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RANDOM WEIGHTED SOBOLEV INEQUALITIES ON \mathbb{R}^d AND APPLICATION TO HERMITE FUNCTIONS

by

Aurélien Poiret, Didier Robert & Laurent Thomann

Abstract. — We extend a randomisation method, introduced by Shiffman-Zelditch and developed by Burq-Lebeau on compact manifolds for the Laplace operator, to the case of \mathbb{R}^d with the harmonic oscillator. We construct measures, thanks to probability laws which satisfy the concentration of measure property, on the support of which we prove optimal weighted Sobolev estimates on \mathbb{R}^d . This construction relies on accurate estimates on the spectral function in a non-compact configuration space. As an application, we show that there exists a basis of Hermite functions with good decay properties in $L^\infty(\mathbb{R}^d)$, when $d \geq 2$.

1. Introduction and results

1.1. Introduction. — During the last years several papers show some basic results concerning P.D.E. and Sobolev spaces can be strikingly improved using randomization techniques. In particular Burq-Lebeau developed in [2] a randomisation method based on the Laplace operator on a compact Riemannian manifold, and showed that almost surely, a function enjoys better Sobolev estimates than expected, using ideas of Shiffman-Zelditch [18]. This approach depends heavily on spectral properties of the operator one considers. In this paper we are interested in estimates in Sobolev spaces based on the harmonic oscillator in $L^2(\mathbb{R}^d)$

$$H = -\Delta + |x|^2 = \sum_{j=1}^d (-\partial_j^2 + x_j^2).$$

We get optimal stochastic weighted Sobolev estimates on \mathbb{R}^d using the Burq-Lebeau method. Indeed we show that there is a unified setting for these results, including the case of compact manifolds. We also make the following extension: In [2], the construction of the measures relied on Gaussian random variables, while in our work we consider general random variable which satisfy concentration

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of measure estimates (including discrete random variables, see Section 2). However, we obtain the optimal estimates only in the case of the Gaussians.

We will see that the extension from a compact manifold to an operator on \mathbb{R}^d with discrete spectrum is not trivial because of the complex behaviour of the spectral function on a non-compact configuration space.

In our forthcoming paper [15], we will give some applications to the well-posedness of nonlinear Schrödinger equations with Sobolev regularity below the optimal deterministic index.

Most of the results stated here can be extended to more general Schrödinger Hamiltonians $-\Delta + V(x)$ with confining potentials V . This will be detailed in [16].

Let $d \geq 2$. We want to define probability measures on finite dimensional subspaces $\mathcal{E}_h \subset L^2(\mathbb{R}^d)$, based on spectral projections with respect to H . We denote by $\{\varphi_j, j \geq 1\}$ an orthonormal basis of $L^2(\mathbb{R}^d)$ of eigenvectors of H (the Hermite functions), and we denote by $\{\lambda_j, j \geq 1\}$ the non decreasing sequence of eigenvalues (each is repeated according to its multiplicity): $H\varphi_j = \lambda_j\varphi_j$.

For $h > 0$, we define the interval $I_h = [\frac{a_h}{h}, \frac{b_h}{h}[$ and we assume that a_h and b_h satisfy, for some $a, b, D > 0, \delta \in [0, 1]$,

$$(1.1) \quad \lim_{h \rightarrow 0} a_h = a, \quad \lim_{h \rightarrow 0} b_h = b, \quad 0 < a \leq b \quad \text{and} \quad b_h - a_h \geq Dh^\delta,$$

with any $D > 0$ if $\delta < 1$ and $D \geq 2$ in the case $\delta = 1$. This condition ensures that N_h , the number (with multiplicities) of eigenvalues of H in I_h tends to infinity when $h \rightarrow 0$. Indeed, we can check that $N_h \sim ch^{-d}(b_h - a_h)$, in particular $\lim_{h \rightarrow 0} N_h = +\infty$, since $d \geq 2$. In the sequel, we write $\Lambda_h = \{j \geq 1, \lambda_j \in I_h\}$ and $\mathcal{E}_h = \text{span}\{\varphi_j, j \in \Lambda_h\}$, so that $N_h = \#\Lambda_h = \dim \mathcal{E}_h$. Finally, we denote by $\mathbf{S}_h = \{u \in \mathcal{E}_h : \|u\|_{L^2(\mathbb{R}^d)} = 1\}$ the unit sphere of \mathcal{E}_h .

Let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence so that there exist $K_0, K_1 > 0$

$$(1.2) \quad \frac{K_1}{N_h} \sum_{j \in \Lambda_h} |\gamma_j|^2 \leq |\gamma_n|^2 \leq \frac{K_0}{N_h} \sum_{j \in \Lambda_h} |\gamma_j|^2, \quad \forall n \in \Lambda_h, \quad \forall h \in]0, 1].$$

This so-called ‘‘squeezing’’ condition means that on each level of energy $\lambda_n, n \in \Lambda_h$, the coefficients $|\gamma_k|$ have almost the same size. For instance (1.2) holds if there exists $(d_h)_{h \in]0, 1]}$ so that $\gamma_n = d_h$ for all $n \in \Lambda_h$.

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\{X_n, n \geq 1\}$ be independent standard complex Gaussians $\mathcal{N}_{\mathbb{C}}(0, 1)$. In fact, in our work we will consider more general probability laws, which satisfy concentration of measure estimates (see Assumption 1), but for sake of clarity, we first state the results in this particular case. If $(\gamma_n)_{n \in \mathbb{N}}$ satisfies (1.2), we define the random vector in \mathcal{E}_h

$$v_\gamma(\omega) := v_{\gamma, h}(\omega) = \sum_{j \in \Lambda_h} \gamma_j X_j(\omega) \varphi_j.$$

We define a probability measure $\mathbf{P}_{\gamma,h}$ on \mathbf{S}_h by: for all measurable and bounded function $f : \mathbf{S}_h \rightarrow \mathbb{R}$

$$\int_{\mathbf{S}_h} f(u) d\mathbf{P}_{\gamma,h}(u) = \int_{\Omega} f \left(\frac{v_{\gamma}(\omega)}{\|v_{\gamma}(\omega)\|_{L^2(\mathbb{R}^d)}} \right) d\mathbb{P}(\omega).$$

We can check that in the isotropic case ($\gamma_j = \frac{1}{\sqrt{N}}$ for all $j \in \Lambda_h$), $\mathbf{P}_{\gamma,h}$ is the uniform probability on \mathbf{S}_h (see Appendix C).

Finally, let us recall the definition of harmonic Sobolev spaces for $s \geq 0$, $p \geq 1$.

$$(1.3) \quad \begin{aligned} \mathcal{W}^{s,p} &= \mathcal{W}^{s,p}(\mathbb{R}^d) = \{u \in L^p(\mathbb{R}^d), H^{s/2}u \in L^p(\mathbb{R}^d)\}, \\ \mathcal{H}^s &= \mathcal{H}^s(\mathbb{R}^d) = \mathcal{W}^{s,2}. \end{aligned}$$

The natural norms are denoted by $\|u\|_{\mathcal{W}^{s,p}}$ and up to equivalence of norms we have for $1 < p < +\infty$

$$\|u\|_{\mathcal{W}^{s,p}} = \|H^{s/2}u\|_{L^p} \equiv \|(-\Delta)^{s/2}u\|_{L^p} + \|\langle x \rangle^s u\|_{L^p}.$$

1.2. Main results of the paper. —

1.2.1. Estimates for frequency localised functions. — Our first result gives properties of the elements on the support of $\mathbf{P}_{\gamma,h}$, which are high frequency localised functions. Namely

Theorem 1.1. — *Let $d \geq 2$. Assume that $0 \leq \delta < 2/3$ in (1.1). Then there exist $0 < C_0 < C_1$, $c_1 > 0$ and $h_0 > 0$ such that for all $h \in]0, h_0]$.*

$$\mathbf{P}_{\gamma,h} \left[u \in \mathbf{S}_h : C_0 |\log h|^{1/2} \leq \|u\|_{\mathcal{W}^{d/2,\infty}(\mathbb{R}^d)} \leq C_1 |\log h|^{1/2} \right] \geq 1 - h^{c_1}.$$

Moreover the estimate from above is satisfied for any $\delta \geq 1$ with D large enough.

It is clear that under condition (1.2), there exist $0 < C_2 < C_3$, so that for all $u \in \mathbf{S}_h$, and $s \geq 0$

$$C_2 h^{-s/2} \leq \|u\|_{\mathcal{H}^s(\mathbb{R}^d)} \leq C_3 h^{-s/2},$$

since all elements of \mathbf{S}_h oscillate with frequency $h^{-1/2}$. Thus Theorem 1.1 shows a gain of $d/2$ derivatives in L^∞ , and this induces a gain of d derivatives compared to the usual deterministic Sobolev embeddings. This can be compared with the results of [2] where the authors obtain a gain of $d/2$ derivatives on compact manifolds: this comes from different behaviours of the spectral function, see Section 3. Notice that the bounds in Theorem 1.1 (and in the results of [2] as well) do not depend on the length of the interval of the frequency localisation I_h (see (1.1)), but only on the size of the frequencies. This is a consequence of the randomisation, and from the bound (3.15).

We will see in Theorem 4.1 that the upper bound in Theorem 1.1 holds for any $0 \leq \delta \leq 1$ and for more general random variables X which satisfy the concentration of measure property. However, to prove the lower bound (see Corollary 4.8), we have to restrict to the case of Gaussians: in the general case, under Assumption 1, we do not reach the factor $|\ln h|^{1/2}$. Following the approach of [18, 2], we first prove estimates of $\|u\|_{\mathcal{W}^{d/2,\infty}(\mathbb{R}^d)}$ with large r and uniform constants (see Theorem 4.12), and which are essentially optimal for general random variables (see Theorem 4.13).

The condition $\delta < 2/3$ is needed to prove the lower bound, thanks to a reasonable functional calculus based on the harmonic oscillator (see Appendix B).

Finally we point out that in a very recent paper [6], Feng and Zelditch prove similar estimates for the mean and median for the L^∞ -norm of random holomorphic fields.

1.2.2. Global Sobolev estimates. — Using a dyadic Littlewood-Paley decomposition, we now give general estimates in Sobolev spaces; we refer to Subsection 4.1 for more details. For $s \in \mathbb{R}$, $p, q \in [1, +\infty]$, we define the harmonic Besov space by

$$(1.4) \quad \mathcal{B}_{p,q}^s(\mathbb{R}^d) = \left\{ u = \sum_{n \geq 0} u_n : \sum_{n \geq 0} 2^{nqs/2} \|u_n\|_{L^p(\mathbb{R}^d)}^q < +\infty \right\},$$

where the u_n have frequencies of size $\sim 2^n$. The space $\mathcal{B}_{p,q}^s(\mathbb{R}^d)$ is a Banach space with the norm in $\ell^q(\mathbb{N})$ of $\{2^{ns/2} \|u_n\|_{L^p(\mathbb{R}^d)}\}_{n \geq 0}$.

We assume that γ satisfies (1.2) and

$$\sum_{n \geq 0} |\gamma|_{\Lambda_n} < +\infty.$$

Then we set

$$v_\gamma(\omega) = \sum_{j=0}^{+\infty} \gamma_j X_j(\omega) \varphi_j,$$

so that almost surely $v_\gamma \in \mathcal{B}_{2,1}^0(\mathbb{R}^d)$ and its probability law defines a measure μ_γ in $\mathcal{B}_{2,1}^0(\mathbb{R}^d)$. Notice that we have

$$\mathcal{H}^s(\mathbb{R}^d) \subset \mathcal{B}_{2,1}^0(\mathbb{R}^d) \subset L^2(\mathbb{R}^d), \quad \forall s > 0.$$

We have the following result

Theorem 1.2. — *For every $(s, r) \in \mathbb{R}^2$ such that $r \geq 2$ and $s = d(\frac{1}{2} - \frac{1}{r})$ there exists $c_0 > 0$ such that for all $K > 0$ we have*

$$(1.5) \quad \mu_\gamma \left[u \in \mathcal{B}_{2,1}^0(\mathbb{R}^d) : \|u\|_{\mathcal{W}^{s,r}(\mathbb{R}^d)} \geq K \|u\|_{\mathcal{B}_{2,1}^0(\mathbb{R}^d)} \right] \leq e^{-c_0 K^2}.$$

In particular μ_γ -almost all functions in $\mathcal{B}_{2,1}^0(\mathbb{R}^d)$ are in $\mathcal{W}^{s,r}(\mathbb{R}^d)$.

If γ satisfies (1.2) and the (weaker) condition $\sum_{n \geq 0} |\gamma|_{\Lambda_n}^2 < +\infty$, then μ_γ defines a probability measure on $L^2(\mathbb{R}^d)$ and we can prove the estimate

$$(1.6) \quad \mu_\gamma \left[u \in L^2(\mathbb{R}^d) : \|u\|_{\mathcal{W}^{s,r}(\mathbb{R}^d)} \geq K \|u\|_{L^2(\mathbb{R}^d)} \right] \leq e^{-c_0 K^2},$$

with $s = d(\frac{1}{2} - \frac{1}{r})$ when $r < +\infty$ and $s < d/2$ in the case $r = +\infty$. From this result it is easy to deduce space-time estimates (Strichartz) for the linear flow $e^{-itH}u$, which can be used to study the nonlinear problem. This will be pursued in [15].

1.2.3. An application to Hermite functions. — Similarly to [2], the previous results give some information on Hilbertian bases. We prove that there exists a basis of Hermite functions with good decay properties.

Theorem 1.3. — *Let $d \geq 2$. Then there exists a Hilbertian basis of $L^2(\mathbb{R}^d)$ of eigenfunctions of the harmonic oscillator H denoted by $(\varphi_n)_{n \geq 1}$ such that $\|\varphi_n\|_{L^2(\mathbb{R}^d)} = 1$ and so that for some $M > 0$ and all $n \geq 1$,*

$$(1.7) \quad \|\varphi_n\|_{L^\infty(\mathbb{R}^d)} \leq Mn^{-1/4}(1 + \log n)^{1/2}.$$

In other words, all elements of this basis are decreasing in $L^\infty(\mathbb{R}^d)$ norm. This seems not known before for $d \geq 2$.

We refer to Theorem 5.1 for a more quantitative result, and where we prove that for a natural probability measure, almost all Hermite basis satisfies the property of Theorem 1.3 (see also Corollary 4.14). For the proof of this result, we need the finest randomisation with $\delta = 1$ and $D = 2$ in (1.1), so that $\mathbf{P}_{\gamma, h}$ is a probability measure on each eigenspace.

The result of Theorem 1.3 does not hold true in dimension $d = 1$. Indeed, in this case one can prove the optimal bound (see [11])

$$(1.8) \quad \|\varphi_n\|_{L^\infty(\mathbb{R})} \leq Cn^{-1/12}.$$

We stress that we don't now any explicit example of $(\varphi_n)_{n \geq 1}$ which satisfy the conclusion of the Theorem. For instance, the basis obtained by tensorisation of the 1D basis does not realise (1.7) because of (1.8).

Let us compare (1.7) with the general known bounds on Hermite functions. We have $H\varphi_n = \lambda_n\varphi_n$, with $\lambda_n \sim cn^{1/d}$, therefore (1.7) can be rewritten

$$(1.9) \quad \|\varphi_n\|_{L^\infty(\mathbb{R}^d)} \leq M\lambda_n^{-\frac{d}{4}}(1 + \log \lambda_n)^{1/2}.$$

For a general basis with $d \geq 2$, Koch and Tataru [11] (see also [12]) prove that

$$\|\varphi_n\|_{L^\infty(\mathbb{R}^d)} \leq C\lambda_n^{\frac{d}{4} - \frac{1}{2}}.$$

which shows that (1.9) induces a gain of $d - 1$ derivatives compared to the general case. Observe also that the basis of radial Hermite functions does not satisfy (1.7) in dimension $d \geq 2$. As in [2, Théorème 8], it is likely that the log term in (1.7) can not be avoided.

1.3. Notations and plan of the paper. —

Notations. — *In this paper $c, C > 0$ denote constants the value of which may change from line to line. These constants will always be universal, or uniformly bounded with respect to the other parameters. We denote by $H = -\Delta + |x|^2 = \sum_{j=1}^d (-\partial_j^2 + x_j^2)$ the harmonic oscillator on \mathbb{R}^d , and for $s \geq 0$ we define the Sobolev space \mathcal{H}^s by the norm $\|u\|_{\mathcal{H}^s} = \|H^{s/2}u\|_{L^2(\mathbb{R}^d)} \approx \|u\|_{H^s(\mathbb{R}^d)} + \|\langle x \rangle^s u\|_{L^2(\mathbb{R}^d)}$. More generally, we define the spaces $\mathcal{W}^{s,p}$ by the norm $\|u\|_{\mathcal{W}^{s,p}} = \|H^{s/2}u\|_{L^p(\mathbb{R}^d)}$. We write $L^{r,s}(\mathbb{R}^d) = L^r(\mathbb{R}^d, \langle x \rangle^s dx)$, and its norm $\|u\|_{r,s}$.*

The rest of the paper is organised as follows. In Section 2 we describe the general probabilistic setting and we prove large deviation estimates on Hilbert spaces. In Section 3 we state crucial estimates on the spectral function of the harmonic oscillator. Section 4 is devoted to the proof of weighted Sobolev estimates and of the main results. In Section 5 we prove Theorem 1.3.

