Reaction-diffusion waves with nonlinear boundary conditions

Narcisa Apreutesei, Vitaly Volpert

NETWORKS AND HETEROGENEOUS MEDIA doi:10.3934/nhm.2013.8.23

Volume **8**, Number **1**, March **2013** pp. **23–35**

REACTION-DIFFUSION WAVES WITH NONLINEAR BOUNDARY CONDITIONS

Narcisa Apreutesei

Department of Mathematics, "Gheorghe Asachi" Technical University
Bd. Carol. I, 700506 Iasi, Romania

Vitaly Volpert

Institut Camille Jordan, UMR 5208 CNRS, University Lyon 1
69622 Villeurbanne, France

Abstract. A reaction-diffusion equation with nonlinear boundary condition is considered in a two-dimensional infinite strip. Existence of waves in the bistable case is proved by the Leray-Schauder method.

1. **Formulation of the problem.** Reaction-diffusion problems with nonlinear boundary conditions arise in various applications. In physiology, such problems describe in particular development of atherosclerosis and other inflammatory diseases [3]. In this context, nonlinear boundary conditions show the influx of white blood cells from blood flow into the tissue where the inflammation occurs. Among other possible applications, let us indicate molecular transport through biological membrane where some molecules can amplify their own transport opening membrane channels, as it is the case, for example, with calcium induced calcium release [1].

In this work we consider the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u + f(u), \tag{1}$$

with nonlinear boundary conditions:

$$y = 0 : \frac{\partial u}{\partial y} = 0, \quad y = 1 : \frac{\partial u}{\partial y} = g(u). \tag{2}$$

Here f and g are sufficiently smooth functions, $-\infty < x < \infty, 0 < y < 1$. We will study the existence of travelling wave solutions of this problem, that is of solutions of the equation

$$\Delta u + c\frac{\partial u}{\partial x} + f(u) = 0 \tag{3}$$

with the same boundary conditions. Here c is an unknown constant, the wave speed, and the variable x in equation (3) is identified with the variable $x - ct$ in equation (1).

We assume that $f(u_\pm) = 0$, $g(u_\pm) = 0$ for some u_+ and u_-, and

$$f'(u_\pm) < 0, \quad g'(u_\pm) < 0. \tag{4}$$

2010 *Mathematics Subject Classification.* Primary: 35K57; Secondary: 35J60.

Key words and phrases. Travelling waves, nonlinear boundary conditions, existence, Leray-Schauder method.

We will also assume that these functions have a single zero u_0 in the interval $u_+ < u < u_-$, $f(u_0) = g(u_0) = 0$, and $f'(u_0) > 0, g'(u_0) > 0$. We will look for solutions of problem (3), (2) with the limits

$$\lim_{x\to\pm\infty} u(x,y) = u_\pm, \quad 0 < y < 1. \tag{5}$$

A reaction-diffusion system with nonlinear boundary conditions suggested as a model of atherosclerosis was studied in [3] in the monostable case. The scalar equation with nonlinear boundary condition and with $f(u) \equiv 0$ was considered in [4]. However, behavior of solutions at infinity in [4] is not specified.

In this work we study problem (3), (2) in the bistable case (see (4)). The wave existence and uniqueness are proved by the Leray-Schauder method. We use the topological degree for elliptic problems in unbounded domains and obtain a priori estimates of solutions. Though the conditions on the functions f and g are rather restrictive, this formulation provides an interesting model problem that allows us to develop the method to prove wave existence applicable in more general cases.

2. **Solutions in the cross-section.** We will look for solutions of problem (3), (2) having limits as $x \to \pm\infty$. The limiting functions are solutions of the following problem in the interval $0 \le y \le 1$:

$$u'' + f(u) = 0, \tag{6}$$

$$u'(0) = 0, \ \ u'(1) = g(u(1)), \tag{7}$$

where prime denotes the derivative with respect to y.

We begin with the case where $f(u) \equiv 0$. Then from the first boundary condition in (7) we obtain $u = \text{const}$, from the second one, $g(u) = 0$. Denote a zero of the function g by u^*. Let us analyze the eigenvalue problem

$$v'' = \lambda v, \ \ v'(0) = 0, \ \ v'(1) = g'(u^*)v(1). \tag{8}$$

Since the principal eigenvalue of this problem is real [6] (in fact, they are all real because the problem is self-adjoint), it is sufficient for what follows to consider real λ. It can be easily verified that $\lambda = 0$ is not an eigenvalue of this problem if $g'(u^*) \neq 0$. Let $\lambda > 0$. Set $\mu = \sqrt{\lambda}$. Then from the equation and the first boundary condition we obtain

$$v(y) = k(e^{\mu y} + e^{-\mu y}).$$

From the second boundary condition it follows that

$$\mu = g'(u^*) \, \frac{e^\mu + e^{-\mu}}{e^\mu - e^{-\mu}} \, .$$

This equation has a positive solution for $g'(u^*) > 0$, that is for $u^* = u_0$. All eigenvalues are negative for $u^* = u_\pm$ since $g'(u_\pm) < 0$.

If $f(u)$ is different from zero, then corresponding eigenvalue problem, instead of (8), writes

$$v'' + f'(u^*)v = \lambda v, \ \ v'(0) = 0, \ \ v'(1) = g'(u^*)v(1). \tag{9}$$

If $f'(u^*) > 0$, then the principal eigenvalue of this problem is greater than the principal eigenvalue of problem (8), and it remains positive. This is the case for $u^* = u_0$. If $u^* = u_\pm$, then the eigenvalues are negative.

