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► **To cite this version:**

Bertrand Cottenceau, Laurent Hardouin, Jean-Louis Boimond. Modeling and Control for Balanced Timed and Weighted Event Graphs in Dioids. 2012, pp.x-x. hal-00846179

HAL Id: hal-00846179

<https://hal.science/hal-00846179>

Submitted on 27 Aug 2013

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Modeling and Control for Balanced Timed and Weighted Event Graphs in Dioids

Bertrand Cottenceau, Laurent Hardouin, Jean-Louis Boimond

Abstract

The class of Timed Event Graphs (TEGs) has widely been studied for the last 30 years thanks to an algebraic approach known as the theory of Max-Plus linear systems. In particular, the modeling of TEGs via formal power series has led to input-output descriptions for which some model matching control problems have been solved. In the context of manufacturing applications, the controllers obtained by these approaches have the effect of regulating material flows in order to decrease internal congestions and intermediate stocks. The objective of this work is to extend the class of systems for which a similar control synthesis is possible. To this end, we define first a subclass of timed Petri nets that we call Balanced Timed and Weighted Event Graphs (B-TWEGs). B-TWEGs can model synchronisation and delays (B-TWEGs contains TEGs) and can also describe some dynamic phenomena such as batching and event duplications. Their behavior is described by some rational compositions of four elementary operators γ^n , δ^t , μ_m and β_b on a dioid of formal power series. Then, we show that the series associated to B-TWEGs have a three dimensional graphical representation with a property of ultimate periodicity. This modeling allows us to show that B-TWEGs can be handled thanks to finite and canonical forms. Therefore, the existing results on control synthesis, in particular the model matching control problem, have a natural application in that framework.

Index Terms

Discrete-Event Systems, Timed and Weighted Event Graphs, Dioids, Formal Power Series, Residuation, Three Dimensional Representation, Controller Synthesis.

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I. INTRODUCTION

Since the beginning of the 80s, it has been known that the class of Timed Event Graphs (TEGs) can be studied thanks to linear models in some specific algebraic structures called dioids (or idempotent semirings) [22][5][1][9] [16]. Among different representations, a specific approach lies on an operatorial description of such systems. By denoting Σ the semimodule of counter functions¹, one can describe the behavior of a TEG by combining two shift operators (see [5]) denoted respectively $\gamma, \delta : \Sigma \rightarrow \Sigma$

$$\gamma : (\gamma x)(t) = x(t) + 1 \quad \delta : (\delta x)(t) = x(t - 1)$$

The input-output behavior of a TEG is then described by a matrix the entries of which are some elements of the rational closure² of the set $\{\varepsilon, e, \gamma, \delta\}$, *i.e.* the transfer matrix of a TEG can be written with a finite composition of these operators. Moreover, due to some fundamental equivalences such as $\gamma^n \oplus \gamma^{n'} = \gamma^{\min(n, n')}$ and $\delta^t \oplus \delta^{t'} = \delta^{\max(t, t')}$, a rational expression has a canonical form which is ultimately periodic [9][18][10]. In other words, we can manipulate TEG transfer as periodic formal series in two variables γ and δ , with some simplification rules, within a dioid called $\mathcal{M}_{\text{in}}^{\text{ax}}[[\gamma, \delta]]$ [5][1]. On the one hand, this fact has made it possible to elaborate software tools to compute the transfer matrix of any TEG [13] [8]. On the other hand, such an input-output model is well suited to address some model matching control problems [7] [20] [17] [15]. By analogy with the classical control theory, controllers are computed in order to achieve, for the closed-loop system, some prescribed performances. In a manufacturing production context, the controller describes how to manage the input of raw parts into the production line in order to achieve some performance. The controllers obtained by this approach lead to improve the internal flows of products by decreasing internal stocks.

The main objective of this work is to study a class of systems greater than those described by TEGs, but with similar algebraic tools. We focus here on the class of Timed and Weighted Event Graphs (TWEGs). They correspond to TEGs the arcs of which are valued by some positive integers. The arcs valuations express how many tokens the firing events consume/produce in the graph. The modeling power is greatly increased by the introduction of these valuations since in

¹A counter function $x : \mathbb{Z} \rightarrow \mathbb{Z}, t \mapsto x(t)$ gives the cumulative number of occurrences of the events labeled x at date t . Such a function plays the role of signal.

²Where ε (resp. e) is the null (resp. neutral) operator.

addition to synchronisations and delays, TWEGs can also describe batch constitution (several successive input events are necessary to release one output event) and duplication (one input event instantaneously produces several output events). These situations are usual in manufacturing systems (lot making, batch splitting) and cannot be accurately modeled with ordinary TEGs.

In literature, TWEGs have been studied both as a modeling tool for manufacturing systems and as a model for computation in the field of concurrent applications. In both domain, TWEGs can describe the scheduling of tasks with precedence constraints and also parallel executions. The analysis and the optimization problems associated to TWEGs aim at checking or enforcing the liveness (can an event be executed an infinite number of times in the model), and also at evaluating the throughput (number of events/unit of time) of the system, in particular for cyclic scheduling (see [21] [2]). When applied to concurrent programming, real-time and embedded systems, an equivalent graphical model called Synchronous Data-Flow (SDF) is generally used (see [19][24][11]).

In order to adapt the control problems described in [7] and [20] to the context of TWEGs, an input-output representation (transfer function) is necessary. The model proposed here seems to be well adapted to that aim. Our work is in the spirit of [4] where a class of Fluid TWEGs is analyzed thanks to a dioid of formal power series. The authors introduce a multiplier operator denoted μ that models the effect of the graph valuations. They thus obtain a necessary and sufficient condition under which a Fluid TWEG can be reduced to a Fluid TEG. This gives a way to "linearize" some TWEGs. This work has been extended in several papers, for instance in [14] where some hybrid Fluid/Discrete TWEGs are considered. In the context of SDF modeling, the studies developed in [11] and [12] are very close to that approach too.

We focus here on the *discrete* functioning of TWEGs. As in [5] and [4], one uses the classical shift operators γ^n and δ^t to describe event-shift and time-shift, but we also introduce two additional ones denoted β_b and μ_m that represent respectively a batch operation (which is modeled by an integer division³ on a counter variable) and a duplication phenomenon (multiplier operator), $\forall x \in \Sigma$

$$\beta_b : (\beta_b x)(t) = \lfloor x(t)/b \rfloor \quad \mu_m : (\mu_m x)(t) = x(t) \times m.$$

³ $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x .

The behavior of a TWEG denoted \mathcal{G} can be described by a rational expression on the set of operators $O_{M,B} = \{\varepsilon, e, \gamma, \delta, \mu_2, \mu_3, \dots, \mu_M, \beta_2, \dots, \beta_B\}$, where M is the maximal multiplier value and B the maximal batch value of \mathcal{G} . But, it is not clear whether, for the general case, there exists a canonical form for a rational expression on $O_{M,B}$: the formal computation of the behavior of a TWEG does not necessarily lead to a unique expression. In particular, the non unitary TWEGs seem to be difficult to handle.

We show that for a subclass of TWEGs that we call Balanced Timed and Weighted Event Graphs (B-TWEGs), the rational expressions generated have a *canonical periodic* form. The class of B-TWEGs corresponds to TWEGs such that parallel paths have the same gain⁴. For these systems, the transfer relation can be expressed, on a dioid of series denoted $\mathcal{E}[[\delta]]$, by an ultimately periodic power series in one variable δ with coefficients in a dioid \mathcal{E} of event operators⁵. The construction of $\mathcal{E}[[\delta]]$ is done so as to include dioid $\mathcal{M}_{\text{in}}^{\text{ax}}[[\gamma, \delta]]$ [5] [1]. The main feature is that the graphical representation of series in $\mathcal{E}[[\delta]]$ is three-dimensional : two dimensions to describe event operators in \mathcal{E} and a third dimension for time shift operators. As in $\mathcal{M}_{\text{in}}^{\text{ax}}[[\gamma, \delta]]$, the graphical representation of series in $\mathcal{E}[[\delta]]$ helps us to understand the simplification rules on operators generated by B-TWEGs. Since the input-output behavior of a B-TWEG can be described by a periodic formal series, the existing results on control synthesis can be directly applied. It is only necessary to express the residuation of the product by the elementary operators.

The paper is organised as follows. In section 2, the subclass of Balanced Timed and Weighted Event Graphs is first defined. Then, the modeling via an operatorial description is presented. Section 3 is devoted to define the dioid of formal series denoted $\mathcal{E}[[\delta]]$ and to associate a 3D graphical representation. In section 4, the result concerning the periodicity of B-TWEGs' transfer series is stated. Eventually, the question of control synthesis is addressed in section 5 after some reminders on the residuation theory and its application to dioid $\mathcal{E}[[\delta]]$.

II. BALANCED TIMED AND WEIGHTED EVENT GRAPHS (B-TWEGS)

A. Definitions

Weighted Event Graphs (WEGs) constitute a subclass of generalized Petri Nets given by a set of places $P = \{p_1, \dots, p_m\}$ and a set of transitions $T = \{t_1, \dots, t_n\}$ (see [23] for a survey on

⁴Thus, we also reduce the problems of liveness.

⁵Operators that act only on the event numbering

Petri nets). An event graph cannot describe concurrency phenomenon, then every place $p_k \in P$ is defined between an input transition t_i and an output transition t_o . The arcs $t_i \rightarrow p_k$ and $p_k \rightarrow t_o$ are oriented and valued⁶ by strictly positive integers denoted respectively $w_i(p_k)$ and $w_o(p_k)$. A transition without input (resp. output) place is called a source or input (resp. sink or output) transition. An initial marking (a set of initial tokens depicted with black dots) denoted $M_0(p_k)$ is associated to each place $p_k \in P$. A given transition t_j is said *enabled* as soon as each input place p_l contains at least $w_o(p_l)$ tokens. A transition can be fired only if it is enabled. At each firing of a transition, $w_o(p_l)$ tokens are removed from each input place p_l , and $w_i(p_k)$ tokens are added to each output place p_k .

Example 1: For the WEG depicted on Fig. 1, t_1 (resp. t_3) is an input (resp. output) transition. The initial marking is given by $M_0(p_1) = 1$ and $M_0(p_2) = 2$. All arcs are assumed to be 1-valued except when mentioned, for instance $w_i(p_1) = 2$ and $w_o(p_1) = 3$. Transition t_3 is enabled when place p_1 has 3 tokens and place p_2 has two tokens. The firing of transition t_1 adds 2 tokens in place p_1 .

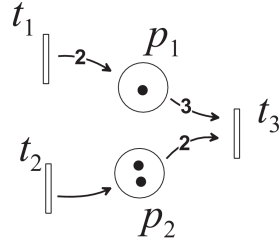


Figure 1. Weighted Event Graph

Definition 1 (Gain of a path): The gain of an elementary (oriented) path $t_i \rightarrow p_k \rightarrow t_o$ is defined as $\Gamma(t_i, p_k, t_o) \triangleq w_i(p_k)/w_o(p_k) \in \mathbb{Q}$. For a general path π passing through places p_i , the gain corresponds to the product of elementary paths, *i.e.* $\Gamma(\pi) = \prod_{p_j \in \pi} w_i(p_j)/w_o(p_j)$.

Definition 2 (Neutral and Balanced WEG): A WEG is said *neutral* if all its circuits have a gain of 1. A WEG is said *balanced* if $\forall t_i, t_j \in T$, all the paths from t_i to t_j have the same gain.

⁶From a graphical point of view, the valuations are depicted directly on the arcs

Remark 1: A balanced WEG is necessarily neutral. In [21], a WEG which is neutral and strongly connected is said unitary. It is important to note that the strongly connectedness is not required for the class of Balanced WEG.

