}p\|}{\|\lambda\|\sqrt{\pi\delta_L}\sqrt{d}^{n+1}}$. From

Lemma 2.3.6 and the relation (40), we deduce that for every $x \in \mathbb{R}X \backslash Crit(p)$ and every $(K,\lambda) \in \mathrm{Gr}(n-k, \ker d_{|x}p) \times \mathbb{R}(E \otimes L^d)^*_{|x} \backslash \{0\}$,

$$\begin{aligned} & \int_{\rho_x^{-1}(K,\lambda)} |\det d_{|\sigma} ev^{\perp}_{(\sigma,x)}|^{-1} |\det d_{|\sigma} \rho_x^{\perp}|^{-1} d\mu_{\mathbb{R}}(\sigma) \\ \underset{d\to\infty}{\sim} & \int_{\rho_x^{-1}(K,\lambda)} |\det d_{|\sigma} ev^{H}_{(\sigma,x)}|^{-1} |\det d_{|\sigma} \rho_x^{H}|^{-1} d\mu_{\mathbb{R}}(\sigma). \end{aligned}$$

Thus, from Lemmas 3.3.1, 3.3.2 and 3.2.2, $\mathbb{E}(\nu_i)$ converges to

$$\int_{M_{k-1}(\mathbb{R})} |\det(a^i_j)_{2\leqslant i,j\leqslant k}|^{n-k+2} d\mu(a^i_j) \int_{Sym_{\mathbb{R}}(i,n-k-i)} |\det(\tilde{a}^1_{lm})_{k+1\leqslant l,m\leqslant n}| d\mu(\tilde{a}^1_{lm}) ...$$
$$... \int_{\mathrm{Gr}(n-k,\ker d_{|x}p)\times\mathbb{R}(E\otimes L^d)^*_{|x}\backslash\{0\}} \frac{(a^1_1)^2 e^{-(a^1_1)^2}}{\|\lambda\|^k} \frac{dK \wedge d\lambda}{\sqrt{\pi}^{(n-k)(k-1)+k}},$$

where the convergence is dominated by a function in $L^1(\mathbb{R}X, dvol_{h_L})$, see Remark 3.3.3. We deduce that $\mathbb{E}(\nu_i)$ gets equivalent to

$$\frac{\|d_{|x}p\|^2}{\delta_L d^{n+1} \sqrt{\pi}^{k+2}} V_{k-1,n-1} E_{k-1}(|\det|^{n-k+2}) e_{\mathbb{R}}(i, n-k-i) \Big(\int_{\mathbb{R}(E\otimes L^d)^*_{|x}\backslash\{0\}} \frac{e^{-(a^1_1)^2}}{\|\lambda\|^{k+2}} d\lambda \Big) dvol_{h_L}.$$

Now,

$$\begin{aligned} \frac{\|d_{|x}p\|^2}{\pi\delta_L d^{n+1}} \int_{\mathbb{R}(E\otimes L^d)^*_{|x}\backslash\{0\}} \frac{e^{-(a^1_1)^2}}{\|\lambda\|^{k+2}} d\lambda &= \frac{Vol(S^{k-1}) \|d_{|x}p\|^2}{\pi\delta_L d^{n+1}} \int_0^{+\infty} \frac{e^{-(a^1_1)^2}}{\|\lambda\|^3} d\|\lambda\| \\ &= Vol(S^{k-1}) \int_0^{+\infty} e^{-r^2} r dr = \frac{1}{2} Vol(S^{k-1}). \end{aligned}$$

Since $Vol(S^{k-1}) = \frac{2\sqrt{\pi}^k}{\Gamma(k/2)}$, we finally deduce that $\mathbb{E}(\nu_i)$ weakly converges to

$$\frac{1}{\Gamma(k/2)} V_{k-1,n-1} E_{k-1}(|\det|^{n-k+2}) e_{\mathbb{R}}(i, n-k-i) dvol_{h_L},$$

where the convergence is dominated by a function in $L^1(\mathbb{R}X, dvol_{h_L})$. □

3.4.2 The case $k = n$

When the rank of E equals the dimension of X, the vanishing locus of a generic section σ of $\mathbb{R}H^0(X, E \otimes L^d)$ is a finite set of points. We set $\nu = \frac{1}{\sqrt{d}^n} \sum_{x\in\mathbb{R}C_\sigma} \delta_x$, and define the incidence variety as

$$\mathcal{I} = \{(\sigma, x) \in (\mathbb{R}H^0(X, E \otimes L^d) \backslash \mathbb{R}\Delta_d) \times \mathbb{R}X \,|\, \sigma(x) = 0\}.$$