3. **Fredholm property.** Consider the operator corresponding to problem (3), (2) and linearized about a solution $u(x,y)$:

$$Av = \Delta v + c\frac{\partial v}{\partial x} + a(x,y)v, \quad (x,y) \in \Omega, \tag{10}$$

$$Bv = \begin{cases} \frac{\partial v}{\partial y} & , \quad y=0 \\ \frac{\partial v}{\partial y} - b(x)v & , \quad y=1 \end{cases}, \tag{11}$$

where $\Omega = \{-\infty < x < \infty, \ \ 0 < y < 1\}$, and

$$a(x,y) = f'(u(x,y)), \quad b(x) = g'(u(x,1)).$$

The operator $L = (A, B)$ acts from the space $E = C^{2+\alpha}(\bar{\Omega})$ into the space $F = C^{\alpha}(\bar{\Omega}) \times C^{1+\alpha}(\partial\Omega)$. Consider the limiting operators

$$A^{\pm}v = \Delta v + c\frac{\partial v}{\partial x} + a_{\pm}v, \quad (x,y) \in \Omega, \tag{12}$$

$$B^{\pm}v = \begin{cases} \frac{\partial v}{\partial y} & , \quad y=0 \\ \frac{\partial v}{\partial y} - b_{\pm}v & , \quad y=1 \end{cases} \tag{13}$$

and the corresponding equations

$$A^{\pm}v = 0, \quad B^{\pm}v = 0. \tag{14}$$

Here

$$a_{\pm} = \lim_{x\to\pm\infty} a(x,y) = f'(u_{\pm}) < 0, \quad b_{\pm} = \lim_{x\to\pm\infty} b(x) = g'(u_{\pm}) < 0.$$

Denote by $\tilde{v}(\xi, y)$ the partial Fourier transform of $v(x,y)$ with respect to x. Then from (14) we obtain

$$\tilde{v}'' + (-\xi^2 + ci\xi + a_{\pm})\tilde{v} = 0, \quad 0 < y < 1, \tag{15}$$

$$\tilde{v}'(\xi, 0) = 0, \ \ \tilde{v}'(\xi, 1) = b_{\pm}\tilde{v}(\xi, 1). \tag{16}$$

Since, according to Section 2, the eigenvalue problem

$$v'' + a_{\pm}v = \lambda v, \ \ 0 < y < 1, \quad v'(0) = 0, \ \ v'(1) = b_{\pm}v(1)$$

has nonzero solutions only for real negative λ, then it follows that for each $\xi \in \mathbb{R}$, problem (15), (16) has only zero solution. Hence $v(x,y) \equiv 0$, and thus we have proved that limiting problems do not have nonzero bounded solutions. This is also true for the formally adjoint operator. Therefore the operator L satisfies the Fredholm property. It remains also true if the operator acts from $W^{2,2}_{\infty}(\Omega)$ into $L^2_{\infty}(\Omega) \times W^{1/2,2}_{\infty}(\partial\Omega)$ ([7], page 163) where the ∞-spaces are defined as follows. Let E be a Banach space with the norm $\|\cdot\|$ and ϕ_i be a partition of unity. Then E_{∞} is the space of functions for which the expression

$$\|u\|_{\infty} = \sup_i \|u\phi_i\|$$

is bounded. This is the norm in this space.

Theorem 3.1. *If conditions (4), (5) are satisfied, then the operator $L = (A, B)$ acting from $C^{2+\alpha}(\bar{\Omega})$ into $F = C^{\alpha}(\bar{\Omega}) \times C^{1+\alpha}(\partial\Omega)$ or from $W^{2,2}_{\infty}(\Omega)$ into $L^2_{\infty}(\Omega) \times W^{1/2,2}_{\infty}(\partial\Omega)$ satisfies the Fredholm property.*

4. **Properness and topological degree.** Consider the nonlinear operator in the domain Ω

$$T_0(w) = \Delta w + c\frac{\partial w}{\partial x} + f(w), \quad (x,y) \in \Omega, \tag{17}$$

and the boundary operator

$$Q_0(w) = \left\{ \begin{array}{ccc} \frac{\partial w}{\partial y} & , & y = 0 \\ \frac{\partial w}{\partial y} - g(w) & , & y = 1 \end{array} \right. . \tag{18}$$

Let $w = u+\psi$, where $\psi(x)$ is an infinitely differentiable function such that $\psi(x) = u_+$ for $x \geq 1$ and $\psi(x) = u_-$ for $x \leq -1$. Set

$$T(u) = T_0(u+\psi) = \Delta u + c\frac{\partial u}{\partial x} + f(u+\psi) + \psi'' + c\psi', \quad (x,y) \in \Omega, \tag{19}$$

$$Q(u) = Q_0(u+\psi) = \left\{ \begin{array}{ccc} \frac{\partial u}{\partial y} & , & y = 0 \\ \frac{\partial u}{\partial y} - g(u+\psi) & , & y = 1 \end{array} \right. . \tag{20}$$

We consider the operator $P = (T, Q)$ acting in weighted spaces,

$$P = (T,Q) : W^{2,2}_{\infty,\mu}(\Omega) \to L^2_{\infty,\mu}(\Omega) \times W^{1/2,2}_{\infty,\mu}(\partial\Omega).$$

with the weight function $\mu(x) = \sqrt{1+x^2}$. The norm in the weighted space is defined as follows:

$$\|u\|_{\infty,\mu} = \|u\mu\|_\infty .$$

It is proved that under condition (4) the operator P is proper in the weighted spaces and that the topological degree can be defined [7].