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2. A general setting for probabilistic smoothing estimates

Our aim in this section is to unify several probabilistic approaches to improve smoothing estimates established for dispersive equations. This setting is inspired by papers of Burq-Lebeau [2], Burq-Tzvetkov [3, 4] and their collaborators.

2.1. The concentration of measure property. —

Definition 2.1. — We say that a probability measure ν on \mathbb{R}^N satisfies the concentration of measure property if there exists constants $c, C > 0$ independent of $N \in \mathbb{N}$ such that for all Lipschitz and convex function $F : \mathbb{R}^N \rightarrow \mathbb{R}$

$$(2.1) \quad \nu \left[X \in \mathbb{R}^N : |F(X) - \mathbb{E}(F(X))| \geq r \right] \leq c e^{-\frac{Cr^2}{\|F\|_{Lip}^2}}, \quad \forall r > 0,$$

where $\|F\|_{Lip}$ is the best constant so that $|F(X) - F(Y)| \leq \|F\|_{Lip} \|X - Y\|_{\ell^2}$.

For a comprehensive study of these phenomena, we refer to the book of Ledoux [13]. Notice that one of the main features of (2.1) is that the bound is independent of the dimension of space, which enables to take N large.

Typically, in our applications, F will be a norm in \mathbb{R}^N .

Let us give some significative examples of such measures.

- If ν is a probability measure on \mathbb{R}^N which satisfies a Log-Sobolev estimate with constant $C^* > 0$, then (2.1) is satisfied for all Lipschitz function $F : \mathbb{R}^N \rightarrow \mathbb{R}$ (see [1, Théorème 7.4.1, page 123]). Recall that a probability measure ν on \mathbb{R}^N satisfies a Log-Sobolev estimate if there exists $C > 0$ so that for all $f \in \mathcal{C}_b(\mathbb{R}^N)$

$$(2.2) \quad \int_{\mathbb{R}^N} f^2 \ln \left(\frac{f^2}{\mathbb{E}(f^2)} \right) d\nu(x) \leq C \int_{\mathbb{R}^N} |\nabla f|^2 d\nu(x), \quad \mathbb{E}(f^2) = \int_{\mathbb{R}^N} f^2 d\nu(x).$$

Such a property is usually difficult to check. See [1] for more details. Notice that the convexity of F is not needed.

- A probability measure of the form $d\nu(x) = c_{\alpha, N} \exp(-\sum_{j=1}^N |x_j|^\alpha) dx$, $x \in \mathbb{R}^N$, satisfies (2.1) if and only if $\alpha \geq 2$ (see [1, page 109]).

• Assume that ν is a measure on \mathbb{R}^N with bounded support, then ν satisfies the concentration of measure property. This is the Talagrand theorem [19] (see also [20] for an introduction to the topic).

Assumption 1. — Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\{X_n, n \geq 1\}$ be a sequence of independent, identically distributed, real or complex random values. In the sequel we can assume that they are real with the identification $\mathbb{C} \approx \mathbb{R}^2$. Moreover, we assume that for all $n \geq 1$,

- (i) The joint law ν of X_n satisfies the concentration of measure property in the sense of Definition 2.1.
- (ii) The r.v. X_n is centred: $\mathbb{E}(X_n) = 0$.
- (iii) The r.v. X_n is normalized: $\mathbb{E}(X_n^2) = 1$.

Under Assumption 1, for all $n \geq 1$, and $\varepsilon > 0$ small enough

$$(2.3) \quad \mathbb{E}(e^{\varepsilon X_n^2}) < +\infty.$$

Indeed, by Definition 2.1 with $F(X) = X_n$

$$\mathbb{E}(e^{\varepsilon X_n^2}) = \int_0^{+\infty} \nu_n(e^{C\varepsilon X_n^2} > \lambda) d\lambda = 1 + \int_1^{+\infty} \nu_n(|X_n| > \sqrt{\frac{\ln \lambda}{\varepsilon}}) d\lambda \leq 1 + 2 \int_1^{+\infty} \lambda^{-\frac{1}{\varepsilon C}} d\lambda < +\infty.$$

Next, with the inequality $s|x| \leq \varepsilon x^2/2 + s^2/(2\varepsilon)$, we obtain that for all $s \in \mathbb{R}$, $\mathbb{E}(e^{sX_n}) \leq C e^{Cs^2}$ which in turn implies (see [14, Proposition 46]) that there exists $C > 0$ so that for all $s \in \mathbb{R}$

$$(2.4) \quad \mathbb{E}(e^{sX_n}) \leq e^{Cs^2}.$$

Remark 2.2. — Condition (2.4) is weaker than (2.2): a family of independent centred r.v. $\{X_n, n \geq 1\}$ which satisfies (2.4) does not necessarily satisfy (2.1) for all Lipschitz function F . Indeed, using Kolmogorov estimate, one can prove (see [13]) that condition (2.1) is equivalent to

$$(2.5) \quad \int_{\mathbb{R}^d} e^{sF} d\nu \leq e^{Cs^2 \|F\|_{Lip}^2}, \quad \forall s \in \mathbb{R},$$

for all Lipschitz function F with ν -mean 0.

We conclude with the elementary property

Lemma 2.3. — Assume that $\{X_n\}$ satisfies (2.4) and that $\{\alpha_j, 1 \leq j \leq N\}$ are real numbers such that $\sum_{1 \leq j \leq N} \alpha_j^2 \leq 1$. Then $X := \sum_{1 \leq j \leq N} \alpha_j X_j$ satisfies (2.4) with the same constant C .

Proof. — It is a direct application of (2.5) with $F(X) = \sum_{j=1}^N \alpha_j X_j$. □

2.2. Probabilities on Hilbert spaces. — In this sub-section \mathcal{K} is a separable complex Hilbert space and K is a self-adjoint, positive operator on \mathcal{K} with a compact resolvent. We denote by $\{\varphi_j, j \geq 1\}$ an orthonormal basis of eigenvectors of K , $K\varphi_j = \lambda_j\varphi_j$, and $\{\lambda_j, j \geq 1\}$ is the non decreasing sequence of eigenvalues of K (each is repeated according to its multiplicity). Then we get a natural scale of Sobolev spaces associated with K defined for $s \geq 0$ by $\mathcal{K}^s = \text{Dom}(K^{s/2})$.

Now we want to introduce probability measures on these spaces and on some finite dimensional spaces of \mathcal{K} .

Let us describe in our setting the randomization technique deeply used by Burq-Tzvetkov in [3]. Let $\gamma = \{\gamma_j\}_{j \geq 1}$ a sequence of complex numbers such that $\sum_{j \geq 1} \lambda_j^s |\gamma_j|^2 < +\infty$.

We denote by $v_\gamma^0 = \sum_{j \geq 1} \gamma_j \varphi_j \in \mathcal{K}^s$, and we define the random vector $v_\gamma(\omega) = \sum_{j \geq 1} \gamma_j X_j(\omega) \varphi_j$. We have $\mathbb{E}(\|v_\gamma\|_{\mathcal{K}}^2) < +\infty$, therefore $v_\gamma \in \mathcal{K}^s$, a.s. We define the measure μ_γ on \mathcal{K}^s as the law of the random vector v_γ .

2.2.1. The Kakutani theorem. — The following proposition gives some properties of the measures μ_γ (see [5] for more details).

Proposition 2.4. — *Assume that all random variables X_j have the same law ν .*

- (i) *If the support of ν is \mathbb{R} and if $\gamma_j \neq 0$ for all $j \geq 1$ then the support of μ_γ is \mathcal{K}^s .*
- (ii) *If for some $\varepsilon > 0$ we have $v_\gamma^0 \notin \mathcal{K}^{s+\varepsilon}$ then $\mu_\gamma(\mathcal{K}^{s+\varepsilon}) = 0$.*
- (iii) *Assume that we are in the particular case where $d\nu(x) = c_\alpha e^{-|x|^\alpha} dx$ with $\alpha \geq 2$. Let $\gamma = \{\gamma_j\}$ and $\beta = \{\beta_j\}$ be two complex sequences and assume that*

$$(2.6) \quad \sum_{j \geq 1} \left(\left| \frac{\gamma_j}{\beta_j} \right|^{\alpha/2} - 1 \right)^2 = +\infty.$$

Then the measures μ_γ and μ_β are mutually singular, i.e there exists a measurable set $A \subset \mathcal{H}^s$ such that $\mu_\gamma(A) = 1$ and $\mu_\beta(A) = 0$.

We give the proof of (iii) in Appendix A.

We shall see now that condition (1.2) can be perturbed so that Proposition 2.4 gives us an infinite number of mutually singular measures on \mathcal{K}^s .

Lemma 2.5. — *Let γ satisfying (1.2) and $\delta = \{\delta_n\}_{n \geq 1}$ such that $|\delta_n| \leq \varepsilon |\gamma_n|$ for every $n \geq n_0$. Then for every $\varepsilon \in [0, \sqrt{2} - 1[$, the sequence $\gamma + \delta$ satisfies (1.2) (with a new constant K_ε).*

We do not give the details of the proof. From this Lemma and Proposition 2.4 we get an infinite number of measures μ_γ with γ satisfying (1.2). Let ε_j be any sequence such that $\sum_{j \geq 1} \varepsilon_j^2 = +\infty$ and $\limsup \varepsilon_j < \sqrt{2} - 1$ and denote by $\varepsilon \otimes \gamma$ the sequence $\varepsilon_j \gamma_j$. Then μ_γ and $\mu_{\gamma + \varepsilon \otimes \gamma}$ are mutually singular.

2.2.2. Measures on the sphere \mathbf{S}_h . — Now we consider finite dimensional subspaces \mathcal{E}_h of \mathcal{K} defined by spectral localizations depending on a small parameter $0 < h \leq 1$ (h^{-1} is a measure of energy for the quantum Hamiltonian K). In the sequel, we use the notations $I_h = [\frac{a_h}{h}, \frac{b_h}{h}[$, N_h , Λ_h and \mathcal{E}_h introduced in Section 1.1, and we assume that (1.1) is satisfied. Observe that \mathcal{E}_h is the spectral subspace of K in the interval I_h : $\mathcal{E}_h = \Pi_h \mathcal{K}$ where Π_h is the orthogonal projection on \mathcal{K} . For simplicity, we sometimes denote by $N = N_h$, $\Lambda = \Lambda_h$, \dots , with implicit dependence in h . Our goal is to find uniform estimates in $h \in]0, h_0[$ for a small constant $h_0 > 0$.

Let us consider the random vector in \mathcal{E}_h

$$(2.7) \quad v_\gamma(\omega) := v_{\gamma,h}(\omega) = \sum_{j \in \Lambda} \gamma_j X_j(\omega) \varphi_j,$$

and assume that (1.2) is satisfied. In the sequel we denote by $|\gamma|_\Lambda^2 = \sum_{n \in \Lambda} \gamma_n^2$.

Now we consider probabilities on the unit sphere \mathbf{S}_h of the subspaces \mathcal{E}_h . The random vector v_γ in (2.7) defines a probability measure $\nu_{\gamma,h}$ on \mathcal{E}_h . Then we can define a probability measure $\mathbf{P}_{\gamma,h}$ on \mathbf{S}_h as the image of by $v \mapsto \frac{v}{\|v\|}$. Namely, we have for every Borel and bounded function f on \mathbf{S}_h ,

$$(2.8) \quad \int_{\mathbf{S}_h} f(w) \mathbf{P}_{\gamma,h}(dw) = \int_{\mathcal{E}_h} f\left(\frac{v}{\|v\|_{\mathcal{K}}}\right) \nu_{\gamma,h}(dv) = \int_{\Omega} f\left(\frac{v_\gamma(\omega)}{\|v_\gamma(\omega)\|_{\mathcal{K}}}\right) \mathbb{P}(d\omega).$$

Remark that we have

$$\|v_\gamma(\omega)\|_{\mathcal{K}}^2 = \sum_{j \in \Lambda} |\gamma_j|^2 |X_j(\omega)|^2$$

and

$$\mathbb{E}(\|v_\gamma\|_{\mathcal{K}}^2) = \sum_{j \in \Lambda} |\gamma_j|^2 = |\gamma|_\Lambda^2.$$

Let us detail two particular cases of interest:

- If $|\gamma_n| = \frac{1}{\sqrt{N}}$ for all $j \in \Lambda$ and if X_n follows the complex normal law $\mathcal{N}_{\mathbb{C}}(0, 1)$ then $\mathbf{P}_{\gamma,h}$ is the uniform probability on \mathbf{S}_h considered in [2]. This follows from (2.8) and property of Gaussian laws.

- Assume that for all $n \in \mathbb{N}$, $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = 1/2$, then $\mathbf{P}_{\gamma,h}$ is a convex sum of 2^N Dirac measures. Indeed we have $\|v_\gamma(\omega)\|_{\mathcal{K}}^2 = \sum_{j \in \Lambda} |\gamma_j|^2 = |\gamma|_\Lambda^2$. Denote by $(\varepsilon^{(k)})_{1 \leq k \leq 2^N}$ all the sequences so that $\varepsilon_j^{(k)} = \pm 1$ for all $1 \leq j \leq N$, and set

$$\Phi_k = \frac{1}{|\gamma|} \sum_{j \in \Lambda} \gamma_j \varepsilon_j^{(k)} \varphi_j, \quad 1 \leq k \leq 2^N.$$

Then

$$\mathbf{P}_{\gamma,h} = \frac{1}{2^N} \sum_{k=1}^{2^N} \delta_{\Phi_k}.$$

To get an optimal lower bound for L^∞ estimates we shall need a stronger normal concentration estimate than estimate given in (2.1). Hence we make the following assumptions:

Assumption 2. — *We assume that*

- (i) *The random variables X_j are standard independent Gaussians $\mathcal{N}_{\mathbb{C}}(0, 1)$.*
- (ii) *The sequence γ satisfies (1.2).*

Let L be a linear form on \mathcal{E}_h , and denote by $e_L = \sum_{j \in \Lambda_h} |L(\varphi_j)|^2$. The main result of this section is the following

Theorem 2.6. — *Let L be a linear form on \mathcal{E}_h . Under Assumption 1, there exist $C_1, C_2, c_1, c_2 > 0$ and $\varepsilon_0, h_0 > 0$ so that*

$$(2.9) \quad \mathbf{P}_{\gamma, h} \left[u \in \mathbf{S}_h : |L(u)| \geq t \right] \leq C_2 e^{-c_2 \frac{N}{e_L} t^2}, \quad \forall t \geq 0, \quad \forall h \in]0, h_0],$$

and such that

$$(2.10) \quad C_1 e^{-c_1 \frac{N}{e_L} t^2} \leq \mathbf{P}_{\gamma, h} \left[u \in \mathbf{S}_h : |L(u)| \geq t \right], \quad \forall t \in \left[0, \varepsilon_0 \frac{\sqrt{e_L}}{\sqrt{N}} \right], \quad \forall h \in]0, h_0].$$

Moreover, if Assumption 2 is satisfied, there exist $C_1, C_2, c_1, c_2, \varepsilon_0, h_0 > 0$ so that

$$(2.11) \quad C_1 e^{-c_1 \frac{N}{e_L} t^2} \leq \mathbf{P}_{\gamma, h} \left[u \in \mathbf{S}_h : |L(u)| \geq t \right] \leq C_2 e^{-c_2 \frac{N}{e_L} t^2}, \quad \forall t \in [0, \varepsilon_0 \sqrt{e_L}], \quad \forall h \in]0, h_0].$$

Since $\mathbf{P}_{\gamma, h}$ is supported by \mathbf{S}_h , the bounds in the previous result don't depend on $|\gamma|_\Lambda$. The restriction on $t \geq 0$ in (2.11) is natural, because by the Cauchy-Schwarz inequality we have

$$|L(u)| \leq \sqrt{e_L}, \quad \forall u \in \mathbf{S}_h.$$

In the applications we give, there is some embedding $\mathcal{K}^s \rightarrow C(M)$, for $s > 0$ large enough, where M is a metric space. We have $\mathcal{E} \subseteq \bigcap_{s \in \mathbb{R}} \mathcal{K}^s$, thus we can consider the Dirac evaluation linear form $\delta_x(v) = v(x)$. In this case we have $e_L = \sum_{j \in \Lambda} |\varphi_j(x)|^2 = e_x$, which is usually called the spectral function of K in the interval I .