The projections π_1 and π_2 are defined by (31) and (32). As before, for every $(\sigma_0, x_0) \in (\mathbb{R}H^0(X, E\otimes L^d)\backslash\mathbb{R}\Delta_d) \times \mathbb{R}X$, π_1 is invertible in a neighbourhood $\mathbb{R}U$ of σ_0, defining an evaluation map at the critical point

$$ev_{(\sigma_0,x_0)} : \sigma \in \mathbb{R}U \mapsto \pi_2 \circ \pi_1^{-1}(\sigma) = x \in \mathbb{R}C_\sigma \cap \mathbb{R}V,$$

where $\mathbb{R}V$ denotes a neighbourhood of x_0 in $\mathbb{R}X$, compare §2.4.2 of [14]. We denote by $d_{|\sigma_0}ev^{\perp}_{(\sigma_0,x_0)}$ the restriction of its differential map $d_{|\sigma_0}ev_{(\sigma_0,x_0)}$ at σ_0 to the orthogonal complement of $\pi_1(\pi_2^{-1}(x_0))$ in $\mathbb{R}H^0(X, E\otimes L^d)$. Then, from Proposition 3.2.1,

$$\mathbb{E}(\nu) = \frac{1}{\sqrt{d}^n}(\pi_2)_*(\pi_1^* d\mu_{\mathbb{R}})_{|x} = \frac{1}{\sqrt{\pi d}^n}\int_{\pi_1(\pi_2^{-1}(x))} |\det d_{|\sigma}ev^{\perp}_{(\sigma,x)}|^{-1} d\mu_{\mathbb{R}}(\sigma) dvol_{h_L}.$$

The space $H_x = \langle (\sigma_0^i)_{1\leqslant i\leqslant k}\rangle$ is a complement to $T_\sigma\pi_1(\pi_2^{-1}(x))$ in $\mathbb{R}H^0(X, E\otimes L^d)$ and in the decomposition (41), $a_0^i = 0$ for every $i = 1, \cdots, k$. The tangent space of $\mathcal{I}$ at (σ, x) reads

$$T_{(\sigma,x)}\mathcal{I} = \{(\dot\sigma, \dot x) \in \mathbb{R}H^0(X, E\otimes L^d)\times T_x\mathbb{R}X \,|\, \dot\sigma(x) + \nabla_{|x}\sigma(\dot x) = 0\}.$$

As in the proof of Lemma 3.3.2, we deduce that the Jacobian determinant of the map

$$\dot\sigma \in H_x \mapsto \dot x = -(\nabla_{|x}\sigma^{\perp})^{-1}(\dot\sigma(x)) \in T_x\mathbb{R}X$$

equals $\sqrt{\pi^n d^n}|\det(a_j^i)_{1\leqslant i,j\leqslant n}|(1+o(1))$, so that

$$|\det d_{|\sigma}ev^H_{(\sigma,x)}|^{-1} = \sqrt{\pi d^n}|\det(a_j^i)_{1\leqslant i,j\leqslant n}|(1+o(1)),$$

where the $o(1)$ term is uniformly bounded over $\mathbb{R}X$. From lemma 2.3.6 we deduce that $\mathbb{E}(\nu)$ gets equivalent to

$$\Big(\int_{M_n(\mathbb{R})} |\det(a_j^i)_{1\leqslant i,j\leqslant n}| d\mu(a_j^i)\Big) dvol_{h_L} = E_n(|\det|) dvol_{h_L}.$$

Formula (15.4.12) of [23], see Remark 3.1.1, now gives

$$E_n(|\det|) = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(1/2)} = \frac{1}{Vol_{FS}(\mathbb{R}P^n)},$$

see Remark 2.14 of [14], hence the result. □

3.5 Equidistribution of critical points in the complex case

Let X be a smooth complex projective manifold of dimension n, (L, h_L) be a holomorphic Hermitian line bundle of positive curvature ω over X and (E, h_E) be a rank k holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n$. For every $d > 0$, we denote by L^d the dth tensor power of L and by h^d the induced Hermitian metric on L^d. We denote by $H^0(X, L^d)$ its complex vector space of global holomorphic sections and by N_d the dimension of $H^0(X, L^d)$. We denote then by $\langle ., .\rangle$ the L^2-Hermitian product on this vector space, defined by the relation

$$\forall \sigma, \tau \in H^0(X, L^d), \langle \sigma, \tau\rangle = \int_X h^d(\sigma, \tau) dx. \tag{45}$$

