Event-triggered control with LQ optimality guarantees for saturated linear systems

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Abstract: Given a predesigned linear state feedback law for a linear plant ensuring (global) exponential stability of the linear closed loop, together with a certain level of performance, we address the problem of recovering (local or global) exponential stability and performance in the presence of plant input saturation and of a communication channel between the controller output and the saturated plant input. To this aim, we adopt Lyapunov-based techniques which combine generalized sector conditions to deal with the saturation nonlinearity, and event-triggered techniques to deal with the communication channel. The arising analysis yields an event-triggered algorithm to update the saturated plant-input based on conditions involving the closed-loop state. The proposed Lyapunov formulation leads to numerically tractable conditions that guarantee local (or global) exponential stability of the origin of the sampled-data system with an estimate of the domain of attraction.

Keywords: event-based control, input saturation, linear quadratic performance

1. INTRODUCTION

In recent years, the study of sampled-data systems has provided several techniques of dealing with linear or nonlinear systems (see, for example, Nešić and Teel [2004], Fiter et al. [2012] Seuret [2012] and references therein). Among them, an interesting method so-called Event-Based Control suggests to adapt the sampling sequence to some events related to the state of the system (see for example Hespanha et al. [2007], Zampieri [2008], Tabuada [2007]). In this situation, the controlled system works in continuous-time whereas the controller provides a discrete-time input during a sampling period. Hence, the problem of the design of an event-triggered algorithm can be first rewritten as the stability study of a system with a mixed continuous/discrete dynamics (also called hybrid system), as considered e.g. in Donkers and Heemels [2010], Lehmann et al. [2012] or in Goebel et al. [2009, 2012], Prieur et al. [2007, 2010] in a different context.
Another important feature when dealing with the stability/performance analysis of control design problems resides in the presence of limitations of the actuator. It is now well known that the presence of saturation may cause loss in performance, even unstable behavior (see, for example, Tarbouriech et al. [2011] and references therein). At the knowledge of the authors, few results deal with event-based control and saturated system as in Kiener et al. [2013]. In the current paper, extending the results developed in Seuret and Prieur [2011], we use the hybrid framework and the Lyapunov theory to define the update policy to deal with event-triggered control algorithms for linear systems subject to plant input saturation. Hence, given a predesigned linear state feedback law for a linear plant ensuring (global) exponential stability of the linear closed loop together with a performance criterion as LQ cost, we address the problem of recovering (local or global) exponential stability and performance in the presence of plant input saturation and of a communication channel between the controller output and the saturated plant input. To this aim, we adopt Lyapunov-based techniques which combine generalized sector conditions to deal with the saturation nonlinearity and event-triggered techniques to deal with the communication channel. The arising analysis yields three architectures, namely 1) periodic sampling, 2) event-triggered sampling, 3) self-triggered sampling to update the saturated plant input based on conditions involving the state of the closed-loop system. The proposed Lyapunov formulation leads to numerically tractable conditions that guarantee local (or global) exponential stability of the origin of the sampled-data system with an estimate of the domain of attraction and LQ performance.
The paper is organized as follows. Section 2 describes the problem and presents the hybrid frameworks on which is based our technique. Section 3 is dedicated to the main results, addressing the three techniques 1) periodic sampling, 2) event-triggered sampling and 3) self-triggered sampling. Section 4 proposes an illustrative example allowing to point out the trade-off between the size of the region of attraction of the origin, the desired LQ performance index and the number of updates to be performed. Finally, Section 5 ends the paper with concluding remarks. Note

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that, for space limitations, the proofs of the theorems are omitted.

Notation. Throughout the article, the sets $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ and $\mathbb{S}^n$ denote respectively the sets of positive integers, positive scalars, n-dimensional vectors, $n \times n$ matrices and symmetric matrices of $\mathbb{R}^{n\times n}$. The notation $|\cdot|$ stands for the Euclidean norm. Given a compact set $\mathcal{A}$, the notation $|x|_{\mathcal{A}} = \min\{|x-y|,\ y \in \mathcal{A}]\}$ indicates the distance of the vector x to the set $\mathcal{A}$. The superscript 'T' stands for matrix transposition. A function μ is said to be of class $\mathcal{K}_\infty$ if it is continuous, zero at zero, increasing and unbounded. The symbols I and 0 represent the identity and the zero matrices of appropriate dimensions. For a given strictly positive integer m, define the set $\mathcal{S}_m = \{1, \dots, m\}$. For any $j \in \mathcal{S}_m$, define the set $\mathcal{S}_m^j$ of all possible sequences of j distinct elements of $\mathcal{S}_m$.