For example, one can consider the Laplace-Beltrami operator on compact Riemannian manifolds, namely $K = -\Delta$ and $\mathcal{K}^s = \mathcal{H}^s(M)$ are the usual Sobolev spaces: this is the framework of [2]. In Section 3 we will apply the result of Theorem 2.6 to the Harmonic oscillator $K = -\Delta + |x|^2$ on \mathbb{R}^d . In this latter case \mathcal{K}^s is the weighted Sobolev space

$$\mathcal{K}^s = \left\{ u \in \mathcal{H}^s(\mathbb{R}^d), |x|^s u \in L^2(\mathbb{R}^d) \right\}, \quad s \geq 0.$$

Remark 2.7. — In the particular case where $\mathbf{P}_{\gamma, h}$ is the uniform probability on \mathbf{S}_h , we have the explicit explicit computation.

$$\mathbf{P}_{\gamma, h} \left[u \in \mathbf{S}_h : |L(u)| \geq t \right] = \Phi \left(\frac{r}{\sqrt{e_L}} \right),$$

where

$$(2.12) \quad \Phi(t) = \mathbb{I}_{[0,1[}(t)(1-t^2)^{N_h-1},$$

and (2.9) follows directly. For a proof, see [2] or in Appendix C of this paper for an alternative argument.

For the proof of Theorem 2.6 we will need the following result.

Proposition 2.8. — *Let L be a linear form on \mathcal{E}_h . Then we have the large deviation estimate*

$$\mathbb{P}\left[\omega \in \Omega : |L(v_\gamma)| \geq t\right] \leq 4e^{-\kappa_1 \frac{N}{e_L |\gamma|_\Lambda^2} t^2},$$

where $\kappa_1 = \frac{\kappa_0}{4K_1}$. As a consequence, if $\nu_{\gamma,h}$ denotes the probability law of v_γ , then

$$\nu_{\gamma,h}\left[w \in \mathcal{E}_h : |L(w)| \geq t\right] \leq 4e^{-\kappa_1 \frac{N}{e_L |\gamma|_\Lambda^2} t^2}.$$

Proof. — We have

$$L(v_\gamma) = \sum_{j \in \Lambda} \gamma_j X_n(\omega) L(\varphi_j).$$

It is enough to assume that $L(v_\gamma)$ is real and to estimate $\mathbb{P}[L(v_\gamma) \geq t]$. Using the Markov inequality, we have for all $s > 0$

$$\mathbb{P}[L(v_\gamma) \geq t] \leq e^{-st} \mathbb{E}(e^{sL(v_\gamma)}),$$

and thanks (1.2) we have

$$\sum_{j \in \Lambda} |\gamma_j L(\varphi_j)|^2 \leq K_1 \frac{|\gamma|_\Lambda^2}{N} \sum_{j \in \Lambda} |L(\varphi_j)|^2.$$

Using Lemma 2.3 we get

$$\mathbb{P}[L(v_\gamma) \geq t] \leq e^{-st} e^{\kappa_0 K_1 \frac{e_L}{N} |\gamma|_\Lambda^2 s^2},$$

and with the choice $s = \frac{\kappa_0}{2} \frac{tN}{K_1 e_L |\gamma|_\Lambda^2}$ we obtain $\mathbb{P}[L(v_\gamma) \geq t] \leq e^{-\kappa_1 \frac{N}{|\gamma|_\Lambda^2 e_L} t^2}$. \square

It will be useful to show that $\|v_\gamma(\omega)\|_{\mathcal{K}}^2$ is close to its expectation for large N .

Lemma 2.9. — *Let γ satisfying the squeezing condition (1.2). Then there exists $c_0 > 0$ (depending only on K_0 and K_1) such that for every $\varepsilon > 0$*

$$\mathbb{P}\left[\omega \in \Omega : \left|\|v_\gamma(\omega)\|_{\mathcal{K}}^2 - |\gamma|_\Lambda^2\right| > \varepsilon\right] \leq 2e^{-\frac{\varepsilon c_0 N}{|\gamma|_\Lambda^2}}.$$

Proof. — It is enough to consider the real case, so we assume that γ_n and X_n are real and $\{X_n, n \geq 1\}$ have a common law ν . We also assume that $|\gamma|_\Lambda^2 = 1$.

We have

$$\|v_\gamma(\omega)\|_{\mathcal{K}}^2 = \sum_{j \in \Lambda} |\gamma_j|^2 X_j^2(\omega) := M_N(\omega).$$

From large number law, $\|v_\gamma(\omega)\|_{\mathcal{K}}^2$ converges to 1 a.s. To estimate the tail we use the Chernoff large deviation principle. This applies because from (2.3) we know that $f(s) := \mathbb{E}(e^{sX_1^2})$ is \mathcal{C}^2 in $] -\infty, s_0[$ for some $s_0 > 0$.

We reproduce here a well known computation in large deviation theory. Define the cumulant function $g(s) = \log(f(s))$ which is well defined for $s < s_0$. Now, since the X_j are i.i.d., for $t, s \geq 0$ we have

$$\begin{aligned} \mathbb{P}[M_N > t] &= \mathbb{P}[e^{sNM_N} > e^{sNt}] \leq \mathbb{E}(e^{sNM_N})e^{-sNt} \\ &= \prod_{j \in \Lambda} e^{-(Ns|\gamma_j|^2 t - g(Ns|\gamma_j|^2))}. \end{aligned}$$

Next, apply the Taylor formula to g at 0: $g(0) = 0$, $g'(0) = 1$ so $t\tau - g(\tau) = (t-1)\tau + \mathcal{O}(\tau^2)$, hence there exists $s_1 > 0$ such that for $0 \leq \tau \leq s_1$, $t\tau - g(\tau) \geq (t-1)\frac{\tau}{2}$. Then, with $t = 1 + \varepsilon$, and since $N|\gamma_j|^2 \leq K_0$ we get

$$\mathbb{P}[M_N > 1 + \varepsilon] \leq \prod_{j \in \Lambda} e^{-\varepsilon N s |\gamma_j|^2 / 2} = e^{-\varepsilon s N / 2},$$

provided $s > 0$ is small enough, but independent of $\varepsilon > 0$ and $N \geq 1$. The same computation applied to $-M_N$ gives as well $\mathbb{P}[M_N < 1 - \varepsilon] \leq e^{-\varepsilon c_0 N}$. \square

Proof of (2.9). — By homogeneity, we can assume that $|\gamma|_\Lambda = 1$. Denote by

$$(2.13) \quad A = \{ \omega \in \Omega : | \|v_\gamma(\omega)\|_{\mathcal{K}}^2 - 1 | \leq 1/2 \}.$$

By the Cauchy-Schwarz inequality, for all $u \in \mathbf{S}_h$, we obtain $|L(u)| \leq e_L^{1/2}$. Thus in the sequel we can assume that $t \leq e_L^{1/2}$. Then, from Proposition 2.8 and Lemma 2.9 we have

$$(2.14) \quad \begin{aligned} \mathbf{P}_{\gamma,h} [u \in \mathbf{S}_h : |L(u)| \geq t] &= \mathbb{P}[\omega \in \Omega : |L(v(\omega))| \geq t \|v(\omega)\|_{L^2}] \\ &= \mathbb{P}[(|L(v(\omega))| \geq t \|v(\omega)\|_{L^2}) \cap A] + \mathbb{P}[(|L(v(\omega))| \geq t \|v(\omega)\|_{L^2}) \cap A^c]. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{P}_{\gamma,h} [u \in \mathbf{S}_h : |L(u)| \geq t] &\leq \mathbb{P}[|L(v(\omega))| \geq t/2] + \mathbb{P}(A^c) \\ &\leq C_1 e^{-c_1 \frac{N}{e_L} t^2} + 2e^{-c_2 N} \leq C e^{-c \frac{N}{e_L} t^2}, \end{aligned}$$

which implies (2.9). \square

We now turn to the proof of (2.10). To begin with, we state the following elementary lemma

Lemma 2.10. — *Consider a sequence of random variables, so that for all $N \geq 1$, $c_0 \leq \|Y_N\|_2 \leq C_0$ and $\|Y_N\|_{2+\delta} \leq C$ for some $\delta > 0$. Then there exists $\eta > 0$ and $\varepsilon > 0$ so that for all $N \geq 1$*

$$\mathbb{P}(\omega \in \Omega : |Y_N| > \eta) > \varepsilon.$$

Proof. — We proceed by contradiction: Assume that for all $k \geq 1$ and $s > 0$ there exists $N(s, k) \geq 1$ so that $\mathbb{P}(\omega \in \Omega : |Y_{N(s,k)}| > s) \leq 1/k$, and we assume that $N(s, k)$ is minimal for this property. Fix $A > 0$, and let $s > A$. Then $\mathbb{P}(\omega \in \Omega : |Y_{N(A,k)}| > s) \leq \mathbb{P}(\omega \in \Omega : |Y_{N(A,k)}| > A) \leq 1/k$, therefore, we can assume that the sequence $N(s, k)$ is independent of $s \in [A, +\infty)$.

By Markov, for all $s \geq A$, we get

$$s\mathbb{P}(\omega \in \Omega : |Y_{N(A,k)}| > s) \leq s \frac{\|Y_{N(A,k)}\|_{2+\delta}^{2+\delta}}{\langle s \rangle^{2+\delta}} \leq \frac{C}{\langle s \rangle^{1+\delta}} \in L^1(\mathbb{R}),$$

thus by the Lebesgue theorem,

$$\int_A^{+\infty} s\mathbb{P}(\omega \in \Omega : |Y_{N(A,k)}| > s) ds \longrightarrow 0,$$

when $k \longrightarrow +\infty$.

Next, we have

$$\int_0^A s\mathbb{P}(\omega \in \Omega : |Y_{N(A,k)}| > s) ds \leq \int_0^A s ds \leq A^2.$$

We now use the formula

$$c_0^2 \leq \|Y_{N(A,k)}\|_2^2 = 2 \int_0^{+\infty} s\mathbb{P}(\omega \in \Omega : |Y_{N(A,k)}| > s) ds,$$

which yields a contradiction, if A is fixed small enough, and $k \rightarrow +\infty$. \square

As a consequence, we have

Lemma 2.11. — *We suppose that Assumption 1 is satisfied. Then there exist $C_1 > 0, c_1 > 0, h_0 > 0, \varepsilon_0 > 0$ such that*

$$\mathbb{P}\left[\omega \in \Omega : |L(v_\gamma(\omega))| \geq t\right] \geq C_1 e^{-c_1 \frac{N}{e_L |\gamma|_\Lambda} t^2}, \quad \forall t \in [0, \varepsilon_0 \frac{\sqrt{e_L} |\gamma|_\Lambda}{\sqrt{N}}], \quad \forall h \in]0, h_0].$$

Proof. — We apply the result of Lemma 2.10 to the random variable

$$Y_N = \frac{\sqrt{N}}{\sqrt{e_L} |\gamma|_\Lambda} L(v_\gamma) = \frac{\sqrt{N}}{\sqrt{e_L} |\gamma|_\Lambda} \sum_{j \in \Lambda} \gamma_j X_j L(\varphi_j).$$

By (1.2), we clearly have $c_0 \leq \|Y_N\|_2 \leq C_0$. Then, by Khinchin, the condition $\|Y_N\|_{2+\delta} \leq C$ is also satisfied. \square

Proof of (2.10). — We assume that $|\gamma|_\Lambda = 1$, and consider the set A defined in (2.13). Then by (2.14) and the inequality $\mathbb{P}(B \cap A) \geq \mathbb{P}(B) - \mathbb{P}(A^c)$ we get

$$\begin{aligned} \mathbf{P}_{\gamma,h}\left[u \in \mathbf{S}_h : |L(u)| \geq t\right] &\geq \mathbb{P}\left[(|L(v(\omega))| \geq t \|v(\omega)\|_{L^2}) \cap A \right] \\ &\geq \mathbb{P}\left[|L(v(\omega))| \geq 3t/2 \right] - \mathbb{P}(A^c) \\ &\geq C_1 e^{-c_1 \frac{N}{e_L} t^2} - 2e^{-c_2 N}, \end{aligned}$$

where in the last line we used Lemma 2.11 and Lemma 2.9. This yields the result if $t \leq \varepsilon_0 \frac{\sqrt{e_L}}{\sqrt{N}}$ with $\varepsilon_0 > 0$ small enough. \square

We now prove (2.11). To begin with, we can state

Lemma 2.12. — *We suppose that Assumption 2 is satisfied. Then there exist $C_1 > 0, c_1 > 0, h_0 > 0, \varepsilon_0 > 0$ such that*

$$\mathbb{P}\left[\omega \in \Omega : |L(v_\gamma(\omega))| \geq t\right] \geq C_1 e^{-c_1 \frac{N}{e_L |\gamma|_\Lambda^2} t^2}, \quad \forall t \geq 0, \forall h \in]0, h_0].$$

Proof. — Denote by $\gamma \otimes L(\varphi)$ the vector $(\gamma \otimes L(\varphi))_j = \gamma_j L(\varphi_j)$. Observe that, thanks to (1.2),

$$K_1 \frac{|\gamma|_\Lambda^2 e_L}{N} \leq |\gamma \otimes L(\varphi)|^2 = \sum_{j \in \Lambda_h} \gamma_j^2 |L(\varphi_j)|^2 \leq K_0 \frac{|\gamma|_\Lambda^2 e_L}{N}.$$

Then, using the rotation invariance of the Gaussian law and the previous line, we get

$$\begin{aligned} \mathbb{P}\left[\omega \in \Omega : |L(v_\gamma(\omega))| \geq t\right] &= \mathbb{P}\left[\left|\left\langle \frac{\gamma \otimes L(\varphi)}{|\gamma \otimes L(\varphi)|}, X \right\rangle\right| \geq \frac{t}{|\gamma \otimes L(\varphi)|}\right] \\ &= \frac{1}{\sqrt{2\pi}} \int_{|s| \geq \frac{t}{|\gamma \otimes L(\varphi)|}} e^{-s^2/2} ds \\ &\geq C e^{-\frac{cN}{e_L |\gamma|_\Lambda^2} t^2}. \end{aligned}$$

\square

The estimate (2.11) then follows from Lemma 2.12 and with the same argument as for Lemma 2.11.

2.2.3. Concentration phenomenon. — We now state a concentration property for $\mathbf{P}_{\gamma,h}$, inherited from Assumption 1. See [13] for more details on this topic.

Proposition 2.13. — *Suppose that Assumption 1 is satisfied. Then there exist constants $K > 0, \kappa > 0$ (depending only on C^*) such that for every Lipschitz function $F : \mathbf{S}_h \rightarrow \mathbb{R}$ satisfying*

$$|F(u) - F(v)| \leq \|F\|_{Lip} \|u - v\|_{L^2(\mathbb{R}^d)}, \quad \forall u, v \in \mathbf{S}_h,$$

we have

$$(2.15) \quad \mathbf{P}_{\gamma,h}\left[u \in \mathbf{S}_h : |F - \mathcal{M}_F| > r\right] \leq K e^{-\frac{\kappa N r^2}{\|F\|_{Lip}^2}}, \quad \forall r > 0, h \in]0, 1],$$

where \mathcal{M}_F is a median for F .

Recall that a median \mathcal{M}_F for F is defined by

$$\mathbf{P}_{\gamma,h}\left[u \in \mathbf{S}_h : F \geq \mathcal{M}_F\right] \geq \frac{1}{2}, \quad \mathbf{P}_{\gamma,h}\left[u \in \mathbf{S}_h : F \leq \mathcal{M}_F\right] \geq \frac{1}{2}.$$

In Proposition 2.13, the distance in L^2 can be replaced with the geodesic distance d_S on \mathbf{S}_h , since we can check that

$$\|u - v\|_{L^2(\mathbb{R}^d)} \leq d_S(u, v) = 2 \arcsin\left(\frac{\|u - v\|_{L^2(\mathbb{R}^d)}}{2}\right) \leq \frac{\pi}{2} \|u - v\|_{L^2(\mathbb{R}^d)}.$$

When $\mathbf{P}_{\gamma,h}$ is the uniform probability on \mathbf{S}_h , Proposition 2.13 is proved in [13, Proposition 2.10], and the proof can be adapted in the general case (see Appendix D). The factor N in the exponential of r.h.s of (2.15) will be crucial in our application.

3. Some spectral estimates for the harmonic oscillator

Our goal here is to apply the general setting of Section 2 to the harmonic oscillator in \mathbb{R}^d . This way we shall get probabilistic estimates analogous to results proved in [2] for the Laplace operator in a compact Riemannian manifold.