5. **A priori estimates.**

5.1. **Monotonicity.** Consider the problem

$$\Delta u + c\frac{\partial u}{\partial x} + f(u) = 0, \tag{21}$$

$$y = 0 : \frac{\partial u}{\partial y} = 0, \quad y = 1 : \frac{\partial u}{\partial y} = g(u), \tag{22}$$

where $f(u_\pm) = 0$, $g(u_\pm) = 0$. We look for the solutions with the limits

$$\lim_{x \to \pm\infty} u(x,y) = u_\pm, \quad 0 < y < 1 \tag{23}$$

at infinity, $u_- > u_+$.

Lemma 5.1. *Let $U_0(x,y)$ be a solution of problem (21), (22) such that $\frac{\partial U_0}{\partial x} \leq 0$ for all $(x,y) \in \bar{\Omega}$. Then the last inequality is strict.*

Proof. Set $v = -\frac{\partial U_0}{\partial x}$. Then

$$\Delta v + c\frac{\partial v}{\partial x} + f'(U_0)v = 0, \tag{24}$$

$$y = 0 : \frac{\partial v}{\partial y} = 0, \quad y = 1 : \frac{\partial v}{\partial y} = g'(U_0)v. \tag{25}$$

Suppose that

$$\frac{\partial U_0}{\partial x}|_{(x_0,y_0)} = 0$$

for some $(x_0, y_0) \in \bar{\Omega}$. Then

$$v(x,y) \geq 0, \ \ v(x,y) \not\equiv 0, \quad (x,y) \in \bar{\Omega}; \quad v(x_0,y_0) = 0.$$

If $(x_0, y_0) \in \Omega$, then we obtain a contradiction with a positiveness theorem. If $(x_0, y_0) \in \partial\Omega$, then from (25) it follows that $\frac{\partial v(x_0,y_0)}{\partial y} = 0$. This contradicts the Hopf lemma which states that $\frac{\partial v}{\partial y} \neq 0$. □

Lemma 5.2. *Let $u_n(x,y)$ be a sequence of solutions of problem (21), (22) such that $u_n \to U_0$ in $C^1(\bar{\Omega})$, where $U_0(x,y)$ is a solution monotonically decreasing with respect to x. Then for all n sufficiently large $\frac{\partial u_n}{\partial x} < 0$, $(x,y) \in \bar{\Omega}$.*

Proof. Set $v_0 = -\frac{\partial U_0}{\partial x}$. Then v_0 is the eigenfunction of the problem

$$\Delta v + c\frac{\partial v}{\partial x} + f'(U_0)v = \lambda v, \tag{26}$$

$$y = 0 : \frac{\partial v}{\partial y} = 0, \quad y = 1 : \frac{\partial v}{\partial y} = g'(U_0)v \tag{27}$$

corresponding to the zero eigenvalue. Since $v > 0$ in Ω, then $\lambda = 0$ is the principal eigenvalue, that it is real, simple and all other eigenvalues lie in the left-half plane of the complex plane [6].

Similarly, $v_n = -\frac{\partial u_n}{\partial x}$ is the eigenfunction of the problem

$$\Delta v + c\frac{\partial v}{\partial x} + f'(u_n)v = \lambda v, \tag{28}$$

$$y = 0 : \frac{\partial v}{\partial y} = 0, \quad y = 1 : \frac{\partial v}{\partial y} = g'(u_n)v \tag{29}$$

corresponding to the zero eigenvalue. Suppose that u_n are not monotone for n sufficiently large. Then v_n are not positive. Therefore, the principal eigenvalues λ_n of problem (28), (29) are real and positive [6]. Since they are uniformly bounded, then there is a subsequence, which converges to some limiting λ_0. Hence λ_0 is an eigenvalue of problem (26), (27).

If $\lambda_0 > 0$, then we obtain a contradiction with the conclusion above that the principal eigenvalue of problem (26), (27) equals zero. If $\lambda_0 = 0$, then we obtain a contradiction with simplicity of the zero eigenvalue of this problem. Indeed, problems (28), (29) have the zero eigenvalue and the eigenvalue λ_n, which converges to zero. Let us recall that if a bounded domain of the complex plane does not contain the points of the essential spectrum, then the number of eigenvalues contained in this domain together with their multiplicities remains constant under small perturbation of the operator. Hence the zero eigenvalue of problem (26), (27) is not simple. This contradiction proves that the functions u_n are monotone with respect to x. □

We note that this lemma is proved under the assumption that the constant c in problems (26), (27) and (28), (29) is the same, that is the wave speeds for U_0 and u_n are the same. The proof remains similar if they are different and the speed c_n of the wave u_n converges to the speed c_0 of the wave U_0.

5.2. **Wave speed.**

5.2.1. *Sign of the speed.* We will determine the sign of the speed of the wave connecting a stable and an unstable solutions. This result will be used below for estimates of solutions.

