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A moment estimator for the conditional extreme-value index

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Abstract. In extreme value theory, the so-called extreme-value index is a parameter that controls the behavior of a distribution function in its right tail. Knowing this parameter is thus essential to solve many problems related to extreme events. In this paper, the estimation of the extreme-value index is considered in the presence of a random covariate, whether the conditional distribution of the variable of interest belongs to the Fréchet, Weibull or Gumbel max-domain of attraction. The pointwise weak consistency and asymptotic normality of the proposed estimator are established and some illustrations on simulations are provided.

AMS Subject Classifications: 62G05, 62G20, 62G30, 62G32.

Keywords: Extreme-value index, moment estimator, random covariate, consistency, asymptotic normality.

1 Introduction

The problem of studying extreme events arises in many fields of statistical applications, such as climatology, hydrology, actuarial science or finance, to name a few. The pioneering result in extreme value theory, known as the Fisher-Tippett-Gnedenko theorem (see Fisher and Tippett [9] and Gnedenko [13]) states that if (Y_n) is an independent sequence of random copies of a random variable Y such that there exist normalizing nonrandom sequences of real numbers (a_n) and (b_n) , with $a_n > 0$ and such that the sequence

$$\frac{1}{a_n} \left(\max_{1 \leq i \leq n} Y_i - b_n \right)$$

converges in distribution to some nondegenerate limit, then the cumulative distribution function (cdf) of this limit can necessarily be written $y \mapsto G_\gamma(ay + b)$, with $a > 0$ and $b, \gamma \in \mathbb{R}$ where

$$G_\gamma(y) = \begin{cases} \exp(-(1 + \gamma y)^{-1/\gamma}) & \text{if } \gamma \neq 0 \text{ and } 1 + \gamma y > 0, \\ \exp(-\exp(-y)) & \text{if } \gamma = 0. \end{cases}$$

If the aforementioned convergence holds, we shall say that Y (or equivalently, its cdf F) belongs to the max-domain of attraction (MDA) of G_γ , with γ being the extreme-value index of Y , and we write $F \in \mathcal{D}(G_\gamma)$. Clearly, γ drives the behavior of F in its right tail:

- if $\gamma > 0$, namely Y belongs to the Fréchet MDA, then $1 - G_\gamma$ is heavy-tailed, *i.e.* it has a polynomial decay;
- if $\gamma < 0$, namely Y belongs to the Weibull MDA, then $1 - G_\gamma$ is short-tailed, *i.e.* it has a support bounded to the right;
- if $\gamma = 0$, namely Y belongs to the Gumbel MDA, then $1 - G_\gamma$ has an exponential decay.

The knowledge of γ is therefore necessary to tackle a number of problems in extreme value theory, such as the estimation of extreme quantiles of Y , which made its estimation a central topic in the literature. Recent monographs on extreme value theory and especially univariate extreme-value index estimation include Beirlant *et al.* [1] and de Haan and Ferreira [15].

In practical applications, it is often the case that the variable of interest Y can be linked to a covariate X . In this situation, the extreme-value index of the conditional distribution of Y given $X = x$ may depend on x ; the problem is then to estimate the conditional extreme-value index $x \mapsto \gamma(x)$. In most recent works, this problem has been addressed in the “fixed design” case, namely when the covariates are nonrandom. For instance, Smith [19] and Davison and Smith [7] considered a regression model while Hall and Tajvidi [16] used a semiparametric approach in this context; a nonexhaustive list of fully nonparametric methods include Davison and Ramesh [6] for a local polynomial estimator, Chavez-Demoulin and Davison [3] for a method using splines, Gardes and Girard [10] for a moving window approach and Gardes and Girard [11] who used a nearest neighbor approach.

By contrast, the case when the covariate is random, which is very interesting as far as practical applications are concerned, has only been tackled in even newer works. We refer to Wang and Tsai [20] for a maximum likelihood approach, Daouia *et al.* [4] who used a fixed number of nonparametric conditional quantile estimators to estimate the conditional extreme-value index, Goegebeur *et al.* [14] who studied a nonparametric regression estimator and Gardes and Stupfler [12] who introduced a smoothed local Hill estimator. Besides, the method of [4] was recently generalized in Daouia *et al.* [5] to a regression context with response distributions belonging to the general max-domain of attraction: the latter study is the only one in this list which is not restricted to the case of the Fréchet MDA.

The aim of this paper is to introduce a moment estimator of the conditional extreme-value index, working in the three domains of attraction. In Section 2, we define our estimator of the conditional extreme-value index. The pointwise weak consistency and asymptotic normality of the estimator are stated in Section 3. The finite sample performance of the estimator is studied in Section 4. Proofs of the main results are given in Section 5 and those of the auxiliary results are postponed to Section 6.

2 Estimation of the conditional extreme-value index

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be n independent copies of a random pair (X, Y) taking its values in $E \times (0, \infty)$ where E is a metric space endowed with a metric d . For all $x \in E$, we assume that the conditional survival function (csf) $\overline{F}(\cdot|x) = 1 - F(\cdot|x)$ of Y given $X = x$ belongs to $\mathcal{D}(G_{\gamma(x)})$. Specifically, we shall work in the following framework:

(M_1) Y is a positive random variable and for every $x \in E$, there exist a real number $\gamma(x)$ and a positive function $a(\cdot|x)$ such that the left-continuous inverse $U(\cdot|x)$ of $1/\overline{F}(\cdot|x)$, defined by $U(z|x) = \inf\{t \in \mathbb{R} \mid 1/\overline{F}(t|x) \geq z\}$ for every $z \geq 1$, satisfies

$$\forall z > 0, \lim_{t \rightarrow \infty} \frac{U(tz|x) - U(t|x)}{a(t|x)} = \begin{cases} \frac{z^{\gamma(x)} - 1}{\gamma(x)} & \text{if } \gamma(x) \neq 0 \\ \log z & \text{if } \gamma(x) = 0. \end{cases}$$

Model (M_1) is the conditional analogue of the classical extreme-value framework, see for instance [15], p.19. In this model, for every $x \in E$, the function $U(\cdot|x)$ has a positive limit $U(\infty|x)$ at infinity; the function $U(\infty|\cdot)$ is called the conditional right endpoint of Y .

We now introduce our estimator, which is an adaptation of the moment estimator of Dekkers *et al.* [8]. To this end, we let, for an arbitrary $x \in E$ and $h = h(n) \rightarrow 0$, $N_n(x, h)$ be the total number of observations in the closed ball $B(x, h)$ having center x and radius h :

$$N_n(x, h) = \sum_{i=1}^n \mathbb{1}_{\{X_i \in B(x, h)\}} \quad \text{with } B(x, h) = \{x' \in E \mid d(x, x') \leq h\},$$

where $\mathbb{1}_{\{\cdot\}}$ is the indicator function. The purpose of the bandwidth sequence $(h(n))$ is to select those covariates which are close enough to x . Given $N_n(x, h) = p \geq 1$, we let, for $i = 1, \dots, p$, $Z_i = Z_i(x, h)$ be the response variables whose associated covariates $W_i = W_i(x, h)$ belong to the ball $B(x, h)$. Let further $Z_{1,p} \leq \dots \leq Z_{p,p}$ be the related order statistics (this way of denoting order statistics shall be used throughout the paper) and set for $j = 1, 2$

$$M_n^{(j)}(x, k_x, h) = \frac{1}{k_x} \sum_{i=1}^{k_x} [\log(Z_{p-i+1,p}) - \log(Z_{p-k_x,p})]^j$$

if $k_x \in \{1, \dots, p-1\}$ and 0 otherwise. Given $N_n(x, h) = p$, the random variable $M_n^{(j)}(x, k_x, h)$ is then computed by using only the response variables whose values are not less than the random threshold $Z_{p-k_x,p}$ and whose associated covariates belong to a (small) neighborhood of x . For $j = 1$, this statistic is an analogue of Hill's estimator (see Hill [17]) in the presence of a random covariate; see also [11] for a nearest neighbor analogue of this quantity in the fixed design case. Our estimator, in the spirit of [8], is then

$$\widehat{\gamma}_n(x, k_x, h) = \widehat{\gamma}_{n,+}(x, k_x, h) + \widehat{\gamma}_{n,-}(x, k_x, h)$$

where

$$\widehat{\gamma}_{n,+}(x, k_x, h) = M_n^{(1)}(x, k_x, h) \quad \text{and} \quad \widehat{\gamma}_{n,-}(x, k_x, h) = 1 - \frac{1}{2} \left(1 - \frac{[M_n^{(1)}(x, k_x, h)]^2}{M_n^{(2)}(x, k_x, h)} \right)^{-1}$$

if $[M_n^{(1)}(x, k_x, h)]^2 \neq M_n^{(2)}(x, k_x, h)$, with $\widehat{\gamma}_{n,-}(x, k_x, h) = 0$ otherwise.

The assumption that Y is a positive random variable makes the quantities $M_n^{(j)}(x, k_x, h)$ well-defined for every k_x . This simplifies somewhat a couple of technical results (see for instance Lemma 3). We point out that since we shall only compute our estimator using upper order statistics of the Z_i , this hypothesis may be replaced by the assumption $U(\infty|x) > 0$ for every $x \in E$, at the price of extra regularity conditions on the joint cumulative distribution function F of the pair (X, Y) .

3 Main results

3.1 Weak consistency

We first wish to state the pointwise weak consistency of our estimator. To this end, let, for $x \in E$, $n_x = n_x(n, h) = n\mathbb{P}(X \in B(x, h))$ be the average total number of points in the ball $B(x, h)$: we assume that $n_x(n, h) > 0$ for every n . Let $k_x = k_x(n)$ be a sequence of positive integers; furthermore, let $F_h(\cdot|x)$ be the conditional cdf of Y given $X \in B(x, h)$:

$$F_h(y|x) = \mathbb{P}(Y \leq y | X \in B(x, h))$$

and $U_h(\cdot|x)$ be the left-continuous inverse of $1/\overline{F}_h(\cdot|x)$. For $u, v \in [1, \infty)$ such that $u < v$, we introduce the quantity

$$\omega(u, v, x, h) = \sup_{z \in [u, v]} \left| \log \frac{U_h(z|x)}{U(z|x)} \right|.$$

Recall the notation $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$ for $a, b \in \mathbb{R}$. Our consistency result is then:

Theorem 1. *Assume that (M_1) holds. Pick $x \in E$. We assume that $n_x \rightarrow \infty$, $k_x \rightarrow \infty$, $k_x/n_x \rightarrow 0$ and for some $\delta > 0$*

$$\frac{U(n_x/k_x|x)}{a(n_x/k_x|x)} \omega \left(\frac{n_x}{(1+\delta)k_x}, n_x^{1+\delta}, x, h \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (1)$$

Then, setting $\gamma_+(x) = 0 \vee \gamma(x)$ and $\gamma_-(x) = 0 \wedge \gamma(x)$, it holds that

$$\widehat{\gamma}_{n,+}(x, k_x, h) \xrightarrow{\mathbb{P}} \gamma_+(x) \quad \text{and} \quad \widehat{\gamma}_{n,-}(x, k_x, h) \xrightarrow{\mathbb{P}} \gamma_-(x) \quad \text{as } n \rightarrow \infty$$

and therefore $\widehat{\gamma}_n(x, k_x, h) \xrightarrow{\mathbb{P}} \gamma(x)$ as $n \rightarrow \infty$.

Theorem 1 is the conditional analogue of the consistency result proven in [8]; see also [15], Theorem 3.5.2. As far as the conditions of this result are concerned, note that conditions $n_x \rightarrow \infty$, $k_x \rightarrow \infty$ and

$k_x/n_x \rightarrow 0$ are standard hypotheses for the estimation of the conditional extreme-value index: they are the exact analogues of the conditions $n \rightarrow \infty$, $k = k(n) \rightarrow \infty$ and $k/n \rightarrow 0$ needed to ensure the convergence of Hill's estimator. Moreover, condition $n_x \rightarrow \infty$ is necessary to ensure that there are sufficiently many observations close to x , which is a standard assumption in the random covariate case.

Condition (1) is somewhat harder to grasp. To analyze further this hypothesis, we assume that

(A₁) For every $x \in E$, $U(\cdot|x)$ is a continuous increasing function on $[1, \infty)$.

We may now state the following result, which relates the behavior of the function $\log U_h(z|\cdot)$ around x to that of $\log U(z|\cdot)$:

Proposition 1. *Assume that (M₁) and (A₁) hold. Then for every $z > 1$, it holds that*

$$\left| \log \frac{U_h(z|x)}{U(z|x)} \right| \leq \sup_{x' \in B(x, h)} \left| \log \frac{U(z|x')}{U(z|x)} \right|.$$

With this result at hand, we define for $u, v \in [1, \infty)$ such that $u < v$:

$$\Omega(u, v, x, h) = \sup_{z \in [u, v]} \sup_{x' \in B(x, h)} \left| \log \frac{U(z|x')}{U(z|x)} \right|.$$

Proposition 1 entails that

$$\omega \left(\frac{n_x}{(1+\delta)k_x}, n_x^{1+\delta}, x, h \right) \leq \Omega \left(\frac{n_x}{(1+\delta)k_x}, n_x^{1+\delta}, x, h \right).$$

Consequently, if (A₁) is satisfied, a sufficient condition for (1) to hold is

$$\frac{U(n_x/k_x|x)}{a(n_x/k_x|x)} \Omega \left(\frac{n_x}{(1+\delta)k_x}, n_x^{1+\delta}, x, h \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (2)$$

which is an hypothesis on the uniform oscillation of $\log U$ in its second variable. To understand more about this condition, we introduce an additional regularity assumption:

(A₂) The function γ is a continuous function on E .

If we omit the case $\gamma(x) = 0$ of the Gumbel MDA, then under (A₂), condition (2) can be made more explicit:

- If $\gamma(x) > 0$, namely $\overline{F}(\cdot|x)$ belongs to the Fréchet MDA, then Lemma 1.2.9 in [15] entails that $a(\cdot|x)/U(\cdot|x)$ converges to $\gamma(x)$ at infinity. Condition (2) then becomes

$$\Omega \left(\frac{n_x}{(1+\delta)k_x}, n_x^{1+\delta}, x, h \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3)$$

Since γ is continuous, one has $\gamma(x') > 0$ for x' close enough to x . Corollary 1.2.10 in [15] then yields for n large enough and every $x' \in B(x, h)$

$$\forall z \geq 1, U(z|x') = z^{\gamma(x')} L(z|x')$$

where for every $x' \in B(x, h)$, $L(\cdot|x')$ is a slowly varying function at infinity. Letting

$$\forall z \geq 1, L(z|x') = c(z|x') \exp\left(\int_1^z \frac{\Delta(v|x')}{v} dv\right) \quad (4)$$

be Karamata's representation of $L(\cdot|x')$ (see Theorem 1.3.1 in Bingham *et al.* [2]), where $c(\cdot|x')$ is a positive Borel measurable function converging to a positive constant at infinity and $\Delta(\cdot|x')$ is a Borel measurable function converging to 0 at infinity, condition (3) is thus a consequence of the convergences

$$\log n_x \sup_{x' \in B(x, h)} |\gamma(x') - \gamma(x)| \rightarrow 0, \quad (5)$$

$$\sup_{z \in K_{x, \delta}} \sup_{x' \in B(x, h)} |\log c(z|x') - \log c(z|x)| \rightarrow 0, \quad (6)$$

$$\text{and } \log n_x \sup_{z \in K_{x, \delta}} \sup_{x' \in B(x, h)} |\Delta(z|x') - \Delta(z|x)| \rightarrow 0 \quad (7)$$

as $n \rightarrow \infty$, where $K_{x, \delta} = [n_x / [(1 + \delta)k_x], n_x^{1+\delta}]$. Assuming that $L(\cdot|x')$ is normalised for $x' \in B(x, h)$, namely that $c(\cdot|x')$ is constant equal to $c(x') > 0$, condition (6) above reduces to a continuity assumption on the function $\log c$. Besides, if γ and Δ satisfy some sort of Hölder condition, for instance

$$\sup_{x' \in B(x, h)} |\gamma(x') - \gamma(x)| = O(h^\alpha) \quad \text{and} \quad \sup_{z \in K_{x, \delta}} \sup_{x' \in B(x, h)} |\Delta(z|x') - \Delta(z|x)| = O(h^\alpha) \quad (8)$$

for some $\alpha \in (0, 1]$ as $n \rightarrow \infty$, then conditions (5) and (7) become $h^\alpha \log n_x \rightarrow 0$ as $n \rightarrow \infty$. The regularity conditions above are fairly standard when estimating the conditional tail-index in the Fréchet MDA, see for instance [4].