For a WEG, a holding time denoted $\Delta(p_k) \in \mathbb{N}$ can be associated to each place $p_k \in P$. Each token entering in a place p_k has to wait $\Delta(p_k)$ time units before contributing to enable the output transition. A WEG with holding times is called a Timed and Weighted Event Graph (TWEg). Hereafter, we will only consider Balanced Timed and Weighted Event Graphs (in short B-TWEgS).

Example 2: For the TWEg depicted on Fig. 2, holding times are attached to some places: $\Delta(p_1) = 2$, $\Delta(p_6) = 1$, $\Delta(p_4) = 1$ and $\Delta(p_5) = 2$. This is a Balanced TWEg since it is neutral and all the parallel paths from t_1 to t_4 have the same gain equal to $3/2$. For instance, $\Gamma(t_1, p_1, t_2) = 1/2$ and $\Gamma(t_1, p_2, t_3) = 3$.

Remark 2 (Ordinary TEG): If all the existing arcs are 1-valued, the TWEg is said Ordinary, or simply Timed Event Graph (TEG). A TEG is obviously a Balanced TWEg.

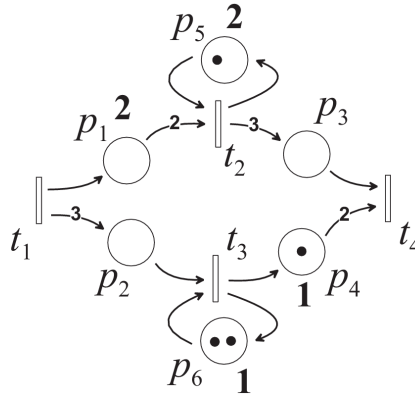


Figure 2. Balanced Timed and Weighted Event Graph

Definition 3 (Earliest Functioning): The *earliest functioning* of a B-TWEg consists in firing transitions⁷ as soon as they are enabled.

⁷Except source transitions.

B. Operatorial representation of B-TWEGs

A dioid (or idempotent semiring) is an algebraic structure with two inner operations, a sum and a product. The sum is commutative, associative and idempotent ($a \oplus a = a$) and the product is associative and distributes over the sum. The neutral elements of these operations are usually denoted ε for the sum, and e for the product. Since the sum is idempotent, a natural order can be associated to a dioid as $a \succeq b \iff a = a \oplus b$. When the sum of any finite or infinite subset of a dioid is defined, and the product distributes over infinite sums, the dioid is said complete. A complete dioid is an ordered set with a complete lattice structure : the inf operator is defined as $a \wedge b = \bigoplus \{x \mid x \oplus a = a \text{ and } x \oplus b = b\}$.

The operatorial representation of TWEGs requires to associate a counter function $x_i : \mathbb{Z} \rightarrow \mathbb{Z} \cup +\infty$ to each transition t_i . The set of counter functions denoted Σ has a semimodule structure for the internal operation $\oplus = \min$ and for the scalar operation defined by $\lambda.x(t) = x(t) + \lambda$. An operator is a map $\mathcal{H} : \Sigma \rightarrow \Sigma$ which is said *linear* if $\forall x, y \in \Sigma$, a) $\mathcal{H}(x \oplus y) = \mathcal{H}(x) \oplus \mathcal{H}(y)$ and b) $\mathcal{H}(\lambda.x) = \lambda.\mathcal{H}(x)$. An operator is said *additive* if only a) is satisfied.

Definition 4 (Dioid \mathcal{O} of additive operators [22]): The set of additive operators on Σ , with the operations defined below, is a non commutative complete dioid denoted $\mathcal{O} : \forall \mathcal{H}_1, \mathcal{H}_2 \in \mathcal{O}$

$$\mathcal{H}_1 \oplus \mathcal{H}_2 \triangleq \forall x \in \Sigma, (\mathcal{H}_1 \oplus \mathcal{H}_2)(x) = \min(\mathcal{H}_1(x), \mathcal{H}_2(x))$$

$$\mathcal{H}_1 \circ \mathcal{H}_2 \triangleq \forall x \in \Sigma, (\mathcal{H}_1 \circ \mathcal{H}_2)(x) = \mathcal{H}_1(\mathcal{H}_2(x))$$

The null operator (neutral for \oplus and absorbing for \circ) is denoted $\varepsilon : \forall x \in \Sigma, (\varepsilon x)(t) = +\infty$ and the unit operator (neutral for \circ) is denoted $e : \forall x \in \Sigma, (ex)(t) = x(t)$.

For the sequel, we will simply denote by $\mathcal{H}x$ (instead of $\mathcal{H}(x)$) the image of the counter $x \in \Sigma$ by the additive operator $\mathcal{H} \in \mathcal{O}$. And we will also often omit the \circ symbol for the product of \mathcal{O} , $\mathcal{H}_1\mathcal{H}_2 = \mathcal{H}_1 \circ \mathcal{H}_2$. Two additive operators $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{O}$ are equal if $\forall x \in \Sigma, \mathcal{H}_1x = \mathcal{H}_2x$.

Definition 5 (Operators for B-TWEGs): The operators found in B-TWEGs are generated from a family of additive operators in \mathcal{O} defined by : let $x \in \Sigma$ be a counter,

$$\tau \in \mathbb{Z}, \delta^\tau : \forall x, (\delta^\tau x)(t) = x(t - \tau)$$

$$\nu \in \mathbb{Z}, \gamma^\nu : \forall x, (\gamma^\nu x)(t) = x(t) + \nu$$

$$b \in \mathbb{N}^*, \beta_b : \forall x, (\beta_b x)(t) = \lfloor x(t)/b \rfloor$$

$$m \in \mathbb{N}^*, \mu_m : \forall x, (\mu_m x)(t) = x(t) \times m.$$

On the basis of these operators, we can remark that the unit operator e has several equivalent expressions : $e = \gamma^0 = \delta^0 = \mu_1 = \beta_1$. Hereafter, operators γ^ν, β_b and μ_m (and their \oplus and \circ compositions) are considered as event operators (in short E-operators).

Proposition 1: The next formal equivalences can be stated

$$\gamma^n \gamma^{n'} = \gamma^{n+n'} \quad \delta^t \delta^{t'} = \delta^{t+t'} \quad (1)$$

$$\gamma^n \oplus \gamma^{n'} = \gamma^{\min(n, n')} \quad \delta^t \oplus \delta^{t'} = \delta^{\max(t, t')} \quad (2)$$

$$\gamma^1 \delta^1 = \delta^1 \gamma^1 \quad \mu_m \delta^1 = \delta^1 \mu_m \quad \beta_b \delta^1 = \delta^1 \beta_b \quad (3)$$

$$\mu_m \gamma^n = \gamma^{m \times n} \mu_m \quad \gamma^n \beta_b = \beta_b \gamma^{n \times b} \quad (4)$$

Proof: For all counter $x \in \Sigma$ we have (1) : $\forall t, (x(t) + n') + n = x(t) + (n' + n)$ and $x(\tau - t - t') = x(\tau - (t + t'))$. (2) : $\forall t, \min(x(t) + n, x(t) + n') = x(t) + \min(n, n')$. Since $\forall t, x(t) \geq x(t - 1)$ (x is monotone non-decreasing), then $\min(x(\tau - t), x(\tau - t')) = x(\tau - \max(t, t'))$. (3) : immediate (4): $m \times (x(t) + n) = m \times x(t) + m \times n$ and $\lfloor x(t)/b \rfloor + n = \lfloor \frac{x(t) + n \times b}{b} \rfloor$. ■

Remark 3: We can note that equalities (2) are those expressed by the simplification rules in $\mathcal{M}_{\text{in}}^{\text{ax}}[\gamma, \delta]$.

Definition 6 (Kleene star): The Kleene star of an operator in \mathcal{O} is defined by : $\forall \mathcal{H} \in \mathcal{O}$,

$$\mathcal{H}^* = \bigoplus_{i \in \mathbb{N}} \mathcal{H}^i = e \oplus \mathcal{H} \oplus \mathcal{H}^2 \oplus \dots$$

with $\mathcal{H}^n = \mathcal{H} \circ \dots \circ \mathcal{H}$ (n times).

Theorem 1: On a complete dioid \mathcal{D} , the implicit equation $x = ax \oplus b$ has $x = a^*b$ as least solution.

Proof: see [1] ■

Theorem 2: For all operator $\mathcal{H} \in \mathcal{O}$, the next equalities are satisfied

$$\mathcal{H} = \mathcal{H}(\delta^{-1})^* = (\delta^{-1})^* \mathcal{H} = (\gamma^1)^* \mathcal{H} = \mathcal{H}(\gamma^1)^*.$$

Proof: Since a counter function x is monotone, then $\forall t, x(t + 1) \geq x(t) \iff \delta^{-1}x \preceq x$. For the same reason, $\forall t, x(t) + 1 \geq x(t) \iff \gamma x \preceq x$. Therefore, $\forall x \in \Sigma, \forall \mathcal{H} \in \mathcal{O}$, $\mathcal{H}(\gamma^1)^* x = \mathcal{H}x = (\gamma^1)^* \mathcal{H}x = \mathcal{H}(\delta^{-1})^* x = (\delta^{-1})^* \mathcal{H}x$. ■

C. Modeling of B-TWEGs

The B-TWEGs are analysed here with the earliest functioning rule (see Def. 3). We can model a path of a B-TWEG by a product of operators in \mathcal{O} , the synchronization of parallel paths by a

sum \oplus of operators, and the circuits by the Kleene star of some operators. Each elementary path $t_i \rightarrow p_k \rightarrow t_j$ of a B-TWEG, where $M_0(p_k)$ is the initial marking of place p_k and $\tau = \Delta(p_k)$ its holding time, can be described by the relation

$$x_j = \beta_{w(p_k, t_j)} \gamma^{M_0(p_k)} \mu_{w(t_i, p_k)} \delta^\tau x_i, \quad (5)$$

where x_i (resp. x_j) is the counter function associated to transition t_i (resp. t_j).

Example 3 (B-TWEG of Fig. 1): We can link the counter functions x_i associated to the transitions t_i of the B-TWEG depicted in Fig. 1 as follows. For the earliest functioning, we have

$$\begin{aligned} x_3(t) &= \min(\lfloor \frac{2 \times x_1(t) + 1}{3} \rfloor, \lfloor \frac{x_2(t) + 2}{2} \rfloor) \\ &= \min(\lfloor \frac{2 \times x_1(t) + 1}{3} \rfloor, \lfloor \frac{x_2(t)}{2} \rfloor + 1) \end{aligned}$$

Therefore, the counter functions are linked by

$$\begin{aligned} x_3 &= \beta_3 \gamma^1 \mu_2 \delta^0 x_1 \oplus \beta_2 \gamma^2 \delta^0 x_2 \\ &= \beta_3 \gamma^1 \mu_2 \delta^0 x_1 \oplus \gamma^1 \beta_2 \delta^0 x_2 \end{aligned}$$

with $\beta_3 \gamma^1 \mu_2 \delta^0, \gamma^1 \beta_2 \delta^0 \in \mathcal{O}$.

Example 4 (B-TWEG of Fig. 2): For the B-TWEG depicted in Fig. 2 and for the earliest functioning, we have

$$\begin{aligned} x_2(t) &= \min(\lfloor \frac{x_1(t-2)}{2} \rfloor, x_2(t-2) + 1) \\ x_3(t) &= \min(x_1(t) \times 3, x_3(t-1) + 2) \end{aligned}$$

Therefore, the counter functions are linked by $x_2 = \beta_2 \delta^2 x_1 \oplus \gamma^1 \delta^2 x_2$ and thanks to Th. 1, $x_2 = (\gamma^1 \delta^2)^* \beta_2 \delta^2 x_1$. Similarly, $x_3 = (\gamma^2 \delta^1)^* \mu_3 x_1$. Finally, the counter function associated to the output transition is $x_4 = \mu_3 x_2 \oplus \beta_2 \gamma^1 \delta^1 x_3 = (\mu_3 (\gamma^1 \delta^2)^* \beta_2 \delta^2 \oplus \beta_2 \gamma^1 \delta^1 (\gamma^2 \delta^1)^* \mu_3) x_1$. The input-output behavior (or transfer function) of the B-TWEG is described by the rational expression $\mu_3 (\gamma^1 \delta^2)^* \beta_2 \delta^2 \oplus \beta_2 \gamma^1 \delta^1 (\gamma^2 \delta^1)^* \mu_3$ in \mathcal{O} .