Lemma 5.3. *Suppose that $f(u_0) = g(u_0) = 0$ for some u_0, $u_+ < u_0 < u_-$, and $f'(u_0) > 0, g'(u_0) > 0$. If a monotone with respect to x function $w(x,y)$ satisfies the problem*

$$\Delta w + c\frac{\partial w}{\partial x} + f(w) = 0, \tag{30}$$

$$y = 0 : \frac{\partial w}{\partial y} = 0, \quad y = 1 : \frac{\partial w}{\partial y} = g(w), \tag{31}$$

$$\lim_{x\to-\infty} w(x,y) = u_-, \quad \lim_{x\to\infty} w(x,y) = u_0, \tag{32}$$

then $c > 0$.

Proof. Let us integrate equation (30) with respect to x from x_0 to ∞ and with respect to $y \in [0,1]$:

$$-\int_0^1 \frac{\partial w(x_0,y)}{\partial x}\, dy + \int_{x_0}^\infty g(w(x,1))dx + c\int_0^1 (u_0 - w(x_0,y))dy + \tag{33}$$
$$\int_{x_0}^\infty \int_0^1 f(w(x,y))dxdy = 0.$$

We can choose x_0 so large that

$$f(w(x,y)) > 0, \;\; g(w(x,y)) > 0, \quad x > x_0, 0 \le y \le 1.$$

Then the second and the last terms in (33) are positive. The first term is positive since $w(x,y)$ is decreasing with respect to x. Since $u_0 - w(x,y) < 0$, then this equality can take place only if $c > 0$. □

Lemma 5.3$'$. *In the conditions of the previous lemma, if the conditions at infinity instead of (32) are*

$$\lim_{x\to-\infty} w(x,y) = u_0, \quad \lim_{x\to\infty} w(x,y) = u_+,$$

then $c < 0$.

The proof of the lemma is similar to the previous one.

5.2.2. *Estimate of the speed.* We begin with an auxiliary result on comparison of solutions. Consider the initial boundary value problems

$$\frac{\partial u}{\partial t} = \Delta u + c\frac{\partial u}{\partial x} + f_i(u), \tag{34}$$

$$y = 0 : \frac{\partial u}{\partial y} = 0, \quad y = 1 : \frac{\partial u}{\partial y} = g_i(u), \tag{35}$$

$$u(x,y,0) = u_i^0(x,y), \tag{36}$$

where $i = 1,2$. We denote solutions of these two problems by $u_1(x,y,t)$ and $u_2(x,y,t)$, respectively.

Lemma 5.4. *If*

$$f_1(u) < f_2(u), \;\; g_1(u) < g_2(u), \;\; u_1^0(x,y) < u_2^0(x,y), \quad \forall u,x,y,$$

then

$$u_1(x,y,t) < u_2(x,y,t), \quad \forall x \in \mathbb{R}, \; 0 < y < 1, \; t \ge 0. \tag{37}$$

Proof. Set $z = u_2 - u_1$. Then

$$\frac{\partial z}{\partial t} = \Delta z + c\frac{\partial z}{\partial x} + a(x,y,t)z + \phi(x,y,t), \tag{38}$$

$$y = 0 : \frac{\partial z}{\partial y} = 0, \quad y = 1 : \frac{\partial z}{\partial y} = b(x,y,t)z + \psi(x,y,t), \tag{39}$$

$$z(x,y,0) = u_2^0(x,y) - u_1^0(x,y). \tag{40}$$

Here

$$f_2(u_2) - f_1(u_1) = f_2(u_2) - f_1(u_2) + f_1(u_2) - f_1(u_1) = a(x,y,t)z + \phi(x,y,t),$$

$$a(x,y,t) = \frac{f_1(u_2) - f_1(u_1)}{u_2 - u_1}, \quad \phi(x,y,t) = f_2(u_2) - f_1(u_2) > 0,$$

$$g_2(u_2) - g_1(u_1) = g_2(u_2) - g_1(u_2) + g_1(u_2) - g_1(u_1) = b(x,y,t)z + \psi(x,y,t),$$

$$b(x,y,t) = \frac{g_1(u_2) - g_1(u_1)}{u_2 - u_1}, \quad \psi(x,y,t) = g_2(u_2) - g_1(u_2) > 0.$$

Suppose that inequality (37) is satisfied for $0 \le t < t_0$ and $z(x_0,y_0,t_0) = 0$ for some x_0, y_0, t_0.

If $0 < y_0 < 1$, then we obtain a contradiction with the positiveness theorem for parabolic equations.

If $y_0 = 0$, then we obtain a contradiction with the fact that if $z(x,y,t_0) \ge 0$ everywhere in the domain and $z(x_0,y_0,t_0) = 0$, then the normal derivative at (x_0,y_0) is negative [2].

Remark that in [2] this theorem (Theorem 14, Chapter II, Section 5) is proved for a positive maximum (negative minimum) under the assumption that $a(x,y,t) \le 0$. It remains valid without this assumption if this maximum (minimum) equals zero. There is also a direct way to obtain a contradiction. Consider a function $\tilde{z}$ defined in the twice wider domain $\tilde{\Omega}$, $-1 \le y \le 1$ such that it coincides with z for $0 \le y \le 1$ and $\tilde{z}(x,y,t) = z(x,-y,t)$ for $-1 \le y \le 0$. The function $\tilde{z}$ is a solution of the parabolic problem in $\tilde{\Omega}$. By virtue of the boundary condition at $y = 0$, it is continuous together with the second derivatives. Since it is nonnegative everywhere in $\tilde{\Omega}$ and $\tilde{z}(x_0,y_0,t_0) = 0$, then we obtain a contradiction with the positiveness theorem.