2. PROBLEM STATEMENT AND SAMPLED-DATA ARCHITECTURES

In this section we present our problem statement and we explain how the feedback system operated via sampled-and-hold or more sophisticated event-triggered sampling can be seen as a hybrid dynamical system using the notation of Goebel et al. [2009, 2012].

2.1 System data

Consider a linear plant with a saturated input

$$\begin{aligned} \dot{x} &= Ax + Bs, \\ s &= \mathrm{sat}(u), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ stand respectively for the state variable and the input vector. The matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant and given and such that the pair (A, B) is controllable.
We suppose that the function $\mathrm{sat}(\cdot)$ in (1) is a decentralized saturation with saturation bounds $u_0 = \left[\, u_{01} \cdots u_{0m} \,\right]^T$, namely $s = \mathrm{sat}(u)$ corresponds to enforcing $s_i = \max(-u_{0i}, \min(u_{0i}, u_i))$, where s_i and u_i denote the i-th components of s and u, respectively, for all $i = 1, \dots, m$. We will also use the decentralized deadzone function $\mathrm{dz}(u) = u - \mathrm{sat}(u)$ in the rest of the paper.

2.2 State feedback design and optimality criterion

In this paper we address the problem of event-triggered implementation of a static state-feedback stabilizing law for plant (1), given by the following equation

$$u = Kx, \tag{2}$$

where the gain $K \in \mathbb{R}^{m\times n}$ should be designed to ensure local asymptotic stability of the zero equilibrium of the arising closed-loop system (1), (2), with a guaranteed region of attraction that will be characterized by requiring that it contains the ball $\mathcal{B}(\alpha) := \{x \in \mathbb{R}^n : |x| \leq \alpha\}$ where $\alpha \in \mathbb{R}$ is a design parameter. Moreover, we require some optimality guarantee in the sense that for any initial condition $x(0) \in \mathcal{B}(\alpha)$, the corresponding (unique) solution to the closed-loop system (1), (2) is required to satisfy an LQ type of bound. More specifically, we provide an upper bound for the worst case cost:

$$J(\alpha) = \max_{x(0)\in\mathcal{B}(\alpha)} \int_0^\infty [x^T(t)Qx(t) + s^T(t)Rs(t)]\mathrm{d}t \tag{3}$$

where $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are symmetric positive definite matrices. Based on the above setting, we consider the following problem in this paper.

Problem 1. Given plant (1) and the linear state feedback law (2), given a scalar $\alpha > 0$, determine the gain K and a sampled-data implementation of the state feedback law (periodic, event-triggered or self-triggered) guaranteeing local exponential stability of the origin for the sampled-data system, with a region of attraction containing $\mathcal{B}(\alpha)$ and possibly an upper bound on the performance index (3).

2.3 Hybrid representation of sampled-data systems

A sampled-data implementation of the feedback law (2) for plant (1) corresponds to breaking the continuous-time closed loop given by $u(t) = Kx(t)$, for all $t \geq 0$ and converting this into a zero order hold $\dot{s} = 0$ combined with the update rule $s^+ = \mathrm{sat}(Kx)$ for s, which should be performed at suitable times according to the specific sampled-data architecture. As mentioned in the introduction, this architecture could be given by 1) *periodic sampling*, 2) *event-triggered sampling*; 3) *self-triggered sampling.* We explain the three architectures below and provide for each on of them a hybrid formulation that uses the hybrid formalism of Goebel et al. [2009], Prieur et al. [2007, 2013].
1) *Periodic sampling* corresponds to performing the update rule $s^+ = \mathrm{sat}(Kx)$ at periodic instants of time. Following, e.g., [Goebel et al., 2012, Example 1.4], the corresponding closed loop can be described by adding a timer $\tau \in \mathbb{R}$ to the hybrid model. Then, for any sampling period $T > 0$, the dynamics of the system can be rewritten as

$$\begin{aligned} &\begin{cases} \dot{x} = Ax + Bs, \\ \dot{s} = 0, & \tau \in [0, T]\ , \\ \dot{\tau} = 1, \end{cases} \\ &\begin{cases} x^+ = x, \\ s^+ = \mathrm{sat}(Kx), & \tau = T\ , \\ \tau^+ = 0, \end{cases} \end{aligned} \tag{4}$$

where $s \in \mathbb{R}^m$ represents the held value of the control input and it appears that the timer τ is confined to the compact set $[0, T]$.