- If $\gamma(x) < 0$, namely $\overline{F}(\cdot|x)$ belongs to the Weibull MDA, then according to Lemma 1.2.9 in [15], one has

$$\frac{U(\infty|x) - U(z|x)}{a(z|x)} \rightarrow -\frac{1}{\gamma(x)} \quad \text{as } z \rightarrow \infty.$$

Furthermore, since one has $\gamma(x') < 0$ for x' close enough to x , Corollary 1.2.10 in [15] yields for n large enough and every $x' \in B(x, h)$ that

$$\forall z \geq 1, U(\infty|x') - U(z|x') = z^{\gamma(x')} L(z|x')$$

where for every $x' \in B(x, h)$, $L(\cdot|x')$ is a slowly varying function at infinity. Especially

$$\frac{U(z|x)}{a(z|x)} = -\frac{U(\infty|x)}{\gamma(x)} \frac{z^{-\gamma(x)}}{L(z|x)} (1 + o(1)) \quad \text{as } z \rightarrow \infty.$$

Consequently, in this framework, condition (2) becomes

$$\frac{(n_x/k_x)^{-\gamma(x)}}{L(n_x/k_x|x)} \Omega\left(\frac{n_x}{(1+\delta)k_x}, n_x^{1+\delta}, x, h\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (9)$$

Write then for an arbitrary $z > 1$ and for $x' \in B(x, h)$

$$\log U(z|x') - \log U(z|x) = \log \frac{U(\infty|x')}{U(\infty|x)} + \log \left(\frac{1 - [U(\infty|x')]^{-1} z^{\gamma(x')} L(z|x')}{1 - [U(\infty|x)]^{-1} z^{\gamma(x)} L(z|x)} \right). \quad (10)$$

The first term on the right-hand side in (10) is readily controlled if the conditional right endpoint $x \mapsto U(\infty|x)$ is a positive Hölder continuous function on E :

$$\sup_{x' \in B(x, h)} \left| \frac{U(\infty|x')}{U(\infty|x)} - 1 \right| = O \left(\sup_{x' \in B(x, h)} |U(\infty|x') - U(\infty|x)| \right) = O(h^\beta) \quad (11)$$

say, with $\beta \in (0, 1]$. The second one can be bounded from above as follows: since $n_x/k_x \rightarrow \infty$ and $z^{\gamma(x)}L(z|x) \rightarrow 0$ as $z \rightarrow \infty$ (see Proposition 1.5.1 in [2]), we can write for n large enough

$$\begin{aligned} \left| \frac{1 - [U(\infty|x')]^{-1} z^{\gamma(x')} L(z|x')}{1 - [U(\infty|x)]^{-1} z^{\gamma(x)} L(z|x)} - 1 \right| &\leq 2 \left| [U(\infty|x)]^{-1} z^{\gamma(x)} L(z|x) - [U(\infty|x')]^{-1} z^{\gamma(x')} L(z|x') \right| \\ &\leq 2 \frac{z^{\gamma(x)} L(z|x)}{U(\infty|x)} \left| 1 - \frac{[U(\infty|x')]^{-1} z^{\gamma(x')} L(z|x')}{z^{\gamma(x)} L(z|x)} \right| \end{aligned} \quad (12)$$

for every $z \geq n_x / [(1 + \delta)k_x]$. Note now that for every $z \in K_{x, \delta}$ we have

$$\left| \log \left(\frac{z^{\gamma(x')} L(z|x')}{z^{\gamma(x)} L(z|x)} \right) \right| \leq (1 + \delta) \log n_x \sup_{x' \in B(x, h)} |\gamma(x') - \gamma(x)| + \sup_{z \in K_{x, \delta}} \sup_{x' \in B(x, h)} \left| \log \frac{L(z|x')}{L(z|x)} \right|.$$

Using Karamata's representation of $L(\cdot|x')$ (see (4)) and assuming that for some $\alpha \in (0, 1]$

$$\sup_{x' \in B(x, h)} |\gamma(x') - \gamma(x)| = O(h^\alpha), \quad (13)$$

$$\sup_{z \in K_{x, \delta}} \sup_{x' \in B(x, h)} |\log c(z|x') - \log c(z|x)| = O(h^\alpha), \quad (14)$$

$$\text{and } \sup_{z \in K_{x, \delta}} \sup_{x' \in B(x, h)} |\Delta(z|x') - \Delta(z|x)| = O(h^\alpha) \quad (15)$$

as $n \rightarrow \infty$, then using the inequality $|e^t - 1| \leq |t|e^{|t|}$ valid for every $t \in \mathbb{R}$, it is readily seen that if $h^\alpha \log n_x \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\sup_{z \in K_{x, \delta}} \sup_{x' \in B(x, h)} \left| \frac{z^{\gamma(x')} L(z|x')}{z^{\gamma(x)} L(z|x)} - 1 \right| = O(h^\alpha \log n_x). \quad (16)$$

Note that conditions (13) and (15) are exactly (8); condition (14) is a strengthened version of (6), which reduces to a simple Hölder condition if the slowly varying function $L(\cdot|x')$ is assumed to be normalised for every $x' \in B(x, h)$. Equations (11), (12) and (16) now entail

$$\sup_{z \in K_{x, \delta}} \sup_{x' \in B(x, h)} \left| \frac{1 - [U(\infty|x')]^{-1} z^{\gamma(x')} L(z|x')}{1 - [U(\infty|x)]^{-1} z^{\gamma(x)} L(z|x)} - 1 \right| = O \left((h^\alpha \log n_x \vee h^\beta) \sup_{z \in K_{x, \delta}} z^{\gamma(x)} L(z|x) \right).$$

Potter bounds for the regularly varying function $z \mapsto z^{\gamma(x)}L(z|x)$ (see Theorem 1.5.6 in [2]) yield

$$\limsup_{n \rightarrow \infty} \sup_{z \in K_{x, \delta}} \frac{z^{\gamma(x)} L(z|x)}{(n_x/k_x)^{\gamma(x)} L(n_x/k_x|x)} < \infty$$

so that

$$\sup_{z \in K_{x, \delta}} \sup_{x' \in B(x, h)} \left| \frac{1 - [U(\infty|x')]^{-1} z^{\gamma(x')} L(z|x')}{1 - [U(\infty|x)]^{-1} z^{\gamma(x)} L(z|x)} - 1 \right| = O \left((h^\alpha \log n_x \vee h^\beta) \frac{L(n_x/k_x|x)}{(n_x/k_x)^{-\gamma(x)}} \right). \quad (17)$$

Finally, use together (10), (11) and (17) to get

$$\Omega \left(\frac{n_x}{(1 + \delta)k_x}, n_x^{1+\delta}, x, h \right) = O \left(h^\beta \vee \left[(h^\alpha \log n_x) \frac{L(n_x/k_x|x)}{(n_x/k_x)^{-\gamma(x)}} \right] \right). \quad (18)$$

Equation (18) makes it clear that in this case, condition (9) shall be satisfied provided it holds that $h^\alpha \log n_x \rightarrow 0$ (which was already required in the Fréchet MDA) and

$$\frac{(n_x/k_x)^{-\gamma(x)}}{L(n_x/k_x|x)} h^\beta \rightarrow 0 \text{ as } n \rightarrow \infty.$$

We can conclude that compared to the case of the Fréchet MDA, there is an additional condition for the pointwise consistency of our estimator to hold in the Weibull MDA. This condition compares the oscillation of the conditional right endpoint to the proportion of order statistics used in the expression of the estimator.

3.2 Asymptotic normality

To prove a pointwise asymptotic normality result for our estimator, we need to introduce a second-order condition on the function $U(\cdot|x)$:

(M_2) Condition (M_1) holds and for every $x \in E$, there exist a real number $\rho(x) \leq 0$ and a function $A(\cdot|x)$ of constant sign converging to 0 at infinity such that the function $U(\cdot|x)$ satisfies

$$\forall z > 0, \lim_{t \rightarrow \infty} \frac{\frac{U(tz|x) - U(t|x)}{a(t|x)} - \frac{z^{\gamma(x)} - 1}{\gamma(x)}}{A(t|x)} = H_{\gamma(x), \rho(x)}(z)$$

where

$$H_{\gamma(x), \rho(x)}(z) = \int_1^z r^{\gamma(x)-1} \left[\int_1^r s^{\rho(x)-1} ds \right] dr.$$

Hypothesis (M_2) is the conditional analogue of the classical second-order condition on U , see for instance Definition 2.3.1 and Corollary 2.3.4 in [15]: the parameter $\rho(x)$ is the so-called second-order parameter of Y given $X = x$. Note that Theorem 2.3.3 in [15] shows that the function $|A(\cdot|x)|$ is regularly varying at infinity with index $\rho(x)$. Moreover, as shown in Lemma B.3.16 therein, if (M_2) holds with $\gamma(x) \neq \rho(x)$ and $\rho(x) < 0$ if $\gamma(x) > 0$, then defining $q(\cdot|x) = a(\cdot|x)/U(\cdot|x)$, a second-order condition also holds for the function $\log U(\cdot|x)$, namely:

$$\forall z > 0, \lim_{t \rightarrow \infty} \frac{\frac{\log U(tz|x) - \log U(t|x)}{q(t|x)} - \frac{z^{\gamma(x)} - 1}{\gamma(x)}}{Q(t|x)} = H_{\gamma(x), \rho'(x)}(z) \quad (19)$$

with

$$\rho'(x) = \begin{cases} \rho(x) & \text{if } \gamma(x) < \rho(x) \leq 0 \\ \gamma(x) & \text{if } \rho(x) < \gamma(x) \leq 0 \\ -\gamma(x) & \text{if } 0 < \gamma(x) < -\rho(x) \text{ and } \ell(x) \neq 0 \\ \rho(x) & \text{if } (0 < \gamma(x) < -\rho(x) \text{ and } \ell(x) = 0) \text{ or } 0 < -\rho(x) \leq \gamma(x) \end{cases}$$

where we have defined

$$\ell(x) = \lim_{t \rightarrow \infty} \left(U(t|x) - \frac{a(t|x)}{\gamma(x)} \right)$$

and $Q(\cdot|x)$ has ultimately constant sign, converges to 0 at infinity and is such that $|Q(\cdot|x)|$ is regularly varying at infinity with index $\rho'(x)$; note that Lemma B.3.16 in [15] entails that one can choose

$$Q(t|x) = \begin{cases} A(t|x) & \text{if } \gamma(x) < \rho(x) \leq 0 \\ & \text{if } \rho(x) < \gamma(x) \leq 0 \\ \gamma_+(x) - \frac{a(t|x)}{U(t|x)} & \text{or } 0 < \gamma(x) < -\rho(x) \text{ and } \ell(x) \neq 0 \\ & \text{or } 0 < \gamma(x) = -\rho(x) \\ \frac{\rho(x)}{\gamma(x) + \rho(x)} A(t|x) & \text{if } (0 < \gamma(x) < -\rho(x) \text{ and } \ell(x) = 0) \text{ or } 0 < -\rho(x) < \gamma(x). \end{cases}$$

Besides, if $\gamma(x) > 0$ and $\rho(x) = 0$, then according to Lemma B.3.16 in [15], one has

$$\forall z > 0, \lim_{t \rightarrow \infty} \frac{\frac{\log U(tz|x) - \log U(t|x)}{q(t|x)} - \log z}{Q(t|x)} = 0 \quad (20)$$

for every $Q(\cdot|x)$ such that $A(t|x) = O(Q(t|x))$ as $t \rightarrow \infty$; especially, we can and will take $Q(\cdot|x) = A(\cdot|x)$ in this case.

We can now state the asymptotic normality of our estimator.

Theorem 2. *Assume that (M_2) holds. Pick $x \in E$. We assume that $n_x \rightarrow \infty$, $k_x \rightarrow \infty$, $k_x/n_x \rightarrow 0$, $\sqrt{k_x} Q(n_x/k_x|x) \rightarrow \lambda(x) \in \mathbb{R}$ and for some $\delta > 0$*

$$\sqrt{k_x} \frac{U(n_x/k_x|x)}{a(n_x/k_x|x)} \omega \left(\frac{n_x}{(1+\delta)k_x}, n_x^{1+\delta}, x, h \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (21)$$

Then if $\gamma(x) \neq \rho(x)$, it holds that $\sqrt{k_x}(\widehat{\gamma}_n(x, k_x, h) - \gamma(x))$ is asymptotically Gaussian with mean $\lambda(x)B(\gamma(x), \rho(x))$ and variance $V(\gamma(x))$ where we have set

$$B(\gamma(x), \rho(x)) = \begin{cases} \frac{(1-\gamma(x))(1-2\gamma(x))}{(1-\gamma(x)-\rho(x))(1-2\gamma(x)-\rho(x))} & \text{if } \gamma(x) < \rho(x) \leq 0 \\ \frac{\gamma(x)(1+\gamma(x))}{(1-\gamma(x))(1-3\gamma(x))} & \text{if } \rho(x) < \gamma(x) \leq 0 \\ -\frac{\gamma(x)}{(1+\gamma(x))^2} & \text{if } 0 < \gamma(x) < -\rho(x) \text{ and } \ell(x) \neq 0 \\ \frac{\gamma(x) - \gamma(x)\rho(x) + \rho(x)}{\rho(x)(1-\rho(x))^2} & \text{if } 0 < \gamma(x) < -\rho(x) \text{ and } \ell(x) = 0 \\ & \text{or } 0 < -\rho(x) \leq \gamma(x) \\ 1 & \text{if } 0 = \rho(x) < \gamma(x) \end{cases}$$

and

$$V(\gamma(x)) = \begin{cases} \gamma^2(x) + 1 & \text{if } \gamma(x) \geq 0 \\ \frac{(1 - \gamma(x))^2(1 - 2\gamma(x))(1 - \gamma(x) + 6\gamma^2(x))}{(1 - 3\gamma(x))(1 - 4\gamma(x))} & \text{if } \gamma(x) < 0. \end{cases}$$

Theorem 2 is the conditional analogue of the asymptotic normality result stated in [8]; see also [15], Theorem 3.5.4. In particular, the asymptotic bias and variance of our estimator are similar to those obtained in the univariate setting. Note that in this result, contrary to the asymptotic normality result of [12], we do not condition on the value of $N_n(x, h)$. Besides, condition $\sqrt{k_x} Q(n_x/k_x|x) \rightarrow \lambda(x) \in \mathbb{R}$ as $n \rightarrow \infty$ in Theorem 2 is a standard condition needed to control the bias of the estimator. Finally, hypothesis (21) can be replaced by an hypothesis on the uniform relative oscillation of the function $\log U$ in its second argument, see Proposition 1, which in turn can be made explicit if suitable regularity conditions are satisfied, see Section 3.1.