If $y_0 = 1$, then $\frac{\partial z(x_0,y_0,t_0)}{\partial y} \le 0$ since $z(x,y,t_0) \ge 0$. On the other hand, from the boundary condition we get

$$\frac{\partial z(x_0,y_0,t_0)}{\partial y} = \psi(x_0,y_0,t_0) > 0.$$

This contradiction proves the lemma. □

Suppose that problem (21)-(23) has a solution $w_0(x,y)$. In order to estimate the corresponding speed c from above, we consider two initial boundary value problems. The first one

$$\frac{\partial u}{\partial t} = \Delta u + f(u), \tag{41}$$

$$y = 0 : \frac{\partial u}{\partial y} = 0, \quad y = 1 : \frac{\partial u}{\partial y} = g(u), \tag{42}$$

$$u(x,y,0) = w_0(x,y) \tag{43}$$

has a solution $u_1(x,y,t) = w_0(x - ct, y)$. The second one

$$\frac{\partial u}{\partial t} = \Delta u + F(u), \tag{44}$$

$$y = 0 : \frac{\partial u}{\partial y} = 0, \quad y = 1 : \frac{\partial u}{\partial y} = G(u) \tag{45}$$

will be constructed in such a way that its solution estimates from above the solution of the first one. Its initial condition will be specified below. Set

$$F(u) = k(u - u_0), \quad G(u) = k(u - u_0),$$

where u_0 is the zero of the functions $f(u)$ and $g(u)$, $u_+ < u_0 < u_-$ and a positive constant k is sufficiently large to provide the estimates

$$f(u) < F(u), \quad g(u) < G(u), \quad u > u_0. \tag{46}$$

We look for a solution of problem (44), (45) in the form

$$u_2(x, y, t) = u_0 + v(x, y, t).$$

Then v satisfies the problem

$$\frac{\partial v}{\partial t} = \Delta v + kv, \tag{47}$$

$$y = 0 : \frac{\partial v}{\partial y} = 0, \quad y = 1 : \frac{\partial v}{\partial y} = kv. \tag{48}$$

Let $v(x, y, t) = w_1(x - c_1 t, y)$. Then

$$\Delta w_1 + c_1 \frac{\partial w_1}{\partial x} + kw_1 = 0, \tag{49}$$

$$y = 0 : \frac{\partial w_1}{\partial y} = 0, \quad y = 1 : \frac{\partial w_1}{\partial y} = kw_1. \tag{50}$$

We seek solution of this problem in the form

$$w_1(x, y) = e^{-\mu x}\, \omega(y) \tag{51}$$

with some real positive μ and a positive twice continuously differentiable function $\omega(y)$. Hence, ω satisfies

$$\omega'' + (\mu^2 - c_1\mu + k)\omega = 0, \quad \omega'(0) = 0, \quad \omega'(1) = k\omega(1).$$

We note that the principal eigenvalue (i.e., maximal) of the problem

$$U'' = \lambda U, \quad U'(0) = 0, \quad U'(1) = kU(1)$$

is positive. Denote it by λ_0. Then the equation

$$\mu^2 - c_1\mu + k = -\lambda_0$$

has a positive solution μ for some c_1 sufficiently large. Therefore we constructed a solution $w_1(x, y)$ in the form (51).

Since solution $w_0(x, y)$ has limits $u_\pm$ as $x \to \pm\infty$, then we can choose a number h such that

$$w_0(x, y) < u_0 + w_1(x + h, y), \quad -\infty < x < \infty, \ \ 0 < y < 1.$$

We consider problem (44), (45) with the initial condition

$$u_2(x, y, 0) = u_0 + w_1(x + h, y).$$

Its solution is

$$u_2(x, y, t) = u_0 + w_1(x + h - c_1 t, y).$$

Since $u_1(x, y, 0) < u_2(x, y, 0)$, then by virtue of (46) and Lemma 5.4, we obtain

$$u_1(x, y, t) < u_2(x, y, t), \quad -\infty < x < \infty, \ \ 0 < y < 1, \ \ t \geq 0.$$

Hence, $c < c_1$. Thus, we have estimated the speed c of the solution w of problem (21)-(23) from above. Similarly it can be estimated from below. We have proved the following lemma.

Lemma 5.5. *If problem (21)-(23) has a solution w, then the value of the speed admits the estimate $|c| \le M$, where the constant M depends only on $\max_{u\in[u_+,u_-]}$ $|f'(u)|$, $|g'(u)|$.*

Remark 5.6. Intermediate solution is u_0. There are no other intermediate solutions if $f(u)$ is sufficiently small. It follows from the implicit function theorem.

5.2.3. *Functionalization of the parameter.* Let $w_0(x, y)$ be a solution of problem (21)-(23). Then the functions

$$w_h(x, y) = w_0(x + h, y), \quad h \in \mathbb{R}$$

are also solutions of this problem. The existence of the family of solutions does not allow one to use directly the topological degree because there is a zero eigenvalue of the linearized problem and a uniform a priori estimate of solutions in the weighted spaces does not occur.