Theorem 3 (Transfer matrix of a B-TWEG): The behavior of a B-TWEG is described by a matrix the elements of which belong to the rational closure of the set of operators $O_{M,B} = \{\varepsilon, e, \gamma^1, \delta^1, \mu_2, \dots, \mu_M, \beta_2, \dots, \beta_B\}$ where $B = \max_i w_o(p_i)$ and $M = \max_i w_i(p_i)$ with $p_i \in P$.

Proof: For each place p_k we associate an operator $\mu_m \gamma^n \beta_b \delta^t$ (see (5)). Then, the different graph compositions (parallel, serial, loop) are expressed by operations in $\{\oplus, \circ, *\}$. Since a B-TWEG is a finite graph, the rationality is straightforward. \blacksquare

III. THREE DIMENSIONAL REPRESENTATION OF OPERATORS

According to (3) in Prop. 1, operator δ^1 can commute with any simple or composed event operator. For instance, $\delta^1\gamma^1\delta^2\mu_3\beta_2\delta^1 = \gamma^1\mu_3\beta_2\delta^4 = \delta^4\gamma^1\mu_3\beta_2$. Hence, in every finite composition (product) of elementary operators in $\{\delta^t, \gamma^n, \mu_m, \beta_b\}$, we can factorize the time-shift operator. Therefore, the rational expressions on $\mathcal{O}_{M,B}$ can be considered as formal power series in one variable δ where coefficients are some event operators. Moreover, in the particular case of B-TWEGs, the generated event operators have a canonical form.

A. Bi-dimensional representation of E-operators

1) *Event operators*: The set of operators generated by sum and composition of operators in γ^n , μ_m and β_b has a dioid structure.

Definition 7 (Dioid of E-operators \mathcal{E}): We denote by \mathcal{E} the dioid of operators obtained by sums and compositions of operators in $\{\varepsilon, e, \gamma^n, \mu_m, \beta_b\}$, with $n \in \mathbb{Z}$, and $m, b \in \mathbb{N}^*$. The elements of \mathcal{E} are called E-operators hereafter.

Dioid \mathcal{E} is a complete subdioid of \mathcal{O} (additive operators). Since the \circ operation is not commutative on \mathcal{E} , checking the equality of two E-operators is not immediate. Nevertheless, the comparison of E-operators is possible thanks to an associate map called *operator function*. Since an E-operator $w \in \mathcal{E}$ induces modifications only on the event numbering (no time shift), we can describe its behavior by the means of a counter-to-counter function denoted $\mathcal{F}_w : \mathbb{Z} \rightarrow \mathbb{Z}, k_i \mapsto k_o$ which maps an input counter value to an output counter value. For an E-operator, this input-output relation does not depend on time. An E-operator can be considered as an *instantaneous* system. For instance, the γ^2 E-operator is described by $\mathcal{F}_{\gamma^2}(k_i) = k_i + 2$. This function can be interpreted as follows : for the γ^2 E-operator, if k_i input events have occurred at date t , then $k_i + 2$ output events have occurred at this date. Similarly, E-operator $\mu_2\beta_3\gamma^1$ is described by the function $\mathcal{F}_{\mu_2\beta_3\gamma^1}(k_i) = \lfloor (k_i + 1)/3 \rfloor \times 2$ (see Fig. 3). Function \mathcal{F}_w gives an unambiguous representation of E-operator w . Moreover, we have $\mathcal{F}_{w_1 \oplus w_2} = \min(\mathcal{F}_{w_1}, \mathcal{F}_{w_2})$ and $\mathcal{F}_{w_1 \circ w_2} = \mathcal{F}_{w_1} \circ \mathcal{F}_{w_2}$. On the graphical representation, the axis are labeled by I-Count (Input Count) and O-Count (Output Count).

The equality of E-operators can be checked thanks to the operator function : $w_1, w_2 \in \mathcal{E}, w_1 = w_2 \iff \mathcal{F}_{w_1} = \mathcal{F}_{w_2}$. For instance, we can graphically check (see Fig. 4) the equality $\mu_3\beta_2\gamma^1 \oplus \gamma^2\mu_3\beta_2 = \beta_2\gamma^1\mu_3$, even if Prop. 1 does not give all the formal equalities necessary to transform

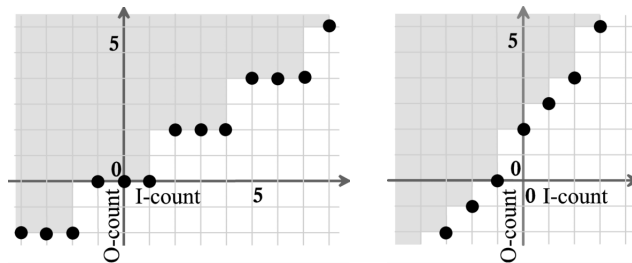


Figure 3. Representation of $\mathcal{F}_{\mu_2\beta_3\gamma^1}$ and $\mathcal{F}_{\gamma^2\beta_3\mu_4}$

the left hand part into the right hand part of the equality. In the right side of Fig. 4, we have depicted $\mathcal{F}_{\beta_2\gamma^1\mu_3}$ with grey dots, and on the left side, $\mathcal{F}_{\mu_3\beta_2\gamma^1}$ is depicted with black dots and $\mathcal{F}_{\gamma^2\mu_3\beta_2}$ with white dots.

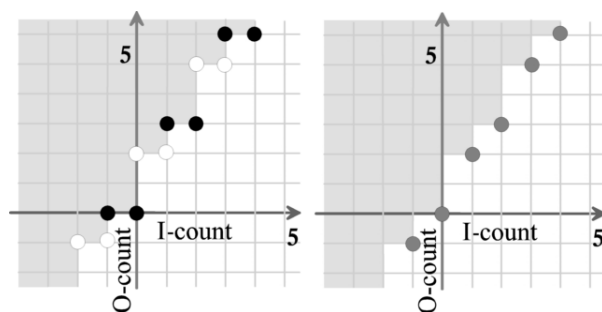


Figure 4. Representation of $\mathcal{F}_{\mu_3\beta_2\gamma^1} \oplus \mathcal{F}_{\gamma^2\mu_3\beta_2} = \mathcal{F}_{\beta_2\gamma^1\mu_3}$

2) *Graphical considerations:* The operator function leads to a natural bi-dimensional graphical representation of E-operators. Some features have to be kept in mind.

Partial order on \mathcal{E} : the comparison of two E-operators is graphically interpreted as follows

$$\begin{aligned}
 w_1 \preceq w_2 &\iff w_1 \oplus w_2 = w_2 \\
 &\iff \min(\mathcal{F}_{w_1}, \mathcal{F}_{w_2}) = \mathcal{F}_{w_2} \\
 &\iff \text{epigraph}(\mathcal{F}_{w_1}) \subset \text{epigraph}(\mathcal{F}_{w_2})
 \end{aligned}$$

Graphically, the sum of two E-operators amounts to do the union of their epigraphs⁸. On Fig. 3 and Fig. 4 the epigraph corresponds to the gray zone.

⁸ $\text{epigraph}(\mathcal{F}_{w_1}) \triangleq (k_i, k) \in \mathbb{Z}^2$ s.t. $k \geq \mathcal{F}_{w_1}(k_i)$.

Left and right product by γ^n : For $w \in \mathcal{E}$,

$$\mathcal{F}_{\gamma^n w} \iff \mathcal{F}_w \text{ vertically shifted of } n \text{ units to the top}$$

$$\mathcal{F}_{w\gamma^{n'}} \iff \mathcal{F}_w \text{ horizontally shifted of } n' \text{ units to the left .}$$

This last feature must be kept in mind when we consider ultimately periodic series that have a 3-D representation.

3) *Periodic E-operators*: The elementary E-operators γ^n, μ_m, β_b are described by periodic⁹ operator functions, i.e. the associate operator function satisfies $\forall k_i \in \mathbb{Z}, \mathcal{F}(k_i + n) = \mathcal{F}(k_i) + n'$.

For E-operators γ^n, μ_m and β_b we obtain

$$\mathcal{F}_{\gamma^n}(0) = n, \mathcal{F}_{\gamma^n}(k_i + 1) = \mathcal{F}_{\gamma^n}(k_i) + 1$$

$$\mathcal{F}_{\mu_m}(0) = 0, \mathcal{F}_{\mu_m}(k_i + 1) = \mathcal{F}_{\mu_m}(k_i) + m$$

$$0 \leq k_i < b, \mathcal{F}_{\beta_b}(k_i) = 0, \mathcal{F}_{\beta_b}(k_i + b) = \mathcal{F}_{\beta_b}(k_i) + 1$$

Operators γ^n and μ_m are 1-periodic, and operator β_b is b -periodic. The set of periodic E-operators is denoted \mathcal{E}_{per} .

Definition 8 (Gain of $w \in \mathcal{E}_{per}$): Let $w \in \mathcal{E}_{per}$ be a k -periodic E-operator s.t. $\mathcal{F}_w(k_i + k) = \mathcal{F}_w(k_i) + k'$. The gain¹⁰ of w is defined as $\Gamma(w) = k'/k$. It is the average slope of \mathcal{F}_w .

Proposition 2: Let $w_1, w_2 \in \mathcal{E}_{per}$ be two periodic E-operators. We have

$$w_1 w_2 \in \mathcal{E}_{per} \text{ and } \Gamma(w_1 w_2) = \Gamma(w_1) \times \Gamma(w_2) \quad (6)$$

$$\text{if } \Gamma(w_1) = \Gamma(w_2) \text{ then } w_1 \oplus w_2 \in \mathcal{E}_{per} \quad (7)$$

$$\text{if } \Gamma(w_1) = \Gamma(w_2) \text{ then } w_1 \wedge w_2 \in \mathcal{E}_{per} \quad (8)$$

Proof: The periodic operator functions satisfy $\mathcal{F}_{w_1}(k_i + k_1) = \mathcal{F}_{w_1}(k_i) + k'_1$ and $\mathcal{F}_{w_2}(k_i + k_2) = \mathcal{F}_{w_2}(k_i) + k'_2$. Hence, $\mathcal{F}_{w_2}(k_i + k_1.k_2) = \mathcal{F}_{w_2}(k_i) + k_1.k'_2$ and $\mathcal{F}_{w_1}(\mathcal{F}_{w_2}(k_i + k_1.k_2)) = \mathcal{F}_{w_1}(\mathcal{F}_{w_2}(k_i) + k_1.k'_2) = \mathcal{F}_{w_1}(\mathcal{F}_{w_2}(k_i)) + k'_1.k'_2 = \mathcal{F}_{w_1 w_2}(k_i) + k'_1.k'_2$. Therefore, operator $w_1 w_2$ is a periodic operator the gain of which is $(k'_1.k'_2)/(k_1.k_2)$. For the sum of periodic operators with the same gain, we can write both operators with the same periodicity: $\mathcal{F}_{w_1}(k_i + k_1.k_2) = \mathcal{F}_{w_1}(k_i) + k'_1.k_2$ and $\mathcal{F}_{w_2}(k_i + k_1.k_2) = \mathcal{F}_{w_2}(k_i) + k'_2.k_1$ with $k'_1.k_2 = k_1.k'_2$ (since both operators have the

⁹More exactly, they are only quasi periodic

¹⁰A path of a B-TWEG the gain of which is g is described by an E-operator the gain of which is g too.

same gain). Hence, the min of these two operator functions is also periodic. By symmetry, the max (\wedge) of two periodic E-operators with the same gain is also periodic. ■

Remark 4: Due to the structural definition of the subclass of B-TWEGs (see Def. 2), the parallel paths have the same gain. The periodicity of E-operators is therefore kept by the structural compositions of Balanced TWEGs. Said differently, the E-operators generated by B-TWEGs are periodic.