4 Simulation study

To have an idea of how our estimator behaves on a finite sample situation, we carried out a simulation study in the case $E = [0, 1]$ equipped with the standard Euclidean distance with a covariate X which is uniformly distributed on E . Furthermore, we let γ be the positive function defined by

$$\forall x \in [0, 1], \gamma(x) = \frac{2}{3} + \frac{1}{3} \sin(2\pi x).$$

We consider three different models for the csf of Y given $X = x$: the first one is

$$\forall y > 0, \bar{F}_1(y|x) = (1 + y^{-\tau})^{1/\tau\gamma(x)},$$

where the parameter τ is chosen to be independent of x and its value is picked in the set $\{-1.2, -1, -0.8\}$. In other words, Y given $X = x$ is Burr type XII distributed; note that in this case the csf $\bar{F}_1(\cdot|x)$ belongs to the Fréchet MDA for every $x \in E$, the conditional extreme-value index is $\gamma(x)$ and the conditional second-order parameter is $\rho(x) = \tau\gamma(x)$ (see [1], p.93). The second model is

$$\forall y \in [0, g(x)], \bar{F}_2(y|x) = \int_{y/g(x)}^1 t^{1/\gamma(x)-1} (1-t)^{1/\gamma(x)-1} dt$$

where the frontier function g is defined by

$$\forall x \in [0, 1], g(x) = 1 - c + 8cx(1 - x)$$

with the constant $c > 0$ being picked in the set $\{0.1, 0.2, 0.3\}$. In this case, given $X = x$, $Y/g(x)$ is a Beta($1/\gamma(x)$, $1/\gamma(x)$) random variable: this conditional model is contained in the Weibull MDA with the conditional extreme-value index being $-\gamma(x)$. The final model is

$$\forall y > 0, \bar{F}_3(y|x) = \int_{\log y}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2(x)}} \exp\left(-\frac{(t - \mu(x))^2}{2\sigma^2(x)}\right) dt$$

where μ and σ are the functions defined by

$$\forall x \in [0, 1], \mu(x) = \frac{2}{3} + \frac{1}{3} \sin(2\pi x) \quad \text{and} \quad \sigma(x) = 0.7 + 2.4x(1-x).$$

In this model, Y given $X = x$ has a log-normal distribution with parameters $\mu(x)$ and $\sigma^2(x)$, which is an example of a conditional distribution belonging to the Gumbel MDA.

The aim of this simulation study is to estimate the conditional extreme-value index on a grid of points $\{x_1, \dots, x_M\}$ of $[0, 1]$. We need to choose two parameters: the bandwidth h and the number of upper order statistics k_x . We use a selection procedure which was introduced in [12], which we recall for the sake of completeness.

- 1) For every bandwidth h in a grid $\{h_1, \dots, h_P\}$ of possible values of h , we first make a preliminary choice of k_x : in what follows, we let $\hat{\gamma}_{i,j}(k) = \hat{\gamma}_n(x_i, k, h_j)$. For each $i \in \{1, \dots, M\}$, $j \in \{1, \dots, P\}$ and $k \in \{q_{i,j} + 1, \dots, N_n(x_i, h_j) - q_{i,j}\}$, where $q_{i,j} = \lfloor N_n(x_i, h_j)/10 \rfloor \vee 1$, we introduce the set $E_{i,j,k} = \{\hat{\gamma}_{i,j}(\ell), \ell \in \{k - q_{i,j}, \dots, k + q_{i,j}\}\}$. We compute the variance of the set $E_{i,j,k}$ for every possible value of k and we record the number $K_{i,j}$ for which this variance is minimal. More precisely,

$$K_{i,j} = \arg \min_k \frac{1}{2q_{i,j} + 1} \sum_{\ell=k-q_{i,j}}^{k+q_{i,j}} \left(\hat{\gamma}_{i,j}(\ell) - \bar{\gamma}_{i,j}(k) \right)^2 \quad \text{with} \quad \bar{\gamma}_{i,j}(k) = \frac{1}{2q_{i,j} + 1} \sum_{\ell=k-q_{i,j}}^{k+q_{i,j}} \hat{\gamma}_{i,j}(\ell).$$

We record the value $k_{i,j}$ such that $\hat{\gamma}_{i,j}(k_{i,j})$ is the median of the set $E_{i,j,K_{i,j}}$. For the sake of simplicity, the estimate $\hat{\gamma}_{i,j}(k_{i,j})$ will be denoted by $\tilde{\gamma}_{i,j}$.

- 2) We now select the bandwidth h : let q' be a positive integer such that $2q' + 1 < P$. For each $i \in \{1, \dots, M\}$ and $j \in \{q' + 1, \dots, P - q'\}$, let $F_{i,j} = \{\tilde{\gamma}_{i,\ell}, \ell \in \{j - q', \dots, j + q'\}\}$ and compute the standard deviation $\sigma_i(j)$ of $F_{i,j}$:

$$\sigma_i(j) = \left[\frac{1}{2q' + 1} \sum_{\ell=j-q'}^{j+q'} (\tilde{\gamma}_{i,\ell} - \bar{\gamma}_{i,j})^2 \right]^{1/2} \quad \text{with} \quad \bar{\gamma}_{i,j} = \frac{1}{2q' + 1} \sum_{\ell=j-q'}^{j+q'} \tilde{\gamma}_{i,\ell}.$$

Our stability criterion is then the average of these quantities over the grid $\{x_1, \dots, x_M\}$:

$$\bar{\sigma}(j) = \frac{1}{M} \sum_{i=1}^M \sigma_i(j).$$

We next record the integer j^* such that $\bar{\sigma}(j^*)$ is the first local minimum of the application $j \mapsto \bar{\sigma}(j)$ which is less than the average value of $\bar{\sigma}$. In other words, $j^* = q' + 1$ if $\bar{\sigma}$ is increasing, $j^* = P - q'$ if $\bar{\sigma}$ is decreasing and

$$j^* = \min \left\{ j \text{ such that } \bar{\sigma}(j) \leq \bar{\sigma}(j-1) \wedge \bar{\sigma}(j+1) \text{ and } \bar{\sigma}(j) \leq \frac{1}{P-2q'} \sum_{\ell=q'+1}^{P-q'} \bar{\sigma}(\ell) \right\} \quad (22)$$

otherwise, where we extend $\bar{\sigma}$ by setting $\bar{\sigma}(q') = \bar{\sigma}(q' + 1)$ and $\bar{\sigma}(P - q' + 1) = \bar{\sigma}(P - q')$.

The selected bandwidth is then independent of x and is given by $h^* = h_{j^*}$ where j^* is defined in (22). The selected number of upper order statistics is given, for $x = x_i$, by $k_{x_i}^* = k_{i,j^*}$. The main idea of this procedure is that the bandwidth and the number of upper order statistics are selected in order to satisfy a stability criterion. This estimation procedure is carried out on $N = 100$ independent samples of size $n = 500$. The conditional extreme-value index is estimated on a grid of $M = 50$ evenly spaced points in $[0, 1]$. Regarding the selection procedure, $P = 25$ values of h ranging from 0.05 to 0.3 are tested; the parameter q' is set to 1.

To have an idea of our estimator behaves compared to another estimator in the conditional extreme-value index estimation literature, we introduce the estimator $\tilde{\gamma}_D = \hat{\gamma}_n^{RP,1}$ of [5]. Let K be the triweight kernel:

$$K(t) = \frac{35}{32}(1 - t^2)^3 \mathbb{1}_{[-1,1]}(t).$$

Let $\widehat{F}(\cdot, h|x)$ be the empirical kernel estimator of the csf:

$$\widehat{F}(y, h|x) = \frac{\frac{1}{nh^d} \sum_{i=1}^n K\left(\frac{x - X_i}{h}\right) \mathbb{1}_{\{Y_i > y\}}}{\frac{1}{nh^d} \sum_{i=1}^n K\left(\frac{x - X_i}{h}\right)}$$

and let $\widehat{q}_n(\cdot, h|x)$ be the generalized inverse of $\widehat{F}(\cdot, h|x)$: for $\alpha \in (0, 1)$,

$$\widehat{q}_n(\alpha, h|x) = \inf\{y > 0, \widehat{F}(y, h|x) \leq \alpha\}.$$

The quantity $\widehat{q}_n(\cdot, h|x)$ is the empirical estimator of the conditional quantile function. The estimator $\tilde{\gamma}_D$ is then

$$\tilde{\gamma}_D(x, \alpha_{n,x}, h) = \frac{1}{-\log 3} \log \left(\frac{\widehat{q}_n(\alpha_{n,x}, h|x) - \widehat{q}_n(\alpha_{n,x}/3, h|x)}{\widehat{q}_n(\alpha_{n,x}/3, h|x) - \widehat{q}_n(\alpha_{n,x}/9, h|x)} \right)$$

where $\alpha_{n,x} \rightarrow 0$ is a nonrandom sequence. This estimator is exactly the estimator $\hat{\gamma}_n^{RP,1}$ of [5] with $J = 3$ and $r = 1/J$: it is a kernel version of the Pickands estimator, see Pickands [18]. To choose the parameters $\alpha_{n,x}$ and h for $\tilde{\gamma}_D$, we restrict our search to a parameter $\alpha_{n,x}$ having the form $k_x/N_n(x_i, h_j)$, so that we are led to a choice of k_x and h just as for our estimator, and we use the procedure detailed above.

We give in Table 1 the empirical mean squared errors (MSEs) of each estimator, averaged over the M points of the grid. Table 1 shows that our estimator outperforms the estimator $\tilde{\gamma}_D$ in terms of MSEs in every case except the Gumbel one. Besides, one can see that in the Fréchet MDA, the MSEs of both estimators increase as $|\rho|$ gets closer to 0, which was expected since ρ controls the rate of convergence in (M_2) : the closer $|\rho|$ is to 0, the slower is this convergence and the harder is the estimation. Some illustrations are given on Figures 1–3, where the estimations corresponding to the median of the MSE are represented in each case for both estimators. One can see on these pictures that our estimator

generally oscillates less than $\tilde{\gamma}_D$; in the case when the conditional survival function belongs to the Fréchet or Weibull MDA, it also does a better job of mimicking the shape of the conditional extreme-value index.

5 Proofs of the main results

5.1 Weak consistency

We start by proving the pointwise weak consistency of our estimator at a point x lying in E . To this end, since the $M_n^{(j)}(x, k_x, h)$ are defined conditionally on the value of the total number $N_n(x, h)$ of covariates belonging to $B(x, h)$, which is random, a natural idea is to condition on this value. A preliminary classical lemma is then required to control this random variable.

Lemma 1. *If $n_x \rightarrow \infty$ as $n \rightarrow \infty$ then for every $\delta > 0$*

$$\sqrt{n_x^{1-\delta}} \left| \frac{N_n(x, h)}{n_x} - 1 \right| \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \rightarrow \infty.$$

From Lemma 1, we deduce that if $I_x = I_x(n) = \mathbb{N} \cap \left[\left(1 - n_x^{-1/4}\right) n_x, \left(1 + n_x^{-1/4}\right) n_x \right]$, it holds that $N_n(x, h)$ lies in I_x with arbitrarily large probability; in other words,

$$\sum_{p \in I_x} \mathbb{P}(N_n(x, h) = p) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Furthermore, since $k_x/n_x \rightarrow 0$, we may and will, in the sequel, take n so large that $k_x < \inf I_x$.

The next step is to show that when n is large, studying the convergence in probability of the quantities $M_n^{(j)}(x, k_x, h)$ amounts to studying the behavior of analogous quantities defined in terms of upper order statistics of a sample of independent and identically distributed random variables having cdf $F(\cdot|x)$. To achieve that we begin by stating a second lemma which gives the conditional distribution of the random variables Z_i .

Lemma 2. *Given $N_n(x, h) = p \geq 1$, the random variables Z_i , $1 \leq i \leq p$, are independent and identically distributed random variables having cdf $F_h(\cdot|x)$.*

Letting T_i , $i \geq 1$ be independent standard Pareto random variables, *i.e.* having cdf $t \mapsto 1 - 1/t$ on $[1, \infty)$, we deduce from this result that the distribution of the random vector (Z_1, \dots, Z_p) given $N_n(x, h) = p \geq 1$ is the distribution of the random vector $(U_h(T_1|x), \dots, U_h(T_p|x))$. In other words, since $U_h(\cdot|x)$ is nondecreasing, we may focus on the behavior in probability of the quantities

$$\mathfrak{M}_{np}^{(j)}(x, k_x, h) = \frac{1}{k_x} \sum_{i=1}^{k_x} [\log U_h(T_{p-i+1,p}|x) - \log U_h(T_{p-k_x,p}|x)]^j$$

for $p > k_x$ and $j = 1, 2$. Lemma 3 below is the desired approximation of the statistics $\mathfrak{M}_{np}^{(j)}(x, k_x, h)$.

Lemma 3. Given $N_n(x, h) = p \geq 1$, one has if $p > k_x$ that

$$\left| \mathfrak{M}_{np}^{(1)}(x, k_x, h) - \mathcal{M}_{np}^{(1)}(x, k_x, h) \right| \leq 2\omega(T_{p-k_x, p}, T_{p, p}, x, h)$$

and

$$\left| \mathfrak{M}_{np}^{(2)}(x, k_x, h) - \mathcal{M}_{np}^{(2)}(x, k_x, h) \right| \leq 4\omega(T_{p-k_x, p}, T_{p, p}, x, h) \left[\omega(T_{p-k_x, p}, T_{p, p}, x, h) + \mathcal{M}_{np}^{(1)}(x, k_x, h) \right]$$

where for $j = 1, 2$, we let $\mathcal{M}_{np}^{(j)}(x, k_x, h) = \frac{1}{k_x} \sum_{i=1}^{k_x} [\log U(T_{p-i+1, p}|x) - \log U(T_{p-k_x, p}|x)]^j$.

The final lemmas are technical results. The first one is a simple lemma we shall repeatedly make use of.

Lemma 4. Let $(R_{ij}), (R'_{ij}), 1 \leq j \leq i$ be triangular arrays of random variables. Assume that there exist $\ell, \ell' \in \mathbb{R}$ and a sequence of non-empty sets (I_n) contained in $\{1, \dots, n\}$ such that for every $\varepsilon > 0$

$$\sup_{p \in I_n} \mathbb{P}(|R_{np} - \ell| > \varepsilon) \rightarrow 0 \quad \text{and} \quad \sup_{p \in I_n} \mathbb{P}(|R'_{np} - \ell'| > \varepsilon) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then

$$\sup_{p \in I_n} \mathbb{P}(|R_{np} + R'_{np} - (\ell + \ell')| > \varepsilon) \rightarrow 0 \quad \text{and} \quad \sup_{p \in I_n} \mathbb{P}(|R_{np}R'_{np} - \ell\ell'| > \varepsilon) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The following result is the main technical tool we shall use to prove our asymptotic results. It is basically a conditional analogue of the additive version of Slutsky's lemma.