Remark 1. As shown in Goebel et al. [2009], the hybrid model (4) expresses the case of periodic sampling. Since $\tau^+ = 0$ across jumps, all solutions have to flow for T ordinary time after each jump. This rules out Zeno solutions and also simplifies the implementation. A drawback is that, in general, the origin of system (4) is not asymptotically stable (although when $A + BK$ is Hurwitz one can show asymptotic stability of the origin for small enough T). Some conservative estimates of the range of values of T preserving asymptotic stability can be computed by using the results in Seuret and Gomes da Silva Jr. [2012]. However, to reduce the average number of samplings per unit of time, one needs to resort to alternative schemes such as the ones described below.

2) *Event-triggered sampling* corresponds to performing the update rule $s^+ = \mathrm{sat}(Kx)$ whenever the augmented state (x, s) belongs to suitable sets that should be designed in such a way to guarantee asymptotic stability. In this case, the sampled-data system does not require a timer and can be written as

$$\begin{cases} \dot{x} = Ax + Bs, \\ \dot{s} = 0, \end{cases} \quad \text{if } (x,s) \in \mathcal{F}_E, \\ \begin{cases} x^+ = x, \\ s^+ \in \mathrm{sat}(Kx), \end{cases} \quad \text{if } (x,s) \in \mathcal{J}_E, \tag{5}$$

where $s \in \mathbb{R}^m$ represents the held value of the control input and $\mathcal{F}_E$ and $\mathcal{J}_E$ are two subsets of $\mathbb{R}^n \times \mathbb{R}^m$ indicating where the solution is allowed to flow and to jump. Sets $\mathcal{F}_E$ and $\mathcal{J}_E$ are respectively called flow and jump sets and are the available degrees of freedom in the control design.

3) *Self-triggered sampling* corresponds to performing the update rule $s^+ = \mathrm{sat}(Kx)$ at times which are determined on-line, only based on the information about the last sample. As compared to the event-triggered case, this is more advantageous because it does not require to permanent monitor the plant state and is useful in situations where sampling the plant state is expensive in some sense. The sampled-data system can then be written as

$$\begin{cases} \dot{x} = Ax + Bs, \\ \dot{\xi} = 0, \\ \dot{s} = 0, \\ \dot{\tau} = 1, \end{cases} \quad (\xi, s, \tau) \in \mathcal{F}_S \cap \mathcal{F}_T, \\ \begin{cases} x^+ = x, \\ \xi^+ = x \\ s^+ = \mathrm{sat}(Kx), \\ \tau^+ = 0, \end{cases} \quad (\xi, s, \tau) \in \mathcal{J}_S \cap \mathcal{J}_T\ , \\ \mathcal{F}_T = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times [0,T],\ \mathcal{J}_T = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \{T\} \tag{6}$$

where the sets $\mathcal{F}_T$, $\mathcal{J}_T$ are introduced to ensure that the timer τ belongs to a compact set and where T would be typically selected as a large timeout constant. The self-triggered nature of the sample appears evident from the fact that the jump/flow condition (namely the structure of the jump and flow sets) only depends on the sampled values (ξ, s, τ) and is independent of the instantaneous value of the state x. Then, this implementation does not require acquiring continuous measurements from the plant. Just as in the previous case, the design of the self-triggered policy amounts to designing the flow and jump sets $\mathcal{F}_S$ and $\mathcal{J}_S$ in (6).

3. MAIN RESULTS

3.1 A saturated LQ design for a linear plants

In this section we introduce a design technique for the gains in (2) in such a way to guarantee a bound on the value of a linear quadratic cost function evaluated along the closed-loop solutions. The arising feedback stabilizing law will be revisited in the next section using an event-triggered implementation.