A k -periodic E-operator $w \in \mathcal{E}_{per}$ can be handled by the means of a finite representation : a pair $(k, k') \in \mathbb{N}^2$ describing the gain $\Gamma(w) = k'/k$ and the values of $\mathcal{F}_w(i)$ for one period $i \in \{0, \dots, k-1\}$. The canonical form of a periodic function is the one for which the period is minimal.

Definition 9 (Canonical form of $w \in \mathcal{E}_{per}$): A periodic E-operator w s.t. $\Gamma(w) = k'/k$ has a canonical form which is given by

$$w = \bigoplus_{i=1}^{i=N} \gamma^{n_i} \nabla_{m|b} \gamma^{n'_i}$$

with $\nabla_{m|b} \triangleq \mu_m \beta_b$, $m/b = k'/k$ and N, b are minimal.

For periodic operators of gain 1, we will also use the simplified notation $\nabla_m \triangleq \nabla_{m|m} = \mu_m \beta_m$. A periodic E-operator can be canonically decomposed on a basis of $\nabla_{m|b}$ operators, right and left shifted by some γ^n operators. Operator $\nabla_{m|b}$ is graphically represented by a staircase function from \mathbb{Z} to \mathbb{Z} (see Fig. 3 for the representation of $\nabla_{2|3} \gamma^1 = \mu_2 \beta_3 \gamma^1$).

As shown in the following example, the canonical form is not necessarily the most concise one.

Example 5: To establish the canonical form of $\gamma^2 \beta_3 \mu_4$, we can graphically represent $\mathcal{F}_{\gamma^2 \beta_3 \mu_4}$ (see Fig. 3). We have, $\Gamma(\gamma^2 \beta_3 \mu_4) = \frac{4}{3}$, $\mathcal{F}_{\gamma^2 \beta_3 \mu_4}(0) = 2$, $\mathcal{F}_{\gamma^2 \beta_3 \mu_4}(1) = 3$, $\mathcal{F}_{\gamma^2 \beta_3 \mu_4}(2) = 4$, $\mathcal{F}_{\gamma^2 \beta_3 \mu_4}(k_i + 3) = \mathcal{F}_{\gamma^2 \beta_3 \mu_4}(k_i) + 4$. The operator function $\mathcal{F}_{\gamma^2 \beta_3 \mu_4}$ can be seen as a min combination $\mathcal{F}_{\gamma^2 \beta_3 \mu_4} = \min(\mathcal{F}_{\gamma^2 \mu_4 \beta_3 \gamma^2}, \mathcal{F}_{\gamma^3 \mu_4 \beta_3 \gamma^1}, \mathcal{F}_{\gamma^4 \mu_4 \beta_3})$. Hence, we have the equality

$$\gamma^2 \beta_3 \mu_4 = \gamma^2 \nabla_{4|3} \gamma^2 \oplus \gamma^3 \nabla_{4|3} \gamma^1 \oplus \gamma^4 \nabla_{4|3}.$$

Translated into a B-TWEG model, the previous equality means that the two B-TWEGs depicted in Fig. 5 are equivalent from an input-output point of view : the same input sequence will produce the same output sequence¹¹.

¹¹This assertion is true only for the earliest functioning.

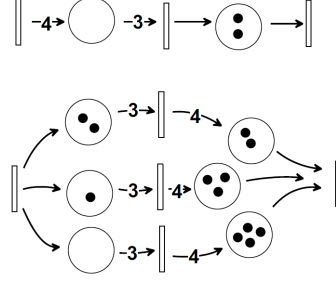


Figure 5. Input-output equivalence for E-operators : $\gamma^2\beta_3\mu_4 = \gamma^2\nabla_{4|3}\gamma^2 \oplus \gamma^3\nabla_{4|3}\gamma^1 \oplus \gamma^4\nabla_{4|3}$.

Remark 5: We can remark that, even if the canonical form is not necessarily the most compact, sometimes the periodicity may also be reduced by \oplus combination. For instance, we let the reader verifying that the following equality is satisfied : $\gamma\mu_2\beta_2\gamma \oplus \gamma^2\mu_2\beta_2 = \gamma\nabla_2\gamma \oplus \gamma^2\nabla_2 = \gamma$.

Remark 6: If $\Gamma(w_1) \neq \Gamma(w_2)$, then $w_1 \oplus w_2$ is not necessarily a periodic operator. Said differently, \mathcal{E}_{per} is not a subdioid of \mathcal{E} .

B. Dioid $\mathcal{E}[[\delta]]$

The previous subsection shows that E-operators generated by B-TWEGs are periodic and have a canonical form. Moreover, all the E-operators commute with the time-shift operator δ^τ (see Prop. 1). Therefore, all the operators generated by a B-TWEG can be described by the means of formal series in one variable δ denoted $\bigoplus_i w_i \delta^{t_i}$, where coefficients w_i are taken in \mathcal{E}_{per} and the exponents are in \mathbb{Z} .

1) *Three Dimensional representation of operators in B-TWEGs:* By analogy with [4], we can describe discrete B-TWEGs as rational combination of periodic E-operators and time-shift operators.

Definition 10 (Dioid $\mathcal{E}[[\delta]]$): The set of formal power series in one variable δ with exponents in \mathbb{Z} and coefficients in the non commutative complete dioid \mathcal{E} , with the simplification rule: $\forall s \in \mathcal{E}[[\delta]]$,

$$s = s(\delta^{-1})^* = (\delta^{-1})^*s, \quad (9)$$

is a non commutative complete dioid denoted $\mathcal{E}[[\delta]]$. A series $s \in \mathcal{E}[[\delta]]$ is written $s = \bigoplus_{t \in \mathbb{Z}} s(t)\delta^t$

with $s(t) \in \mathcal{E}$. For two series $s_1, s_2 \in \mathcal{E}[[\delta]]$:

$$\begin{aligned} (s_1 \oplus s_2)(t) &= s_1(t) \oplus s_2(t) \\ (s_1 \otimes s_2)(t) &= \bigoplus_{\tau+\tau'=t} s_1(\tau) \circ s_2(\tau') \end{aligned}$$

Adding the simplification rule (9) to formal series in δ allows us to assimilate the variable δ in dioid $\mathcal{E}[[\delta]]$ to the time-shift operator $\delta^1 : \Sigma \rightarrow \Sigma, \delta^1 x(t) = x(t-1)$ in dioid \mathcal{O} . Therefore, in dioid $\mathcal{E}[[\delta]]$, all the equalities given in Th. 2 are satisfied.

The series of $\mathcal{E}[[\delta]]$ have a graphical representation which consists in describing for each $t \in \mathbb{Z}$ the value of $s(t) \in \mathcal{E}$. The convention adopted here is to represent s in a 3D basis, where t is described along the z -axis and coefficients $s(t) \in \mathcal{E}$ are represented by their operator function in the $x \times y$ basis (more exactly, by the epigraph of the operator function). Moreover, according to Th. 2, a series is invariant by a product with $(\delta^{-1})^*$ and by $(\gamma^1)^*$. Therefore, if a piece of information of a series s is depicted by a point $p = (x, y, z) \in \mathbb{Z}^3$, then all the points $p' = (x', y', z')$ in the cone described by $x' \leq x, y' \geq y$ and $z' \leq z$ are dominated by p . This domination must be understood in the sense that the information represented by each point of the cone is yet contained in those represented by the vertex p . Therefore, the graphical representation of a series $s \in \mathcal{E}[[\delta]]$ may be seen as an infinite union of cones. We will see later on that the protruding vertices constitute the essential information that we must keep to represent a series in $\mathcal{E}[[\delta]]$.

From an equivalent point of view, each monomial $s(t)\delta^t$ of a series s generates a volume described by $s(t)\delta^t(\delta^{-1})^* = s(t)\delta^t \oplus s(t)\delta^{t-1} \oplus \dots$. For this reason, the 3D representation of a series in $\mathcal{E}[[\delta]]$ is a volume which also looks like a flight of stairs.

Example 6: The simple series (with only one term) $\gamma^2\beta_3\mu_4\delta^5 \in \mathcal{E}[[\delta]]$ is depicted on Fig. 6. The graphical representation of $\mathcal{F}_{\gamma^2\beta_3\mu_4}$ (see Fig. 3) is depicted in a 3D basis at height 5 (value of the time-shift operator). In order to improve the readability of the picture, the 3D representation is truncated to the positive values, *i.e.* to $(x, y, z) \in \mathbb{N}^3$.

Remark 7 (Simplifications): The equivalences given in Th. 2 lead to some rules to simplify series in $\mathcal{E}[[\delta]]$. In the 3D domain, two operators are equal if they have the same representation. Said differently, by considering a series $s = \bigoplus s(t)\delta^t$ of $\mathcal{E}[[\delta]]$, if a term $s(\tau)\delta^\tau$ is not *visible* in the 3D representation of s , then it means that it can be removed from s . For instance, let us consider the series $\alpha = \gamma^2\nabla_2\delta^2 \oplus \gamma^1\nabla_2\gamma^1\delta^5 \oplus \gamma^2\delta^4$. The representation of $\gamma^2\delta^4$ is not visible since it is hidden by those of $\gamma^1\nabla_2\gamma^1\delta^5$ (see Fig. 7). It means that the next simplification applies

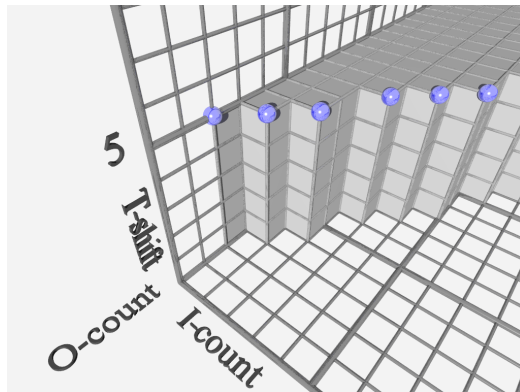


Figure 6. 3D representation of $\gamma^2 \beta_3 \mu_4 \delta^5$

$\alpha = \gamma^2 \nabla_2 \delta^2 \oplus \gamma^1 \nabla_2 \gamma^1 \delta^5 \oplus \gamma^2 \delta^4 = \gamma^2 \nabla_2 \delta^2 \oplus \gamma^1 \nabla_2 \gamma^1 \delta^5$. Finally, the main pieces of information in a series of $\mathcal{E}[\delta]$ are those coded by the protruding vertices (depicted by some balls in the figures).

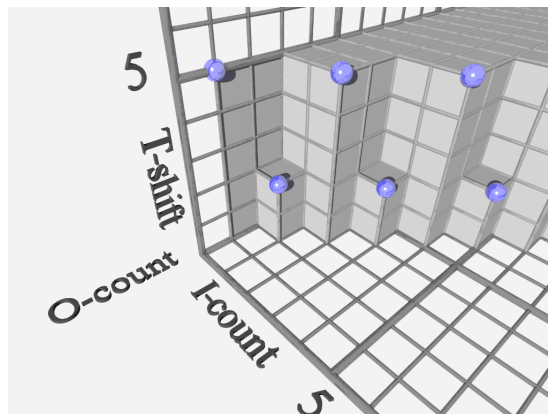


Figure 7. Simplifications in $\mathcal{E}[\delta]$

Due to the specific structure of B-TWEGs, we do not consider the whole set of series of $\mathcal{E}[\delta]$ but only the series the coefficients of which are periodic E-operators. This subset is denoted $\mathcal{E}_{per}[\delta]$.