Lemma 5. Let r be a positive integer, $(S_n) = (S_n^{(1)}, \dots, S_n^{(r)})$ be a sequence of random vectors and $S = (S^{(1)}, \dots, S^{(r)})$ be a random vector. Assume that there exist

1. a triangular array of events $(A_{ij})_{0 \leq j \leq i}$ and a sequence of non-empty sets (I_n) contained in $\{1, \dots, n\}$ such that

- for every n the $A_{np}, 0 \leq p \leq n$ are pairwise disjoint and $\sum_{p=0}^n \mathbb{P}(A_{np}) = 1$;
- it holds that

$$\sum_{p \in I_n} \mathbb{P}(A_{np}) \rightarrow 1 \quad \text{as } n \rightarrow \infty;$$

2. two triangular arrays of random vectors

$$(D_{ij} = (D_{ij}^{(1)}, \dots, D_{ij}^{(r)}))_{1 \leq j \leq i} \quad \text{and} \quad (R_{ij} = (R_{ij}^{(1)}, \dots, R_{ij}^{(r)}))_{1 \leq j \leq i}$$

such that

- for $1 \leq p \leq n$, the distribution of S_n given A_{np} is the distribution of $D_{np} + R_{np}$;
- it holds that for every $t = (t_1, \dots, t_r) \in \mathbb{R}^r$

$$\sup_{p \in I_n} \left| \mathbb{E}[\exp[it' D_{np}]] - \mathbb{E}[\exp[it' S]] \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

where t' is the transpose vector of t ;

- it holds that for every $\varepsilon > 0$ and every $j \in \{1, \dots, r\}$

$$\sup_{p \in I_n} \mathbb{P}(|R_{np}^{(j)}| > \varepsilon) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Then $S_n \xrightarrow{d} S$ as $n \rightarrow \infty$. In particular, if $D_{ij} = 0$ for every $1 \leq j \leq i$, then $S_n \xrightarrow{\mathbb{P}} 0$ as $n \rightarrow \infty$.

The next lemma, which specifies the asymptotic behavior in probability of the order statistic $T_{p-k_x, p}$ uniformly in $p \in I_x$, shall be used several times.

Lemma 6. Assume that $n_x \rightarrow \infty$, $k_x \rightarrow \infty$ and $k_x/n_x \rightarrow 0$ as $n \rightarrow \infty$. Then for every $t > 0$ it holds that

$$\sup_{p \in I_x} \mathbb{P} \left(\left| \frac{k_x}{p} T_{p-k_x, p} - 1 \right| > t \right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Especially, for every $t > 0$

$$\sup_{p \in I_x} \mathbb{P} (T_{p-k_x, p} \leq t) \rightarrow 0 \text{ as } n \rightarrow \infty$$

and for every function φ which is regularly varying at infinity, we have both

$$\sup_{p \in I_x} \mathbb{P} \left(\left| \frac{\varphi(T_{p-k_x, p})}{\varphi(p/k_x)} - 1 \right| > t \right) \rightarrow 0 \text{ and } \sup_{p \in I_x} \mathbb{P} \left(\left| \frac{\varphi(T_{p-k_x, p})}{\varphi(n_x/k_x)} - 1 \right| > t \right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Lemma 7 below is a useful corollary of Rényi's representation (see e.g. [15], p.37).

Lemma 7. For every Borel measurable functions f and g , every $p \geq 2$ and $k \in \{1, \dots, p-1\}$, the random vectors

$$\left(\frac{1}{k} \sum_{i=1}^k f \left(\log \frac{T_{p-i+1, p}}{T_{p-k, p}} \right), \frac{1}{k} \sum_{i=1}^k g \left(\log \frac{T_{p-i+1, p}}{T_{p-k, p}} \right) \right) \text{ and } \left(\frac{1}{k} \sum_{i=1}^k f(\log T_i), \frac{1}{k} \sum_{i=1}^k g(\log T_i) \right)$$

have the same distribution.

The final lemma shows that the asymptotic behavior of the random variables $\mathcal{M}_{np}^{(j)}(x, k_x, h)$ is in some way uniform in $p \in I_x$. Before stating this result, we note that applying Theorem B.2.18 in [15], (M_1) entails that there exists a positive function $q_0(\cdot|x)$ which is equivalent to $a(\cdot|x)/U(\cdot|x)$ at infinity such that the following property holds: for each $\varepsilon > 0$, there exists $t_0 \geq 1$ such that for every $t \geq t_0$ and $z > 0$ with $tz \geq t_0$,

$$\left| \frac{\log U(tz|x) - \log U(t|x)}{q_0(t|x)} - \frac{z^{\gamma_-(x)} - 1}{\gamma_-(x)} \right| \leq \varepsilon z^{\gamma_-(x)} (z^{-\varepsilon} \vee z^\varepsilon). \quad (23)$$

Lemma 8. Assume that (M_1) holds, and $n_x \rightarrow \infty$, $k_x \rightarrow \infty$ and $k_x/n_x \rightarrow 0$ as $n \rightarrow \infty$. Then for every $t > 0$ the convergences

$$\begin{aligned} \sup_{p \in I_x} \mathbb{P} \left(\left| \frac{\mathcal{M}_{np}^{(1)}(x, k_x, h)}{q_0(p/k_x|x)} - \frac{1}{1 - \gamma_-(x)} \right| > t \right) &\rightarrow 0, \\ \text{and } \sup_{p \in I_x} \mathbb{P} \left(\left| \frac{\mathcal{M}_{np}^{(2)}(x, k_x, h)}{q_0^2(p/k_x|x)} - \frac{2}{(1 - \gamma_-(x))(1 - 2\gamma_-(x))} \right| > t \right) &\rightarrow 0 \end{aligned}$$

hold as $n \rightarrow \infty$.

We are now in position to examine the convergence in probability of the statistics $M_n^{(j)}(x, k_x, h)$, of which the consistency of our estimator is a simple corollary.

Proposition 2. *Assume that (M_1) holds, that $n_x \rightarrow \infty$, $k_x \rightarrow \infty$, $k_x/n_x \rightarrow 0$ and for some $\delta > 0$*

$$\frac{U(n_x/k_x|x)}{a(n_x/k_x|x)} \omega \left(\frac{n_x}{(1+\delta)k_x}, n_x^{1+\delta}, x, h \right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Then it holds that

$$\begin{aligned} \frac{U(N_n(x, h)/k_x|x)}{a(N_n(x, h)/k_x|x)} M_n^{(1)}(x, k_x, h) &\xrightarrow{\mathbb{P}} \frac{1}{1 - \gamma_-(x)} \\ \text{and } \left[\frac{U(N_n(x, h)/k_x|x)}{a(N_n(x, h)/k_x|x)} \right]^2 M_n^{(2)}(x, k_x, h) &\xrightarrow{\mathbb{P}} \frac{2}{(1 - \gamma_-(x))(1 - 2\gamma_-(x))} \end{aligned}$$

as $n \rightarrow \infty$.

This result is the analogue of Lemma 3.5.1 in [15] when there is a covariate: of course, a major difference here is that the total number of observations $N_n(x, h)$ is random.

Proof of Proposition 2. We start the proof by remarking that with the notation of (23), applying Lemma 1 yields

$$\frac{q_0(N_n(x, h)/k_x|x)}{a(N_n(x, h)/k_x|x)/U(N_n(x, h)/k_x|x)} \xrightarrow{\mathbb{P}} 1 \text{ as } n \rightarrow \infty. \quad (24)$$

Pick then an arbitrary $t > 0$ and introduce the two events

$$\begin{aligned} A_n^{(1)} &= \left\{ \left| \frac{M_n^{(1)}(x, k_x, h)}{q_0(N_n(x, h)/k_x|x)} - \frac{1}{1 - \gamma_-(x)} \right| > t \right\} \\ \text{and } A_n^{(2)} &= \left\{ \left| \frac{M_n^{(2)}(x, k_x, h)}{q_0^2(N_n(x, h)/k_x|x)} - \frac{2}{(1 - \gamma_-(x))(1 - 2\gamma_-(x))} \right| > t \right\}. \end{aligned}$$

From (24), it is enough to prove that $\mathbb{P}(A_n^{(1)}) \rightarrow 0$ and $\mathbb{P}(A_n^{(2)}) \rightarrow 0$ as $n \rightarrow \infty$.

We start by controlling $\mathbb{P}(A_n^{(1)})$. Note that according to Lemma 2, one has

$$\mathbb{P}(A_n^{(1)} | N_n(x, h) = p) = \mathbb{P} \left(\left| \frac{\mathfrak{M}_{np}^{(1)}(x, k_x, h)}{q_0(p/k_x|x)} - \frac{1}{1 - \gamma_-(x)} \right| > t \right).$$

Moreover, Lemma 3 entails

$$\left| \frac{\mathfrak{M}_{np}^{(1)}(x, k_x, h)}{q_0(p/k_x|x)} - \frac{1}{1 - \gamma_-(x)} \right| \leq \left| \frac{\mathcal{M}_{np}^{(1)}(x, k_x, h)}{q_0(p/k_x|x)} - \frac{1}{1 - \gamma_-(x)} \right| + \frac{\omega(T_{p-k_x, p}, T_{p, p}, x, h)}{q_0(p/k_x|x)}.$$

Introducing for an arbitrary $t' > 0$

$$u_{np}^{(1,1)} = \mathbb{P} \left(\left| \frac{\mathcal{M}_{np}^{(1)}(x, k_x, h)}{q_0(p/k_x|x)} - \frac{1}{1 - \gamma_-(x)} \right| > t' \right) \text{ and } u_{np}^{(1,2)} = \mathbb{P} \left(\frac{\omega(T_{p-k_x, p}, T_{p, p}, x, h)}{q_0(p/k_x|x)} > t' \right),$$

Lemmas 4 and 5 with $A_{np} = \{N_n(x, h) = p\}$ make it enough to prove that $u_{np}^{(1,j)} \rightarrow 0$ as $n \rightarrow \infty$ uniformly in the integers $p \in I_x$ for every $j \in \{1, 2\}$.

To control $u_{np}^{(1,1)}$ we apply Lemma 8 to obtain the convergence

$$\sup_{p \in I_x} u_{np}^{(1,1)} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (25)$$

To control $u_{np}^{(1,2)}$ we recall that the function $q_0(\cdot|x)$ is regularly varying at infinity with index $\gamma_-(x)$ so that we can apply a uniform convergence result (see *e.g.* Theorem 1.5.2 in [2]) to get

$$\sup_{p \in I_x} \left| \frac{q_0(n_x/k_x|x)}{q_0(p/k_x|x)} - 1 \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Especially, for n large enough, recalling that $q_0(\cdot|x)$ and $a(\cdot|x)/U(\cdot|x)$ are equivalent at infinity, we have

$$\sup_{p \in I_x} \frac{1}{q_0(p/k_x|x)} \leq 2 \frac{U(n_x/k_x|x)}{a(n_x/k_x|x)}. \quad (26)$$

This inequality gives for n sufficiently large

$$\sup_{p \in I_x} u_{np}^{(1,2)} \leq \sup_{p \in I_x} \mathbb{P} \left(\frac{U(n_x/k_x|x)}{a(n_x/k_x|x)} \omega(T_{p-k_x,p}, T_{p,p}, x, h) > \frac{t'}{2} \right).$$

Using condition (1), we get for n large enough

$$\sup_{p \in I_x} u_{np}^{(1,2)} \leq \sup_{p \in I_x} \mathbb{P} \left(\left\{ T_{p-k_x,p} < \frac{n_x}{(1+\delta)k_x} \right\} \cup \{ T_{p,p} > n_x^{1+\delta} \} \right).$$

Because the random variables T_i are independent standard Pareto random variables, one has for n sufficiently large

$$\begin{aligned} \sup_{p \in I_x} u_{np}^{(1,2)} &\leq \sup_{p \in I_x} \mathbb{P} \left(T_{p-k_x,p} < \frac{n_x}{(1+\delta)k_x} \right) + \sup_{p \in I_x} (1 - n_x^{-1-\delta})^p \\ &\leq \sup_{p \in I_x} \mathbb{P} \left(T_{p-k_x,p} < \frac{n_x}{(1+\delta)k_x} \right) + (1 - n_x^{-1-\delta})^{n_x/2} \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned} \quad (27)$$

by Lemma 6. Collecting (25) and (27) shows that $\mathbb{P}(A_n^{(1)}) \rightarrow 0$ as $n \rightarrow \infty$.

Let us now consider $\mathbb{P}(A_n^{(2)})$. Applying Lemma 2, one has

$$\mathbb{P}(A_n^{(2)} | N_n(x, h) = p) = \mathbb{P} \left(\left| \frac{\mathfrak{M}_{np}^{(2)}(x, k_x, h)}{q_0^2(p/k_x|x)} - \frac{2}{(1-\gamma_-(x))(1-2\gamma_-(x))} \right| > t \right).$$

Lemma 3 yields

$$\begin{aligned} \left| \frac{\mathfrak{M}_{np}^{(2)}(x, k_x, h)}{q_0^2(p/k_x|x)} - \frac{2}{(1-\gamma_-(x))(1-2\gamma_-(x))} \right| &\leq \left| \frac{\mathcal{M}_{np}^{(2)}(x, k_x, h)}{q_0^2(p/k_x|x)} - \frac{2}{(1-\gamma_-(x))(1-2\gamma_-(x))} \right| \\ &+ \left[\frac{\omega(T_{p-k_x,p}, T_{p,p}, x, h)}{q_0(p/k_x|x)} \right]^2 \\ &+ \frac{\omega(T_{p-k_x,p}, T_{p,p}, x, h)}{q_0^2(p/k_x|x)} \mathcal{M}_{np}^{(1)}(x, k_x, h). \end{aligned}$$

Letting for an arbitrary $t' > 0$

$$\begin{aligned} u_{np}^{(2,1)} &= \mathbb{P} \left(\left| \frac{\mathcal{M}_{np}^{(2)}(x, k_x, h)}{q_0^2(p/k_x|x)} - \frac{2}{(1-\gamma_-(x))(1-2\gamma_-(x))} \right| > t' \right), \\ u_{np}^{(2,2)} &= \mathbb{P} \left(\left[\frac{\omega(T_{p-k_x,p}, T_{p,p}, x, h)}{q_0(p/k_x|x)} \right]^2 > t' \right) \\ \text{and } u_{np}^{(2,3)} &= \mathbb{P} \left(\frac{\omega(T_{p-k_x,p}, T_{p,p}, x, h)}{q_0^2(p/k_x|x)} \mathcal{M}_{np}^{(1)}(x, k_x, h) > t' \right), \end{aligned}$$

