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Two-dimensional von Neumann–Wigner potentials with a multiple positive eigenvalue

R.G. Novikov ^{*} I.A. Taimanov [†] S.P. Tsarev [‡]

Let H be a Schrödinger operator

$$H = -\Delta + U$$

with a potential $U(x)$, on \mathbb{R}^N , decaying at infinity. The potential U is called a *von Neumann–Wigner* potential if H has a positive eigenvalue with an eigenfunction from $L_2(\mathbb{R}^N)$, i.e. there is a point of discrete spectrum which is embedded into the absolutely continuous spectrum.

The very first example of such a potential was constructed by von Neumann and Wigner [1]. They found a three-dimensional rotation-symmetric nonsingular potential $U(r)$ with the following asymptotic behavior (a computational mistake in [1] reported later is corrected here):

$$U(r) = -\frac{8 \sin 2r}{r} + O(r^{-2}) \quad \text{as } r = |x| \rightarrow \infty, x \in \mathbb{R}^3.$$

The Schrödinger operators with $U(x) = o(1/|x|)$ as $x \rightarrow \infty$ have no positive eigenvalues [2].

In the present note we construct multiparametric families of explicit two-dimensional potentials which decay as $1/|x|$ and have a multiple positive eigenvalue. To our knowledge these are the first examples of such potentials.

We use the method introduced in [3, 4] where two-dimensional Schrödinger operators with fast decaying potentials and multidimensional kernels were constructed. This method is based on the Moutard transformation, of two-dimensional Schrödinger operators, which is as follows: let ω be a formal solution of the equation

$$H\omega = (-\Delta + U(x, y))\omega = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \quad (1)$$

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The Moutard transformation corresponding to H and ω gives a new Schrödinger operator

$$\tilde{H} = -\Delta + \tilde{U}, \quad \tilde{U} = U - 2\Delta \log \omega$$

such that if φ satisfies $H\varphi = 0$, then a function θ determined modulo $\frac{\text{const}}{\omega}$ by the consistent system

$$(\omega\theta)_x = -\omega^2 \left(\frac{\varphi}{\omega}\right)_y, \quad (\omega\theta)_y = \omega^2 \left(\frac{\varphi}{\omega}\right)_x, \quad (2)$$

satisfies

$$\tilde{H}\theta = 0.$$

So there are maps

$$U \rightarrow M_\omega(U) = U - 2\Delta \log \omega, \quad \varphi \rightarrow S_\omega(\varphi) = \left\{ \theta + \frac{C}{\omega}, C \in \mathbb{C} \right\}.$$

Let us consider an operator H and a pair of solutions to (1): ω_1 and ω_2 . For every $\theta_1 \in S_{\omega_1}(\omega_2)$ there is a function (a result of double iteration of the Moutard transformation)

$$\hat{U} = M_{\theta_1} M_{\omega_1}(U) - U = -2\Delta \log(\theta_1 \omega_1). \quad (3)$$

The result of this iteration depends on the choice of $\theta_1 \in S_{\omega_1}(\omega_2)$, i.e., on the integration constant C in (2). Moreover the functions

$$\psi_1 = \frac{1}{\theta_1}, \quad \psi_2 = \frac{\omega_2}{\omega_1 \theta_1}$$

satisfy the equation

$$(-\Delta + M_{\theta_1} M_{\omega_1}(U))\psi = 0.$$

In contrast to [3, 4] where such a double iteration was applied to the case $U = 0$, we apply it to the constant potential $U = -k^2, k \in \mathbb{R}$. Therewith ω_1 and ω_2 have to satisfy to the Helmholtz equation

$$-\Delta\omega = k^2\omega.$$

A large set of solutions to this equation is given by functions of the form

$$\text{Re} \left[\frac{\partial^m}{\partial \lambda^m} e^{i\frac{k}{2}(\lambda z + \frac{\bar{z}}{\lambda})} \right], \quad \lambda \in \mathbb{C}, \quad m = 0, 1, 2, \dots, \quad (4)$$

and their linear combinations.

For simplicity we consider the case $k^2 = 1$ and demonstrate the method by one explicit example.

Theorem 1 *Let $U = -1$ and*

$$\omega_1 = x^2 \cos y - y \sin y + y^2 \sin x + x \cos x, \quad \omega_2 = 4(y \cos x + x \sin y), \quad x, y \in \mathbb{R}.$$

Then the two-dimensional potential \widehat{U} takes the form

$$\widehat{U} = \frac{P}{Q^2}$$

where

$$\begin{aligned} Q = \omega_1 \theta_1 = & -x^4 - y^4 - 4x^2y \sin x \sin y + x^2(-8 \cos y \sin x - 2 \sin^2 y - 1) + \\ & + 4xy^2 \cos x \cos y - 16xy \cos x \sin y + 2x \cos x(-8 \cos y - \sin x) + \\ & + y^2(-8 \cos y \sin x + 2 \sin^2 x - 3) + 2y \sin y(\cos y + 8 \sin x) + \\ & + 16 \cos y \sin x + \sin^2 x - \sin^2 y + 4C + 1, \end{aligned}$$

and P is a polynomial in x, y and in sines and cosines of x and y :

$$\begin{aligned} P = & 16(x^6y \sin x \sin y - x^5y^2 \cos x \cos y + \\ & + x^2y^5 \sin x \sin y - xy^6 \cos x \cos y) + (\dots) \end{aligned}$$

where by dots we denote lower order terms in x and y . The functions ψ_1 and ψ_2 take the form

$$\psi_1 = \frac{\omega_1}{Q}, \quad \psi_2 = \frac{\omega_2}{Q}$$

and satisfy the equation

$$\widehat{H}\psi = \psi \quad \text{with } \widehat{H} = -\Delta + \widehat{U}.$$

Let us assume that C is negative and $|C|$ is sufficiently large, then Q has no zeroes and therefore the potential \widehat{U} and the functions ψ_1 and ψ_2 are smooth. We have

$$\widehat{U} = O\left(\frac{1}{r}\right), \quad \psi_1 = O\left(\frac{1}{r^2}\right), \quad \psi_2 = O\left(\frac{1}{r^3}\right), \quad \text{as } r = \sqrt{x^2 + y^2} \rightarrow \infty. \quad (5)$$

and therefore ψ_1 and ψ_2 lie in $L_2(\mathbb{R}^2)$ and are linearly independent eigenfunctions of the operator $\widehat{H} = -\Delta + \widehat{U}$ with the eigenvalue $E = 1$.

Using various linear combinations of the solutions (4) one can easily construct multiparametric families of similar two-dimensional potentials \widehat{U} with the asymptotics (5) and solutions ψ_i at the energy level $E = k^2$ and moreover to improve the decay of the eigenfunctions ψ_i . It is impossible to improve the decay of potentials due to already mentioned Kato's theorem [2].

We guess that by applying multiple iterations one may obtain such potentials with positive eigenvalues with higher multiplicity.

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