Definition 11 (Balanced series in $\mathcal{E}_{per}[\delta]$): A series $s = \bigoplus s(t) \delta^t \in \mathcal{E}_{per}[\delta]$ is said balanced if all its coefficients $s(t) \in \mathcal{E}_{per}$ have the same gain. The gain of s is denoted $\Gamma(s)$ and corresponds to the gain of all its coefficients. A balanced series is said *conservative* if $\Gamma(s) = 1$.

2) *Polynomials in $\mathcal{E}_{per}[[\delta]]$* : The series that can be described by finite sums $\bigoplus_{i=1}^T s(t_i)\delta^{t_i}$ are called polynomials. Balanced polynomials have a canonical form. According to remark 7, it consists in only keeping for each monomial $w_i\delta^{t_i}$ the information that is not yet contained in monomials $w_j\delta^{t_j}$ such that $t_j > t_i$. As said in remark 7, it amounts to keeping protruding vertices.

Definition 12: Let us consider a balanced polynomial $p = \bigoplus_{i=1}^{i=T} w_i\delta^{t_i} \in \mathcal{E}_{per}[[\delta]]$ with $w_i \in \mathcal{E}_{per}$. The canonical form of p is such that $\forall i, t_i < t_{i+1}$ and

$$w_i = \bigwedge_w \{w \oplus \bigoplus_{j>i} w_j = \bigoplus_{k \geq i} w_k\} \quad (10)$$

Expression (10) conveys the fact that we only want to keep the essential information, *i.e.* coefficient w_i only keeps the information not yet contained in the coefficients w_j with $j > i$.

Example 7: The canonical form is obtained thanks to a backward analysis starting from the monomial with the greatest exponent. For polynomial $p = \delta^2 \oplus (\nabla_3\gamma^2 \oplus \gamma^2\nabla_3)\delta^4 \oplus \nabla_3\gamma^2\delta^7$ depicted on Fig. 8, we obtain the next simplifications. The monomial $\nabla_3\gamma^2\delta^4$ is not visible (since $\nabla_3\gamma^2\delta^4 \preceq \nabla_3\gamma^2\delta^7$), so it can be removed from p . Then, the monomial δ^2 has a non canonical expanded form $\delta^2 = (\gamma^2\nabla_3 \oplus \gamma\nabla_3\gamma \oplus \nabla_3\gamma^2)\delta^2$. The only part of the dynamic of δ^2 which is not yet described by $\gamma^2\nabla_3\delta^4 \oplus \nabla_3\gamma^2\delta^7$ is described by the operator $\gamma\nabla_3\gamma\delta^2$. Finally, we have

$$\begin{aligned} p &= \delta^2 \oplus (\nabla_3\gamma^2 \oplus \gamma^2\nabla_3)\delta^4 \oplus \nabla_3\gamma^2\delta^7 \\ &= \gamma\nabla_3\gamma\delta^2 \oplus \gamma^2\nabla_3\delta^4 \oplus \nabla_3\gamma^2\delta^7 \end{aligned}$$

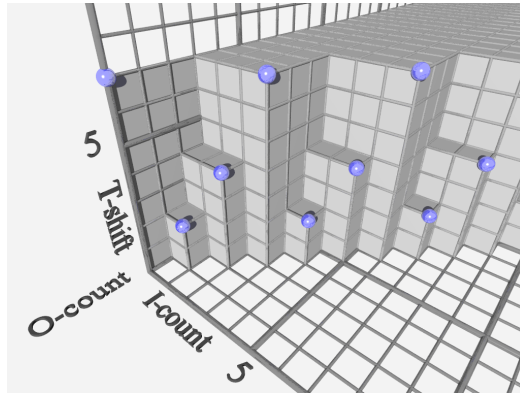


Figure 8. Polynomial $\gamma\nabla_3\gamma\delta^2 \oplus \gamma^2\nabla_3\delta^4 \oplus \nabla_3\gamma^2\delta^7$

IV. B-TWEGS ARE DESCRIBED BY ULTIMATELY PERIODIC SERIES OF $\mathcal{E}_{per}[\delta]$

In this section, we show that the behavior of a B-TWEG is described by ultimately periodic and balanced series of $\mathcal{E}_{per}[\delta]$. This result has to be compared to the well known result for ordinary Timed Event Graphs : the entries of the transfer matrix of a TEG are some ultimately periodic series of $\mathcal{M}_{in}^{ax}[\gamma, \delta]$. For TEGs, operations (and algorithms) on ultimately periodic series of $\mathcal{M}_{in}^{ax}[\gamma, \delta]$ have already been studied in [5] [1] [9] [10] [8] [13].

Since we consider B-TWEGs, only balanced series of $\mathcal{E}_{per}[\delta]$ are considered hereafter.

Definition 13 (Ultimately periodic series of $\mathcal{E}_{per}[\delta]$): A balanced series $s \in \mathcal{E}_{per}[\delta]$ is said ultimately periodic if it can be written as $s = p \oplus q(\gamma^\nu \delta^\tau)^*$, where p and q are balanced polynomials such that $\Gamma(p) = \Gamma(q)$,

$$p = \bigoplus_{i=1..n} w_i \delta^{t_i} \quad q = \bigoplus_{j=1..N} W_j \delta^{T_j},$$

$w_i, W_j \in \mathcal{E}_{per}$.

The property of periodicity has a natural graphical interpretation. For the 3D representation of s , the representation of $q(\gamma^\nu \delta^\tau)^* = q \oplus q\gamma^\nu \delta^\tau \oplus q\gamma^{2\nu} \delta^{2\tau} \oplus \dots$ is a periodic staircase. The polynomial q is depicted as a group of steps that is repeated periodically (we have the same steps but shifted by τ units to the top and by ν units toward the decreasing I-count values).

Example 8: Fig 9 gives the graphical description of $s = \gamma^2 \nabla_{3|2} \delta^3 \oplus \gamma^4 \nabla_{3|2} \gamma^1 \delta^4 \oplus [(\gamma^6 \nabla_{3|2} \gamma^1 \oplus \gamma^7 \nabla_{3|2}) \delta^6 \oplus \gamma^7 \nabla_{3|2} \gamma^1 \delta^7](\gamma^4 \delta^3)^*$. From the T-shift value equals to 6, we have the same two-steps repeated each 3 units to the top but shifted by 4 units toward the decreasing I-count values.

Remark 8: The periodic form is not unique. For instance, $s = p \oplus q(\gamma^\nu \delta^\tau)^*$ and $s = p \oplus q \oplus q\gamma^\nu \delta^\tau (\gamma^\nu \delta^\tau)^*$ are two different ultimately periodic forms of the same series.

Remark 9: Balanced polynomials in $\mathcal{E}_{per}[\delta]$ can always be considered as ultimately periodic series since $(\gamma^1 \delta^0)^* = e$.

Even if the product of $\mathcal{E}[\delta]$ is not commutative, an ultimately periodic balanced series of $\mathcal{E}_{per}[\delta]$ has two periodic forms.

Proposition 3 (Left/Right periodicity): An ultimately (right) periodic series $s = p \oplus q(\gamma^\nu \delta^\tau)^*$ in $\mathcal{E}_{per}[\delta]$ has also an ultimately left periodic form $s = p \oplus (\gamma^{\nu'} \delta^{\tau'})^* q'$ where q' is a balanced polynomial. The left (resp. right) asymptotic slope is defined as $\sigma_l(s) = \tau'/\nu'$ (resp. $\sigma_r(s) = \tau/\nu$), and the next equality is satisfied $\Gamma(s) = \sigma_r(s)/\sigma_l(s)$.

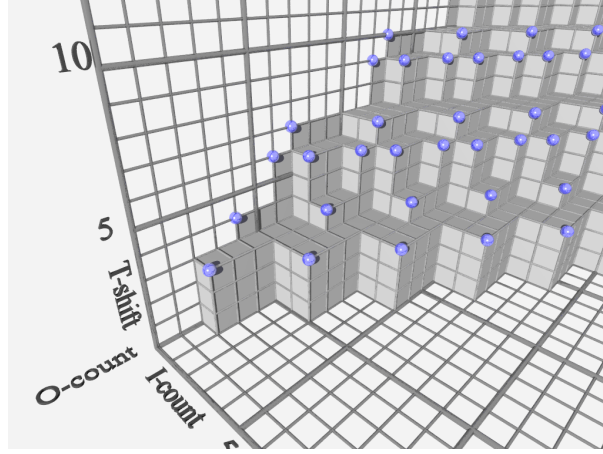


Figure 9. Ultimately periodic series in $\mathcal{E}_{per}[\delta]$

Proof: Let $\Gamma(s) = k'/k$ be the gain of s . The coefficients of polynomial $q = \bigoplus w_j \delta^{t_j}$ in their canonical form are given by $w_j = \bigoplus_i \gamma^{n_{ij}} \nabla_{m_j|b_j} \gamma^{n'_{ij}}$ with $k'/k = m_j/b_j$. Let us remark that thanks to (4), $\nabla_{m_j|b_j} \gamma^{b_j} = \mu_{m_j} \beta_{b_j} \gamma^{b_j} = \mu_{m_j} \gamma^1 \beta_{b_j} = \gamma^{m_j} \mu_{m_j} \beta_{b_j} = \gamma^{m_j} \nabla_{m_j|b_j}$. More generally, $\nabla_{m_j|b_j} \gamma^{\alpha b_j} = \gamma^{\alpha m_j} \nabla_{m_j|b_j}$. Therefore, if we take $B = \text{lcm}(b_j)$ and $M = B \cdot k'/k$, then $\forall i, j, \gamma^{n_{ij}} \nabla_{m_j|b_j} \gamma^{n'_{ij}} \gamma^B = \gamma^M \gamma^{n_{ij}} \nabla_{m_j|b_j} \gamma^{n'_{ij}}$, and consequently $\forall i, w_i \gamma^B = \gamma^M w_i$. Since we can develop $(\gamma^\nu \delta^\tau)^* = (e \oplus \gamma^\nu \delta^\tau \oplus \dots \oplus \gamma^{(B-1)\nu} \delta^{(B-1)\tau}) (\gamma^{B\nu} \delta^{B\tau})^*$, then

$$\begin{aligned} q(\gamma^\nu \delta^\tau)^* &= q(e \oplus \dots \oplus \gamma^{(B-1)\nu} \delta^{(B-1)\tau}) (\gamma^{B\nu} \delta^{B\tau})^* \\ &= q(\gamma^{B\nu} \delta^{B\tau})^* (e \oplus \dots \oplus \gamma^{(B-1)\nu} \delta^{(B-1)\tau}) \\ &= (\gamma^{M\nu} \delta^{B\tau})^* q(e \oplus \dots \oplus \gamma^{(B-1)\nu} \delta^{(B-1)\tau}) \\ &= (\gamma^{M\nu} \delta^{B\tau})^* q' \end{aligned}$$

Finally, $\sigma_r(s) = \tau/\nu$ and $\sigma_l(s) = (B\tau)/(M\nu)$ and $\sigma_r(s)/\sigma_l(s) = \Gamma(s) = k'/k$. ■

Example 9: For the series depicted on Fig. 9, a left and a right forms are given by

$$\begin{aligned} s &= \gamma^2 \nabla_{3|2} \delta^3 \oplus \gamma^4 \nabla_{3|2} \gamma^1 \delta^4 \oplus \\ &\quad [(\gamma^6 \nabla_{3|2} \gamma^1 \oplus \gamma^7 \nabla_{3|2}) \delta^6 \oplus \gamma^7 \nabla_{3|2} \gamma^1 \delta^7] (\gamma^4 \delta^3)^* \\ &= \gamma^2 \nabla_{3|2} \delta^3 \oplus \gamma^4 \nabla_{3|2} \gamma^1 \delta^4 \oplus \\ &\quad (\gamma^6 \delta^3)^* [(\gamma^6 \nabla_{3|2} \gamma^1 \oplus \gamma^7 \nabla_{3|2}) \delta^6 \oplus \gamma^7 \nabla_{3|2} \gamma^1 \delta^7]. \end{aligned}$$

The left and right slopes are $\sigma_l(s) = 3/6$ and $\sigma_r(s) = 3/4$. They are respectively the asymptotic slope of the 3D representation in the $y \times z$ direction (see the plan $x = 0$ in Fig. 9) and in the $x \times z$ direction.