The associated Gaussian measure is denoted by $\mu_{\mathbb{C}}$. It is defined, for every open subset U of $H^0(X, L^d)$, by

$$\mu_{\mathbb{C}}(U) = \frac{1}{\pi^{N_d}}\int_U e^{-\|\sigma\|^2} d\sigma, \tag{46}$$

where $d\sigma$ denotes the Lebesgue measure of $H^0(X, L^d)$. For every $d > 0$, we denote by Δ^d the discriminant hypersurface of $H^0(X, E \otimes L^d)$, that is the set of sections $\sigma \in H^0(X, E \otimes L^d)$ which do not vanish transversally. For every $\sigma \in H^0(X, E \otimes L^d) \backslash \{0\}$, we denote by C_σ the vanishing locus of σ in X. For every $\sigma \in H^0(X, E \otimes L^d) \backslash \Delta^d$, C_σ is then a smooth codimension k complex submanifold of X. We equip X with a Lefschetz pencil $p : X \dashrightarrow \mathbb{C}P^1$. We then denote, for every $d > 0$, by Δ_p^d the set of sections $\sigma \in H^0(X, E \otimes L^d))$ such that $\sigma \in \Delta^d$, or C_σ intersects the critical locus of p, or the restriction of p to C_σ is not a Lefschetz pencil. For d large enough, this extended discriminant locus is of measure 0 for the measure $\mu_{\mathbb{C}}$.

For every $\sigma \in H^0(X, E \otimes L^d) \backslash \Delta_p^d$, we denote by $Crit(p_{|C_\sigma})$ the set of critical points of the restriction of p to C_σ and set, for $1 \leqslant k \leqslant n-1$,

$$\nu(C_\sigma) = \frac{1}{d^n} \sum_{x \in Crit(p_{|C_\sigma})} \delta_x, \tag{47}$$

where δ_x denotes the Dirac measure of X at the point x. When $k = n$, $\nu(C_\sigma) = \frac{1}{d^n} \sum_{x \in C_\sigma} \delta_x$.

Theorem 3.5.1 *Let X be a smooth complex projective manifold of dimension n, (L, h_L) be a holomorphic Hermitian line bundle of positive curvature ω over X and (E, h_E) be a rank k holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n$. Let $p : X \dashrightarrow \mathbb{C}P^1$ be a Lefschetz pencil. Then, the measure $\mathbb{E}(\nu)$ defined by (47) weakly converges to $\binom{n-1}{k-1}\omega^n$ as d grows to infinity.*

When $k = 1$, Theorem 3.5.1 reduces to Theorem 3 of [15], see also Theorem 1.3 of [14].

Proof. The proof goes along the same lines as the one of Theorem 3.1.2, so we only give a sketch of it. Firstly, the analogue of Proposition 3.2.1 provides

$$\mathbb{E}(\nu) = \frac{1}{d^n} (\pi_2)_* (\pi_1^* d\mu_{\mathbb{C}}),$$

and at every point $x \in X \backslash (\mathrm{Crit}(p) \cup \mathrm{Base}(p))$, where $\mathrm{Base}(p)$ denotes the base locus of p,

$$(\pi_2)_* (\pi_1^* d\mu_{\mathbb{C}})_{|x} = \frac{1}{\pi^n} \int_{\pi_1(\pi_2^{-1}(x))} |\det d_{|\sigma} ev^\perp_{(\sigma,x)}|^{-2} d\mu_{\mathbb{R}}(\sigma) \frac{\omega^n}{n!},$$

see Proposition 2.10 of [14]. Choosing complex coefficients in decomposition (41), Lemmas 3.3.1 and 3.3.2 remain valid in the complex setting, see Remark 2.3.8. We deduce that