In the following, we consider the Hamiltonian $H = -\Delta + V(x)$ with $V(x) = |x|^2$, $x \in \mathbb{R}^d$ for $d \geq 2$. For this model, all the necessary spectral estimates are already known. More general confining potentials V shall be considered in the forthcoming paper [16].

A first and basic ingredient in probabilistic approaches of weighted Sobolev spaces is a good knowledge concerning the asymptotic behavior of eigenvalues and eigenfunctions of H . The eigenvalues of this operator are the $\{2(j_1 + \dots + j_d) + d, j \in \mathbb{N}^d\}$, and we can order them in a non decreasing sequence $\{\lambda_j, j \in \mathbb{N}\}$, repeated according to their multiplicities. We denote by $\{\varphi_j, j \in \mathbb{N}\}$ an orthonormal basis in $L^2(\mathbb{R}^d)$ of eigenfunctions (the Hermite functions), so that $H\varphi_j = \lambda_j\varphi_j$. The spectral function is then defined as $\pi_H(\lambda; x, y) = \sum_{\lambda_j \leq \lambda} \varphi_j(x)\overline{\varphi_j(y)}$ (recall that this definition does not depend on the choice of $\{\varphi_j, j \in \mathbb{N}\}$). When the energy λ is localized in $I \subseteq \mathbb{R}^+$ we denote by $\Pi_H(I)$ the spectral projector of H on I . The range $\mathcal{E}_H(I)$ of $\Pi_H(I)$ is spanned by $\{\varphi_j; \lambda_j \in I\}$ and $\Pi_H(I)$ has an integral kernel given by

$$\pi_H(I; x, y) = \sum_{[j: \lambda_j \in I]} \varphi_j(x)\overline{\varphi_j(y)}.$$

We will also use the notation $\mathcal{E}_H(\lambda) = \mathcal{E}_H([0, \lambda])$, $N_H(\lambda) = \dim[\mathcal{E}_H(\lambda)]$.

3.1. Interpolation inequalities. — We begin with some general interpolation results which will be needed in the sequel. In \mathbb{R}^d , the spectral function $\pi_H(\lambda; x, x)$ is fast decreasing for $|x| \rightarrow +\infty$ so it is natural to work with weighted L^p norms. We denote by $\langle x \rangle^s = (1 + |x|^2)^{s/2}$ and introduce the following Lebesgue space with weight

$$L^{p,s}(\mathbb{R}^d) = \left\{ u, \text{ Lebesgue measurable} : \int |u(x)|^p \langle x \rangle^s dx < +\infty \right\} = L^p(\mathbb{R}^d, \langle x \rangle^s dx),$$

endowed with its natural norm, which we denote by $\|u\|_{p,s}$. For $p = \infty$, we set $\|u\|_{\infty,s} = \sup_{x \in \mathbb{R}^d} \langle x \rangle^s |u(x)|$.

The following interpolation inequalities hold true. Let $1 \leq p_1 \leq p \leq p_0 \leq +\infty$ and $\kappa \in]0, 1[$ such that $\frac{1}{p} = \frac{\kappa}{p_1} + \frac{1-\kappa}{p_0}$. Then for $p_0 < +\infty$ we have

$$(3.1) \quad \|u\|_{L^{p,s}(\mathbb{R}^d)} \leq (\|u\|_{L^{p_0,s_0}(\mathbb{R}^d)})^{1-\kappa} (\|u\|_{L^{p_1,s_1}(\mathbb{R}^d)})^\kappa, \quad \text{with} \quad s = \frac{p_1 - p}{p_1 - p_0} s_0 + \frac{p_0 - p}{p_0 - p_1} s_1.$$

In the case $p_0 = +\infty$, we have

$$(3.2) \quad \|u\|_{L^{p,s}(\mathbb{R}^d)} \leq \left(\sup_{\mathbb{R}^d} \langle x \rangle^{s_0} |u(x)| \right)^{1-p_1/p} (\|u\|_{L^{p_1,s_1}(\mathbb{R}^d)})^{p_1/p}, \quad \text{with } s = (p - p_1)s_0 + s_1.$$

3.2. Rough estimates of the harmonic oscillator. — We recall here some more or less standard properties stated in [10]. To begin with, we state a "soft" Sobolev inequality.

Lemma 3.1. — *For all $u \in \mathcal{E}_H(I)$*

$$(3.3) \quad |u(x)| \leq (\pi_H(I; x, x))^{1/2} \|u\|_{L^2(\mathbb{R}^d)}.$$

Proof. — We have

$$u(x) = \Pi u(x) = \int_{\mathbb{R}^d} \pi_H(I; x, y) u(y) dy.$$

Using the Cauchy-Schwarz inequality

$$(3.4) \quad |u(x)| \leq \left(\int_{\mathbb{R}^d} |\pi_H(I; x, y)|^2 dy \right)^{1/2} \|u\|_{L^2(\mathbb{R}^d)}.$$

Now we use that $\Pi_H(I)$ is an orthonormal projector.

$$(3.5) \quad \pi_H(I; x, y) = \int_{\mathbb{R}^d} \pi_H(I; x, z) \pi_H(I; z, y) dz \quad \text{and} \quad \pi_H(I; x, y) = \overline{\pi_H(I; y, x)}.$$

Finally, from (3.4) and (3.5) with $y = x$ we get (3.3). □

The next result gives a bound on π_H .

Lemma 3.2. — *The following bound holds true*

$$(3.6) \quad \pi_H(\lambda; x, x) \leq C \lambda^{d/2} \exp\left(-c \frac{|x|^2}{\lambda}\right), \quad \forall x \in \mathbb{R}^d, \lambda \geq 1.$$

Proof. — Let $K(t; x, y)$ be the heat kernel of e^{-tH} . It is given by the following Mehler formula ⁽¹⁾

$$(3.7) \quad K(t; x, y) = (2\pi \sinh 2t)^{-d/2} \exp\left(-\frac{\tanh t}{4} |x + y|^2 - \frac{|x - y|^2}{4 \tanh t}\right).$$

So we have

$$(3.8) \quad K(t; x, x) = \int_{\mathbb{R}} e^{-t\mu} d\pi_H(\mu; x, x) = (2\pi \sinh 2t)^{-d/2} \exp(-|x|^2 \tanh t).$$

We set $t = \lambda^{-1}$, integrate in μ on $[0, \lambda]$ and get

$$\pi_H(\lambda; x, x) \leq eK(\lambda^{-1}; x, x).$$

Assuming $\lambda \geq \lambda_0$, λ_0 large enough, we easily see that (3.6) is a consequence of (3.8). □

⁽¹⁾The Mehler formula can also be obtained from Fourier transform computation of the Weyl symbol of e^{-tH} (see [17] Exercise IV-2).

Let $u \in \mathcal{E}_H(\lambda)$. From (3.3) and (3.6) we get

$$|u(x)| \leq C\lambda^{d/4} \exp\left(-c\frac{|x|^2}{2\lambda}\right) \|u\|_{L^2(\mathbb{R}^d)},$$

where $c, C > 0$ do not depend on $x \in \mathbb{R}^d$ nor $\lambda \geq 1$.

Remark 3.3. — From (3.6), we can deduce that for every $\theta > 0$ there exists $C_\theta > 0$ such that

$$\pi_H(\lambda; x, x) \leq C_\theta \lambda^{(d+\theta)/2} \langle x \rangle^{-\theta},$$

which by (3.3) implies with the semiclassical parameter $h = \lambda^{-1}$

$$\langle x \rangle^{\theta/2} h^{(d+\theta)/4} |u(x)| \leq C_\theta \|u\|_{L^2(\mathbb{R}^d)}, \quad \forall u \in \mathcal{E}_H(h^{-1}).$$

We can easily see that this uniform estimate is true for $u \in \mathcal{E}(I_h)$ where $I_h = [\frac{a}{h}, \frac{b}{h}]$ with $a < b$. For smaller energy intervals we can get much better estimates, as we will see in Lemma 3.5.

Remark 3.4. — Let us compare the previous results with the case of a compact Riemannian manifold M , and when $H = -\Delta$ is the Laplace operator. We have the uniform Hörmander estimate [7]:

$$(3.9) \quad \pi_H(\lambda; x, x) = c_d(x) \lambda^{d/2} + \mathcal{O}(\lambda^{(d-1)/2}),$$

where $0 < c_d(x)$ is a continuous function on M . Thus from (3.4) and (3.9) we get for some constant $C_S > 0$,

$$\|u\|_{L^\infty(M)} \leq C_S \lambda^{d/4} \|u\|_{L^2(M)}, \quad \forall u \in \mathcal{E}(\lambda).$$

Let us emphasize here that it results from the uniform Weyl law (3.9) that $\pi_H(\lambda; x, x)$ has an upper bound and a lower bound of order $\lambda^{d/2}$. For confining potentials like V the behavior of $\pi_H(\lambda; x, x)$ is much more complicated because of the turning points: $\{|x|^2 = \lambda\}$. This behavior was analyzed in [10].

3.3. More refined estimates for the spectral function. — From the Weyl law for the harmonic oscillator we have

$$N_H(\lambda) = c_d \lambda^d + \mathcal{O}(\lambda^{d-1}), \quad c_d > 0,$$

we deduce that if (1.1) is satisfied with $\delta = 1$ then we have

$$(3.10) \quad \alpha h^{-d} (b_h - a_h) \leq N_h \leq \beta h^{-d} (b_h - a_h), \quad \alpha > 0, \beta > 0.$$

The main result of this section is the following lemma. It is a consequence of the work of Karadzhov [10]. It could also be deduced from the papers by Koch, Tataru and Zworski ([11, 12]) and it is also related, after rescaling, with results obtained by Ivrii [8, Theorem 4.5.4].

Lemma 3.5. — *Let $d \geq 2$ and assume that $|\mu| \leq c_0$, $p \geq 1$ and $\theta \geq 0$. Then there exists $C > 0$ so that for all $\lambda \geq 1$*

$$\|\pi_H(\lambda + \mu; x, x) - \pi_H(\lambda; x, x)\|_{L^{p, (p-1)\theta}(\mathbb{R}^d)} \leq C\lambda^\alpha,$$

with $\alpha = \frac{d}{2}(1 + \frac{1}{p}) - 1 + \frac{\theta}{2}(1 - \frac{1}{p})$.

Proof. — Recall the following estimates proved in [10, Theorem 4]: For $d \geq 2$ and $x \in \mathbb{R}$

$$(3.11) \quad |\pi_H(\lambda + \mu; x, x) - \pi_H(\lambda; x, x)| \leq C\lambda^{d/2-1}, \quad \lambda \geq 1, |\mu| \leq 1.$$

and for every $\varepsilon_0 > 0$ and every $N \geq 1$ there exists $C_{\varepsilon_0, N}$ such that

$$(3.12) \quad \pi_H(\lambda; x, x) \leq C_{\varepsilon_0, N}|x|^{-N}, \quad \text{for } |x|^2 \geq (1 + \varepsilon_0)\lambda.$$

From (3.11) we get that for every $C_0 > 0$ there exists $C > 0$ such that

$$(3.13) \quad |\pi_H(\lambda + \mu; x, x) - \pi_H(\lambda; x, x)| \leq C(1 + |\mu|)\lambda^{d/2-1}, \quad \lambda \geq 1, |\mu| \leq C_0\lambda.$$

Then from (3.13) and (3.12) we get that for every $\theta \geq 0$ there exists C such that

$$(3.14) \quad |\pi_H(\lambda + \mu; x, x) - \pi_H(\lambda; x, x)| \leq C(1 + |\mu|)\lambda^{d/2-1+\theta/2}\langle x \rangle^{-\theta}, \quad \lambda \geq 1, |\mu| \leq C_0\lambda.$$

Therefore, by (3.12), to get the result of Lemma 3.5, it is enough to integrate the previous inequality on $|x| \leq c_0\lambda^{1/2}$. \square

From (3.14), we easily get an accurate estimate for the spectral function

$$e_x = \pi_H\left(\frac{b_h}{h}; x, x\right) - \pi_H\left(\frac{a_h}{h}; x, x\right).$$

Lemma 3.6. — Assume that (1.1) is satisfied with $0 < \delta \leq 1$. For any $\theta \geq 0$ there exists $C > 0$ such that

$$(3.15) \quad \langle x \rangle^\theta e_x \leq CN_h h^{(d-\theta)/2}.$$

Using (3.3) and interpolation inequalities we get Sobolev type inequalities for $u \in \mathcal{E}_h$, $\theta \geq 0$, $p \geq 2$.

$$(3.16) \quad \|u\|_{L^\infty, \theta/2(\mathbb{R}^d)} \leq C \left(N_h h^{(d-\theta)/2}\right)^{1/2} \|u\|_{L^2(\mathbb{R}^d)},$$

which in turn implies, by (3.1)

$$(3.17) \quad \|u\|_{L^{p, \theta(p/2-1)}(\mathbb{R}^d)} \leq C \left(N_h h^{(d-\theta)/2}\right)^{\frac{1}{2}-\frac{1}{p}} \|u\|_{L^2(\mathbb{R}^d)}.$$

By (3.10), the previous inequality can be written as

$$\|u\|_{L^{p, \theta(p/2-1)}(\mathbb{R}^d)} \leq C(b_h - a_h)^{\frac{1}{2}-\frac{1}{p}} h^{-(\frac{d+\theta}{2})(\frac{1}{2}-\frac{1}{p})} \|u\|_{L^2(\mathbb{R}^d)}, \quad \forall p \in [2, +\infty], \forall \theta \in [0, d].$$

Remark 3.7. — For similar bounds for eigenfunctions or quasimodes, we refer to [12].

4. Probabilistic weighted Sobolev estimates

We apply here the general probabilistic setting of Section 2 when $K = H$ is the harmonic oscillator, $\mathcal{K} = L^2(\mathbb{R}^d)$ and $\{\varphi_j, j \in \mathbb{N}\}$ an orthonormal basis of Hermite functions. Recall that \mathbf{S}_h is the unit sphere of the complex Hilbert space \mathcal{E}_h , identified with \mathbb{C}^N or \mathbb{R}^{2N} , and that $\mathbf{P}_{\gamma, h}$ is the probability on \mathbf{S}_h defined as in Section 2.

We divide this section in two parts: in the first part, under Assumption 1, we establish upper bounds and in the second part we obtain lower bounds, but only in the case of Gaussian random variables (Assumption 2), and under the condition $0 \leq \delta < 2/3$.

4.1. Upper bounds. — We suppose here that Assumption 1 and (1.1) with $0 \leq \delta \leq 1$ are satisfied. Our result is the following

Theorem 4.1. — *There exists $h_0 \in]0, 1]$, $c_2 > 0$ and $C > 0$ such that if $c_1 = d(1 + d/4)$, we have*

$$(4.1) \quad \mathbf{P}_{\gamma, h} \left[u \in \mathbf{S}_h : h^{-\frac{d-\theta}{4}} \|u\|_{L^\infty, \theta/2(\mathbb{R}^d)} > \Lambda \right] \leq Ch^{-c_1} e^{-c_2 \Lambda^2}, \quad \forall \Lambda > 0, \forall h \in]0, h_0].$$

Proof. — We adapt here the argument of [2]. To begin with, by (3.15) and (2.9), there exists $c_2 > 0$ such that for every $\theta \in [0, d]$, every $x \in \mathbb{R}^d$, and every $\Lambda > 0$ we have

$$(4.2) \quad \mathbf{P}_{\gamma, h} \left[u \in \mathbf{S}_h : \langle x \rangle^{\frac{\theta}{2}} h^{-\frac{d-\theta}{4}} |u(x)| > \Lambda \right] \leq e^{-c_2 \Lambda^2}.$$

Now, we will need a covering argument. Our configuration space is not compact but using (3.12) we have, for every $u \in \mathbf{S}_h$,

$$|u(x)| \leq C_N |x|^{-N}, \quad \text{for } |x| \geq (1 + \varepsilon_0)h^{-1/2}.$$

So choosing $R > 0$ large enough it is sufficient to estimate u inside the box $B_{R_h} = \{x \in \mathbb{R}^d, |x|_\infty \leq Rh^{-1/2}\}$. We divide B_{R_h} in small boxes of side with length τ small enough. We use the gradient estimate

$$|\nabla_x u(x)| \leq Ch^{-1/2-d/4}, \quad \forall u \in \mathbf{S}_h,$$

and (4.2) at the center of each small box to get the result.