As for series in $\mathcal{M}_{\text{in}}^{\text{ax}}[[\gamma, \delta]]$, the ultimately periodic series of $\mathcal{E}_{\text{per}}[[\delta]]$ have different expressions. But, we can provide two canonical forms, left and right periodic, where the periodicity is minimal.

Definition 14 (Canonical forms): An ultimately periodic series of $\mathcal{E}_{\text{per}}[[\delta]]$ has a left and a right canonical forms for which the degree of p is minimal and the value of ν (resp. ν') is minimal.

The main result concerning the class of B-TWEGs is that they structurally keep the ultimate periodicity property. To obtain this result, one has to analyze how the sum, the product and the Kleene star operations behave on ultimately periodic series in $\mathcal{E}_{\text{per}}[[\delta]]$.

First, we recall a result given in [5, Lemma 6] and detailed in [9, Lemma 4.1.4]. This result is stated in $\mathcal{M}_{\text{in}}^{\text{ax}}[[\gamma, \delta]]$ and is still valid in $\mathcal{E}_{\text{per}}[[\delta]]$ since $\mathcal{M}_{\text{in}}^{\text{ax}}[[\gamma, \delta]]$ is a subdioid.

Lemma 1: For given $\nu, \tau, \nu', \tau', \alpha, T, \alpha', T'$ some positive integers, if $\tau/\nu > \tau'/\nu'$ then the periodic series $\gamma^\alpha \delta^T (\gamma^\nu \delta^\tau)^*$ is asymptotically greater than $\gamma^{\alpha'} \delta^{T'} (\gamma^{\nu'} \delta^{\tau'})^*$, say

$$\begin{aligned} \exists N : \forall n' \geq N, \exists n \text{ s.t.} \\ \gamma^\alpha \delta^T (\gamma^\nu \delta^\tau)^n \succeq \gamma^{\alpha'} \delta^{T'} (\gamma^{\nu'} \delta^{\tau'})^{n'}. \end{aligned}$$

Proposition 4: Let us consider two ultimately right periodic series of $\mathcal{E}_{\text{per}}[[\delta]]$ denoted $s_1 = p_1 \oplus q_1 (\gamma^{\nu_1} \delta^{\tau_1})^*$ and $s_2 = p_2 \oplus q_2 (\gamma^{\nu_2} \delta^{\tau_2})^*$.

(a) If $\Gamma(s_1) = \Gamma(s_2)$ then $s_1 \oplus s_2$ is an ultimately periodic series of $\mathcal{E}_{\text{per}}[[\delta]]$ such that

$$\begin{aligned} \sigma_r(s_1 \oplus s_2) &= \max(\sigma_r(s_1), \sigma_r(s_2)) \\ \sigma_l(s_1 \oplus s_2) &= \max(\sigma_l(s_1), \sigma_l(s_2)) \end{aligned}$$

(b) $s_1 \otimes s_2$ is an ultimately periodic series s.t. $\Gamma(s_1 \otimes s_2) = \Gamma(s_1) \times \Gamma(s_2)$

$$\begin{aligned} \sigma_r(s_1 \otimes s_2) &= \max(\sigma_r(s_2), \Gamma(s_2) \times \sigma_r(s_1)) \\ \sigma_l(s_1 \otimes s_2) &= \max(\sigma_l(s_1), \sigma_l(s_2)/\Gamma(s_1)) \end{aligned}$$

Proof: These results come from a direct adaptation of results given in [9] for periodic series in $\mathcal{M}_{\text{in}}^{\text{ax}}[[\gamma, \delta]]$. We only give the main ideas.

Outline of Proof for (a) : according to Lemma 1, if $(\tau_1/\nu_1) > (\tau_2/\nu_2)$, then the simple periodic series $\nabla_{m|b} \gamma^{\alpha_1} \delta^{T_1} (\gamma^{\nu_1} \delta^{\tau_1})^*$ is asymptotically greater than $\nabla_{m|b} \gamma^{\alpha_2} \delta^{T_2} (\gamma^{\nu_2} \delta^{\tau_2})^*$. We can choose m and b such that $m/b = \Gamma(s_1) = \Gamma(s_2)$ and some integers $\alpha_1, T_1, \alpha_2, T_2$ such that we obtain two approximations of $q_1 (\gamma^{\nu_1} \delta^{\tau_1})^*$ and $q_2 (\gamma^{\nu_2} \delta^{\tau_2})^*$ satisfying

$$\begin{aligned} \nabla_{m|b} \gamma^{\alpha_1} \delta^{T_1} (\gamma^{\nu_1} \delta^{\tau_1})^* &\preceq q_1 (\gamma^{\nu_1} \delta^{\tau_1})^* \\ \nabla_{m|b} \gamma^{\alpha_2} \delta^{T_2} (\gamma^{\nu_2} \delta^{\tau_2})^* &\succeq q_2 (\gamma^{\nu_2} \delta^{\tau_2})^* \end{aligned}$$

By applying Lemma 1, we obtain that $q_1(\gamma^{\nu_1}\delta^{\tau_1})^*$ is asymptotically greater than $q_2(\gamma^{\nu_2}\delta^{\tau_2})^*$. Therefore, series s_1 is asymptotically greater than s_2 , and then $s_1 \oplus s_2$ is asymptotically periodic with the periodicity of s_1 . When $(\tau_1/\nu_1) = (\tau_2/\nu_2)$, $s_1 \oplus s_2$ is also asymptotically periodic.

Outline of Proof for (b) : We can write s_1 and s_2 with their right and left forms :

$$\begin{aligned} s_1 \otimes s_2 &= (p_1 \oplus q_1(\gamma^{\nu_1}\delta^{\tau_1})^*) \\ &\quad \otimes (p_2 \oplus (\gamma^{\nu_2}\delta^{\tau_2})^* q_2') \\ &= p_1 p_2 \oplus p_1 (\gamma^{\nu_2}\delta^{\tau_2})^* q_2' \\ &\quad \oplus q_1 (\gamma^{\nu_1}\delta^{\tau_1})^* p_2 \\ &\quad \oplus q_1 (\gamma^{\nu_1}\delta^{\tau_1})^* (\gamma^{\nu_2}\delta^{\tau_2})^* q_2' \end{aligned}$$

Series $p_1(\gamma^{\nu_2}\delta^{\tau_2})^* q_2'$ and $q_1(\gamma^{\nu_1}\delta^{\tau_1})^* p_2$ are finite sums of periodic series, due to (a), the result is periodic. The last term $(\gamma^{\nu_1}\delta^{\tau_1})^* (\gamma^{\nu_2}\delta^{\tau_2})^*$ is also a ultimately periodic series in $\mathcal{M}_{\text{in}}^{\text{ax}}[[\gamma, \delta]]$ (see [9]), and therefore in $\mathcal{E}[[\delta]]$ too. ■

Let us now focus on the behavior of circuits in B-TWEGs. They are algebraically described by Kleene star operations on series of $\mathcal{E}_{\text{per}}[[\delta]]$.

Proposition 5: Let $p = \bigoplus_{i=1}^{i=N} w_i \delta^{t_i}$ be a conservative balanced polynomial ($\forall i, \Gamma(w_i) = 1$). Then, series p^* is a conservative and ultimately periodic series of $\mathcal{E}_{\text{per}}[[\delta]]$.

Proof: The complete proof is detailed in [6]. ■

Proposition 6: Let $s = p \oplus q(\gamma^\nu \delta^\tau)^*$ be a conservative ($\Gamma(s) = \Gamma(p) = \Gamma(q) = 1$) ultimately periodic series in $\mathcal{E}_{\text{per}}[[\delta]]$. Then s^* is a conservative ultimately periodic series.

Proof: We can write $p = \bigoplus_i \gamma^{n_i} \nabla_M \gamma^{n_i} \delta^{t_i}$ and $q = \bigoplus_j \gamma^{N_j} \nabla_M \gamma^{N_j} \delta^{T_j}$. If we take $r = \gamma^{M\nu} \delta^{M\tau}$, then monomial r commutes with p and q , i.e. $pr = rp$ and $qr = rq$. First, we can write

$$(\gamma^\nu \delta^\tau)^* = (e \oplus \gamma^\nu \delta^\tau \oplus \gamma^{2\nu} \delta^{2\tau} \oplus \dots \oplus \gamma^{(M-1)\nu} \delta^{(M-1)\tau}) r^*.$$

Then, series s can be written $s = p \oplus q(e \oplus \gamma^\nu \delta^\tau \oplus \dots \oplus \gamma^{(M-1)\nu} \delta^{(M-1)\tau}) r^* = p \oplus q' r^*$. Moreover, r also commutes with q' , i.e. $q' r = r q'$. Since, $(a \oplus b)^* = a^* (b a^*)^*$, therefore $s^* = (p \oplus q' r^*)^* = p^* (q' r^* p^*)^*$. Since $rp = pr$, then $r^* p^* = (r \oplus p)^*$ (see [5, Lemma3]). Finally, one also have $(ab^*)^* = e \oplus a(a \oplus b)^*$, therefore, we can write $s^* = p^* (q' (r \oplus p)^*)^* = p^* (e \oplus q' (q' \oplus r \oplus p)^*)$. Since $(q' \oplus r \oplus p)$ is a conservative polynomial, then $(q' \oplus r \oplus p)^*$ is a periodic series (see Prop. 5). Since the product of periodic series is periodic too, the Kleene star of a conservative and ultimately periodic series is an ultimately periodic series. ■

Proposition 7 (Transfer of a B-TWEG): The transfer matrix of a B-TWEG is composed of ultimately periodic series of $\mathcal{E}_{per}[[\delta]]$.

Proof: We recall first that all the elementary operators $\gamma^n, \delta^t, \mu_m$ and β_b can be considered as ultimately periodic series. Then, due to the specific structure of B-TWEGs, the modeling by series in $\mathcal{E}_{per}[[\delta]]$ is such that:

- the sum (\oplus) of series in $\mathcal{E}_{per}[[\delta]]$ are necessarily done on series with the same gain (balanced property). The periodicity is kept by the balanced synchronization (see Prop. 4)
- the product of ultimately periodic series is done when we the serial composition of systems arises, and the product keeps the periodicity property (see Prop. 4)
- the Kleene star is done only on conservative ultimately periodic series since the loops of a B-TWEG are neutral. (see Prop. 6)

■

Remark 10 (Liveness): The liveness of a B-TWEG depends on the initial marking of the circuits. If a B-TWEG is not alive, then the transfer relation computed in $\mathcal{E}_{per}[[\delta]]$ will contain some degenerate periodic series such as $\bigoplus_i w_i \delta^{t_i} \oplus W(\delta^1)^* = p \oplus W(\delta^1)^*$, where w_i and W are periodic E-operators in \mathcal{E}_{per} . The last monomial $W(\delta^1)^*$, that can be considered as $W\delta^{+\infty}$, describes the situation where, after a finite number of output events, the B-TWEG is definitely blocked, and the system can not release output event anymore : some events are infinitely delayed. Prop. 7 considers the cases of ultimately blocked B-TWEGs as some degenerate ultimately periodic cases.

V. CONTROL OF B-TWEGS

The input-output model obtained in the previous section for B-TWEGs allows us to consider some model matching control problems such as the ones studied in [7] [20] [17] [15]. We only need to express the residuation of the product in $\mathcal{E}_{per}[[\delta]]$. The first step is to express the residuation of the product in \mathcal{E}_{per} .