In order to overcome this difficulty, we replace the unknown parameter c, the wave speed, by a functional $c(w_h)$ [5]. This functional determines a function of h, $s(h) = c(w_h)$. We will construct this functional in such a way that $s'(h) < 0$ and $s(h) \to \pm\infty$ as $h \to \mp\infty$. Then instead of the family of solutions we obtain a single solution for the value of h for which $c = s(h)$.

Let

$$\rho(w_h) = \int_\Omega (w_0(x + h, y) - u_+) r(x) dx dy,$$

where $r(x)$ is an increasing function satisfying the conditions:

$$r(-\infty) = 0, \quad r(+\infty) = 1, \quad \int_{-\infty}^{0} r(x) dx < \infty.$$

Since $w_0(x, y)$ is a decreasing function of x, then $\rho(w_h)$ is a decreasing function of h, and

$$\rho(w_h) \to \left\{ \begin{array}{ccc} 0 & , & h \to +\infty \\ +\infty & , & h \to -\infty \end{array} \right. .$$

Hence the function $s(h) = c(w_h) = \ln \rho(w_h)$ possesses the required properties.

5.3. **Estimates of solutions.** We consider next the problem

$$\Delta u + c\frac{\partial u}{\partial x} + f_\tau(u) = 0, \tag{52}$$

$$y = 0 : \frac{\partial u}{\partial y} = 0, \quad y = 1 : \frac{\partial u}{\partial y} = g_\tau(u), \tag{53}$$

$$u(\pm\infty, y) = u_\pm, \tag{54}$$

where the functions f and g depend on the parameter $\tau \in [0, 1]$. Denote by w_τ a solution of this problem. We need to obtain a uniform estimate of the solution $u_\tau = w_\tau - \psi$ in the norm of the space $W^{2;2}_{\infty,\mu}(\Omega)$. Let us note that the Hölder norm $C^{2+\alpha}(\bar{\Omega})$, $0 < \alpha < 1$ of the solution is uniformly bounded. Hence the norm $W^{2;2}_{\infty}(\Omega)$ is also uniformly bounded. However, the boundedness of the norm in the weighted space does not follow from this and should be proved. In order to obtain the estimate, it is sufficient to prove that the solution is bounded in the weighted space, that is

$$\sup_{(x,y)\in\Omega} |(w_\tau(x,y)-\psi(x))\mu(x)| \le M \tag{55}$$

with some constant M independent of τ. If this estimate is satisfied, then the derivatives of the solution up to the order two are also bounded. Indeed, the function $u_\tau = w_\tau - \psi$ satisfies the problem

$$\Delta u + c\frac{\partial u}{\partial x} + f(u+\psi) + \psi'' + c\psi' = 0,$$

$$y=0:\ \frac{\partial u}{\partial y} = 0, \quad y=1:\ \frac{\partial u}{\partial y} = g(u+\psi).$$

Then the function $v_\tau = u_\tau\mu$ satisfies the problem

$$\Delta v + (c-2\mu_1)\frac{\partial v}{\partial x} + (-c\mu_1 + 2\mu_1^2 - \mu_2)v + (f(u+\psi)-f(\psi))\mu + (\psi''+c\psi'+f(\psi))\mu = 0, \tag{56}$$

$$y=0:\ \frac{\partial v}{\partial y} = 0, \quad y=1:\ \frac{\partial v}{\partial y} = (g(u+\psi)-g(\psi))\mu + g(\psi)\mu, \tag{57}$$

where

$$\mu_1 = \frac{\mu'}{\mu}\ , \quad \mu_2 = \frac{\mu''}{\mu}$$

are bounded infinitely differentiable functions converging to zero at infinity. Since

$$|(f(u+\psi)-f(\psi))\mu| \le \sup_s |f'(s)||u\mu|, \quad |(g(u+\psi)-g(\psi))\mu| \le \sup_s |g'(s)||u\mu|,$$

then, by virtue of (55), the functions

$$\Phi(u,x) = (f(u+\psi)-f(\psi))\mu + (\psi''+c\psi'+f(\psi))\mu,$$

$$\Psi(u,x) = (g(u+\psi)-g(\psi))\mu + g(\psi)\mu$$

are bounded together with their second derivatives. Therefore solutions of problem (56), (57) are uniformly bounded in the space $C^{2+\alpha}(\Omega)$. Then the norm $W^{2,2}_\infty(\Omega)$ is also bounded.

It remains to prove estimate (55). We will consider here solutions monotone with respect to x. Consider first of all the behavior of solutions at the vicinity of infinity. By virtue of the Fredholm property, $|w_\tau(x,y)-u_\pm|$ decay exponentially as $x \to \pm\infty$. The decay rate is determined by the principal eigenvalue of the corresponding operators in the cross-section of the cylinder. They can be estimated independently of τ.

Let $\epsilon > 0$ be small enough, $x = N_-(\tau)$ and $x = N_+(\tau)$ be such that $|w_\tau(x,y) - u_+| \le \epsilon$ for $x \ge N_+(\tau)$ and $|w_\tau(x,y)-u_-| \le \epsilon$ for $x \le N_-(\tau)$. For a polynomial weight function $\mu(x)$ there exists a constant K independent of $\tau \in [0,1]$ such that

$$|w_\tau(x,y) - u_\pm|\mu(x) \le K, \quad x > N_+(\tau) \text{ or } x < N_-(\tau), \quad \tau\in[0,1].$$

Since the functions $w_\tau(x,y)$ are uniformly bounded, then (55) will follow from the uniform boundedness of the values $N_\pm(\tau)$.