For $x, x' \in \mathbb{R}^d$ we have

$$|\langle x \rangle^{\theta/2} u(x) - \langle x' \rangle^{\theta/2} u(x')| \leq C(\langle x \rangle^{\theta/2} |u(x) - u(x')| + \langle x \rangle^{\theta/2} |x - x'| |u(x')|).$$

Let $\{Q_\tau\}_{\tau \in A}$ be a covering of B_{R_h} with small boxes Q_τ with center x_τ and side length τ small enough. Then for every $x \in Q_\tau$ we have

$$(4.3) \quad h^{(\theta-d)/4} |\langle x \rangle^{\theta/2} u(x) - \langle x_\tau \rangle^{\theta/2} u(x_\tau)| \leq C\tau h^{-1/2-d/4}.$$

We choose

$$(4.4) \quad \tau \approx \frac{\varepsilon \Lambda}{2C} h^{1/2+d/4}$$

and $h_\varepsilon > 0$ such that

$$(4.5) \quad |x|_\infty > Rh^{-1/2} \Rightarrow h^{(\theta-d)/4} \langle x \rangle^{\theta/2} |u(x)| \leq \frac{\varepsilon \Lambda}{2}, \quad \forall h \in]0, h_\varepsilon].$$

Then using (4.2), (4.3), (4.4) and (4.5) we get

$$(4.6) \quad \mathbf{P}_{\gamma, h} \left[u \in \mathbf{S}_h : h^{-\frac{d-\theta}{4}} \|u\|_{L^\infty, \theta/2(\mathbb{R}^d)} > \Lambda \right] \leq \#A e^{-c_2(1-\varepsilon)^2 \Lambda^2}, \quad \forall \Lambda > 0, \forall h \in]0, h_\varepsilon].$$

Using now that $\#A \approx Ch^{-c_1}$ with $c_1 = d(1 + d/4)$ we get (4.1) from (4.6). \square

We can deduce probabilistic estimates for the derivatives as well. Recall that the Sobolev spaces $\mathcal{W}^{s,p}(\mathbb{R}^d)$ are defined in (1.3).

Corollary 4.2. — For any multi index $\alpha, \beta \in \mathbb{N}^d$ there exists \tilde{c}_2 such that

$$\mathbf{P}_{\gamma, h} \left[u \in \mathbf{S}_h : h^{\frac{|\alpha|+|\beta|}{2}-\frac{d}{4}} \|x^\alpha \partial_x^\beta u\|_{L^\infty(\mathbb{R}^d)} > \Lambda \right] \leq Ch^{-c_1} e^{-\tilde{c}_2 \Lambda^2}, \quad \forall \Lambda > 0, \forall h \in]0, h_0].$$

In particular we have, for every $s > 0$,

$$\mathbf{P}_{\gamma, h} \left[u \in \mathbf{S}_h : h^{\frac{s}{2}-\frac{d}{4}} \|u\|_{W^{s, \infty}(\mathbb{R}^d)} > \Lambda \right] \leq Ch^{-c_1} e^{-\tilde{c}_2 \Lambda^2}, \quad \forall \Lambda > 0, \forall h \in]0, h_0].$$

Proof. — We apply (4.1) using that from the spectral localization of $u \in \mathcal{E}_h$ we have

$$\|x^\alpha \partial_x^\beta u\|_{L^2(\mathbb{R}^d)} \leq Ch^{-\frac{|\alpha|+|\beta|}{2}} \|u\|_{L^2(\mathbb{R}^d)},$$

$$\|H^s u\|_{L^2(\mathbb{R}^d)} \leq Ch^{-s/2} \|u\|_{L^2(\mathbb{R}^d)}.$$

□

The following corollary shows that we get a probabilistic Sobolev estimate improving the deterministic one (3.16) with probability close to one as $h \rightarrow 0$. The improvement is "almost" of order $N_h^{1/2}$ or $((b_h - a_h)h^{-d})^{1/2}$. Choosing $\Lambda = \sqrt{-K \log h}$ for $K > 0$ we get

Corollary 4.3. — Let $c_1, c_2 > 0$ be the constants given by Theorem 4.1. Then for every $K > \frac{c_1}{c_2}$ we have

$$\mathbf{P}_{\gamma, h} \left[u \in \mathbf{S}_h : \|u\|_{L^{\infty, \theta/2}(\mathbb{R}^d)} > Kh^{\frac{d-\theta}{4}} |\log h|^{1/2} \right] \leq h^{Kc_2 - c_1}, \quad \forall h \in]0, h_0], \forall \theta \in [0, d].$$

$$\mathbf{P}_{\gamma, h} \left[u \in \mathbf{S}_h : \|u\|_{W^{s, \infty}(\mathbb{R}^d)} > Kh^{\frac{d}{4}-\frac{s}{2}} |\log h|^{1/2} \right] \leq h^{Kc_2 - c_1}, \quad \forall h \in]0, h_0], \forall s \geq 0.$$

Let us give now an application to a probabilistic Sobolev embedding for the Harmonic oscillator. We shall use a Littlewood-Paley decomposition with $h_j = 2^{-j}$. Let φ a C^∞ real function on \mathbb{R} such that $\varphi(t) = 0$ for $t \leq a$, $\varphi(t) = 1$ for $t \leq b/2$ with $0 < a < b/2$. Define $\psi_{-1}(t) = 1 - \varphi(t)$, $\psi_j(t) = \varphi(h_j t) - \varphi(h_{j+1} t)$ for $j \in \mathbb{N}$. Notice that the support of ψ_j is in $[\frac{a}{h_j}, \frac{b}{h_j}]$. For every distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ we have the Littlewood-Paley decomposition

$$u = \sum_{j \geq -1} u_j, \quad \text{with} \quad u_j = \sum_{k \in \mathbb{N}} \psi_j(\lambda_k) \langle u, \varphi_k \rangle \varphi_k$$

and we have $u_j \in \mathcal{E}_{h_j}$.

The Besov spaces for the Harmonic are naturally defined as follows: if $p, r \in [1, \infty]$ and $s \in \mathbb{R}$, $u \in \mathcal{B}_{p, r}^s$ if and only if

$$\|u\|_{\mathcal{B}_{p, r}^s} := \left(\sum_{j \geq -1} 2^{jsr/2} \|u_j\|_{L^p(\mathbb{R}^d)}^r \right)^{1/r} < +\infty.$$

We shall use here the spaces $\mathcal{B}_{2, \infty}^s$. For every $s > 0$ we have

$$\mathcal{B}_{2, \infty}^s \subseteq L^2(\mathbb{R}^d) \subseteq \mathcal{B}_{2, \infty}^0.$$

Another scale of spaces is defined as

$$\mathcal{G}^m = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : \sum_{j \geq 1} j^m \|u_j\|_{L^2(\mathbb{R}^d)} < +\infty \right\}, \quad m \geq 0.$$

Then for every $s > 0$, $m \geq 0$ we have $\mathcal{B}_{2,\infty}^s \subseteq \mathcal{G}^m \subseteq L^2(\mathbb{R}^d)$.

It is not difficult to see that \mathcal{G}^m can be compared with the domain in $L^2(\mathbb{R}^d)$ of the operator $\log^s H$. This domain is denoted by \mathcal{H}_{\log}^s , the norm being the graph norm. For every $s > 1/2$ we have

$$\mathcal{H}_{\log}^{m+s} \subset \mathcal{G}^m \subset \mathcal{H}_{\log}^m.$$

Notice that we do not need that the energy localizations ψ_j are smooth and we can define the same spaces with $\psi(t) = \mathbb{I}_{[1,2]}(t)$ so that the energy intervals $[2^j, 2^{j+1}]$ are disjoint.

Let us now define probabilities on \mathcal{G}^m as we did for Sobolev spaces \mathcal{H}^s . Let γ_j be a sequence of complex numbers satisfying (1.2) and such that

$$(4.7) \quad \sum_{j \geq 0} j^m |\gamma_{\Lambda_j}| < +\infty,$$

where $\Lambda_j = \Lambda_{h_j}$ and

$$v_\gamma^0 = \sum_{j \geq 0} \gamma_j \varphi_j, \quad v_\gamma(\omega) = \sum_{j \geq 0} \gamma_j X_j(\omega) \varphi_j,$$

so that v_γ is a.s. in \mathcal{G}^m and its probability law defines a measure μ_γ^m in \mathcal{G}^m . This measure satisfies also the following properties as in Proposition 2.4.

- (i) If the support of ν is \mathbb{R} and if $\gamma_j \neq 0$ for all $j \geq 1$ then the support of μ_γ^m is \mathcal{G}^m .
- (ii) If $u_\gamma^0 \in \mathcal{G}^m$ and $v_\gamma^0 \notin \mathcal{G}^s$ where $s > m$ then $\mu_\gamma^m(\mathcal{G}^s) = 0$. In particular $\mu_\gamma^m(\mathcal{H}^s) = 0$ for every $s > 0$.
- (iii) Under the assumptions (iii) in Proposition 2.4 we can construct singular measures μ_γ^m and μ_β^m .

Now we can state the following corollary of Theorem 4.1.

Corollary 4.4. — *Suppose that γ satisfies (1.2) with $a < b$ and (4.7) with $m = 1/2$. Then for the measure $\mu_\gamma^{1/2}$ almost all functions in the space $\mathcal{G}^{1/2}$ are in the space $\mathcal{C}_H^{[d/2]}$ where*

$$\mathcal{C}_H^\ell(\mathbb{R}^d) = \left\{ u \in \mathcal{C}^\ell(\mathbb{R}^d) : \|x^\alpha \partial_x^\beta u\|_{L^\infty(\mathbb{R}^d)} < +\infty, \quad \forall |\alpha| + |\beta| \leq \ell \right\}.$$

In particular if $v_\gamma^0 \in \mathcal{H}^{s_0}$, $s_0 > 0$ and if $v_\gamma^0 \notin \mathcal{H}^s$, $s > s_0$, then we have $\mu_\gamma^{1/2}(\mathcal{B}_{2,\infty}^\sigma) = 1$ for every $\sigma > 0$ and we have an a.s. embedding of the Besov space $\mathcal{B}_{2,\infty}^\sigma$ in $\mathcal{C}_H^{[d/2]}$.

Proof. — Let $u = \sum_{n \geq -1} u_n \in \mathcal{G}^{1/2}$ with $u_n \in \mathcal{E}_{h_n}$. For $\kappa > 0$ (chosen large enough) denote by

$$B_n^\kappa = \left\{ v \in \mathcal{E}_{h_n} : \|x^\alpha \partial_x^\beta v\|_{L^\infty(\mathbb{R}^d)} \leq \kappa \sqrt{n} \|v\|_{L^2(\mathbb{R}^d)}, \quad \forall |\alpha| + |\beta| \leq [d/2] \right\}.$$

We have, using Corollary 4.2

$$\nu_{\gamma,n}(B_n^\kappa) \geq 1 - e^{-n(c_2\kappa^2 - c_1)}.$$

So if $B^\kappa = \{u \in \mathcal{G}^{1/2} : u_0 \in \mathcal{E}_{h_0}, u_n \in B_n^\kappa, \forall n \geq 1\}$, then we have

$$\mu_\gamma^{1/2}(B^\kappa) \geq \prod_{n \geq 1} (1 - e^{-n(c_2\kappa^2 - c_1)}) \geq 1 - \varepsilon(\kappa)$$

with $\lim_{\kappa \rightarrow +\infty} \varepsilon(\kappa) = 0$. More precisely we have $\varepsilon(\kappa) \approx e^{-c\kappa^2}$ for some $c > 0$.

Now if $u \in B^\kappa$ we have

$$\|x^\alpha \partial_x^\beta u\|_{L^\infty(\mathbb{R}^d)} \leq \sum_{n \geq -1} \|x^\alpha \partial_x^\beta u_n\|_{L^\infty(\mathbb{R}^d)} \leq \kappa \sum_{n \geq -1} \sqrt{n} \|u_n\|_{L^2(\mathbb{R}^d)} := \kappa \|u\|_{\mathcal{G}^{1/2}}.$$

So the corollary is proved. \square

Remark 4.5. — In the last corollary, for every $s > 0$ we can choose γ such that $\mu_\gamma^{1/2}(\mathcal{H}^s) = 0$. So the smoothing property is a probabilistic effect similar to the Khinchin inequality.

From the proof we get a more quantitative statement. There exists $c > 0$ such that

$$\mu_\gamma^{1/2} \left[\|u\|_{\mathcal{W}^{d/2,\infty}} \geq \kappa \|u\|_{\mathcal{G}^{1/2}} \right] \leq e^{-c\kappa^2}.$$

Remark 4.6. — The proof of the corollary depends on the squeezing assumption (1.2) on γ . For example if (1.2) is satisfied for $b_h - a_h \approx h$ then we can consider the energy decomposition in intervals $[2n, 2(n+1)[$ instead of the dyadic decomposition. So when applying Theorem 4.1 with h of order $\frac{1}{n}$ we get $h^{-c_1} e^{-c_2 \Lambda^2} = e^{c_1 \log n - c_2 \Lambda^2}$.

Then taking $\Lambda = \kappa \sqrt{\log n}$ with κ large enough, in the construction of B_n^κ we have to replace \sqrt{n} by $\sqrt{\log n}$. In the conclusion the space $\mathcal{G}^{1/2}$ is replaced by $\tilde{\mathcal{G}}^{1/2}$ where

$$\tilde{\mathcal{G}}^m = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : \sum_{j \geq 1} \log^m j \|u_j\|_{L^2(\mathbb{R}^d)} < +\infty \right\}, \quad u_j := \sum_{2j \leq \lambda_n < 2(j+1)} \langle u, \varphi_j \rangle \varphi_j.$$

4.2. Lower bounds in the case of Gaussian random variables. — Here we suppose that the stronger Assumption 2 and (1.1) with $\delta < 2/3$ are satisfied. We are interested to get a lower bound for $\|u\|_{L^\infty, \theta/2(\mathbb{R}^d)}$.

The spectral condition $\delta < 2/3$ is needed here because it seems difficult to estimate from below the variations of the spectral function of the harmonic oscillator in intervals of length $\leq h^{-1/3}$.

A first step is to get two sides weighted L^r estimates for large r which is a probabilistic improvement of (3.17). Denote by

$$(4.8) \quad \beta_{r,\theta} = \frac{d-\theta}{2} \left(1 - \frac{2}{r}\right).$$

Theorem 4.7. — Assume that $\theta \in [0, d]$, and denote by \mathcal{M}_r a median of $\|u\|_{L^{r,\theta(r/2-1)}}$. Then there exist $0 < C_0 < C_1$, $K > 0$, $c_1 > 0$, $h_0 > 0$ such that for all $r \in [2, K \log h]$ and $h \in]0, h_0]$ such that

$$(4.9) \quad \mathbf{P}_{\gamma,h} \left[u \in \mathbf{S}_h : \left| \|u\|_{L^{r,\theta(r/2-1)}} - \mathcal{M}_r \right| > \Lambda \right] \leq 2 \exp \left(-c_2 N_h^{2/r} h^{-\beta_{r,\theta}} \Lambda^2 \right).$$

and where

$$C_0\sqrt{r}h^{\frac{d-\theta}{4}(1-\frac{2}{r})} \leq \mathcal{M}_r \leq C_1\sqrt{r}h^{\frac{d-\theta}{4}(1-\frac{2}{r})}, \quad \forall r \in [2, K \log N].$$

This result shows that $\|u\|_{L^{r,\theta}(r/2-1)}$ has a Gaussian concentration around its median.

From (4.9) we deduce that for every $\kappa \in]0, 1[$, $K > 0$, there exist $0 < C_0 < C_1$, $c_1 > 0$, $h_0 > 0$ such that for all $r \in [2, K|\log h|^\kappa]$, $h \in]0, h_0]$ and $\Lambda > 0$ we have

$$\mathbf{P}_{\gamma,h} \left[u \in \mathbf{S}_h : C_0\sqrt{r}h^{\frac{d-\theta}{4}(1-\frac{2}{r})} \leq \|u\|_{L^{r,\theta}(r/2-1)} \leq C_1\sqrt{r}h^{\frac{d-\theta}{4}(1-\frac{2}{r})} \right] \geq 1 - e^{-c_1|\log h|^{1-\kappa}}.$$

As a consequence of Theorem 4.7, for every $\theta \in [0, d]$ we get a two sides weighted L^∞ estimate showing that Theorem 4.1 and its corollary are sharp.

Corollary 4.8. — *After a slight modification of the constants in Theorem 4.7, if necessary, we get that for all $\theta \in [0, d]$ and $h \in]0, h_0]$*

$$(4.10) \quad \mathbf{P}_{\gamma,h} \left[u \in \mathbf{S}_h : C_0|\log h|^{1/2}h^{(d-\theta)/4} \leq \|u\|_{L^\infty,\theta/2} \leq C_1|\log h|^{1/2}h^{(d-\theta)/4} \right] \geq 1 - h^{c_1}.$$

To prove these results we have to adapt to the unbounded configuration space \mathbb{R}^d the proofs of [2, Theorems 4 and 5] which hold for compact manifolds. The concentration result stated in Proposition 2.13 will prove useful.