A. Residuation in \mathcal{E}_{per}

On a complete dioid, the product is not invertible. But the theory of residuation developed in [3], and applied to idempotent semirings in [1], can be used to find optimal solutions to some inequalities. On a complete dioid, mappings $L_a : x \mapsto ax$ and $R_a : x \mapsto xa$ are residuated. It

means that $\forall b$, $L_a(x) \preceq b$ and $R_a(x) \preceq b$ have maximal solutions, that are respectively denoted $L_a^\sharp(b) = a \backslash b = \bigoplus \{x | ax \preceq b\}$ and $R_a^\sharp(b) = b / a = \bigoplus \{x | xa \preceq b\}$. Mappings L_a^\sharp and R_a^\sharp are said residual mappings of L_a et R_a . When the dioid product is commutative, then $L_a^\sharp = R_a^\sharp$.

Theorem 4 ([3] [1]): On a complete dioid \mathcal{D} ,

$$abx \preceq c \iff x \preceq b \backslash a \backslash c \quad (11)$$

$$xba \preceq c \iff x \preceq c / a / b \quad (12)$$

$$(a \oplus b)x \preceq c \iff a \backslash c \wedge b \backslash c \quad (13)$$

$$x(a \oplus b) \preceq c \iff c / a \wedge c / b \quad (14)$$

The dioid of E-operators denoted \mathcal{E} is complete. It is then possible to define the residual mappings of L_a and R_a on \mathcal{E} . More precisely, concerning the elementary operators of \mathcal{E} , we obtain the following results.

Proposition 8: Let us consider $w \in \mathcal{E}$ an E-operator. We have :

$$\gamma^n \backslash w = \gamma^{-n} w \quad w / \gamma^n = w \gamma^{-n} \quad (15)$$

$$\mu_m \backslash w = \beta_m \gamma^{m-1} w \quad w / \mu_m = w \beta_m \quad (16)$$

$$\beta_b \backslash w = \mu_b w \quad w / \beta_b = w \gamma^{b-1} \mu_b \quad (17)$$

Proof: Since operator γ^n is invertible ($\gamma^n \gamma^{-n} = \gamma^{-n} \gamma^n = e$), then we obtain (15). For (16), the right product by μ_m is invertible since $\beta_m \mu_m = e$. For the left product, the residual mapping satisfies

$$\mu_m \backslash w = \bigoplus \{v \in \mathcal{E} | \mu_m v \preceq w\}.$$

Let us remind that $w_1, w_2 \in \mathcal{E}$, then $w_1 \preceq w_2 \iff \mathcal{F}_{w_1} \geq \mathcal{F}_{w_2}$. Therefore, we also can express the residual mapping as

$$\begin{aligned} \mu_m \backslash w &= \bigoplus \{v \in \mathcal{E} | \mathcal{F}_{\mu_m v} \geq \mathcal{F}_w\} \\ &= \bigoplus \{v \in \mathcal{E} | m \cdot \mathcal{F}_v \geq \mathcal{F}_w\} \\ &= \bigoplus \{v \in \mathcal{E} | \mathcal{F}_v \geq \mathcal{F}_w / m\} \end{aligned}$$

Therefore, the operator function of $\mu_m \backslash w$ satisfies

$$\forall k \in \mathbb{Z}, \mathcal{F}_{\mu_m \backslash w}(k) \geq \mathcal{F}_w(k) / m$$

Since an operator function is defined on \mathbb{Z} , it is equivalent to, $\forall k \in \mathbb{Z}$

$$\begin{aligned}\mathcal{F}_{\mu_m \backslash w}(k) &= \lceil \mathcal{F}_w(k)/m \rceil \\ &= \lfloor (\mathcal{F}_w(k) + m - 1)/m \rfloor\end{aligned}$$

It comes that $\mu_m \backslash w = \beta_m \gamma^{m-1} w$.

For (17), we know that the left product by β_b is invertible. For the right product, we have

$$\begin{aligned}w \not\beta_b &= \bigoplus \{v \in \mathcal{E} \mid \mathcal{F}_{v \beta_b} \geq \mathcal{F}_w\} \\ &= \bigoplus \{v \in \mathcal{E} \mid \forall k \in \mathbb{Z}, \mathcal{F}_v(\lfloor k/b \rfloor) \geq \mathcal{F}_w(k)\}.\end{aligned}$$

Therefore, the operator function of $w \not\beta_b$ has to satisfy the following constraints

$$\begin{aligned}0 \leq k \leq b - 1, & \quad \mathcal{F}_{w \not\beta_b}(0) \geq \mathcal{F}_w(k) \\ b \leq k \leq 2b - 1, & \quad \mathcal{F}_{w \not\beta_b}(1) \geq \mathcal{F}_w(k) \\ 2b \leq k \leq 3b - 1, & \quad \mathcal{F}_{w \not\beta_b}(2) \geq \mathcal{F}_w(k) \\ & \dots\end{aligned}$$

Since \mathcal{F}_w is a not decreasing function, $\mathcal{F}_{w \not\beta_b}$ satisfies

$$\begin{aligned}\mathcal{F}_{w \not\beta_b}(0) &= \mathcal{F}_w(b - 1), \\ \mathcal{F}_{w \not\beta_b}(1) &= \mathcal{F}_w(2b - 1) \\ & \dots\end{aligned}$$

i.e. $\mathcal{F}_{w \not\beta_b}(k) = \mathcal{F}_w(b.k + (b - 1))$, which amounts to $w \not\beta_b = w \gamma^{b-1} \mu_b$. ■

Example 10: Let us develop the computation of an example : $(\gamma^1 \mu_2) \backslash (\gamma^2 \beta_3 \mu_4) \in \mathcal{E}$. By applying results from Prop. 8 and from Prop. 1, we obtain

$$\begin{aligned}(\gamma^1 \mu_2) \backslash (\gamma^2 \beta_3 \mu_4) &= \mu_2 \backslash (\gamma^1 \backslash (\gamma^2 \beta_3 \mu_4)) \\ &= \mu_2 \backslash (\gamma^{-1}(\gamma^2 \beta_3 \mu_4)) \\ &= \beta_2 \gamma^1(\gamma^1 \beta_3 \mu_4) = \beta_2 \gamma^2 \beta_3 \mu_4 \\ &= \gamma^1 \beta_2 \beta_3 \mu_4 = \gamma^1 \beta_6 \mu_4 = \gamma^1 \beta_3 \mu_2\end{aligned}$$

Let us note that the canonical form of $\gamma^1 \beta_3 \mu_2$ is $\gamma^1 \mu_2 \beta_3 \gamma^1 \oplus \gamma^2 \mu_2 \beta_3 = \gamma^1 \nabla_{2|3} \gamma^1 \oplus \gamma^2 \nabla_{2|3}$.

Since residuation is not an exact inversion, we can check here that $(\gamma^1 \mu_2)[(\gamma^1 \mu_2) \backslash (\gamma^2 \beta_3 \mu_4)] \neq (\gamma^2 \beta_3 \mu_4)$. Indeed, we obtain

$$\begin{aligned}(\gamma^1 \mu_2)[(\gamma^1 \mu_2) \backslash (\gamma^2 \beta_3 \mu_4)] &= (\gamma^1 \mu_2)[\gamma^1 \beta_3 \mu_2] \\ &= \gamma^3 \nabla_{4|3} \gamma^1 \oplus \gamma^5 \nabla_{4|3},\end{aligned}$$

whereas the canonical form of $\gamma^2 \beta_3 \mu_4$ is $\gamma^2 \nabla_{4|3} \gamma^2 \oplus \gamma^3 \nabla_{4|3} \gamma^1 \oplus \gamma^4 \nabla_{4|3}$.

Proposition 9: Let us consider $w_1, w_2 \in \mathcal{E}_{per}$. Then $w_2 \backslash w_1$ and $w_1 \not\backslash w_2$ are also periodic E-operators such that $\Gamma(w_2 \backslash w_1) = \Gamma(w_1)/\Gamma(w_2)$ and $\Gamma(w_1 \not\backslash w_2) = \Gamma(w_1)/\Gamma(w_2)$.

Proof: Thanks to th. 4 and Prop. 8, and since we can write periodic E-operators as finite sums, $w_1 = \bigoplus_i \gamma^{n_i} \nabla_{m|b} \gamma^{n'_i}$ and $w_2 = \bigoplus_j \gamma^{n_j} \nabla_{M|B} \gamma^{n'_j}$, then

$$\begin{aligned} w_2 \backslash w_1 &= [\bigoplus_j \gamma^{n_j} \nabla_{M|B} \gamma^{n'_j}] \backslash [\bigoplus_i \gamma^{n_i} \nabla_{m|b} \gamma^{n'_i}] \\ &= \bigwedge_j \left([\gamma^{n_j} \nabla_{M|B} \gamma^{n'_j}] \backslash [\bigoplus_i \gamma^{n_i} \nabla_{m|b} \gamma^{n'_i}] \right) \\ &= \bigwedge_j \left(\bigoplus_i \gamma^{-n'_j} \mu_B \beta_M \gamma^{M-1} \gamma^{-n_j} \gamma^{n_i} \nabla_{m|b} \gamma^{n'_i} \right) \end{aligned}$$

It is then a finite inf of periodic E-operators, that is also a periodic E-operator thanks to Prop. 2. ■

B. Residuation in $\mathcal{E}_{per}[[\delta]]$

Thanks to (13) and (14), we can express the residuation of the product of balanced polynomials. Let $p_1 = \bigoplus w_{1_i} \delta^{t_{1_i}}$ and $p_2 = \bigoplus w_{2_j} \delta^{t_{2_j}}$ be two balanced polynomials in $\mathcal{E}_{per}[[\delta]]$. Then, we can write $p_2 \backslash p_1$ and $p_1 \not\backslash p_2$ as

$$\begin{aligned} p_2 \backslash p_1 &= (\bigoplus_j w_{2_j} \delta^{t_{2_j}}) \backslash [\bigoplus_i w_{1_i} \delta^{t_{1_i}}] \\ &= \bigwedge_j \left((w_{2_j} \delta^{t_{2_j}}) \backslash [\bigoplus_i w_{1_i} \delta^{t_{1_i}}] \right) \\ &= \bigwedge_j [\bigoplus_i (w_{2_j} \backslash w_{1_i}) \delta^{t_{1_i} - t_{2_j}}] \end{aligned}$$

and

$$p_1 \not\backslash p_2 = \bigwedge_j [\bigoplus_i (w_{1_i} \not\backslash w_{2_j}) \delta^{t_{1_i} - t_{2_j}}] .$$

The computation of operations \backslash and $\not\backslash$ on polynomials lies on the residuation of coefficients in \mathcal{E}_{per} , and it is then equivalent to an infimum operation on some polynomials in $\mathcal{E}_{per}[[\delta]]$. When we extend the computation of operations \backslash and $\not\backslash$ to ultimately periodic series of $\mathcal{E}_{per}[[\delta]]$, we obtain the next result that we can not show here. As for polynomials, the residuation of two periodic series is equivalent to compute an infimum of a finite set of ultimately periodic series.

Assertion 1: Let s_1 and s_2 be two ultimately periodic series of $\mathcal{E}_{per}[[\delta]]$. If $\sigma_r(s_1) \geq \sigma_r(s_2)$ then $s_1 \not\backslash s_2$ and $s_2 \backslash s_1$ are ultimately periodic series of $\mathcal{E}_{per}[[\delta]]$ such that

$$\begin{aligned} \sigma_r(s_1 \not\backslash s_2) &= \sigma_r(s_1) \\ \Gamma(s_1 \not\backslash s_2) &= \Gamma(s_1)/\Gamma(s_2) \\ \sigma_r(s_2 \backslash s_1) &= \sigma_r(s_1) \\ \Gamma(s_2 \backslash s_1) &= \Gamma(s_1)/\Gamma(s_2) \end{aligned}$$

If $\sigma_r(s_1) < \sigma_r(s_2)$ then $s_1 \not\backslash s_2 = s_2 \backslash s_1 = \varepsilon$.