Lemmas 4 and 5 with $A_{np} = \{N_n(x, h) = p\}$ make it enough to prove that $u_{np}^{(2,j)} \rightarrow 0$ as $n \rightarrow \infty$ uniformly in the integers $p \in I_x$ for every $j \in \{1, 2, 3\}$. We start by noting that Lemma 8 leads to

$$\sup_{p \in I_x} u_{np}^{(2,1)} \rightarrow 0 \text{ as } n \rightarrow \infty \quad (28)$$

and since

$$u_{np}^{(2,2)} = \mathbb{P} \left(\frac{\omega(T_{p-k_x, p}, T_{p, p}, x, h)}{q_0(p/k_x|x)} > \sqrt{t'} \right),$$

this term is similar to $u_{np}^{(1,2)}$ and therefore we obtain from (27) that

$$\sup_{p \in I_x} u_{np}^{(2,2)} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (29)$$

Finally, the obvious inequality

$$\begin{aligned} \frac{\omega(T_{p-k_x, p}, T_{p, p}, x, h)}{q_0^2(p/k_x|x)} \mathcal{M}_{np}^{(1)}(x, k_x, h) &\leq \left[\frac{\omega(T_{p-k_x, p}, T_{p, p}, x, h)}{q_0(p/k_x|x)} \right] \left| \frac{\mathcal{M}_{np}^{(1)}(x, k_x, h)}{q_0(p/k_x|x)} - \frac{1}{1 - \gamma_-(x)} \right| \\ &+ \left[\frac{\omega(T_{p-k_x, p}, T_{p, p}, x, h)}{q_0(p/k_x|x)} \right] \frac{1}{1 - \gamma_-(x)} \end{aligned}$$

together with (25), (27) and Lemma 4 entails

$$\sup_{p \in I_x} u_{np}^{(2,3)} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (30)$$

Collecting (28), (29) and (30) shows that $\mathbb{P}(A_n^{(2)}) \rightarrow 0$ as $n \rightarrow \infty$ which completes the proof. \blacksquare

Proof of Theorem 1. Using Lemma 1.2.9 in [15] yields $a(t|x)/U(t|x) \rightarrow \gamma_+(x)$ as $t \rightarrow \infty$. Applying Proposition 2, we get

$$\widehat{\gamma}_{n,+}(x, k_x, h) \xrightarrow{\mathbb{P}} \gamma_+(x) \text{ and } \widehat{\gamma}_{n,-}(x, k_x, h) \xrightarrow{\mathbb{P}} \gamma_-(x) \text{ as } n \rightarrow \infty.$$

The result then follows from summing these two convergences. \blacksquare

We conclude this section by proving Proposition 1.

Proof of Proposition 1. We introduce the functions \overline{F}_{\min} and \overline{F}_{\max} defined by

$$\overline{F}_{\min}(z|x) = \inf_{x' \in B(x, h)} \overline{F}(z|x') \text{ and } \overline{F}_{\max}(z|x) = \sup_{x' \in B(x, h)} \overline{F}(z|x').$$

The inequality

$$\forall v, \alpha > 0, \forall x' \in B(x, h), \overline{F}(v + \alpha|x') \leq \overline{F}(v|x')$$

shows that these functions are nonincreasing. With this definition, we get

$$\overline{F}_{\min}(z|x) \leq \overline{F}_h(z|x) = \frac{\mathbb{E}(\overline{F}(z|X) \mathbb{1}_{\{X \in B(x, h)\}})}{\mathbb{P}(X \in B(x, h))} \leq \overline{F}_{\max}(z|x)$$

We now remark that because (A_1) holds, the functions $1/\overline{F}(\cdot|x)$ and $U(\cdot|x)$ are in fact true continuous inverses of each other for every $x \in E$. This entails

$$\overline{F}_{\max} \left(\sup_{x' \in B(x, h)} U(z|x') \middle| x \right) \leq \sup_{x' \in B(x, h)} \overline{F}(U(z|x')|x') = \frac{1}{z}$$

and likewise

$$\bar{F}_{\min} \left(\inf_{x' \in B(x,h)} U(z|x') \middle| x \right) \geq \inf_{x' \in B(x,h)} \bar{F}(U(z|x')|x') = \frac{1}{z}.$$

An easy consequence of these two inequalities is that

$$\inf_{x' \in B(x,h)} U(z|x') \leq U_h(z|x) \leq \sup_{x' \in B(x,h)} U(z|x')$$

and therefore, because the logarithm function is increasing:

$$\inf_{x' \in B(x,h)} \log \frac{U(z|x')}{U(z|x)} \leq \log \frac{U_h(z|x)}{U(z|x)} \leq \sup_{x' \in B(x,h)} \log \frac{U(z|x')}{U(z|x)}.$$

This entails

$$\begin{aligned} \left| \log \frac{U_h(z|x)}{U(z|x)} \right| &= \left[\log \frac{U_h(z|x)}{U(z|x)} \right] \vee \left[-\log \frac{U_h(z|x)}{U(z|x)} \right] \\ &\leq \left[\sup_{x' \in B(x,h)} \log \frac{U(z|x')}{U(z|x)} \right] \vee \left[-\inf_{x' \in B(x,h)} \log \frac{U(z|x')}{U(z|x)} \right]. \end{aligned} \quad (31)$$

The obvious inequality

$$-\left| \log \frac{U(z|x')}{U(z|x)} \right| \leq \log \frac{U(z|x')}{U(z|x)} \leq \left| \log \frac{U(z|x')}{U(z|x)} \right|$$

leads to

$$\sup_{x' \in B(x,h)} \log \frac{U(z|x')}{U(z|x)} \leq \sup_{x' \in B(x,h)} \left| \log \frac{U(z|x')}{U(z|x)} \right| \quad (32)$$

and

$$-\inf_{x' \in B(x,h)} \log \frac{U(z|x')}{U(z|x)} \leq \sup_{x' \in B(x,h)} \left| \log \frac{U(z|x')}{U(z|x)} \right|. \quad (33)$$

Collecting (31), (32) and (33) concludes the proof. \blacksquare

5.2 Asymptotic normality

We proceed by proving the pointwise asymptotic normality of the estimator at a point $x \in E$ when condition (M_2) holds. We shall use the same ideas as in the proof of Proposition 2 to examine the asymptotic behavior of the statistics $M_n^{(j)}(x, k_x, h)$: if $\gamma(x) \neq \rho(x)$ and $\rho(x) < 0$ if $\gamma(x) > 0$, then from (19) and Theorem 2.3.6 in [15], there exist functions $q_0(\cdot|x)$ and $Q_0(\cdot|x)$ which are equivalent to $q(\cdot|x)$ and

$$\frac{1}{\rho'(x)} Q(\cdot|x) \mathbb{1}_{\{\rho'(x) < 0\}} + Q(\cdot|x) \mathbb{1}_{\{\rho'(x) = 0\}} \quad (34)$$

respectively at infinity such that for every $\varepsilon > 0$ there exists $t_0 \geq 1$ such that for every $t \geq t_0$ and $z > 0$ with $tz \geq t_0$,

$$\left| \frac{\frac{\log U(tz|x) - \log U(t|x)}{q_0(t|x)} - \frac{z^{\gamma-(x)} - 1}{\gamma-(x)}}{Q_0(t|x)} - \psi_{\gamma-(x), \rho'(x)}(z) \right| \leq \varepsilon z^{\gamma-(x) + \rho'(x)} (z^\varepsilon \vee z^{-\varepsilon}) \quad (35)$$

where

$$\psi_{\gamma_-(x), \rho'(x)}(z) = \begin{cases} \frac{z^{\gamma_-(x) + \rho'(x)} - 1}{\gamma_-(x) + \rho'(x)} & \text{if } \rho'(x) < 0, \\ \frac{z^{\gamma_-(x)} \log z}{\gamma_-(x)} & \text{if } \gamma_-(x) < \rho'(x) = 0, \\ \frac{1}{2}(\log x)^2 & \text{if } \gamma_-(x) = \rho'(x) = 0. \end{cases}$$

Besides, if $\gamma(x) > 0$ and $\rho(x) = 0$, then recalling the equality $q(\cdot|x) = a(\cdot|x)/U(\cdot|x)$, we get from Lemma B.3.16 in [15] that

$$q(t|x) - \gamma(x) = Q(t|x)(1 + o(1)) \quad \text{as } t \rightarrow \infty. \quad (36)$$

Equation (20) thus yields

$$\forall z > 0, \lim_{t \rightarrow \infty} \frac{\log U(tz|x) - \log U(t|x) - \gamma(x) \log z}{Q(t|x)} = \log z.$$

We may now apply Theorem B.2.18 in [15] to obtain that for every $\varepsilon > 0$ there exists $t_0 \geq 1$ such that for every $t \geq t_0$ and $z > 0$ with $tz \geq t_0$,

$$\left| \frac{\log U(tz|x) - \log U(t|x) - \gamma(x) \log z}{Q(t|x)} - \log z \right| \leq \varepsilon(z^\varepsilon \vee z^{-\varepsilon}). \quad (37)$$

Using together (36), (37) and the fact that the function $z \mapsto (z^\varepsilon \vee z^{-\varepsilon})^{-1} \log z$ is bounded on $(0, \infty)$, we get that for every $\varepsilon > 0$ there exists $t_0 \geq 1$ (possibly different) such that for every $t \geq t_0$ and $z > 0$ with $tz \geq t_0$,

$$\left| \frac{\frac{\log U(tz|x) - \log U(t|x)}{q(t|x)} - \log z}{Q(t|x)} \right| \leq \varepsilon(z^\varepsilon \vee z^{-\varepsilon}).$$

The following result is the analogue of Lemma 3.5.5 in [15] when there is a random covariate: define

$$\mathcal{V}(\gamma(x)) = \frac{1}{(1 - \gamma_-(x))^2(1 - 2\gamma_-(x))} \begin{pmatrix} 1 & \frac{4}{1 - 3\gamma_-(x)} \\ \frac{4}{1 - 3\gamma_-(x)} & \frac{4(5 - 11\gamma_-(x))}{(1 - 2\gamma_-(x))(1 - 3\gamma_-(x))(1 - 4\gamma_-(x))} \end{pmatrix}$$

and note that if $\gamma(x) > 0$ then

$$\mathcal{V}(\gamma(x)) = \begin{pmatrix} 1 & 4 \\ 4 & 20 \end{pmatrix}.$$

Lemma 9. *Assume that (M_2) holds, that $n_x \rightarrow \infty$, $k_x \rightarrow \infty$, $k_x/n_x \rightarrow 0$, $\sqrt{k_x}Q(n_x/k_x|x) \rightarrow \lambda(x) \in \mathbb{R}$ and for some $\delta > 0$*

$$\sqrt{k_x} \frac{U(n_x/k_x|x)}{a(n_x/k_x|x)} \omega \left(\frac{n_x}{(1 + \delta)k_x}, n_x^{1+\delta}, x, h \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

- *If $\gamma(x) \neq \rho(x)$ and $\rho(x) < 0$ if $\gamma(x) > 0$, it holds that the distribution of the random vector*

$$\sqrt{k_x} \left(\frac{\mathfrak{M}_{np}^{(1)}(x, k_x, h)}{q_0(T_{p-k_x, p}|x)} - \frac{1}{1 - \gamma_-(x)}, \frac{\mathfrak{M}_{np}^{(2)}(x, k_x, h)}{q_0^2(T_{p-k_x, p}|x)} - \frac{2}{(1 - \gamma_-(x))(1 - 2\gamma_-(x))} \right)$$

is the distribution of a random vector $(D_{np}^{(1)} + R_{np}^{(1)}, D_{np}^{(2)} + R_{np}^{(2)})$ where

– the triangular array $(D_{ij}^{(1)}, D_{ij}^{(2)})_{1 \leq j \leq i}$ is such that for every $(t_1, t_2) \in \mathbb{R}^2$,

$$\sup_{p \in I_x} \left| \mathbb{E}[\exp[i(t_1 D_{np}^{(1)} + t_2 D_{np}^{(2)})]] - \mathbb{E}[\exp[i(t_1 P_1 + t_2 P_2)]] \right| \rightarrow 0 \text{ as } n \rightarrow \infty$$

where (P_1, P_2) is a Gaussian random vector having mean $(m^{(1)}(x), m^{(2)}(x))$ with

$$m^{(1)}(x) = \lambda(x) \left(\frac{\mathbb{1}_{\{\rho'(x) < 0\}}}{\rho'(x)} + \mathbb{1}_{\{\rho'(x) = 0\}} \right) \mathbb{E}(\psi_{\gamma_-(x), \rho'(x)}(T))$$

$$\text{and } m^{(2)}(x) = 2\lambda(x) \left(\frac{\mathbb{1}_{\{\rho'(x) < 0\}}}{\rho'(x)} + \mathbb{1}_{\{\rho'(x) = 0\}} \right) \mathbb{E} \left(\frac{T^{\gamma_-(x)} - 1}{\gamma_-(x)} \psi_{\gamma_-(x), \rho'(x)}(T) \right)$$

and covariance matrix $\mathcal{V}(\gamma(x))$;

– the triangular arrays of random variables $(R_{ij}^{(1)})_{1 \leq j \leq i}$ and $(R_{ij}^{(2)})_{1 \leq j \leq i}$ are such that for every $\varepsilon > 0$ and $j \in \{1, 2\}$,

$$\sup_{p \in I_x} \mathbb{P}(|R_{np}^{(j)}| > \varepsilon) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

• If $\gamma(x) > 0$ and $\rho(x) = 0$, it holds that the distribution of the random vector

$$\sqrt{k_x} \left(\frac{\mathfrak{M}_{np}^{(1)}(x, k_x, h)}{q(T_{p-k_x, p}|x)} - 1, \frac{\mathfrak{M}_{np}^{(2)}(x, k_x, h)}{q^2(T_{p-k_x, p}|x)} - 2 \right)$$

is the distribution of a random vector $(D_{np}^{(1)} + R_{np}^{(1)}, D_{np}^{(2)} + R_{np}^{(2)})$ where

– the triangular array $(D_{ij}^{(1)}, D_{ij}^{(2)})_{1 \leq j \leq i}$ is such that for every $(t_1, t_2) \in \mathbb{R}^2$,

$$\sup_{p \in I_x} \left| \mathbb{E}[\exp[i(t_1 D_{np}^{(1)} + t_2 D_{np}^{(2)})]] - \mathbb{E}[\exp[i(t_1 P_1 + t_2 P_2)]] \right| \rightarrow 0 \text{ as } n \rightarrow \infty$$

where (P_1, P_2) is a Gaussian centered random vector having covariance matrix $\mathcal{V}(\gamma(x))$;

– the triangular arrays of random variables $(R_{ij}^{(1)})_{1 \leq j \leq i}$ and $(R_{ij}^{(2)})_{1 \leq j \leq i}$ are such that for every $\varepsilon > 0$ and $j \in \{1, 2\}$,

$$\sup_{p \in I_x} \mathbb{P}(|R_{np}^{(j)}| > \varepsilon) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This result paves the way for a proof of Theorem 2.