Proof of Theorem 4.7. — Denote by $F_r(u) = \|u\|_{L^{r,\theta}(r/2-1)}$ and by \mathcal{M}_r its median. Thanks to (3.17) we have the Lipschitz estimate

$$|F_r(u) - F_r(v)| \leq C \left(N_h h^{\frac{d-\theta}{2}} \right)^{\frac{1}{2}-\frac{1}{r}} \|u - v\|_{L^2(\mathbb{R}^d)}, \quad \forall u, v \in \mathbf{S}_h.$$

Therefore, by (2.15) and (4.8), we have for some $c_2 > 0$

$$(4.11) \quad \mathbf{P}_{\gamma,h} \left[u \in \mathbf{S}_h : |F_r - \mathcal{M}_r| > \Lambda \right] \leq 2 \exp \left(-c_2 N_h^{2/r} h^{-\beta_{r,\theta}} \Lambda^2 \right).$$

The next step is to estimate \mathcal{M}_r . Denote by $\mathcal{A}_r^r = \mathbf{E}_h(F_r^r)$ the moment of order r and compute, with $s = \theta(r/2 - 1)$,

$$\begin{aligned} \mathcal{A}_r^r &= \mathbf{E}_h \left(\int_{\mathbb{R}^d} \langle x \rangle^s |u(x)|^r dx \right) \\ &= r \int_{\mathbb{R}^d} \langle x \rangle^s \left(\int_0^{+\infty} s^{r-1} \mathbf{P}_{\gamma,h} \left[u \in \mathbf{S}_h : |u(x)| > s \right] ds \right) dx. \end{aligned}$$

Thus by (2.11) we get

$$C_1 r \int_{\mathbb{R}^d} \langle x \rangle^s \left(\int_0^{\varepsilon_0 \sqrt{e_x}} s^{r-1} e^{-c_1 \frac{N}{e_x} s^2} ds \right) dx \leq \mathcal{A}_r^r \leq C_2 r \int_{\mathbb{R}^d} \langle x \rangle^s \left(\int_0^{+\infty} s^{r-1} e^{-c_2 \frac{N}{e_x} s^2} ds \right) dx.$$

Performing the change of variables $t = c \frac{N}{e_x} s^2$ we obtain that there exist $C_1, C_2 > 1$ such that

$$(4.12) \quad C_1 r N^{-r/2} \left(\int_{\mathbb{R}^d} \langle x \rangle^s e_x^{r/2} dx \right) \int_0^{\varepsilon N} t^{r/2-1} e^{-t} dt \leq \mathcal{A}_r^r \leq C_2 r N^{-r/2} \left(\int_{\mathbb{R}^d} \langle x \rangle^s e_x^{r/2} dx \right) \Gamma(r/2),$$

with $\varepsilon = c_1 \varepsilon_0^2$. We need to estimate the term $\int_0^{\varepsilon N} t^{r/2-1} e^{-t} dt$ from below. Using the elementary estimate

$$\int_T^{+\infty} t^{r/2-1} e^{-t} dt \leq T^{r/2} e^{1-T} \Gamma(r/2), \quad T \geq 1,$$

we get that there exists $\varepsilon_1 > 0$ such that for N large and $r \leq \varepsilon_1 \frac{N}{\log N}$ then we have

$$\int_0^{\varepsilon N} t^{r/2-1} e^{-t} dt \geq \frac{\Gamma(r/2)}{2}.$$

So we get the expected lower bound, $\forall r \in [1, \varepsilon_1 \frac{N}{\log N}]$,

$$e^{-r/2} C^{-1} r \left(\int_{\mathbb{R}^d} \langle x \rangle^s e_x^{r/2} dx \right) N^{-r/2} \Gamma(r/2) \leq \mathcal{A}_r \leq C_2 r N^{-r/2} \left(\int_{\mathbb{R}^d} \langle x \rangle^s e_x^{r/2} dx \right) \Gamma(r/2).$$

and where $\Gamma(r/2)$ can be estimated thanks to the Stirling formula: there exist $0 < C_0 < C_1$ such that

$$(C_0 r)^{r/2} \leq \Gamma(r/2) \leq (C_1 r)^{r/2}, \quad \forall r \geq 1.$$

Now we need the following lemma which will be proven in Appendix B. The upper bound can be seen as an application of Lemma 3.5 with $\lambda = h^{-1}$ and $\mu = (b_h - a_h) h^{-1} \sim N_h h^{d-1}$.

Lemma 4.9. — *There exist $0 < C_0 < C_1$ and $h_0 > 0$ such that*

$$C_0 N_h h^{\beta_{2p,\theta}} \leq \left(\int_{\mathbb{R}^d} \langle x \rangle^{\theta(p-1)} e_x^p dx \right)^{1/p} \leq C_1 N_h h^{\beta_{2p,\theta}},$$

for every $p \in [1, \infty[$ and $h \in]0, h_0]$ where $\beta_{r,\theta} = \frac{d-\theta}{2} (1 - \frac{2}{r})$.

From this lemma we get

$$(4.13) \quad C_0 \sqrt{r h^{\beta_{r,\theta}}} \leq \mathcal{A}_r \leq C_1 \sqrt{r h^{\beta_{r,\theta}}}, \quad \forall r \geq 2, h \in]0, h_0].$$

Now we have to compare \mathcal{A}_r and the median \mathcal{M}_r . We have

$$\begin{aligned} |\mathcal{A}_r - \mathcal{M}_r|^r &= \left| \|F_r\|_{L^r(\mathbf{s}_h)} - \|\mathcal{M}_r\|_{L^r(\mathbf{s}_h)} \right|^r \\ &\leq \|F_r - \mathcal{M}_r\|_{L^r(\mathbf{s}_h)}^r = r \int_0^\infty s^{r-1} \mathbf{P}_{\gamma,h} [|F_r - \mathcal{M}_r| > s] ds. \end{aligned}$$

Then using the large deviation estimate (2.15) we get

$$|\mathcal{A}_r - \mathcal{M}_r| \leq C N^{-1/r} \sqrt{r h^{\beta_{r,\theta}}}, \quad \forall r \geq 2.$$

Choosing $r \leq K \log N$, ($K < 1$) and N large, from (4.13) we obtain

$$(4.14) \quad C_0 \sqrt{r h^{\beta_{r,\theta}}} \leq \mathcal{M}_r \leq C_1 \sqrt{r h^{\beta_{r,\theta}}}, \quad \forall r \in [2, K \log N]$$

and the proof of Theorem 4.7 follows using (4.14) and (4.11). \square

Remark 4.10. — The upper-bound in Lemma 4.9 is true for $\delta = 1$. This is proved in Appendix B.

Now let us prove Corollary 4.8.

Proof of Corollary 4.8. — For simplicity we assume that $\theta = d$. Using (4.1) it is enough to prove that there exist $C_0 > 0$, $h_0 > 0$, $c_1 > 0$ such that

$$(4.15) \quad \mathbf{P}_{\gamma,h} \left[u \in \mathbf{S}_h : \|u\|_{L^\infty,d/2} \leq C_0 |\log h|^{1/2} \right] \leq h^{c_1}, \quad \forall h \in]0, h_0].$$

Let $u \in \mathbf{S}_h$, then by (3.2) we have the interpolation inequality

$$\|u\|_{L^{r,d(r/2-1)}(\mathbb{R}^d)} \leq \|u\|_{L^\infty,d/2}^{1-2/r}.$$

So we get

$$\mathbf{P}_{\gamma,h} \left[u \in \mathbf{S}_h : \|u\|_{L^\infty,d/2} \leq C_0 |\log h|^{1/2} \right] \leq \mathbf{P}_{\gamma,h} \left[u \in \mathbf{S}_h : \|u\|_{L^{r,d(r/2-1)}} \leq \left(C_0 |\log h|^{1/2} \right)^{1-2/r} \right],$$

and choosing $r = r_h = \varepsilon_0 |\log h|$ we obtain

$$\mathbf{P}_{\gamma,h} \left[u \in \mathbf{S}_h : \|u\|_{L^\infty,d/2} \leq C_0 |\log h|^{1/2} \right] \leq \mathbf{P}_{\gamma,h} \left[u \in \mathbf{S}_h : \|u\|_{L^{r_h,d(r_h/2-1)}} \leq \left(\frac{C_0}{\sqrt{\varepsilon_0}} r_h^{1/2} \right)^{1-2/r_h} \right].$$

Then choosing $h_0 > 0$, $\frac{C_0}{\sqrt{\varepsilon_0}}$ small enough and $\Lambda = c |\log h|^{1/2}$ we can conclude that (4.15) is satisfied using (4.9). \square

Remark 4.11. — Concerning the mean \mathcal{M}_∞ of $F_\infty(u) := \|u\|_{L^\infty,d/2}$ it results from Corollary 4.8, (4.1) and (3.16) that we have the two sides estimates

$$C_0 |\log h|^{1/2} \leq \mathcal{M}_\infty \leq C_1 |\log h|^{1/2}, \quad \forall h \in]0, h_0].$$

It is not difficult to adapt the proof of (4.9) and (4.10) for the Sobolev norms $\|u\|_{\mathcal{W}^{s,p}(\mathbb{R}^d)}$. It is enough to remark that considering $L_s u(x) = H^{s/2} u(x)$ we have

$$e_{L_s} = \sum_{j \in \Lambda} \lambda_j^s \varphi_j^2(x).$$

But for $j \in \Lambda$, λ_j is of order h^{-1} hence there exists $C > 0$ such that

$$C^{-1} h^{-s} e_x \leq e_{L_s} \leq C h^{-s} e_x.$$

Using this property we easily get the next result, which in particular implies Theorem 1.1. Let $\mathcal{M}_{r,s}$ be the median of $u \mapsto \|u\|_{\mathcal{W}^{s,r}(\mathbb{R}^d)}$, and recall the definition (4.8). Then

Theorem 4.12. — *Let $s \geq 0$. There exist $0 < C_0 < C_1$, $K > 0$, $c_1 > 0$, $h_0 > 0$ such that for all $r \in [2, K |\log h|]$ and $h \in]0, h_0]$*

$$(4.16) \quad \mathbf{P}_{\gamma,h} \left[u \in \mathbf{S}_h : \left| \|u\|_{\mathcal{W}^{s,r}(\mathbb{R}^d)} - \mathcal{M}_{r,s} \right| > \Lambda \right] \leq 2 \exp \left(-c_2 N_h^{2/r} h^{-\beta_{r,0} + s} \Lambda^2 \right).$$

where

$$C_0 \sqrt{r} h^{\frac{\beta_{r,0} - s}{2}} \leq \mathcal{M}_{r,s} \leq C_1 \sqrt{r} h^{\frac{\beta_{r,0} - s}{2}}, \quad \forall r \in [2, K \log N].$$

In particular, for every $\kappa \in]0, 1[$, $K > 0$, there exist $C_0 > 0$, $C_1 > 0$, $c_1 > 0$ such that for every $r \in [2, K|\log h|^\kappa]$ we have

$$\mathbf{P}_{\gamma, h} \left[u \in \mathbf{S}_h : C_0 \sqrt{r} h^{\frac{d}{4}(1-\frac{2}{r})} h^{-\frac{s}{2}} \leq \|u\|_{\mathcal{W}^{s, r}(\mathbb{R}^d)} \leq C_1 \sqrt{r} h^{\frac{d}{4}(1-\frac{2}{r})} h^{-\frac{s}{2}} \right] \geq 1 - e^{-c_1 |\log h|^{1-\kappa}},$$

For $r = +\infty$ we have for all $h \in]0, h_0]$

$$\mathbf{P}_{\gamma, h} \left[u \in \mathbf{S}_h : C_0 |\log h|^{1/2} h^{\frac{d-2s}{4}} \leq \|u\|_{\mathcal{W}^{s, \infty}(\mathbb{R}^d)} \leq C_1 |\log h|^{1/2} h^{\frac{d-2s}{4}} \right] \geq 1 - h^{c_1}.$$

Namely,

$$\|u\|_{\mathcal{W}^{s, r}(\mathbb{R}^d)} \approx h^{-s/2} \|u\|_{L^{r, 0}(\mathbb{R}^d)} + \|u\|_{L^{r, s}(\mathbb{R}^d)},$$

and

$$h^{-s/2} \|u\|_{L^{r, 0}(\mathbb{R}^d)} \sim h^{\frac{d}{4}(1-\frac{2}{r})} h^{-\frac{s}{2}}, \quad \|u\|_{L^{r, s}(\mathbb{R}^d)} \sim h^{\frac{d}{4}(1-\frac{2}{r})} h^{-\frac{s}{2r}}.$$

4.3. Lower bounds in the general case. — Under Assumption 1, we prove a weaker version of Theorem 4.12.

Theorem 4.13. — Suppose that Assumption 1 is satisfied. Let $s \geq 0$, $\kappa \in]0, 1[$, $K > 0$. There exist $0 < C_0 < C_1$, $K > 0$, $c_1 > 0$, $h_0 > 0$ such that for all $r \in [2, K|\log h|^\kappa]$ and $h \in]0, h_0]$

$$\mathbf{P}_{\gamma, h} \left[u \in \mathbf{S}_h : C_0 h^{\frac{d}{4}(1-\frac{2}{r})} h^{-\frac{s}{2}} \leq \|u\|_{\mathcal{W}^{s, r}(\mathbb{R}^d)} \leq C_1 \sqrt{r} h^{\frac{d}{4}(1-\frac{2}{r})} h^{-\frac{s}{2}} \right] \geq 1 - e^{-c_1 |\log h|^{1-\kappa}},$$

For $r = +\infty$ we have for all $h \in]0, h_0]$

$$\mathbf{P}_{\gamma, h} \left[u \in \mathbf{S}_h : C_0 h^{\frac{d-2s}{4}} \leq \|u\|_{\mathcal{W}^{s, \infty}(\mathbb{R}^d)} \leq C_1 |\log h|^{1/2} h^{\frac{d-2s}{4}} \right] \geq 1 - h^{c_1}.$$

Therefore, we have optimal constants in the control of the $\mathcal{W}^{s, r}(\mathbb{R}^d)$ norms when $r < +\infty$ and for general random variables which satisfy the concentration property, but when $r = +\infty$ we lose the factor $|\log h|^{1/2}$ in the lower bound.

Proof. — We can follow the main lines of the proof of Theorem 4.12. Here compared to (4.12) we get

$$\begin{aligned} \mathcal{A}_r^r &\geq C r N^{-r/2} \left(\int_{\mathbb{R}^d} \langle x \rangle^s e_x^{r/2} dx \right) \int_0^\varepsilon t^{r/2-1} e^{-t} dt \\ &\geq C N^{-r/2} \left(\int_{\mathbb{R}^d} \langle x \rangle^s e_x^{r/2} dx \right) \varepsilon^{r/2}, \end{aligned}$$

and this explains the loss of the factor \sqrt{r} . □

4.4. Global probabilistic L^p -Sobolev estimates. — Here we extend the L^∞ - random estimates obtained before to the L^r -spaces for any real $r \geq 2$, and we prove Theorem 1.2. Let us recall the definition (1.4) of the Besov spaces, where we use the notations of Subsection 4.1 for the dyadic Littlewood-Paley decomposition.

Proof of Theorem 1.2. — Recall that for every $\sigma > m$ we can choose γ such that $\mu_\gamma(\mathcal{H}^\sigma) = 0$. Denote by $F_{r,s}(u) = \|u\|_{\mathcal{W}^{s,r}}$. The Lipschitz norm of $F_{r,s}$ satisfies

$$\|F_{r,s}\|_{Lip} \leq Ch^{-s+d(\frac{1}{2}-\frac{1}{r})} N_h^{\frac{1}{2}-\frac{1}{r}}.$$

Let us denote by $\mathcal{M}_{r,s}$ the median of $F_{r,s}$ on the sphere \mathbf{S}_h for the probability $\mathbf{P}_{\gamma,h}$ and by $\mathcal{A}_{r,s}^r$ the mean of $F_{r,s}^r$. From Proposition 2.13 we have, for some $0 < c_0 < c_1$,

$$(4.17) \quad \mathbf{P}_{\gamma,h} \left[u \in \mathbf{S}_h : |F_{r,s} - \mathcal{M}_{r,s}| > K \right] \leq \exp \left(-c_1 N \frac{K^2}{\|F_{r,s}\|_{Lip}^2} \right) \leq \exp \left(-c_0 N^{1/r} K^2 \right).$$

With the same computations as for (4.14) we get

$$(4.18) \quad \mathcal{A}_{r,s} \approx \sqrt{r} \quad \text{and} \quad |\mathcal{A}_{r,s} - \mathcal{M}_{r,s}| \lesssim \sqrt{r} N^{-1/r}.$$

These formulas are obtained from (2.9) applied to the linear form $L_s u := H^s u(x)$ noticing that

$$e_{L_s} = \sum_{j \in \Lambda_h} |H^s \varphi_j(x)|^2 \approx h^{-2s} e_x.$$

Then taking $c_0 > 0$ small enough that we have

$$(4.19) \quad \nu_{\gamma,h} \left[v \in \mathcal{E}_h : \|v\|_{\mathcal{W}^{s,r}} \geq K \|v\|_{L^2(\mathbb{R}^d)} \right] \leq \exp \left(-c_0 N^{2/r} K^2 \right), \quad \forall K \geq 1.$$

Then from (4.19) we proceed as for the proof of Corollary 4.4. For simplicity we consider here the usual Littlewood-Paley decomposition. Then we have $N^{2/r} \approx 2^{2nd/r}$. So the end of the proof follows by considering

$$B_n^k = \{v \in \mathcal{E}_n : \|v\|_{\mathcal{W}^{s,r}} \leq K \|v\|_{L^2(\mathbb{R}^d)}\}.$$

So for a fixed $r \geq 2$ we infer (1.5) from (4.17) and (4.18), taking $c_0 > 0$ small enough, we get

$$\mu_\gamma \left(\prod_{n \geq 0} B_n^K \right) \geq 1 - e^{-c_0 K^2}.$$

□

Using the isometry $u \mapsto H^{-m/2}u$ between $\mathcal{B}_{2,1}^s$ and $\mathcal{B}_{2,1}^{m+s}$ for all real $m \geq 0$, we can get the following corollary to Theorem 1.2.