First, let us note that the difference between them is uniformly bounded. Indeed, if this is not the case and $N_+(\tau)-N_-(\tau)\to\infty$ as $\tau\to\tau_0$ for some τ_0, then there are two solutions of problem (52), (53) for $\tau=\tau_0$, w_1 and w_2 with the limits

$$w_1(x,y) \to \left\{ \begin{array}{ccc} u_- & , & x\to-\infty \\ u_0 & , & x\to+\infty \end{array} \right. , \quad w_2(x,y) \to \left\{ \begin{array}{ccc} u_0 & , & x\to-\infty \\ u_+ & , & x\to+\infty \end{array} \right. .$$

These solutions are obtained as limits of the solution w_τ as $\tau \to \tau_0$. In order to obtain them, consider a sequence of functions $w_{\tau_k}(x,y)$, $\tau_k \to \tau_0$ and two sequences of shifted functions: $w_{\tau_k}(x+N_-(\tau_k),y)$ and $w_{\tau_k}(x+N_+(\tau_k),y)$. The first sequence gives in the limit the first solution, the second limit gives the second solution. The existence of the limits as $x \to \pm\infty$ follows from the monotonicity of solutions.

The existence of such solutions contradicts Lemmas 5.3 and 5.3' since the first one affirms that the speed is positive while the second one that it is negative.

Next, if one of the values $|N_\pm(\tau)|$ tends to infinity as $\tau \to \tau_0$, then the modulus $|c(w_h)|$ of the functional introduced in Section 5.2.3 also tends to infinity as $\tau \to \tau_0$. This contradicts a priori estimates of the wave speed. Thus, we have proved the following theorem.

Theorem 5.7. *If there exists a solution w_τ of problem (52)-(54) monotone with respect to x for some $\tau \in [0,1]$, then the norm $\|w_\tau - \psi\|_{W^{2,2}_{\infty,\mu}(\Omega)}$ is bounded independently of τ and of the solution w_τ.*

6. **Leray-Schauder method.**

6.1. **Model problem.**

The problem

$$\Delta w + c\frac{\partial w}{\partial x} + f(w) = 0, \tag{58}$$

$$y = 0 : \frac{\partial w}{\partial y} = 0, \quad y = 1 : \frac{\partial w}{\partial y} = 0, \tag{59}$$

$$w(\pm\infty, y) = u_\pm, \tag{60}$$

where we put 0 instead of $g(w)$ in the boundary condition, has a one-dimensional solution $w_0(x)$. The existence of such solution for the one-dimensional scalar equation is well known (see, e.g., [5]). Its uniqueness as solution of the two-dimensional problem is proved in the following lemma.

Lemma 6.1. *There exists a unique monotone in x solution of problem (58)-(60) up to translation in space.*

Proof. Suppose that there exist two different monotone solutions of problem (58)-(60), (w_1, c_1) and (w_2, c_2). We recall that the corresponding values of the speed c can be different. Consider the equation

$$\frac{\partial v}{\partial t} = \Delta v + c_1\frac{\partial v}{\partial x} + f(v) \tag{61}$$

with the boundary condition (59). The function $w_1(x,y)$ is a stationary solution of this problem. It is proved in [6] that it is globally stable with respect to all initial conditions $v(x,y,0)$, which are monotone with respect to x and such that the norm $\|v(x,y,0) - w_1(x,y)\|_{L^2(\Omega)}$ is bounded.

Consider the initial condition $v(x,y,0) = w_2(x,y)$. It is monotone and the L^2 norm of the difference $w_2 - w_1$ is bounded since these functions approach exponentially their limits at infinity. According to the stability result, the solution converges to $w_1(x+h,y)$ with some h. On the other hand, the solution writes $u(x,y,t) = w_2(x-(c_2-c_1)t, y)$, and it cannot converge to w_1. This contradiction proves the lemma. $\square$

We consider next the problem (52)-(54) and the corresponding operators

$$T_\tau(u) = \Delta u + c(u+\psi)\frac{\partial u}{\partial x} + f_\tau(u+\psi) + \psi'', \quad (x,y) \in \Omega, \tag{62}$$

$$Q_\tau(u) = \begin{cases} \frac{\partial u}{\partial y} & , \quad y=0 \\ \frac{\partial u}{\partial y} - g_\tau(u+\psi) & , \quad y=1 \end{cases}, \tag{63}$$

$$P_\tau = (T_\tau, Q_\tau) : W^{2,2}_{\infty,\mu}(\Omega) \to L^2_{\infty,\mu}(\Omega) \times W^{1/2,2}_{\infty,\mu}(\partial\Omega).$$

Suppose that $g_\tau(u) \equiv 0$ for $\tau = 0$. Then the equation

$$P_\tau(u) = 0 \tag{64}$$

has a unique solution $u_0 = w_0 - \psi$ for $\tau = 0$. The index of this solution, that is the topological degree of this operator with respect to a small neighborhood of the solution, equal 1. Indeed, the index equals $(-1)^\nu$, where the ν is the number of positive eigenvalues of the linearized operator [5], [7]. In the case under consideration, the linearized operator has all eigenvalues in the left half-plane [6].

6.2. **Wave existence.** We can now formulate the main result of this work.

