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► **To cite this version:**

Roman Novikov, Iskander Taimanov, Sergey Tsarev. Two-dimensional von Neumann–Wigner potentials with a multiple positive eigenvalue. *Functional Analysis and Its Applications*, 2014, 48 (4), pp.295-297. hal-00846557

HAL Id: hal-00846557

<https://hal.science/hal-00846557v1>

Submitted on 19 Jul 2013

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Two-dimensional von Neumann–Wigner potentials with a multiple positive eigenvalue

R.G. Novikov ^{*} I.A. Taimanov [†] S.P. Tsarev [‡]

Let H be a Schrödinger operator

$$H = -\Delta + U$$

with a potential $U(x)$, on \mathbb{R}^N , decaying at infinity. The potential U is called a *von Neumann–Wigner* potential if H has a positive eigenvalue with an eigenfunction from $L_2(\mathbb{R}^N)$, i.e. there is a point of discrete spectrum which is embedded into the absolutely continuous spectrum.

The very first example of such a potential was constructed by von Neumann and Wigner [1]. They found a three-dimensional rotation-symmetric nonsingular potential $U(r)$ with the following asymptotic behavior (a computational mistake in [1] reported later is corrected here):

$$U(r) = -\frac{8 \sin 2r}{r} + O(r^{-2}) \quad \text{as } r = |x| \rightarrow \infty, x \in \mathbb{R}^3.$$

The Schrödinger operators with $U(x) = o(1/|x|)$ as $x \rightarrow \infty$ have no positive eigenvalues [2].

In the present note we construct multiparametric families of explicit two-dimensional potentials which decay as $1/|x|$ and have a multiple positive eigenvalue. To our knowledge these are the first examples of such potentials.

We use the method introduced in [3, 4] where two-dimensional Schrödinger operators with fast decaying potentials and multidimensional kernels were constructed. This method is based on the Moutard transformation, of two-dimensional Schrödinger operators, which is as follows: let ω be a formal solution of the equation

$$H\omega = (-\Delta + U(x, y))\omega = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \quad (1)$$

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The Moutard transformation corresponding to H and ω gives a new Schrödinger operator

$$\tilde{H} = -\Delta + \tilde{U}, \quad \tilde{U} = U - 2\Delta \log \omega$$

such that if φ satisfies $H\varphi = 0$, then a function θ determined modulo $\frac{\text{const}}{\omega}$ by the consistent system

$$(\omega\theta)_x = -\omega^2 \left(\frac{\varphi}{\omega}\right)_y, \quad (\omega\theta)_y = \omega^2 \left(\frac{\varphi}{\omega}\right)_x, \quad (2)$$

satisfies

$$\tilde{H}\theta = 0.$$

So there are maps

$$U \rightarrow M_\omega(U) = U - 2\Delta \log \omega, \quad \varphi \rightarrow S_\omega(\varphi) = \left\{ \theta + \frac{C}{\omega}, C \in \mathbb{C} \right\}.$$

Let us consider an operator H and a pair of solutions to (1): ω_1 and ω_2 . For every $\theta_1 \in S_{\omega_1}(\omega_2)$ there is a function (a result of double iteration of the Moutard transformation)

$$\hat{U} = M_{\theta_1} M_{\omega_1}(U) - U = -2\Delta \log(\theta_1 \omega_1). \quad (3)$$

The result of this iteration depends on the choice of $\theta_1 \in S_{\omega_1}(\omega_2)$, i.e., on the integration constant C in (2). Moreover the functions

$$\psi_1 = \frac{1}{\theta_1}, \quad \psi_2 = \frac{\omega_2}{\omega_1 \theta_1}$$

satisfy the equation

$$(-\Delta + M_{\theta_1} M_{\omega_1}(U))\psi = 0.$$

In contrast to [3, 4] where such a double iteration was applied to the case $U = 0$, we apply it to the constant potential $U = -k^2, k \in \mathbb{R}$. Therewith ω_1 and ω_2 have to satisfy to the Helmholtz equation

$$-\Delta\omega = k^2\omega.$$

A large set of solutions to this equation is given by functions of the form

$$\text{Re} \left[\frac{\partial^m}{\partial \lambda^m} e^{i\frac{k}{2}(\lambda z + \frac{\bar{z}}{\lambda})} \right], \quad \lambda \in \mathbb{C}, \quad m = 0, 1, 2, \dots, \quad (4)$$

and their linear combinations.