Corollary 4.14. — *Let $m \geq 0$ and assume that γ satisfies (1.2) and*

$$\sum_{n \geq 0} 2^{nm} |\gamma|_n < +\infty.$$

Then for $s = d(\frac{1}{2} - \frac{1}{r}) + m$ and $r \geq 2$, we have

$$\mu_\gamma \left[u \in \mathcal{B}_{2,1}^m : \|u\|_{\mathcal{W}^{m+s,r}} \geq K \|u\|_{\mathcal{B}_{2,1}^m} \right] \leq e^{-c_0 K^2}.$$

5. Application to Hermite functions

We turn to the proof of Theorem 1.3 and we can follow the main lines of [2, Section 3]. We use here the upper bounds estimates of Section 4.1 in their full strength. Firstly, we assume that for all $j \in \Lambda_h$, $\gamma_j = N_h^{-1/2}$ and that $X_j \sim \mathcal{N}_{\mathbb{C}}(0, 1)$, so that $\mathbf{P}_h := \mathbf{P}_{\gamma, h}$ is the uniform probability on \mathbf{S}_h . We set $h_k = 1/k$ with $k \in \mathbb{N}^*$, and

$$a_{h_k} = 2 + dh_k, \quad b_{h_k} = 2 + (2 + d)h_k.$$

Then (1.1) is satisfied with $\delta = 1$ and $D = 2$. In particular, each interval

$$I_{h_k} = \left[\frac{a_{h_k}}{h_k}, \frac{b_{h_k}}{h_k} \right] = [2k + d, 2k + d + 2[$$

only contains the eigenvalue $\lambda_k = 2k + d$ with multiplicity $N_{h_k} \sim ck^{d-1}$, and \mathcal{E}_{h_k} is the corresponding eigenspace of the harmonic oscillator H . We can identify the space of the orthonormal basis of \mathcal{E}_{h_k} with the unitary group $U(N_{h_k})$ and we endow $U(N_{h_k})$ with its Haar probability measure ρ_k . Then the space \mathcal{B} of the Hilbertian bases of eigenfunctions of H in $L^2(\mathbb{R}^d)$ can be identified with

$$\mathcal{B} = \times_{k \in \mathbb{N}} U(N_{h_k}),$$

which can be endowed with the measure

$$d\rho = \otimes_{k \in \mathbb{N}} d\rho_k.$$

Denote by $B = (\varphi_{k, \ell})_{k \in \mathbb{N}, \ell \in \llbracket 1, N_{h_k} \rrbracket} \in \mathcal{B}$ a typical orthonormal basis of $L^2(\mathbb{R}^d)$ so that for all $k \in \mathbb{N}$, $(\varphi_{k, \ell})_{\ell \in \llbracket 1, N_{h_k} \rrbracket} \in U(N_{h_k})$ is an orthonormal basis of \mathcal{E}_{h_k} .

Then the main result of the section is the following, which implies Theorem 1.3.

Theorem 5.1. — *Let $d \geq 2$. Then, if $M > 0$ is large enough, there exist $c, C > 0$ so that for all $r > 0$*

$$\rho \left[B = (\varphi_{k, \ell})_{k \in \mathbb{N}, \ell \in \llbracket 1, N_{h_k} \rrbracket} \in \mathcal{B} : \exists k, \ell; \|\varphi_{k, \ell}\|_{\mathcal{W}^{d/2, \infty}(\mathbb{R}^d)} \geq M(\log k)^{1/2} + r \right] \leq Ce^{-cr^2}.$$

We will need the following result

Proposition 5.2. — *Let $d \geq 2$. Then, if $M > 0$ is large enough, there exist $c, C > 0$ so that for all $r > 0$ and $k \geq 1$*

$$(5.1) \quad \rho_k \left[B_k = (\psi_{\ell})_{\ell \in \llbracket 1, N_{h_k} \rrbracket} \in U(N_{h_k}) : \exists \ell \in \llbracket 1, N_{h_k} \rrbracket; \|\psi_{\ell}\|_{\mathcal{W}^{d/2, \infty}(\mathbb{R}^d)} \geq M(\log k)^{1/2} + r \right] \leq Ck^{-2}e^{-cr^2}.$$

Proof. — The proof is similar to the proof of [2, Proposition 3.2]. We observe that for any $\ell_0 \in \llbracket 1, N_{h_k} \rrbracket$, the measure ρ_k is the image measure of \mathbf{P}_{h_k} under the map

$$U(N_{h_k}) \ni B_k = (\psi_{\ell})_{\ell \in \llbracket 1, N_{h_k} \rrbracket} \longmapsto \psi_{\ell_0} \in \mathbf{S}_{h_k}.$$

Then we use that $\mathbf{S}_{h_k} \subset \mathcal{E}_{h_k}$ is an eigenspace and by Theorem 4.1 we obtain that for all $\ell_0 \in \llbracket 1, N_{h_k} \rrbracket$

$$\begin{aligned} \rho_k \left[B_k = (\psi_\ell)_{\ell \in \llbracket 1, N_{h_k} \rrbracket} \in U(N_{h_k}) : \|\psi_{\ell_0}\|_{\mathcal{W}^{d/2, \infty}(\mathbb{R}^d)} \geq M(\log k)^{1/2} + r \right] \\ = \mathbf{P}_{h_k} \left[u \in \mathbf{S}_{h_k} : \|u\|_{\mathcal{W}^{d/2, \infty}(\mathbb{R}^d)} \geq M(\log k)^{1/2} + r \right] \\ = \mathbf{P}_{h_k} \left[u \in \mathbf{S}_{h_k} : k^{d/4} \|u\|_{L^{\infty, 0}(\mathbb{R}^d)} \geq M(\log k)^{1/2} + r \right] \\ \leq C k^{c_1 - M^2 c_2} e^{-c_2 r^2}, \end{aligned}$$

where $c_1, c_2 > 0$ are given by Theorem 4.1. As a consequence, (5.1) is bounded by $C k^c e^{-c_2 r^2}$, with $c = c_1 - M^2 c_2 + d - 1$ which implies the result. \square

Proof of Theorem 5.1. — We set

$$\mathcal{F}_{k,r} = \left\{ B_k = (\psi_\ell)_{\ell \in \llbracket 1, N_{h_k} \rrbracket} \in U(N_{h_k}) : \forall \ell \in \llbracket 1, N_{h_k} \rrbracket; \|\psi_\ell\|_{\mathcal{W}^{d/2, \infty}(\mathbb{R}^d)} \leq M(\log k)^{1/2} + r \right\},$$

and $\mathcal{F}_r = \bigcap_{k \geq 1} \mathcal{F}_{k,r}$. Then for all $r > 0$

$$\rho(\mathcal{F}_r^c) \leq \sum_{k \geq 1} \rho_k(\mathcal{F}_{k,r}^c) \leq C \sum_{k \geq 1} k^{-2} e^{-cr^2} = C' e^{-cr^2},$$

and this completes the proof. \square

We have the following consequence of the previous results.

Corollary 5.3. — *For ρ -almost all orthonormal basis $(\varphi_{k,\ell})_{k \in \mathbb{N}, \ell \in \llbracket 1, N_{h_k} \rrbracket}$ of eigenfunctions of H we have*

$$\|\varphi_{k,\ell}\|_{L^\infty(\mathbb{R}^d)} \leq (M+1)k^{-\frac{d}{4}}(1 + \log k)^{1/2}, \quad \forall k \in \mathbb{N}, \forall \ell \in \llbracket 1, N_{h_k} \rrbracket.$$

Proof. — Apply (5.1) with $r = (\log k)^{1/2}$ and denote, for $k \geq 2$, Ω_k the event

$$\Omega_k = \left\{ B = (\varphi_{k,\ell}), \exists \ell \in \llbracket 1, N_{h_k} \rrbracket, \|\varphi_{k,\ell}\|_{L^\infty(\mathbb{R}^d)} \geq (M+1)k^{-d/4}(\log k)^{1/2} \right\}.$$

We have $\rho(\Omega_k) \leq \frac{C}{k^2}$. Therefore from the Borel-Cantelli Lemma we have $\rho[\limsup_{k \rightarrow +\infty} \Omega_k] = 0$ and this gives the corollary. \square

Appendix A

Proof of Proposition 2.4 (iii)

Proof. — Denote by $f_\gamma(x) = \frac{c_\alpha}{\gamma} e^{-(\frac{|x|}{\gamma})^\alpha}$, $\gamma > 0$. We have, with obvious identifications,

$$\mu_\gamma = \otimes_{j \geq 0} (f_{\gamma_j} dx).$$

Denote by

$$\pi_j = \int_{\mathbb{R}} \left(\frac{f_{\gamma_j}}{f_{\beta_j}} \right)^{1/2} f_{\beta_j} dx.$$

According to the main result of [9] the measures μ_γ and μ_β are mutually singular if the infinite product $\prod_{j \geq 0} \pi_j$ is divergent. From elementary computations we get

$$\pi_j = \left(\frac{1}{2} \left(\frac{\gamma_j}{\beta_j} \right)^{\alpha/2} + \frac{1}{2} \left(\frac{\beta_j}{\gamma_j} \right)^{\alpha/2} \right)^{-1/\alpha}.$$

- If π_j has not 1 as limit then the product is divergent.
- If π_j has 1 as limit then the infinite product is divergent if $\sum_{j \geq 0} (\pi_j^{-\alpha} - 1) = +\infty$. So, using that

$$\frac{1}{2} \left(x + \frac{1}{x} \right) = 1 + \frac{1}{2} (1 - x)^2 + \mathcal{O}(1 - x)^3,$$

we see that the infinite product is divergent if (2.6) is satisfied. \square

Appendix B

L^p weighted spectral estimates for the Harmonic oscillator

Our goal here is to give a self-contained proof of Lemma 4.9. It could be proved using the semi-classical functional calculus for pseudo-differential operators [17], but for the harmonic oscillator it is possible to use the exact Mehler formula and elementary properties of Hermite functions to get the result.

B.1. A functional calculus with parameter for the Harmonic oscillator. — The starting point is the inverse Fourier transform

$$f(H) = \frac{1}{2\pi} \int e^{itH} \hat{f}(t) dt,$$

where f is in the Schwartz space $\mathcal{S}(\mathbb{R})$.

We want estimates for the integral kernel $K_f(x, y)$ of $f(H)$. To do that it is convenient to first compute the Weyl symbol $W_{f(H)}(x, \xi)$ of $f(H)$ and use that

$$K_f(x, y) = (2\pi)^{-d} \int_{\mathbb{R}^d} W_{f(H)}\left(\frac{x+y}{2}, \xi\right) e^{i(x-y) \cdot \xi} d\xi.$$

For basic properties about the Weyl calculus see for example [17]. The unitary operator e^{itH} has an explicit Weyl symbol $w(t, x, \xi)$:

$$(B.1) \quad w(t, x, \xi) = \frac{1}{(\cos t)^d} e^{i \tan t (|x|^2 + |\xi|^2)}, \quad \text{for } |t| < \frac{\pi}{2}.$$

Formula (B.1) can be easily proved from the Mehler formula (3.7) and also directly (see [17], Exercise IV).

Let us introduce a cutoff $\chi \in C^\infty(\mathbb{R})$, $\chi(t) = 1$ for $|t| < \varepsilon_0$, $\chi(t) = 0$ for $|t| > 2\varepsilon_0$ with $0 < \varepsilon_0 < \pi/4$. Denote by

$$R_f = \frac{1}{2\pi} \int e^{itH} (1 - \chi(t)) \hat{f}(t) dt$$

and

$$\tilde{W}_f(x, \xi) = \frac{1}{2\pi} \int_{\mathbb{R}} w(t, x, \xi) \chi(t) \hat{f}(t) dt.$$

We apply these formulas to give estimates with $f_h(s) = f(hs)$ where $h > 0$ is a small parameter. We begin with an estimate for the remainder term for the kernel $KR_{f_h}(x, y)$ of the operator R_{f_h} .

Lemma B.1. — *There exists $M_0 > 0$ such that for every $M \geq 1$ there exists $C_M > 0$ such that*

$$(B.2) \quad |KR_{f_h}(x, y)| \leq C_M h^M (\langle x \rangle \langle y \rangle)^{-\frac{M}{2} - d - 1} \|f\|_{M+M_0}, \quad \forall h \in]0, 1], \quad \forall x, y \in \mathbb{R}^d,$$

$$\text{where } \|f\|_m = \sup_{j+k \leq m, t \in \mathbb{R}} \left| t^j \frac{d^k}{dt^k} \hat{f}(t) \right|.$$

Proof. — Denote by $\hat{g}_h(t) = (1 - \chi(t)) \hat{f}(\frac{t}{h})$. So we have $R_{f,h} = g_h(H)$ and for every $M, M' \geq 1$,

$$|\mu^M g_h(\mu)| \leq \int_{|t| \geq \varepsilon_0} \left| \frac{d^M}{dt^M} \hat{g}_h(t) \right| dt \leq C_{M, M'}(f) h^{M'}.$$

So we have

$$|KR_{f_h}(x, y)| = \left| \sum_j g_h(\lambda_j) \varphi_j(x) \overline{\varphi_j(y)} \right| \leq Ch^M \left(\sum_j \lambda_j^{-M} |\varphi_j(x)|^2 \right)^{1/2} \left(\sum_j \lambda_j^{-M} |\varphi_j(y)|^2 \right)^{1/2}.$$

Recall the Sobolev estimate in the harmonic spaces: for every $s > \frac{d}{2} + r$ there exists $C = C_{sr}$ such that

$$\langle x \rangle^r |u(x)| \leq C \|u\|_{\mathcal{H}^s}, \quad \forall u \in \mathcal{H}^s(\mathbb{R}^d).$$

So we get, for $s > \frac{d}{2} + r$,

$$|KR_{f_h}(x, y)| \leq Ch^M (\langle x \rangle \langle y \rangle)^{-r} \sum_j \lambda_j^{s-M}.$$

Using that $\lambda_j \approx j^{1/d}$ and choosing $r = \frac{M}{2} + d + 1$ we get (B.2). \square

Our aim is to estimate the kernel of $f\left(\frac{H-\nu\lambda}{\mu}\right)$ for large λ , $|\mu| \geq D\lambda^{1-\delta}$ where $D > 0$ and $\delta < 2/3$. The parameter ν is fixed in an interval $[\nu_0, \nu_1]$, where $0 < \nu_0 < \nu_1$. All our estimates will be uniform in ν , so for convenience we shall take $\nu = 1$.

Denote by $g_{\lambda, \mu}(s) = f\left(\frac{s-\lambda}{\mu}\right)$ so we have $\hat{g}_{\lambda, \mu}(t) = \mu e^{-it\lambda} \hat{f}(\mu t)$. We consider the dilated Weyl symbol: $W_{\lambda, \mu}(x, \xi) = \tilde{W}_{g_{\lambda, \mu}}(\sqrt{\lambda}x, \sqrt{\lambda}\xi)$. Then we have

$$(B.3) \quad W_{\lambda, \mu}(x, \xi) = \frac{\mu}{2\pi} \int_{\mathbb{R}} e^{i\lambda\Phi(t, x, \xi)} \frac{\chi(t)}{\cos(t)^d} \hat{f}(\mu t) dt,$$

with the phase $\Phi(t, x, \xi) = \tan t(|x|^2 + |\xi|^2) - t$.