Proof of Theorem 2. According to Lemma 2, the distribution of $(\widehat{\gamma}_{n,+}(x, k_x, h), \widehat{\gamma}_{n,-}(x, k_x, h))$ given $N_n(x, h) = p$ is that of

$$\left(\mathfrak{M}_{np}^{(1)}(x, k_x, h), 1 - \frac{1}{2} \left(1 - \frac{[\mathfrak{M}_{np}^{(1)}(x, k_x, h)]^2}{\mathfrak{M}_{np}^{(2)}(x, k_x, h)} \right)^{-1} \right).$$

Arguing along the first lines of the proof of Theorem 3.5.4 in [15] and applying Lemmas 4 and 5 with $A_{np} = \{N_n(x, h) = p\}$ together with Lemma 9 and the continuous mapping theorem, we then get that

$$\begin{pmatrix} \sqrt{k_x} (\widehat{\gamma}_{n,+}(x, k_x, h) - \gamma_+(x)) \\ \sqrt{k_x} (\widehat{\gamma}_{n,-}(x, k_x, h) - \gamma_-(x)) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \gamma_+(x)P_1 + \frac{\lambda(x)}{1 - \gamma_-(x)} (\mathbb{1}_{\{\gamma(x) > \rho(x) = 0\}} - \mathbb{1}_{\{\rho(x) < \gamma(x) \leq 0\}}) \\ (1 - 2\gamma_-(x))(1 - \gamma_-(x))^2 \left[\left(\frac{1}{2} - \gamma_-(x) \right) P_2 - 2P_1 \right] \end{pmatrix}$$

as $n \rightarrow \infty$, where (P_1, P_2) is the limit vector in Lemma 9. The result thus follows from Lemma 9 and some straightforward but lengthy computations. \blacksquare

6 Proofs of the auxiliary results

Proof of Lemma 1. The proof is a straightforward consequence of the fact that $N_n(x, h)$ is a binomial random variable with parameters n and $\mathbb{P}(X \in B(x, h))$ and of Chebyshev's inequality. ■

Proof of Lemma 2. If $(z_1, \dots, z_p) \in (0, 1)^p$, then since the random pairs (X_i, Y_i) have the same distribution, it holds that

$$\mathbb{P}\left(\bigcap_{i=1}^p \{Z_i \leq z_i\}, N_n(x, h) = p\right) = \binom{n}{p} \mathbb{P}\left(\bigcap_{i=1}^p \{Y_i \leq z_i, X_i \in B(x, h)\}\right).$$

Using the definition of $F_h(\cdot|x)$ and the independence of the random pairs (X_i, Y_i) , $i = 1, \dots, p$ leads to

$$\mathbb{P}\left(\bigcap_{i=1}^p \{Z_i \leq z_i\}, N_n(x, h) = p\right) = \left[\prod_{i=1}^p F_h(z_i|x)\right] \times \left[\binom{n}{p} \prod_{i=1}^p \mathbb{P}(X_i \in B(x, h))\right].$$

The identity

$$\mathbb{P}(N_n(x, h) = p) = \binom{n}{p} \prod_{i=1}^p \mathbb{P}(X_i \in B(x, h))$$

makes it clear that given $N_n(x, h) = p$, the Z_i , $i = 1, \dots, p$ are independent and identically distributed random variables having cdf $F_h(\cdot|x)$, which is the result. ■

Proof of Lemma 3. We start by writing the obvious inequality

$$|\log U_h(T_{p-i+1,p}|x) - \log U(T_{p-i+1,p}|x)| \leq \omega(T_{p-k_x,p}, T_{p,p}, x, h) \quad (38)$$

valid for every $i \in \{1, \dots, k_x + 1\}$. The first part of the result is then a straightforward consequence of (38) and the triangle inequality. To prove the second part, note that according to (38), for every $i = 1, \dots, k_x$

$$\begin{aligned} & \left| \left[\log \frac{U_h(T_{p-i+1,p}|x)}{U_h(T_{p-k_x,p}|x)} \right]^2 - \left[\log \frac{U(T_{p-i+1,p}|x)}{U(T_{p-k_x,p}|x)} \right]^2 \right| \\ & \leq 2\omega(T_{p-k_x,p}, T_{p,p}, x, h) \left| \log \frac{U_h(T_{p-i+1,p}|x)}{U_h(T_{p-k_x,p}|x)} + \log \frac{U(T_{p-i+1,p}|x)}{U(T_{p-k_x,p}|x)} \right| \\ & \leq 4\omega(T_{p-k_x,p}, T_{p,p}, x, h) \left[\omega(T_{p-k_x,p}, T_{p,p}, x, h) + \log \frac{U(T_{p-i+1,p}|x)}{U(T_{p-k_x,p}|x)} \right]. \end{aligned}$$

The result on $\mathfrak{M}_{np}^{(2)}(x, k_x, h)$ then follows from the triangle inequality and from summing the above inequalities for $i = 1, \dots, k_x$. ■

Proof of Lemma 4. Recall the obvious inequalities

$$\mathbb{P}(|R_{np} + R'_{np} - (\ell + \ell')| > \varepsilon) \leq \mathbb{P}(|R_{np} - \ell| > \varepsilon/2) + \mathbb{P}(|R'_{np} - \ell'| > \varepsilon/2)$$

and

$$\begin{aligned} \mathbb{P}(|R_{np}R'_{np} - \ell\ell'| > \varepsilon) & \leq \mathbb{P}\left(|R_{np} - \ell| > \sqrt{\varepsilon/3}\right) + \mathbb{P}\left(|R'_{np} - \ell'| > \sqrt{\varepsilon/3}\right) \\ & \quad + \mathbb{P}(\ell'|R_{np} - \ell| > \varepsilon/3) + \mathbb{P}(\ell|R'_{np} - \ell'| > \varepsilon/3). \end{aligned}$$

Using the hypotheses on (R_{ij}) and (R'_{ij}) concludes the proof. ■

Proof of Lemma 5. Start by writing, for every $t = (t_1, \dots, t_r) \neq (0, \dots, 0)$

$$\begin{aligned} \mathbb{E}[\exp(it'S_n)] - \mathbb{E}[\exp(it'S)] &= \left[\mathbb{E}[\exp(it'S_n)|A_{n0}] - \mathbb{E}[\exp(it'S)] \right] \mathbb{P}(A_{n0}) \\ &+ \sum_{p=1}^n \left[\mathbb{E}[\exp(it'S_n)|A_{np}] - \mathbb{E}[\exp(it'S)] \right] \mathbb{P}(A_{np}). \end{aligned}$$

Pick an arbitrary $\delta > 0$: for n large enough, the triangle inequality yields

$$\left| \mathbb{E}[\exp(it'S_n)] - \mathbb{E}[\exp(it'S)] \right| \leq \frac{\delta}{2} + \sup_{p \in I_n} \left| \mathbb{E}[\exp[it'(D_{np} + R_{np})]] - \mathbb{E}[\exp(it'S)] \right|. \quad (39)$$

We now split the term on the right-hand side as

$$\begin{aligned} \sup_{p \in I_n} \left| \mathbb{E}[\exp[it'(D_{np} + R_{np})]] - \mathbb{E}[\exp(it'S)] \right| &\leq \sup_{p \in I_n} \left| \mathbb{E}[\exp[it'(D_{np} + R_{np})]] - \mathbb{E}[\exp[it'D_{np}]] \right| \\ &+ \sup_{p \in I_n} \left| \mathbb{E}[\exp[it'D_{np}]] - \mathbb{E}[\exp[it'S]] \right|. \end{aligned} \quad (40)$$

The second term of the above inequality is controlled using the hypothesis on the array (D_{ij}) : we have for n sufficiently large

$$\sup_{p \in I_n} \left| \mathbb{E}[\exp[it'D_{np}]] - \mathbb{E}[\exp[it'S]] \right| \leq \frac{\delta}{4}. \quad (41)$$

Besides, using once again the triangle inequality entails, if $\|t\|_\infty = \max_{1 \leq j \leq r} |t_j|$,

$$\begin{aligned} \left| \mathbb{E}[\exp[it'(D_{np} + R_{np})]] - \mathbb{E}[\exp[it'D_{np}]] \right| &\leq \mathbb{E} \left[\left| \exp(it'R_{np}) - 1 \right| \mathbb{1}_{\{\max_{1 \leq j \leq r} |R_{np}^{(j)}| \leq \delta/8r\|t\|_\infty\}} \right] \\ &+ 2\mathbb{P} \left(\max_{1 \leq j \leq r} |R_{np}^{(j)}| > \delta/8r\|t\|_\infty \right). \end{aligned}$$

Applying the mean value theorem to the function $z \mapsto e^{iz}$ and using the hypothesis on the array (R_{np}) leads to

$$\sup_{p \in I_n} \left| \mathbb{E}[\exp[it'(D_{np} + R_{np})]] - \mathbb{E}[\exp[it'D_{np}]] \right| \leq \frac{\delta}{4} \quad (42)$$

for n large enough. Collecting (39), (40), (41) and (42) makes it clear that

$$\left| \mathbb{E}[\exp(it'S_n)] - \mathbb{E}[\exp(it'S)] \right| \leq \delta$$

for n large enough. Using the Cramér-Wold device concludes the proof. ■

Proof of Lemma 6. Pick $t \in (0, 1)$, $p \in I_x$ and write

$$\begin{aligned} \mathbb{P} \left(\left| \frac{k_x}{p} T_{p-k_x, p} - 1 \right| > t \right) &= \mathbb{P} \left(\log(T_{p-k_x, p}) - \log \left(\frac{p}{k_x} \right) > \log(1+t) \right) \\ &+ \mathbb{P} \left(\log(T_{p-k_x, p}) - \log \left(\frac{p}{k_x} \right) < \log(1-t) \right) \\ &\leq 2\mathbb{P} \left(\left| \log(T_{p-k_x, p}) - \log \left(\frac{p}{k_x} \right) \right| > \log(1+t) \wedge (-\log(1-t)) \right). \end{aligned}$$

Pick an arbitrary $t' > 0$: it is therefore enough to prove that

$$u_{np} = \mathbb{P} \left(\left| \log(T_{p-k_x, p}) - \log \left(\frac{p}{k_x} \right) \right| > t' \right) \rightarrow 0$$

uniformly in $p \in I_x$ as $n \rightarrow \infty$. To this end, since the random variables $\log T_i$, $1 \leq i \leq p$ are independent standard exponential random variables, we get according to Rényi's representation

$$\log(T_{p-k_x,p}) \stackrel{d}{=} \sum_{j=1}^{p-k_x} \frac{\log T_j}{p-j+1}. \quad (43)$$

Besides, the inequalities

$$\log\left(\frac{p+1}{k_x+1}\right) = \int_{k_x+1}^{p+1} \frac{dv}{v} \leq \sum_{j=1}^{p-k_x} \frac{1}{p-j+1} \leq \int_{k_x}^p \frac{dv}{v} = \log\left(\frac{p}{k_x}\right)$$

yield

$$\left| \log\left(\frac{p}{k_x}\right) - \sum_{j=1}^{p-k_x} \frac{1}{p-j+1} \right| \leq \log\left(\frac{p}{k_x}\right) - \log\left(\frac{p+1}{k_x+1}\right) = \log\left(\frac{1+k_x^{-1}}{1+p^{-1}}\right).$$

Using the classical inequality $\log(1+s) \leq s$ valid for every $s > 0$, we get for n large enough the inequality

$$\sup_{p \in I_x} \left| \log\left(\frac{p}{k_x}\right) - \sum_{j=1}^{p-k_x} \frac{1}{p-j+1} \right| \leq \frac{1}{k_x} \leq \frac{t'}{2}. \quad (44)$$

Applying (43) and (44) then entails for n large enough

$$\sup_{p \in I_x} u_{np} \leq \sup_{p \in I_x} \mathbb{P} \left(\left| \sum_{j=1}^{p-k_x} \frac{\log(T_j) - 1}{p-j+1} \right| > \frac{t'}{2} \right). \quad (45)$$

Furthermore, Chebyshev's inequality and a comparison with an integral give

$$\sup_{p \in I_x} \mathbb{P} \left(\left| \sum_{j=1}^{p-k_x} \frac{\log(T_j) - 1}{p-j+1} \right| > \frac{t'}{2} \right) \leq \frac{4}{t'^2} \sup_{p \in I_x} \sum_{j=1}^{p-k_x} \frac{1}{(p-j+1)^2} \leq \frac{4}{t'^2 k_x} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (46)$$

Collecting (45) and (46) yields the first result. The second result is then a simple consequence of the first result and of the inequality

$$\sup_{p \in I_x} \mathbb{P}(T_{p-k_x,p} \leq t) \leq \sup_{p \in I_x} \mathbb{P}\left(\frac{k_x}{p} T_{p-k_x,p} - 1 < -\frac{1}{2}\right)$$

valid for n large enough. The third result is obtained by noting that since φ is regularly varying at infinity, writing

$$\frac{\varphi(T_{p-k_x,p})}{\varphi(p/k_x)} = \frac{1}{\varphi(p/k_x)} \varphi\left(\frac{p}{k_x} \left\{ \frac{k_x}{p} T_{p-k_x,p} \right\}\right),$$

then Theorem 1.5.2 in [2] shows that there exists $t' > 0$ such that for n large enough

$$\left\{ \left| \frac{k_x}{p} T_{p-k_x,p} - 1 \right| \leq t' \right\} \subset \left\{ \left| \frac{\varphi(T_{p-k_x,p})}{\varphi(p/k_x)} - 1 \right| \leq t \right\}$$

for every $p \in I_x$; the first result then applies to yield

$$\sup_{p \in I_x} \mathbb{P} \left(\left| \frac{\varphi(T_{p-k_x,p})}{\varphi(p/k_x)} - 1 \right| > t \right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Finally, since

$$\sup_{p \in I_x} \left| \frac{p}{n_x} - 1 \right| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

applying once again Theorem 1.5.2 in [2] gives

$$\sup_{p \in I_x} \left| \frac{\varphi(p/k_x)}{\varphi(n_x/k_x)} - 1 \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Using Lemma 4 completes the proof. \blacksquare

Proof of Lemma 7. Use the fact that if T is a standard Pareto random variable, then $\log T$ is a standard exponential random variable and argue along the lines of the proof of Lemma 3.2.3 in [15]. \blacksquare

Proof of Lemma 8. We start by proving the first statement. Pick $\delta, t > 0$ and $\varepsilon \in (0, 1)$ such that

$$\frac{2\varepsilon}{t(1 - (\gamma_-(x) + \varepsilon))} \leq \frac{\delta}{4}. \quad (47)$$