$$\begin{aligned}
\mathbb{E}(\nu) &= \frac{1}{\pi^n d^n} \Big(\int_{\pi_1(\pi_2^{-1}(x))} |\det d_{|\sigma} ev^\perp_{(\sigma,x)}|^{-2} d\mu_{\mathbb{C}}(\sigma) \Big) \frac{\omega^n}{n!} \\
&\underset{d \to \infty}{\sim} \frac{1}{\pi^n d^n} \Big(\int_{\mathrm{Gr}_{\mathbb{C}}(n-k, \ker d_{|x} p) \times (E \otimes L^d)^*_{|x} \backslash \{0\}} e^{-|a_1^1|^2} \frac{dK \wedge d\lambda}{\pi^{(n-k)(k-1)+k}} \cdots \\
&\qquad \cdots \int_{\rho_x^{-1}(K,\lambda)} |\det d_{|\sigma} ev^\perp_{(\sigma,x)}|^{-2} |\det d_{|\sigma} \rho_x^\perp|^{-2} d\mu_{\mathbb{C}}(\sigma) \Big) \frac{\omega^n}{n!},
\end{aligned}$$

with $|a_1^1|$ given by (40), see Lemma 2.3.6 as before. Here, $\mathrm{Gr}_{\mathbb{C}}(n-k, \ker d_{|x} p)$ denotes the Grassmann manifold of $n-k$-dimensional complex linear subspaces of $\ker d_{|x} p$.

From the complex versions of Lemma 2.3.5 and 2.3.6, see Remark 2.3.8 and the relation (40), we deduce that for every $x \in X \backslash (\mathrm{Crit}(p) \cup \mathrm{Base}(p))$ and every $(K, \lambda) \in \mathrm{Gr}(n-k, \ker d_{|x}p) \times (E \otimes L^d)^*_{|x} \backslash \{0\}$,

$$\begin{aligned} & \int_{\rho_x^{-1}(K,\lambda)} |\det d_{|\sigma} ev^{\perp}_{(\sigma,x)}|^{-2} |\det d_{|\sigma}\rho_x^{\perp}|^{-2} d\mu_{\mathbb{C}}(\sigma) \\ \underset{d\to\infty}{\sim} & \frac{|a_1^1|^4 \pi^n d^n}{\|\lambda\|^{2k}} \int_{M_{k-1}(\mathbb{C})} |\det(a^i_j)_{2\leqslant i,j\leqslant k}|^{2(n-k+2)} d\mu(a^i_j) ... \\ & ... \int_{Sym_{\mathbb{C}}(n-k)} |\det(\tilde{a}^1_{lm})_{k+1\leqslant l,m\leqslant n}|^2 d\mu(\tilde{a}^1_{lm}). \end{aligned}$$

We deduce that $\mathbb{E}(\nu)$ is equivalent to

$$\begin{aligned} & \frac{\|d_{|x}p\|^4}{(\pi\delta_L d^{n+1})^2} \frac{1}{\pi^{(n-k)(k-1)+k}} Vol(\mathrm{Gr}_{\mathbb{C}}(k-1, n-1)) ... \\ & ... \; E^{\mathbb{C}}_{k-1}(|\det|^{2(n-k+2)}) e_{\mathbb{C}}(n-k) \Big(\int_{(E\otimes L^d)^*_{|x}\backslash\{0\}} \frac{e^{-|a_1^1|^2}}{\|\lambda\|^{2(k+2)}} d\lambda \Big) \frac{\omega^n}{n!}, \end{aligned}$$

where $e_{\mathbb{C}}(n-k) = \int_{Sym_{\mathbb{C}}(n-k)} |\det A|^2 d\mu_{\mathbb{C}}(A)$ and

$$E^{\mathbb{C}}_{k-1}(|\det|^{2(n-k+2)}) = \int_{M_{k-1}(\mathbb{C})} |\det A|^{2(n-k+2)} d\mu_{\mathbb{C}}(A).$$

Now,

$$\begin{aligned} \frac{\|d_{|x}p\|^4}{(\pi\delta_L d^{n+1})^2} \int_{(E\otimes L^d)^*_{|x}\backslash\{0\}} \frac{e^{-|a_1^1|^2}}{\|\lambda\|^{2k+4}} d\lambda &= Vol(S^{2k-1}) \frac{\|d_{|x}p\|^4}{(\pi\delta_L d^{n+1})^2} \int_0^{+\infty} \frac{e^{-|a_1^1|^2}}{\|\lambda\|^5} d\|\lambda\| \\ &= Vol(S^{2k-1}) \int_0^{+\infty} e^{-r^2} r^3 dr = \frac{1}{2} Vol(S^{2k-1}). \end{aligned}$$