Lemma B.2. — Assume that $\delta < \frac{2}{3}$. Then for every $N, M \geq 0$ we have

$$(B.4) \quad W_{\lambda, \mu}(x, \xi) = \sum_{j(1-\delta)+k(2-3\delta) < N} c_{k,j} \lambda^k \mu^{-3k-j} (|x|^2 + |\xi|^2)^k f^{(3k+j)} \left(\frac{\lambda}{\mu} (|x|^2 + |\xi|^2 - 1) \right) + \mathcal{O}(\lambda^{-N} (1 + |x|^2 + |\xi|^2)^{-M}),$$

where $c_{k,j}$ are real numbers, $c_{0,0} = 1$.

Proof. — Using that $\partial_t \Phi(t, x, \xi) = (1 + \tan^2 t)(|x|^2 + |\xi|^2) - 1$ and integrating by parts we get that for every M there exists $C_M > 0$ such that for $||x|^2 + |\xi|^2 - 1| \geq 1/2$ we have

$$(B.5) \quad |W_{\lambda, \mu}(x, \xi)| \leq C_M \lambda^{-M\delta} (|x|^2 + |\xi|^2 + 1)^{-M}.$$

So it is enough to estimate $W_{\lambda, \mu}(x, \xi)$ for $|x|^2 + |\xi|^2 \approx 1$.

To do that, we write down

$$e^{i\lambda\Phi(t,x,\xi)} = e^{i\lambda(\tan t - t)(|x|^2 + |\xi|^2)} e^{i\lambda t(|x|^2 + |\xi|^2 - 1)}.$$

Denote by $E_t = (\tan t - t)(|x|^2 + |\xi|^2)$, then we have

$$e^{i\lambda E_t} = \sum_{0 \leq k \leq N} \frac{(i\lambda E_t)^k}{k!} + r_N(i\lambda E_t)$$

where

$$\left| \frac{d^j}{ds^j} r_N(s) \right| \leq \frac{|s|^{N-j}}{(N-j)!}, \quad 0 \leq j \leq N.$$

Lastly, we end up the computation by expanding $(\tan t - t)$ with Taylor

$$(\tan t - t)^k \frac{\chi(t)}{\cos(t)^d} = \sum_{j=0}^{+\infty} d_{k,j} t^{3k+j}.$$

Thus

$$\begin{aligned} W_{\lambda, \mu}(x, \xi) &= \frac{\mu}{2\pi} \sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} \int_{\mathbb{R}} e^{i\lambda t(|x|^2 + |\xi|^2 - 1)} d_{k,j} i^j \lambda^k t^{3k+j} (|x|^2 + |\xi|^2)^k \widehat{f}(\mu t) dt \\ &= \frac{\mu}{2\pi} \sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} \int_{\mathbb{R}} e^{i\lambda t(|x|^2 + |\xi|^2 - 1)} d_{k,j} i^j \lambda^k \mu^{-3k-j} (|x|^2 + |\xi|^2)^k \widehat{f^{(3k+j)}}(\mu t) dt \\ &= \sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} c_{k,j} \lambda^k \mu^{-3k-j} (|x|^2 + |\xi|^2)^k f \left(\frac{\lambda}{\mu} (|x|^2 + |\xi|^2 - 1) \right), \end{aligned}$$

which implies the result with (B.5). □

B.2. Proof of Lemma 4.9. — First remark that the upper-bound is a direct consequence of (3.12) and (3.13) and this hold true for $\delta = 1$. The bound (3.13) being a rather difficult result, we shall prove by the same method the estimate from above and from below for $\delta < 2/3$.

We use here the functional calculus with energy parameter (B.4). Let f be a non negative C^∞ function in $] -2C_0, 2C_0[$ with a compact support, such that $f = 1$ in $[-C_0, C_0]$. We choose two cutoff functions f_\pm with f_+ as above and f_- such that $\text{supp}(f_-) \subseteq]C_1, C_0[$, $f_- = 1$ in $[2C_1, C_0/2]$ where $C_1 < C_0/4$. If $K_{\pm,h}(x, y)$ is the Schwartz kernel of $f_\pm \left(\frac{H-h^{-1}}{\mu} \right)$ ($h = \frac{1}{\lambda}$ is now a small parameter, $\nu \in [\nu_0, \nu_1]$). We have

$$K_{-,h}(x, x) \leq e_x \leq K_{+,h}(x, x).$$

So we have to prove

$$(B.6) \quad C_0 N_h h^{\beta_{2p,\theta}} \leq \left(\int_{\mathbb{R}^d} \langle x \rangle^{\theta(p-1)} K_{\pm,h}(x, x)^p dx \right)^{1/p} \leq C_1 N_h h^{\beta_{2p,\theta}}.$$

Recall that

$$K_{\pm,h}(x, x) = (2\pi)^{-d} \int_{\mathbb{R}^d} W_{\pm,h}(x, \xi) d\xi$$

where $W_{\pm,h}(x, \xi)$ is the Weyl symbol of the operator $f_\pm \left(\frac{H-h^{-1}}{\mu} \right)$. So using (B.4) it is not difficult to see that it is enough to consider only the principal term given by the following formula

$$K_{f,h}^0(x, x) = (2\pi)^{-d} \int_{\mathbb{R}^d} f \left(\frac{|x|^2 + |\xi|^2 - h^{-1}}{\mu} \right) d\xi.$$

We shall detail now the lower-bound; the upper-bound is proved in the same way. Denote by $K_-^0(x) = K_{f,h}^0(x, x)$ and $s = \theta(p-1)$. We have

$$\int_{\mathbb{R}^d} \langle x \rangle^s K_-^0(x)^p dx = (2\pi)^{-dp} \int_{\mathbb{R}^d_x} \langle x \rangle^s \left(\int_{\mathbb{R}^d_\xi} f_- \left(\frac{|x|^2 + |\xi|^2 - h^{-1}}{\mu} \right) d\xi \right)^p dx.$$

Now performing the change of variable $x = h^{-1/2}y$, $\xi = h^{-1/2}\eta$, using property of the support of f_- and integrating in y in the domain $1/2 \leq |y| \leq 2/3$, we obtain

$$\int_{\mathbb{R}^d} \langle x \rangle^s K_-^0(x)^p dx \gtrsim h^{-s/2-(1+p)d/2} (h\mu)^p.$$

Hence we get (B.6) using that $\mu \approx h^{\delta-1}$ so $\mu h \approx h^\delta \approx h^d N_h$.

Appendix C

Proof of (2.12)

To begin with, we identify the complex sphere of \mathbb{C}^N with the real sphere

$$\mathbb{S}^{2N-1} = \{w \in \mathbb{R}^{2N} : w_1^2 + \dots + w_{2N}^2 = 1\} \subset \mathbb{R}^{2N}.$$

Denote by \mathbb{P}_N the uniform probability measure on \mathbb{S}^{2N-1} and by μ_N the Gaussian measure on \mathbb{R}^{2N} of density $d\mu = \frac{1}{(2\pi)^N} \exp\left(-\frac{1}{2} \sum_{j=1}^{2N} x_j^2\right) dx_1 \dots dx_{2N}$. It is easy to check that \mathbb{P}_N is the image measure of μ_N by the map

$$\begin{aligned} G : \mathbb{R}^{2N} &\longrightarrow \mathbb{S}^{2N-1} \\ (x_1, \dots, x_{2N}) &\longmapsto \frac{1}{\sqrt{\sum_{j=1}^{2N} x_j^2}} (x_1, \dots, x_{2N}). \end{aligned}$$

Indeed, $\mu_N \circ G^{-1}$ is a probability measure on \mathbb{S}^{2N-1} which is invariant by the isometries of \mathbb{S}^{2N-1} , therefore $\mathbb{P}_N = \mu_N \circ G^{-1}$. For $t \in [0, 1]$, denote by $\Phi(t) = \mathbb{P}_N(\sqrt{w_1^2 + w_2^2} > t)$, then

$$\begin{aligned} \Phi(t) &= \frac{1}{(2\pi)^N} \int \mathbb{I}_{x_1^2 + x_2^2 > t^2} \sum_{j=1}^{2N} x_j^2 e^{-\frac{1}{2} \sum_{j=1}^{2N} x_j^2} dx_1 \dots dx_{2N} \\ &= \frac{1}{(2\pi)^N} \int \mathbb{I}_{x_1^2 + x_2^2 > \frac{t^2}{1-t^2}} \sum_{j=3}^{2N} x_j^2 e^{-\frac{1}{2} \sum_{j=1}^{2N} x_j^2} dx_1 \dots dx_{2N}. \end{aligned}$$

We make a spherical change of variables $(x_3, \dots, x_{2N}) \mapsto r\sigma$ and the polar change of variables $(x_1, x_2) = (r \cos \theta, r \sin \theta)$. Denote by $s = t/\sqrt{1-t^2}$, thus there exists C_N so that

$$\begin{aligned} \Phi(t) &= C_N \int r^{2N-3} e^{-\frac{1}{2}(\rho^2 + r^2)} \mathbb{I}_{\rho > sr} \rho dr d\rho \\ &= C_N \int_0^{+\infty} r^{2N-3} e^{-\frac{1}{2}(1+s^2)r^2} dr. \end{aligned}$$

Now, by the change of variables $r' = (1+s^2)^{1/2}r$, there exists C_N so that

$$\Phi(t) = C_N (1+s^2)^{-(N-1)} = C_N (1-t^2)^{N-1},$$

and $C_N = \Phi(0) = 1$.

Appendix D

Proof of Proposition 2.13

For simplicity we assume that the random variables, the γ_j and the space \mathcal{E}_h are real, and we identify \mathcal{E}_h with \mathbb{R}^N , endowed with its natural Euclidean norm $|y|_0$. We also consider the γ -dependent norm

$$|y|_\gamma^2 = \frac{1}{N} \sum_{1 \leq j \leq N} \frac{y_j^2}{\gamma_j^2}, \quad y = (y_1, \dots, y_N).$$

Condition (1.2) means that we have

$$\frac{1}{C} |y|_0^2 \leq |y|_\gamma^2 \leq C |y|_0^2.$$

We define a probability measure ν_γ in \mathbb{R}^N as the pull forward of the measure ν in \mathbb{R}^N by the mapping $\varphi: (x_1, \dots, x_N) \mapsto \sqrt{N}(\gamma_1 x_1, \dots, \gamma_N x_N)$. Notice that ν_γ satisfies the concentration property of Definition 2.1.

Now we follow the proof of [13, Proposition 2.10]. Let F be a Lipschitz function on the sphere \mathbf{S}_h , and by homogeneity, it is enough to assume that F is 1-Lipschitz. For $x \in \mathbb{R}^N$, $G(x) = |x|_0 \left(F\left(\frac{x}{|x|_0}\right) - F(x_0) \right)$, where x_0 is a fixed point on \mathbf{S}_h . Thus G satisfies

$$(D.1) \quad |G(x) - G(y)|_0 \leq 2(\pi + 1)|x - y|_0,$$

and

$$(D.2) \quad F\left(\frac{x}{|x|_0}\right) - F\left(\frac{y}{|y|_0}\right) = G\left(\frac{x}{|x|_0}\right) - G\left(\frac{y}{|y|_0}\right).$$

By [13, Corollary 1.5], it is enough to prove that

$$\begin{aligned} \mathbf{P}_{\gamma,h} \otimes \mathbf{P}_{\gamma,h} [u, v \in \mathbf{S}_h : |F(u) - F(v)| \geq 3r] \\ = \nu_\gamma \otimes \nu_\gamma [x, y \in \mathbb{R}^N : \left| F\left(\frac{x}{|x|_0}\right) - F\left(\frac{y}{|y|_0}\right) \right| \geq 3r] \leq C e^{-cNr^2}. \end{aligned}$$

Denote by \mathcal{M}_γ the median of $x \mapsto |x|_0$ with respect to ν_γ . Then by (D.2)

$$\begin{aligned} \nu_\gamma \otimes \nu_\gamma [x, y \in \mathbb{R}^N : \left| F\left(\frac{x}{|x|_0}\right) - F\left(\frac{y}{|y|_0}\right) \right| \geq 3r] \\ = \nu_\gamma \otimes \nu_\gamma [x, y \in \mathbb{R}^N : \left| G\left(\frac{x}{|x|_0}\right) - G\left(\frac{y}{|y|_0}\right) \right| \geq 3r] \\ \leq \nu_\gamma \otimes \nu_\gamma [x, y \in \mathbb{R}^N : \left| G\left(\frac{x}{\mathcal{M}_\gamma}\right) - G\left(\frac{y}{\mathcal{M}_\gamma}\right) \right| \geq r] + 2\nu_\gamma [x \in \mathbb{R}^N : \left| G\left(\frac{x}{|x|_0}\right) - G\left(\frac{x}{\mathcal{M}_\gamma}\right) \right| \geq r]. \end{aligned}$$

By (D.1) we have

$$\left| G\left(\frac{x}{|x|_0}\right) - G\left(\frac{x}{\mathcal{M}_\gamma}\right) \right| \leq 2(\pi + 1) \left| \frac{|x|_0}{\mathcal{M}_\gamma} - 1 \right|,$$

which implies from (2.1) that

$$\nu_\gamma [x \in \mathbb{R}^N : \left| G\left(\frac{x}{|x|_0}\right) - G\left(\frac{x}{\mathcal{M}_\gamma}\right) \right| \geq r] \leq C_1 e^{-c_1 \mathcal{M}_\gamma^2 r^2}.$$

Similarly, by (D.1) and (2.1)

$$\nu_\gamma [x, y \in \mathbb{R}^N : \left| G\left(\frac{x}{\mathcal{M}_\gamma}\right) - G\left(\frac{y}{\mathcal{M}_\gamma}\right) \right| \geq r] \leq C_2 e^{-c_2 \mathcal{M}_\gamma^2 r^2}.$$

To conclude the proof of the proposition we use the following

Lemma D.1. — In \mathbb{R}^N , denote by $\mathcal{M}_\gamma(N)$ the median of $x \mapsto |x|_0$ with respect to ν_γ , and by $\mathcal{A}_\gamma(N)$ its expectation. Then there exist $C, C_1, C_2 > 0$ such that for all $N \geq 1$

$$|\mathcal{M}_\gamma(N) - \mathcal{A}_\gamma(N)| \leq C \quad \text{and} \quad C_1 \sqrt{N} \leq \mathcal{M}_\gamma(N) \leq C_2 \sqrt{N}.$$

Proof. — Here we use the notation $|x|_N := (x_1^2 + \dots + x_N^2)^{1/2}$. By definition of \mathcal{M}_γ , we have for all $t > 0$ and thanks to (2.5)

$$\frac{1}{2} = \mathbf{P}_{\gamma,h} [|x|_N \geq \mathcal{M}_\gamma] \leq e^{-t(\mathcal{M}_\gamma - \mathcal{A}_\gamma)} \mathbf{E} \left[e^{t(|x|_N - \mathcal{A}_\gamma)} \right] \leq e^{-t(\mathcal{M}_\gamma - \mathcal{A}_\gamma)} e^{ct^2}.$$

Then, we choose $t = (\mathcal{M}_\gamma - \mathcal{A}_\gamma)/(2c)$ and get for some $C > 0$, $|\mathcal{M}_\gamma - \mathcal{A}_\gamma| \leq C$. This was the first claim.

Next, by Cauchy-Schwarz, we obtain $\mathcal{A}_\gamma^2(N) \leq \int_{\mathbb{R}^N} |x|_N^2 d\nu(x) = N$. Now we prove that there exists $C > 0$ so that for all $N \geq 1$, $\mathcal{A}_\gamma(N) \geq C\sqrt{N}$. Indeed,

$$\begin{aligned} \mathcal{A}_\gamma(N+1) - \mathcal{A}_\gamma(N) &= \int_{\mathbb{R}^{N+1}} \frac{x_{N+1}^2}{|x|_N + |x|_{N+1}} d\nu(x) \\ &\geq \frac{1}{2} \int_{\mathbb{R}^{N+1}} \frac{x_{N+1}^2}{|x|_{N+1}} d\nu(x) = \frac{\mathcal{A}_\gamma(N)}{2(N+1)}. \end{aligned}$$

This implies that for all $N \geq 1$

$$\mathcal{A}_\gamma(N+1) \geq \left(1 - \frac{1}{2(N+1)}\right)^{-1} \mathcal{A}_\gamma(N) \geq \left(1 + \frac{1}{2(N+1)}\right) \mathcal{A}_\gamma(N),$$

and then $\mathcal{A}_\gamma(N) \geq P_N \mathcal{A}_\gamma(1)$, where

$$\ln P_N = \sum_{k=2}^N \ln\left(1 + \frac{1}{2k}\right) = \frac{1}{2} \ln N + \mathcal{O}(1),$$

which yields the result. □

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