With the notation of (23), letting $B_{np} = \{T_{p-k_x, p} \leq t_0\}$, Lemma 6 shows that $\mathbb{P}(B_{np}) \rightarrow 0$ uniformly in $p \in I_x$ as $n \rightarrow \infty$. For every $p \in I_x$, on the complement B_{np}^c of B_{np} , one can apply (23) to write

$$\frac{\log U(T_{p-i+1, p}|x) - \log U(T_{p-k_x, p}|x)}{q_0(T_{p-k_x, p}|x)} \leq \frac{(T_i^*(p))^{\gamma_-(x)} - 1}{\gamma_-(x)} + \varepsilon (T_i^*(p))^{\gamma_-(x)+\varepsilon} \quad (48)$$

and

$$\frac{(T_i^*(p))^{\gamma_-(x)} - 1}{\gamma_-(x)} - \varepsilon (T_i^*(p))^{\gamma_-(x)+\varepsilon} \leq \frac{\log U(T_{p-i+1, p}|x) - \log U(T_{p-k_x, p}|x)}{q_0(T_{p-k_x, p}|x)} \quad (49)$$

where $T_i^*(p) = T_{p-i+1, p}/T_{p-k_x, p} \geq 1$ for every $p \in I_x$ and $i = 1, \dots, k_x$. Using (48) and (49), the probability of the event

$$C_{np} = \left\{ \left| \frac{\mathcal{M}_{np}^{(1)}(x, k_x, h)}{q_0(T_{p-k_x, p}|x)} - \frac{1}{1 - \gamma_-(x)} \right| > t \right\}$$

is then bounded from above by $\mathbb{P}(B_{np}) + \mathbb{P}(C_{np}^{(1)}) + \mathbb{P}(C_{np}^{(2)}) \leq \delta/2 + \mathbb{P}(C_{np}^{(1)}) + \mathbb{P}(C_{np}^{(2)})$ uniformly in $p \in I_x$ for n large enough, where

$$C_{np}^{(1)} = \left\{ \left| \frac{1}{k_x} \sum_{i=1}^{k_x} \frac{(T_i^*(p))^{\gamma_-(x)} - 1}{\gamma_-(x)} - \frac{1}{1 - \gamma_-(x)} \right| > \frac{t}{2} \right\}$$

and $C_{np}^{(2)} = \left\{ \frac{1}{k_x} \sum_{i=1}^{k_x} (T_i^*(p))^{\gamma_-(x)+\varepsilon} > \frac{t}{2\varepsilon} \right\}.$

Apply Lemma 7 to get for every $p \in I_x$

$$\mathbb{P}(C_{np}^{(1)}) = \mathbb{P} \left(\left| \frac{1}{k_x} \sum_{i=1}^{k_x} \frac{T_i^{\gamma_-(x)} - 1}{\gamma_-(x)} - \frac{1}{1 - \gamma_-(x)} \right| > \frac{t}{2} \right)$$

and $\mathbb{P}(C_{np}^{(2)}) = \mathbb{P} \left(\frac{1}{k_x} \sum_{i=1}^{k_x} T_i^{\gamma_-(x)+\varepsilon} > \frac{t}{2\varepsilon} \right).$

Because

$$\mathbb{E} \left[\frac{T_i^{\gamma_-(x)} - 1}{\gamma_-(x)} \right] = \frac{1}{1 - \gamma_-(x)},$$

Chebyshev's inequality leads to the inequality $\mathbb{P}(C_{np}^{(1)}) \leq \delta/4$ for n large enough, uniformly in $p \in I_x$. Furthermore, since $\varepsilon \in (0, 1)$, using together (47) and Markov's inequality yields $\mathbb{P}(C_{np}^{(2)}) \leq \delta/4$ for every $p \in I_x$. Hence for n large enough the inequality

$$\sup_{p \in I_x} \mathbb{P}(C_{np}) \leq \delta.$$

In other words, it holds that for every $t > 0$

$$\sup_{p \in I_x} \mathbb{P} \left(\left| \frac{\mathcal{M}_{np}^{(1)}(x, k_x, h)}{q_0(T_{p-k_x, p}|x)} - \frac{1}{1 - \gamma_-(x)} \right| > t \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (50)$$

Recall that $q_0(\cdot|x)$ is regularly varying at infinity with index $\gamma_-(x)$ and apply Lemma 6 to get for every $t > 0$

$$\sup_{p \in I_x} \mathbb{P} \left(\left| \frac{q_0(T_{p-k_x, p}|x)}{q_0(p/k_x|x)} - 1 \right| > t \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (51)$$

Finally, writing

$$\begin{aligned} \left| \frac{\mathcal{M}_{np}^{(1)}(x, k_x, h)}{q_0(p/k_x|x)} - \frac{1}{1 - \gamma_-(x)} \right| &\leq \frac{q_0(T_{p-k_x, p}|x)}{q_0(p/k_x|x)} \left| \frac{\mathcal{M}_{np}^{(1)}(x, k_x, h)}{q_0(T_{p-k_x, p}|x)} - \frac{1}{1 - \gamma_-(x)} \right| \\ &\quad + \frac{1}{1 - \gamma_-(x)} \left| \frac{q_0(T_{p-k_x, p}|x)}{q_0(p/k_x|x)} - 1 \right| \end{aligned}$$

and applying Lemma 4 together with (50) and (51) gives the first part of the result. To obtain the second part, square the inequalities (48) and (49) with $\varepsilon < 1/2$ small enough, use the equality

$$\mathbb{E} \left[\frac{T_i^{\gamma_-(x)} - 1}{\gamma_-(x)} \right]^2 = \frac{2}{(1 - \gamma_-(x))(1 - 2\gamma_-(x))}$$

and use the ideas developed for the proof of the first statement. \blacksquare

Proof of Lemma 9. We only consider the case $\gamma(x) \neq \rho(x)$ and $\rho(x) < 0$ if $\gamma(x) > 0$, the proof being entirely similar in the case $\gamma(x) > \rho(x) = 0$. According to Lemma 3, the distribution of the random vector

$$\sqrt{k_x} \left(\frac{\mathfrak{M}_{np}^{(1)}(x, k_x, h)}{q_0(T_{p-k_x, p}|x)} - \frac{1}{1 - \gamma_-(x)}, \frac{\mathfrak{M}_{np}^{(2)}(x, k_x, h)}{q_0^2(T_{p-k_x, p}|x)} - \frac{2}{(1 - \gamma_-(x))(1 - 2\gamma_-(x))} \right)$$

is the distribution of the random vector

$$\sqrt{k_x} \left(\frac{\mathcal{M}_{np}^{(1)}(x, k_x, h)}{q_0(T_{p-k_x, p}|x)} - \frac{1}{1 - \gamma_-(x)}, \frac{\mathcal{M}_{np}^{(2)}(x, k_x, h)}{q_0^2(T_{p-k_x, p}|x)} - \frac{2}{(1 - \gamma_-(x))(1 - 2\gamma_-(x))} \right) + (r_{np}^{(1)}, r_{np}^{(2)}), \quad (52)$$

where

$$\begin{aligned} |r_{np}^{(1)}| &\leq 2\sqrt{k_x} \frac{\omega(T_{p-k_x, p}, T_{p, p}, x, h)}{q_0(T_{p-k_x, p}|x)} \\ \text{and } |r_{np}^{(2)}| &\leq 4\sqrt{k_x} \frac{\omega(T_{p-k_x, p}, T_{p, p}, x, h) \left[\omega(T_{p-k_x, p}, T_{p, p}, x, h) + \mathcal{M}_{np}^{(1)}(x, k_x, h) \right]}{q_0^2(T_{p-k_x, p}|x)}. \end{aligned}$$

Recall that $q_0(\cdot|x)$ is equivalent to $q(\cdot|x)$ at infinity, which is itself regularly varying at infinity with index $\gamma_-(x)$. As a consequence, applying Lemma 6, we get for every $t > 0$ the convergence

$$\sup_{p \in I_x} \mathbb{P} \left(\left| \frac{q_0(T_{p-k_x, p}|x)}{q_0(p/k_x|x)} - 1 \right| > t \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (53)$$

Besides, using (26) and (27) in the proof of Proposition 2, we get for every $t > 0$

$$\sup_{p \in I_x} \mathbb{P} \left(\sqrt{k_x} \frac{\omega(T_{p-k_x, p}, T_{p, p}, x, h)}{q_0(p/k_x|x)} > t \right) \leq \sup_{p \in I_x} \mathbb{P} \left(\sqrt{k_x} \frac{U(n_x/k_x|x)}{a(n_x/k_x|x)} \omega(T_{p-k_x, p}, T_{p, p}, x, h) > \frac{t}{2} \right) \quad (54)$$

for n large enough. Using condition (21), the right-hand side of the above inequality is bounded from above by

$$\sup_{p \in I_x} u_{np}^{(1,2)} \leq \sup_{p \in I_x} \mathbb{P} \left(\left\{ T_{p-k_x, p} < \frac{n_x}{(1+\delta)k_x} \right\} \cup \{T_{p,p} > n_x^{1+\delta}\} \right)$$

for n sufficiently large; consequently (see (27))

$$\sup_{p \in I_x} \mathbb{P} \left(\sqrt{k_x} \frac{U(n_x/k_x|x)}{a(n_x/k_x|x)} \omega(T_{p-k_x, p}, T_{p,p}, x, h) > \frac{t}{2} \right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Inequality (54) thus yields the convergence

$$\sup_{p \in I_x} \mathbb{P} \left(\sqrt{k_x} \frac{\omega(T_{p-k_x, p}, T_{p,p}, x, h)}{q_0(p/k_x|x)} > t \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (55)$$

Applying (25), (53), (55) and Lemma 4 we then obtain for every $t > 0$:

$$\sup_{p \in I_x} \mathbb{P} \left(\left| r_{np}^{(1)} \right| > t \right) \rightarrow 0 \text{ and } \sup_{p \in I_x} \mathbb{P} \left(\left| r_{np}^{(2)} \right| > t \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (56)$$

As in the proof of Lemma 8, letting $B_{np} = \{T_{p-k_x, p} \leq t_0\}$, Lemma 6 shows that $\mathbb{P}(B_{np}) \rightarrow 0$ uniformly in $p \in I_x$ as $n \rightarrow \infty$. Pick $\delta, t > 0$ and choose $\varepsilon \in (0, 1)$ such that

$$\frac{4\varepsilon}{t(1 - (\gamma_-(x) + \rho'(x) + \varepsilon))} \left| \lambda(x) \left(\frac{\mathbb{1}_{\{\rho'(x) < 0\}}}{\rho'(x)} + \mathbb{1}_{\{\rho'(x) = 0\}} \right) \right| \leq \frac{\delta}{4}.$$

For n large enough, one has $\mathbb{P}(B_{np}) \leq \delta/4$ for every $p \in I_x$. Furthermore, on the complement B_{np}^c of B_{np} , one can apply (35) to write

$$\left| \frac{\frac{\log U(T_{p-i+1, p}|x) - \log U(T_{p-k_x, p}|x)}{q_0(T_{p-k_x, p}|x)} - \frac{(T_i^*(p))^{\gamma_-(x)-1}}{\gamma_-(x)}}{Q_0(T_{p-k_x, p}|x)} - \psi_{\gamma_-(x), \rho'(x)}(T_i^*(p)) \right| \leq \varepsilon (T_i^*(p))^{\gamma_-(x) + \rho'(x) + \varepsilon}$$

where $T_i^*(p) = T_{p-i+1, p}/T_{p-k_x, p} \geq 1$ for $p \in I_x$ and $i = 1, \dots, k_x$ as in the proof of Lemma 8.

Consequently, for $i = 1, \dots, k_x$,

$$\begin{aligned} & \frac{(T_i^*(p))^{\gamma_-(x)-1}}{\gamma_-(x)} + Q_0(T_{p-k_x, p}|x) \psi_{\gamma_-(x), \rho'(x)}(T_i^*(p)) - \varepsilon |Q_0(T_{p-k_x, p}|x)| (T_i^*(p))^{\gamma_-(x) + \rho'(x) + \varepsilon} \\ & \leq \frac{\log U(T_{p-i+1, p}|x) - \log U(T_{p-k_x, p}|x)}{q_0(T_{p-k_x, p}|x)} \\ & \leq \frac{(T_i^*(p))^{\gamma_-(x)-1}}{\gamma_-(x)} + Q_0(T_{p-k_x, p}|x) \psi_{\gamma_-(x), \rho'(x)}(T_i^*(p)) + \varepsilon |Q_0(T_{p-k_x, p}|x)| (T_i^*(p))^{\gamma_-(x) + \rho'(x) + \varepsilon}. \end{aligned} \quad (57)$$

Note now that (34) implies that $|Q_0(\cdot|x)|$ is regularly varying at infinity, so that Lemma 6 entails for every $t > 0$

$$\sup_{p \in I_x} \mathbb{P} \left(\left| \frac{Q_0(T_{p-k_x, p}|x)}{Q_0(n_x/k_x|x)} - 1 \right| > t \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (58)$$

Applying Lemma 7 and Chebyshev's inequality leads to

$$\sup_{p \in I_x} \mathbb{P} \left(\left| \frac{1}{k_x} \sum_{i=1}^{k_x} \psi_{\gamma_-(x), \rho'(x)}(T_i^*(p)) - \mathbb{E}(\psi_{\gamma_-(x), \rho'(x)}(T)) \right| > t \right) \rightarrow 0 \text{ as } n \rightarrow \infty \quad (59)$$

for every $t > 0$. Finally, (34) and the hypothesis $\sqrt{k_x} Q(n_x/k_x) \rightarrow \lambda(x)$ as $n \rightarrow \infty$ yield

$$\sqrt{k_x} Q_0(n_x/k_x) \rightarrow \lambda(x) \left(\frac{\mathbb{1}_{\{\rho'(x) < 0\}}}{\rho'(x)} + \mathbb{1}_{\{\rho'(x) = 0\}} \right) \text{ as } n \rightarrow \infty \quad (60)$$

so that collecting (58), (59) and (60) and applying Lemma 4 shows that for every $t > 0$

$$\sup_{p \in I_x} \mathbb{P} \left(\left| \sqrt{k_x} Q_0(T_{p-k_x, p} | x) \frac{1}{k_x} \sum_{i=1}^{k_x} \psi_{\gamma_-(x), \rho'(x)}(T_i^*(p)) - m^{(1)}(x) \right| > t \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (61)$$

Meanwhile, letting

$$C_{np} = \left\{ \left| \frac{Q_0(T_{p-k_x, p} | x)}{Q_0(n_x/k_x | x)} - 1 \right| > \frac{1}{2} \right\}$$

then (58) entails that $\mathbb{P}(C_{np}) \rightarrow 0$ as $n \rightarrow \infty$, uniformly in $p \in I_x$; on C_{np}^c it holds that for n large enough

$$\sqrt{k_x} |Q_0(T_{p-k_x, p} | x)| \leq \frac{3}{2} \sqrt{k_x} |Q_0(n_x/k_x | x)| \leq 2 \left| \lambda(x) \left(\frac{\mathbb{1}_{\{\rho'(x) \leq 0\}}}{\rho'(x)} + \mathbb{1}_{\{\rho'(x) = 0\}} \right) \right|. \quad (62)$$

Therefore, letting n be so large that $\mathbb{P}(C_{np}) \leq \delta/4$ for every $p \in I_x$, Lemma 7 and Markov's inequality together imply that

$$\sup_{p \in I_x} \mathbb{P} \left(\varepsilon \sqrt{k_x} |Q_0(T_{p-k_x, p} | x)| \frac{1}{k_x} \sum_{i=1}^{k_x} (T_i^*(p))^{\gamma_-(x) + \rho'(x) + \varepsilon} > \frac{t}{2} \right) \leq \frac{\delta}{2}. \quad (63)$$

Collecting (57), (61) and (63), we get for n large enough

$$\sup_{p \in I_x} \mathbb{P} \left(\left| \sqrt{k_x} \left[\frac{\mathcal{M}_{np}^{(1)}(x, k_x, h)}{q_0(T_{p-k_x, p} | x)} - \frac{1}{k_x} \sum_{i=1}^{k_x} \frac{(T_i^*(p))^{\gamma_-(x)} - 1}{\gamma_-(x)} \right] - m^{(1)}(x) \right| > t \right) \leq \delta.$$

Recalling (56) and applying Lemma 4, we obtain that

$$\begin{aligned} & \sqrt{k_x} \left(\frac{\mathcal{M}_{np}^{(1)}(x, k_x, h)}{q_0(T_{p-k_x, p} | x)} - \frac{1}{1 - \gamma_-(x)} \right) + r_{np}^{(1)} \\ &= \sqrt{k_x} \left[\frac{1}{k_x} \sum_{i=1}^{k_x} \frac{(T_i^*(p))^{\gamma_-(x)} - 1}{\gamma_-(x)} - \frac{1}{1 - \gamma_-(x)} \right] + m^{(1)}(x) + R_{np}^{(1)} \end{aligned} \quad (64)$$

with $R_{np}^{(1)}$ as in the statement of the result.