C. Example

We will apply our work to obtain an output feedback control for the B-TWEG of Fig. 10. First, we state the transfer relation of Fig. 2 in its canonical form. In example 4 we obtained $x_4 = \mu_3(\gamma^1\delta^2)^*\beta_2\delta^2x_1 \oplus \beta_2\gamma^1\delta^1(\gamma^2\delta^1)^*\mu_3x_1$, *i.e.* $x_4 = Hx_1$. The gain of series H is clearly the gain of all paths from t_1 to t_4 , $\Gamma(H) = 3/2$. Series H is depicted in its 3D representation in Fig. 10. The left and the right canonical forms of H are given below (where coefficients are also described in their canonical form in \mathcal{E}_{per})

$$H = p \oplus q(\gamma^2\delta^3)^* = p \oplus (\gamma^1\delta^1)^*q'$$

with

$$\begin{aligned} p &= \nabla_{3|2}\delta^2 \oplus \gamma^2\nabla_{3|2}\gamma^1\delta^3 \oplus \gamma^3\nabla_{3|2}\delta^4 \\ &\quad \oplus \gamma^4\nabla_{3|2}\gamma^1\delta^5 \oplus (\gamma^5\nabla_{3|2}\gamma^1 \oplus \gamma^6\nabla_{3|2})\delta^6 \\ q &= [(\gamma^6\nabla_{3|2}\gamma^1 \oplus \gamma^8\nabla_{3|2})\delta^7 \\ &\quad \oplus (\gamma^7\nabla_{3|2}\gamma^1 \oplus \gamma^9\nabla_{3|2})\delta^8] \\ q' &= [(\gamma^6\nabla_{3|2}\gamma^1 \oplus \gamma^8\nabla_{3|2})\delta^7] \end{aligned}$$

The left and the right slopes are given by $\sigma_r(H) = 3/2$ and $\sigma_l(H) = 1/1$.

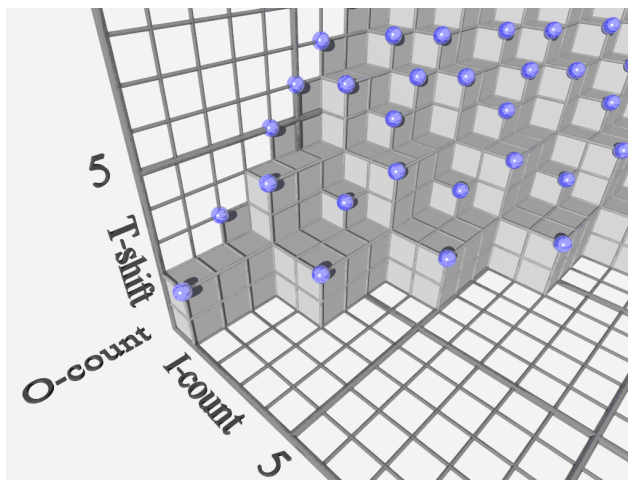


Figure 10. Transfer series of the B-TWEG of Fig. 2

Thanks to results obtained in [7], we can compute the greatest neutral output feedback for the B-TWEG described by the transfer matrix H . From a practical point of view, it is the slowest

controller that we can add between the output and the input so that the closed loop system has the same behavior as the system alone. The benefit from this controller is to reduce the internal stocks as much as possible. By knowing H , this controller is expressed by (see [7]) $\hat{F} = (H \setminus H) \rho H$. For the B-TWEG of Fig. 2, the computation gives

$$\begin{aligned} \hat{F} &= \gamma^3 \nabla_{2|3} \gamma^1 \delta^0 \oplus \gamma^4 \nabla_{2|3} \delta^2 \oplus (\gamma^2 \delta^3)^* [\gamma^6 \nabla_{2|3} \delta^4] \\ &= \gamma^3 \nabla_{2|3} \gamma^1 \delta^0 \oplus \gamma^4 \nabla_{2|3} \delta^2 \oplus [\gamma^6 \nabla_{2|3} \delta^4] (\gamma^3 \delta^3)^*. \end{aligned}$$

The controller is described by an ultimately periodic series the slopes of which are $\sigma_r(\hat{F}) = 3/3$ and $\sigma_l(\hat{F}) = 3/2$. We obtain naturally that $\Gamma(\hat{F}) = 2/3$ is equal to $1/\Gamma(H)$: the supplementary circuit due to the feedback loop is neutral, and the closed-loop system is still a B-TWEG. The transfer series of \hat{F} is described in Fig. 11. Controller \hat{F} also can be described by a B-TWEG which is depicted in Fig. 12. The grey zone corresponds to the realization of controller \hat{F} .

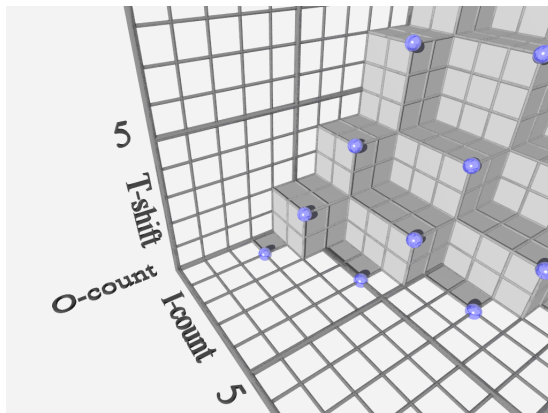


Figure 11. Transfer series of the optimal neutral output feedback for the B-TWEG of Fig. 2

VI. CONCLUSION

This work presents a modeling approach for the class of Balanced Timed and Weighted Event Graphs (B-TWEGs) in a dioid of additive operators. Four elementary operators denoted γ^n , δ^t , μ_m and β_b are necessary to describe the dynamical phenomena modeled by a B-TWEG. The input-output behavior of B-TWEGs can be embedded into some rational formal series in a dioid denoted $\mathcal{E}[[\delta]]$. Each formal series has a natural three dimensional graphical representation which has an ultimate periodicity property. Since a B-TWEG is completely characterized by a

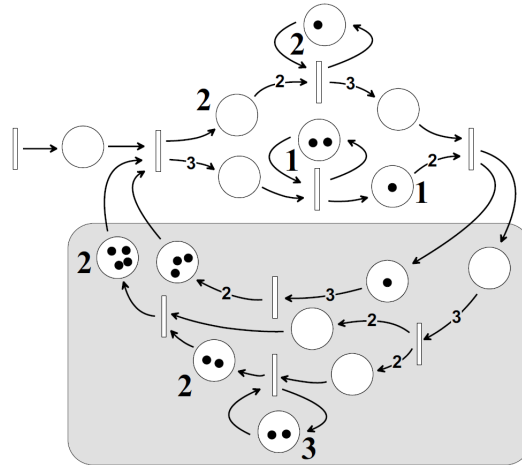


Figure 12. Greatest neutral output feedback

finite series (or a matrix for multivariate systems), some model reference control problems can be stated and solved in that framework. This work provides a natural extension of the max-plus theory for Timed Event Graphs to a class of weighted TEGs.

REFERENCES

- [1] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat. *Synchronization and Linearity: An Algebra for Discrete Event Systems*. John Wiley and Sons, New York, 1992.
- [2] A. Benabid-Najjar, C. Hanen O. Marchetti, and A. Munier-Kordon. Periodic schedules for bounded timed weighted event graphs. *IEEE TAC*, 57:1222–1232, 2012.
- [3] T.S. Blyth and M.F. Janowitz. *Residuation Theory*. Pergamon Press, Oxford, 1972.
- [4] G. Cohen, S. Gaubert, and J.P. Quadrat. Timed event graphs with multipliers and homogeneous min-plus systems. *IEEE TAC*, 43(9):1296 – 1302, September 1998.
- [5] G. Cohen, P. Moller, J.P. Quadrat, and M. Viot. Algebraic Tools for the Performance Evaluation of Discrete Event Systems. *IEEE Proceedings: Special issue on Discrete Event Systems*, 77(1):39–58, January 1989.
- [6] B. Cottenceau, L. Hardouin, and J.-L. Boimond. Input-Output Representation for a Subclass of Timed and Weighted Event Graphs in Dioids. Technical report, LISA Université d’Angers, July 2012.
- [7] B. Cottenceau, L. Hardouin, J.-L. Boimond, and J.-L. Ferrier. Model Reference Control for Timed Event Graphs in Dioids. *Automatica*, vol. 37:1451–1458, 2001.
- [8] B. Cottenceau, M. Lhommeau, L. Hardouin, and J.L. Boimond. Data processing tool for calculation in dioid. In *5th Workshop on Discrete Event Systems*, 2000.
- [9] S. Gaubert. *Théorie des systèmes linéaires dans les dioïdes*. Ph.D. thesis (in French), Ecole des Mines de Paris, Paris, 1992.

- [10] S. Gaubert. Rational series over dioids and discrete event systems. In *Proc. of the 11th Conf. on Anal. and Opt. of Systems: Discrete Event Systems*, number 199 in Lect. Notes. in Control and Inf. Sci, Sophia Antipolis, June 1994. Springer.
- [11] M.C.W. Geilen. Synchronous data flow scenarios. *ACM Transactions on Embedded Computing Systems*, 10(2), December 2010.
- [12] M.C.W. Geilen and S. Stuijk. Worst-case performance analysis of synchronous dataflow scenarios. In *International Conference on Hardware-Software Codesign and System Synthesis*, New-York, 2010.
- [13] M. Mc Gettrick, G. Cohen, S. Gaubert, and J.P. Quadrat. Max-plus toolbox 2.3.3 for scilab. 2005.
- [14] S. Hamaci, J.-L. Boimond, and S. Lahaye. Modeling and Control of Hybrid Timed Event Graphs with Multipliers using (Min,+) Algebra. *Discrete Event Dynamic Systems*, vol. 16, n 2:241–256, 2006.
- [15] L. Hardouin, C. A. Maia, B. Cottenceau, and M. Lhommeau. Observer Design for (max,+) Linear Systems. *IEEE Transaction on Automatic Control*, 55-2, 2010.
- [16] B. Heidergott, G.J. Olsder, and J.van der Woude. *Max Plus at Work - Modelling and Analysis of Synchronized Systems - A Course on Max-Plus Algebra and Its Applications*. Princeton University Press, Princeton, 2006.
- [17] L. Houssin, S. Lahaye, and J.-L. Boimond. Control of (max,+)-linear systems minimizing delays. *Discrete Event Dynamic Systems*, to appear.
- [18] D. Krob and A. Bonnier-Rigny. A complete system of identities for one letter rational expressions with multiplicities in the tropical semiring. *J. Pure Appl. Algebra*, 134:27–50, 1994.
- [19] E.A. Lee and D.G. Messerschmitt. Synchronous data flow. *Proc. IEEE*, 75(9):1235–1245, 1987.
- [20] C.A. Maia, L. Hardouin, R. Santos Mendes, and B. Cottenceau. Optimal Closed-Loop Control for Timed Event Graphs in Dioid. *IEEE Transaction on Automatic Control*, vol. 48:2284–2287, 2003.
- [21] O. Marchetti and A. Munier-Kordon. Complexity results for weighted timed event graphs. *Discrete Optimization*, 7:166–180, 2010.
- [22] P. Moller. *Théorie algébrique des Systèmes à Événements Discrets*. Thèse, École des Mines de Paris, 1988.
- [23] T. Murata. Petri nets: Properties, analysis and applications. *Proceedings of the IEEE*, 1989.
- [24] S. Sriram and S. Bhattacharyya. *Embedded Multiprocessors: Scheduling and Synchronization*. Marcel Dekker, New-York, 2000.