Hence, $\mathbb{E}(\nu)$ is equivalent to

$$\frac{1}{2\pi^{(n-k)(k-1)+k}} Vol(\mathrm{Gr}_{\mathbb{C}}(k-1, n-1)) Vol(S^{2k-1}) \; E^{\mathbb{C}}_{k-1}(|\det|^{2(n-k+2)}) e_{\mathbb{C}}(n-k) \frac{\omega^n}{n!},$$

where $e_{\mathbb{C}}(n-k) = (n-k+1)!$ by Proposition 3.8 of [14], $Vol(S^{2k-1}) = 2\pi^k/(k-1)!$,

$$E^{\mathbb{C}}_{k-1}(|\det|^{2(n-k+2)}) = \frac{\prod_{j=1}^{k-1} \Gamma((n-k+2)+j)}{\prod_{j=1}^{k-1} \Gamma(j)} = \frac{\prod_{j=n-k+3}^{n+1} \Gamma(j)}{\prod_{j=1}^{k-1} \Gamma(j)}$$

by formula 15.4.12 of [23] and

$$Vol(\mathrm{Gr}_{\mathbb{C}}(k-1, n-1)) = \frac{\prod_{j=1}^{k-1} \Gamma(j)}{\prod_{j=n-k+1}^{n-1} \Gamma(j)} \pi^{(k-1)(n-k)}$$

by a computation analogous to the one given in the real case by Remark 3.1.1. We conclude that $\mathbb{E}(\nu)$ weakly converges to $\binom{k-1}{n-1}\omega^n$, where the convergence is dominated by a function in $L^1(X, \frac{\omega^n}{n!})$, for it has poles of order at most $2(n-k)$ near the critical points of p and at most 2 near the base points, see [15]. □

Corollary 3.5.2 *Under the hypotheses of Theorem 3.5.1, for every generic* $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$*, let* $|Crit\ p_{|C_\sigma}|$ *be the number of critical points of* $p_{|C_\sigma}$*. Then,*

$$\frac{1}{d^n}\mathbb{E}(|Crit\ p_{|C_\sigma}|) \underset{d\to\infty}{\sim} \binom{k-1}{n-1}\int_X c_1(L)^n.$$

Proof. Corollary 3.5.2 follows from Theorem 3.5.1 by integration of 1 over X. A direct proof can be given though. The modulus of p is a Morse function on $C_\sigma \backslash (\text{Base}\ (p) \cup F_0 \cup F_\infty)$, where F_0 (resp. F_∞) is the fibre of 0 (resp. of ∞) of $p : X \dashrightarrow \mathbb{C}P^1$. Moreover, the index of every critical point of $|p|$ is $n-k$. As in the proofs of Propositions 1 and 2 in [15], we deduce that $\mathbb{E}(|\text{Crit}\ p_{|C_\sigma}|)$ is equivalent to $|\chi(C_\sigma)|$ as d grows to infinity. Now,

$$\chi(C_\sigma) = \int_{C_\sigma} c_{n-k}(C_\sigma) = \int_X c_{n-k}(C_\sigma) \wedge c_k(E \otimes L^d),$$

while from the adjunction formula, $c(C_\sigma) \wedge c(E \otimes L^d)_{|C_\sigma} = c(X)$. Moreover, for $0 \leqslant i \leqslant k$, $c_i(E \otimes L^d) = \binom{k}{i} d^i c_1(L)^i + o(d^i)$, so that

$$c(E \otimes L^d) = (1 + dc_1(L))^k + o((1 + dc_1(L))^k).$$

From the formula $(1+x)^{-k} = \sum_{j=0}^{\infty}(-1)^j \frac{(k-1+j)!}{j!(k-1)!}x^j$, we then deduce that $c_{n-k}(C_\sigma) = (-1)^{n-k}\binom{n-1}{k-1}d^{n-k}c_1(L)^{n-k} + o(d^{n-k})$ and finally that

$$\chi(C_\sigma) = (-1)^{n-k}\binom{n-1}{k-1}d^n\int_X c_1(L)^n + o(d^n).$$

Hence the result. □

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