To obtain a similar result for $\mathcal{M}_{np}^{(2)}(x, k_x, h)$, we note that using (58), (60), Lemma 7 and Chebyshev's inequality, we have for every $t' > 0$

$$\sup_{p \in I_x} \mathbb{P} \left(\left| 2\sqrt{k_x} Q_0(T_{p-k_x, p} | x) \frac{1}{k_x} \sum_{i=1}^{k_x} \frac{(T_i^*(p))^{\gamma_-(x)} - 1}{\gamma_-(x)} \psi_{\gamma_-(x), \rho'(x)}(T_i^*(p)) - m^{(2)}(x) \right| > t' \right) \rightarrow 0 \quad (65)$$

as $n \rightarrow \infty$. Besides, picking $\delta > 0$ then (62), Lemma 7 and Markov's inequality yield for n large enough

$$\sup_{p \in I_x} \mathbb{P} \left(\varepsilon \sqrt{k_x} |Q_0(T_{p-k_x, p} | x)| \frac{1}{k_x} \sum_{i=1}^{k_x} \frac{(T_i^*(p))^{\gamma_-(x)} - 1}{\gamma_-(x)} (T_i^*(p))^{\gamma_-(x) + \rho'(x) + \varepsilon} > t' \right) \leq \frac{\delta}{4} \quad (66)$$

if $\varepsilon > 0$ is chosen small enough. Using once again (58) and the convergence of $Q_0(\cdot | x)$ to 0, we get

$$\sup_{p \in I_x} \mathbb{P} \left(\sqrt{k_x} Q_0^2(T_{p-k_x, p} | x) > t' \right) \rightarrow 0 \text{ as } n \rightarrow \infty$$

which, together with Lemma 7 and Markov's inequality, entails

$$\sup_{p \in I_x} \mathbb{P} \left(\sqrt{k_x} Q_0^2(T_{p-k_x, p} | x) \frac{1}{k_x} \sum_{i=1}^{k_x} \psi_{\gamma_-(x), \rho'(x)}^2(T_i^*(p)) > t' \right) \rightarrow 0 \quad (67)$$

$$\text{and } \sup_{p \in I_x} \mathbb{P} \left(\sqrt{k_x} Q_0^2(T_{p-k_x, p} | x) \frac{1}{k_x} \sum_{i=1}^{k_x} \varepsilon^2(T_i^*(p))^{2(\gamma_-(x) + \rho'(x) + \varepsilon)} > t' \right) \rightarrow 0 \quad (68)$$

as $n \rightarrow \infty$. Square now the inequalities (57) and use (65), (66), (67) and (68) to obtain for n large enough

$$\sup_{p \in I_x} \mathbb{P} \left(\left| \sqrt{k_x} \left[\frac{\mathcal{M}_{np}^{(2)}(x, k_x, h)}{q_0^2(T_{p-k_x, p} | x)} - \frac{1}{k_x} \sum_{i=1}^{k_x} \left(\frac{(T_i^*(p))^{\gamma_-(x)} - 1}{\gamma_-(x)} \right)^2 \right] - m^{(2)}(x) \right| > t \right) \leq \delta.$$

Finally, recall (56) and apply Lemma 4 to get

$$\begin{aligned} & \sqrt{k_x} \left(\frac{\mathcal{M}_{np}^{(2)}(x, k_x, h)}{q_0^2(T_{p-k_x, p} | x)} - \frac{2}{(1 - \gamma_-(x))(1 - 2\gamma_-(x))} \right) + r_{np}^{(2)} \\ &= \sqrt{k_x} \left[\frac{1}{k_x} \sum_{i=1}^{k_x} \left(\frac{(T_i^*(p))^{\gamma_-(x)} - 1}{\gamma_-(x)} \right)^2 - \frac{2}{(1 - \gamma_-(x))(1 - 2\gamma_-(x))} \right] + m^{(2)}(x) + R_{np}^{(2)} \end{aligned} \quad (69)$$

with $R_{np}^{(2)}$ as in the statement of the result. Letting

$$\begin{aligned} D_{np}^{(1)} &= \sqrt{k_x} \left[\frac{1}{k_x} \sum_{i=1}^{k_x} \frac{(T_i^*(p))^{\gamma_-(x)} - 1}{\gamma_-(x)} - \frac{1}{1 - \gamma_-(x)} \right] + m^{(1)}(x) \\ \text{and } D_{np}^{(2)} &= \sqrt{k_x} \left[\frac{1}{k_x} \sum_{i=1}^{k_x} \left(\frac{(T_i^*(p))^{\gamma_-(x)} - 1}{\gamma_-(x)} \right)^2 - \frac{2}{(1 - \gamma_-(x))(1 - 2\gamma_-(x))} \right] + m^{(2)}(x) \end{aligned}$$

and applying Lemma 7, it is obvious that for fixed n and every $p > k_x$, the random pair $(D_{np}^{(1)}, D_{np}^{(2)})$ has the same distribution as $(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)})$ where

$$\begin{aligned} \mathcal{D}_n^{(1)} &= \sqrt{k_x} \left[\frac{1}{k_x} \sum_{i=1}^{k_x} \frac{T_i^{\gamma_-(x)} - 1}{\gamma_-(x)} - \frac{1}{1 - \gamma_-(x)} \right] + m^{(1)}(x) \\ \text{and } \mathcal{D}_n^{(2)} &= \sqrt{k_x} \left[\frac{1}{k_x} \sum_{i=1}^{k_x} \left(\frac{T_i^{\gamma_-(x)} - 1}{\gamma_-(x)} \right)^2 - \frac{2}{(1 - \gamma_-(x))(1 - 2\gamma_-(x))} \right] + m^{(2)}(x). \end{aligned}$$

Some cumbersome computations show that the random vector $(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)})$ is asymptotically Gaussian with mean $(m^{(1)}(x), m^{(2)}(x))$ and covariance matrix $\mathcal{V}(\gamma(x))$, so that clearly

$$\begin{aligned} & \sup_{p \in I_x} \left| \mathbb{E}[\exp[i(t_1 D_{np}^{(1)} + t_2 D_{np}^{(2)})]] - \mathbb{E}[\exp[i(t_1 P_1 + t_2 P_2)]] \right| \\ &= \left| \mathbb{E}[\exp[i(t_1 \mathcal{D}_n^{(1)} + t_2 \mathcal{D}_n^{(2)})]] - \mathbb{E}[\exp[i(t_1 P_1 + t_2 P_2)]] \right| \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

for every $t_1, t_2 \in \mathbb{R}$. Now, according to (52), (64) and (69) the distribution of the random vector

$$\sqrt{k_x} \left(\frac{\mathfrak{M}_{np}^{(1)}(x, k_x, h)}{q_0(T_{p-k_x, p} | x)} - \frac{1}{1 - \gamma_-(x)}, \frac{\mathfrak{M}_{np}^{(2)}(x, k_x, h)}{q_0^2(T_{p-k_x, p} | x)} - \frac{2}{(1 - \gamma_-(x))(1 - 2\gamma_-(x))} \right)$$

is the distribution of $(D_{np}^{(1)} + R_{np}^{(1)}, D_{np}^{(2)} + R_{np}^{(2)})$, which completes the proof. \blacksquare

References

- [1] Beirlant, J., Goegebeur, Y., Segers, J., Teugels, J. (2004). *Statistics of Extremes*, John Wiley and Sons.
- [2] Bingham, N.H., Goldie, C.M., Teugels, J.L. (1987). *Regular Variation*, Cambridge, U.K.: Cambridge University Press.
- [3] Chavez-Demoulin, V., Davison, A.C. (2005). Generalized additive modelling of sample extremes, *Journal of the Royal Statistical Society, series C* **54**: 207–222.
- [4] Daouia, A., Gardes, L., Girard, S., Lekina, A. (2011). Kernel estimators of extreme level curves, *Test* **20**(2): 311–333.
- [5] Daouia, A., Gardes, L., Girard, S. (2013). On kernel smoothing for extremal quantile regression, *Bernoulli*, to appear.
- [6] Davison, A.C., Ramesh, N.I. (2000). Local likelihood smoothing of sample extremes, *Journal of the Royal Statistical Society, series B* **62**: 191–208.
- [7] Davison, A.C., Smith, R.L. (1990). Models for exceedances over high thresholds, *Journal of the Royal Statistical Society, series B* **52**: 393–442.
- [8] Dekkers, A.L.M., Einmahl, J.H.J., de Haan, L. (1989). A moment estimator for the index of an extreme-value distribution, *Annals of Statistics* **17**(4): 1833–1855.
- [9] Fisher, R.A., Tippett, L.H.C. (1928). Limiting forms of the frequency distribution of the largest or smallest member of a sample, *Proceedings of the Cambridge Philosophical Society* **24**: 180–190.
- [10] Gardes, L., Girard, S. (2008). A moving window approach for nonparametric estimation of the conditional tail index, *Journal of Multivariate Analysis* **99**: 2368–2388.
- [11] Gardes, L., Girard, S. (2010). Conditional extremes from heavy-tailed distributions: an application to the estimation of extreme rainfall return levels, *Extremes* **13**: 177–204.
- [12] Gardes, L., Stupfler, G. (2013). Estimation of the conditional tail-index using a smoothed local Hill estimator, *Extremes*, to appear.
- [13] Gnedenko, B.V. (1943). Sur la distribution limite du terme maximum d’une série aléatoire, *Annals of Mathematics* **44**: 423–453.
- [14] Goegebeur, Y., Guillou, A., Schorgen, A. (2013). Nonparametric regression estimation of conditional tails – the random covariate case, *Statistics*, to appear.
- [15] de Haan, L., Ferreira, A. (2006). *Extreme value theory: An introduction*. Springer, New York.

- [16] Hall, P., Tajvidi, N. (2000). Nonparametric analysis of temporal trend when fitting parametric models to extreme-value data, *Statistical Science* **15**: 153–167.
- [17] Hill, B.M. (1975). A simple general approach to inference about the tail of a distribution, *Annals of Statistics* **3**: 1163–1174.
- [18] Pickands, J. (1975). Statistical inference using extreme order statistics, *Annals of Statistics* **3**: 119–131.
- [19] Smith, R. L. (1989). Extreme value analysis of environmental time series: an application to trend detection in ground-level ozone (with discussion), *Statistical Science* **4**: 367–393.
- [20] Wang, H., Tsai, C.L. (2009). Tail index regression, *Journal of the American Statistical Association* **104**(487): 1233–1240.

Situation	Moment estimator $\hat{\gamma}$	Estimator $\tilde{\gamma}_D$ of Daouia <i>et al.</i>
Model 1		
$\rho = -0.8$	0.1496	0.1962
$\rho = -1$	0.0781	0.1616
$\rho = -1.2$	0.0553	0.1586
Model 2		
$c = 0.1$	0.0686	0.1329
$c = 0.2$	0.0689	0.1257
$c = 0.3$	0.0825	0.1313
Model 3	0.3384	0.2801

Table 1: MSEs associated to the estimators in all cases.

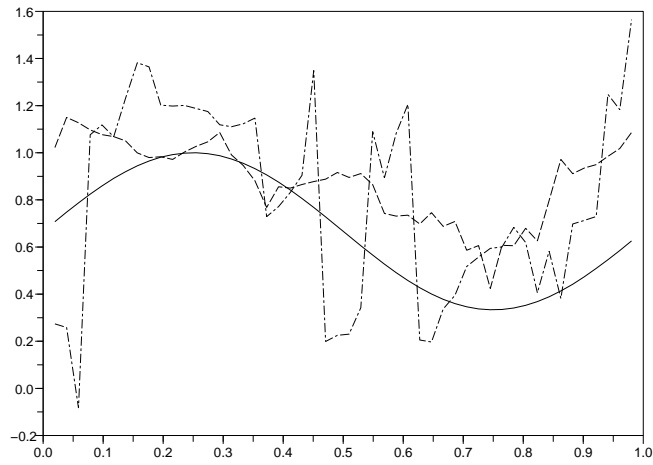


Figure 1: Model 1, case $\rho = -1$: the true function γ (solid line), its estimators $\hat{\gamma}$ (dashed line) and $\tilde{\gamma}_D$ (dashed-dotted line), each corresponding to the median of the MSE.

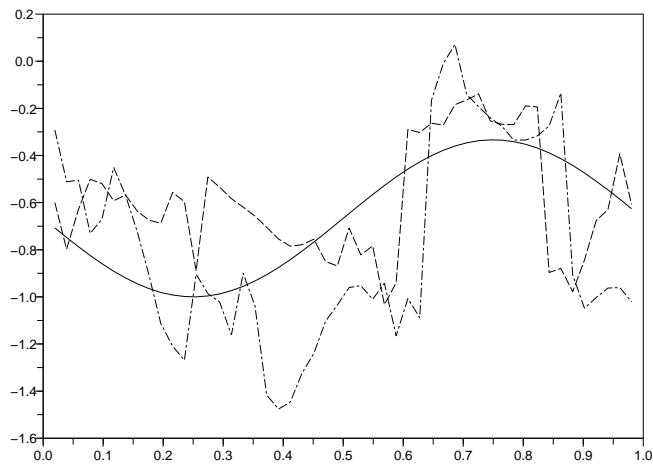


Figure 2: Model 2, case $c = 0.3$: the true function γ (solid line) and its estimators $\hat{\gamma}$ (dashed line) and $\tilde{\gamma}_D$ (dashed-dotted line), each corresponding to the median of the MSE.

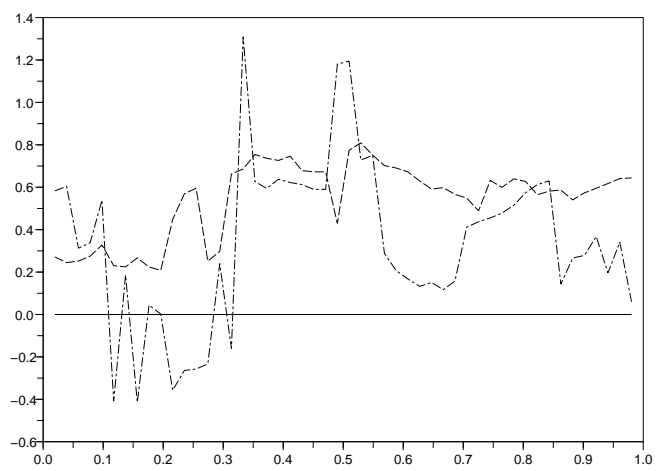


Figure 3: Model 3: the true function γ (solid line) and its estimators $\hat{\gamma}$ (dashed line) and $\tilde{\gamma}_D$ (dashed-dotted line), each corresponding to the median of the MSE.