For simplicity we consider the case $k^2 = 1$ and demonstrate the method by one explicit example.

Theorem 1 *Let $U = -1$ and*

$$\omega_1 = x^2 \cos y - y \sin y + y^2 \sin x + x \cos x, \quad \omega_2 = 4(y \cos x + x \sin y), \quad x, y \in \mathbb{R}.$$

Then the two-dimensional potential \widehat{U} takes the form

$$\widehat{U} = \frac{P}{Q^2}$$

where

$$\begin{aligned} Q = \omega_1 \theta_1 = & -x^4 - y^4 - 4x^2y \sin x \sin y + x^2(-8 \cos y \sin x - 2 \sin^2 y - 1) + \\ & + 4xy^2 \cos x \cos y - 16xy \cos x \sin y + 2x \cos x(-8 \cos y - \sin x) + \\ & + y^2(-8 \cos y \sin x + 2 \sin^2 x - 3) + 2y \sin y(\cos y + 8 \sin x) + \\ & + 16 \cos y \sin x + \sin^2 x - \sin^2 y + 4C + 1, \end{aligned}$$

and P is a polynomial in x, y and in sines and cosines of x and y :

$$\begin{aligned} P = & 16(x^6y \sin x \sin y - x^5y^2 \cos x \cos y + \\ & + x^2y^5 \sin x \sin y - xy^6 \cos x \cos y) + (\dots) \end{aligned}$$

where by dots we denote lower order terms in x and y . The functions ψ_1 and ψ_2 take the form

$$\psi_1 = \frac{\omega_1}{Q}, \quad \psi_2 = \frac{\omega_2}{Q}$$

and satisfy the equation

$$\widehat{H}\psi = \psi \quad \text{with } \widehat{H} = -\Delta + \widehat{U}.$$

Let us assume that C is negative and $|C|$ is sufficiently large, then Q has no zeroes and therefore the potential \widehat{U} and the functions ψ_1 and ψ_2 are smooth. We have

$$\widehat{U} = O\left(\frac{1}{r}\right), \quad \psi_1 = O\left(\frac{1}{r^2}\right), \quad \psi_2 = O\left(\frac{1}{r^3}\right), \quad \text{as } r = \sqrt{x^2 + y^2} \rightarrow \infty. \quad (5)$$

and therefore ψ_1 and ψ_2 lie in $L_2(\mathbb{R}^2)$ and are linearly independent eigenfunctions of the operator $\widehat{H} = -\Delta + \widehat{U}$ with the eigenvalue $E = 1$.

Using various linear combinations of the solutions (4) one can easily construct multiparametric families of similar two-dimensional potentials \widehat{U} with the asymptotics (5) and solutions ψ_i at the energy level $E = k^2$ and moreover to improve the decay of the eigenfunctions ψ_i . It is impossible to improve the decay of potentials due to already mentioned Kato's theorem [2].

We guess that by applying multiple iterations one may obtain such potentials with positive eigenvalues with higher multiplicity.

Acknowledgement. This work was done during the visit of one of the authors (I.A.T.) to Centre de Mathématique Appliquées of Ecole Polytechnique in July, 2013. The work was partially supported by TFP No 14.A18.21.0866 of Ministry of Education and Sciences of Russian Federation (R.G.N.), the interdisciplinary project "Geometrical and algebraic methods of finding explicit solutions to equations of mathematical physics and continuum mechanics" of SB RAS (I.A.T. and S.P.T.) and by the grant 1431/GF of Ministry of Education and Science of Republic of Kazakhstan (I.A.T.).

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