Theorem 6.2. *Let $f(u), g(u) \in C^2[u_+, u_-]$ for some u_+, u_-, and let the following conditions be satisfied:*
1. $f(u_\pm) = 0,\ f'(u_\pm) < 0$, $g(u_\pm) = 0,\ g'(u_\pm) < 0$,
2. $f(u_0) = 0,\ f'(u_0) > 0$, $g(u_0) = 0,\ g'(u_0) > 0$ for some $u_0 \in (u_+, u_-)$, and there are no other zeros of these functions in this interval.
3. the problem

$$\frac{d^2w}{dy^2} + (1-\tau)f(w) = 0, \quad w'(0) = 0, \quad w'(1) = \tau g(w) \tag{65}$$

does not have non-constant solutions $w(y)$ which satisfy $u_+ \le w(y) \le u_-$, $0 < y < 1$ for any $\tau \in [0,1]$.

Then for any $\tau \in [0,1]$ the problem

$$\Delta w + c\frac{\partial w}{\partial x} + (1-\tau)f(w) = 0, \tag{66}$$

$$y = 0 : \frac{\partial w}{\partial y} = 0, \quad y = 1 : \frac{\partial w}{\partial y} = \tau g(w), \tag{67}$$

$$\lim_{x\to\pm\infty} w(x,y) = u_\pm, \quad 0 < y < 1, \tag{68}$$

considered in the domain $\Omega = \{-\infty < x < \infty,\ \ 0 < y < 1\}$, has a unique solution monotone with respect to x.

Proof. The proof of the theorem is based on the Leray-Schauder method. We consider the equation (64). The topological degree for the operator $P_\tau(u)$ is defined (Section 4).

Denote by Γ_m the ensemble of its solutions for all $\tau \in [0,1]$ such that for any $u \in \Gamma_m$ the function $w = u + \psi$ is monotone with respect to x. Let Γ_n be the set of all solutions for which the function $w = u + \psi$ is not monotone with respect to x. Then the distance d between these two sets in the space $E = W^{2,2}_{\infty,\mu}(\Omega)$ is positive. Indeed, suppose that this is not true. Then there exist two sequences $u_k \in \Gamma_m$ and $v_k \in \Gamma_n$ such that $\|u_k - v_k\|_E \to 0$ as $k \to \infty$. From Lemma 5.2 it follows that the functions $w_k = v_k + \psi$ are monotone with respect to x for k sufficiently large. This contradiction shows that the convergence cannot occur.

From Theorem 5.7, applicable for solutions from Γ_m, it follows that the set Γ_m is bounded in E. Moreover, by virtue of properness of the operator P_τ it is compact. Hence there exists a bounded domain $G \subset E$ such that $\Gamma_m \subset G$ and $\Gamma_n \cap \bar{G} = \emptyset$.

Consider the topological degree $\gamma(P_\tau, G)$. Since

$$P_\tau(u) \neq 0, \quad u \in \partial G,$$

then it is well defined. Since $\gamma(P_0, G) = 1$ (Section 6.1), then $\gamma(P_\tau, G) = 1$ for any $\tau \in [0, 1]$. Hence problem (66)-(68) has a solution for any $\tau \in [0, 1]$.

It remains to verify its uniqueness. We recall that

$$\gamma(P_\tau, G) = \sum_i \text{ ind } u_i,$$

where ind u_i is the index of a solution u_i and the sum is taken with respect to all solutions $u_i \in G$. Since $\gamma(P_\tau, G) = 1$ and ind $u_i = 1$ (cf. Section 6.1), then the solution is necessarily unique. □

Acknowledgments. The authors are grateful to the French-Romanian Laboratory "LEA Math-Mode" for the possibility of mutual visits. The work of the first author was supported by the CNCS-UEFISCDI (Romania) project number PN-II-ID-PCE-2011-3-0563. The second author was partially supported by the grant no. 14.740.11.0877 of the Ministry of Education and Research of Russian Federation, "Investigation of Spatial and Temporal Structures in Fluids with Applications to Mathematical Biology".

REFERENCES

[1] A. Fabiato, *Calcium-induced release of calcium from the cardiac sarcoplasmic reticulum*, Am. J. Physiol. Cell. Physiol., **245** (1983), 1–14.

[2] A. Friedman, "Partial Differential Equations of Parabolic Type," Prentice-Hall, Englwood Cliffs, 1964.

[3] N. El Khatib, S. Genieys, B. Kazmierczak and V. Volpert, *Reaction-diffusion model of atherosclerosis development*, J. Math. Biol., **65** (2012), 349–374.

[4] M. Kyed, *Existence of travelling wave solutions for the heat equation in infinite cylinders with a nonlinear boundary condition*, Math. Nachr., **281** (2008), 253–271.

[5] A. Volpert, Vit. Volpert and Vl. Volpert, "Traveling Wave Solutions of Parabolic Systems," Translation of Mathematical Monographs, **140**, Amer. Math. Society, Providence, 1994.

[6] V. Volpert and A. Volpert, *Spectrum of elliptic operators and stability of travelling waves*, Asymptotic Analysis, **23** (2000), 111–134.

[7] V. Volpert, "Elliptic Partial Differential Equations. Volume 1. Fredholm Theory of Elliptic Problems in Unbounded Domains," Birkhäuser, Basel, 2011.

Received January 2012; revised July 2012.

E-mail address: napreut@gmail.com
E-mail address: volpert@math.univ-lyon1.fr