Theorem 1. Given $Q = Q^T > 0$ and $R = R^T > 0$, assume that there exist matrices $W = W^T > 0 \in \mathbb{R}^{n\times n}$, Y, X in $\mathbb{R}^{m\times n}$, a diagonal positive definite matrix $S > 0$ in $\mathbb{R}^{m\times m}$ and positive scalars α, μ satisfying the following linear matrix inequalities:

$$\begin{bmatrix} W & X_i \\ X_i^T & u_{0i}^2 \end{bmatrix} \geq 0\ \forall i = 1, ..., m, \quad \begin{bmatrix} I & \alpha I \\ \alpha I & W \end{bmatrix} \geq 0, \tag{7a}$$

$$\mathrm{He} \begin{bmatrix} AW + BY & BS & 0 & W \\ Y + X & -S & 0 & 0 \\ Y & S & -\mu R^{-1}/2 & 0 \\ 0 & 0 & 0 & -\mu Q^{-1}/2 \end{bmatrix} < 0, \tag{7b}$$

where X_i denotes the i-th column of matrix X. Then, selecting

$$K = YW^{-1} \tag{8}$$

in (2) ensures:

(1) local exponential stability of the origin of system (1), (2) with region of attraction $\mathcal{E}(W^{-1}) = \{x \in \mathbb{R}^n : x^T W^{-1} x \leq 1\}$ which contains the α-ball $\mathcal{B}(\alpha) = \{x \in \mathbb{R}^n : |x| \leq \alpha\}$;
(2) for any initial condition $x(0) \in \mathcal{E}(W^{-1})$, the cost function J in (3) evaluated along the corresponding (unique) solution to (1), (2) satisfies $J \leq \mu$;
(3) global exponential stability of the origin of system (1), (2) if the solution to (7) is such that $X = 0$.

The result of Theorem 1 can be used to determine a suitable trade-off between two conflicing goals: maximize the region of attraction of the closed loop (namely select a large α) and minimize the cost function (namely select a small μ). In particular, one may fix the size α of the guaranteed region of attraction and optimize performance by solving the following LMI eigenvalue problem:

$$\min_{W,X,Y,S,\mu} \mu, \text{ subject to (7)}, \tag{9}$$

or one may fix the desired performance level μ and maximize the size of guaranteed region of attraction by solving the following LMI eigenvalue problem:

$$\max_{W,X,Y,S,\alpha} \alpha, \text{ subject to (7)}. \tag{10}$$

In particular, either of the two optimizations above can be used to compute a curve corresponding to the boundary feasibility set for the LMI constraints (7) on the (α, μ) plane (see Section 4).

It should be emphasized that from the point of view of the achievable region of attraction, conditions (7) are not overly conservative. Indeed, it is known (see, e.g., Sontag [1984]) that global exponential stability (GES) can only be achieved from a bounded input if the plant has its poles in the open left half plane (namely it is already GES with zero input). Moreover, any plant with an exponentially unstable pole has a bounded controllability region (namely the set of initial conditions that can be driven to zero by a suitable – but bounded – input). Then the best that one can achieve from a bounded input is GES if A in (1) is Hurwitz, semiglobal exponential stability (SGES) if A has only eigenvalues with nonpositive real part and local results otherwise. These results are achievable with the construction in Theorem 1 as stated in the next proposition whose proof can be carried out using similar reasonings to [Teel, 1995, Lemma 3.1].

Proposition 1. Consider the LMI constraints (7). Then the following holds:

(1) if A in (1) is Hurwitz, then there is a feasible solution with $X = 0$;
(2) if A in (1) has no eigenvalues with positive real part, then for each (arbitrarily large) $\alpha > 0$ there is a feasible solution;
(3) in all cases, there exists a small enough $\alpha > 0$ leading to a feasible solution.

Moreover, given two positive scalars $\alpha_1 \leq \alpha_2$, if the corresponding optimization (9) has solutions μ_1^* and μ_2^*, then $\mu_1^* \leq \mu_2^*$. Finally given two positive scalars $\mu_1 \leq \mu_2$, if the corresponding optimization (10) has solutions α_1^*, α_2^*, then $\alpha_1^* \leq \alpha_2^*$.

Note that from item 3 of Theorem 1, the statement of item 1 of Proposition 1 implies that there is a solution inducing GES; from item 1 of Theorem 1, the statement of item 2 of Proposition 1 implies that for any arbitrarily large compact subset of $\mathbb{R}^n$ there is a solution inducing LES with region of attraction containing that set (semiglobal exponential stability).

3.2 Event-triggered control design

In this section we will propose an event-triggered implementation of the state feedback design of Theorem 1 in such a way to still ensure that $\mathcal{B}(\alpha)$ be contained in the region of attraction and that some relaxed LQ property is ensured. In particular, given a feasible solution to conditions (7) of Theorem 1, consider the arising Lyapunov matrix $P = W^{-1}$ and the controller gain $K = YW^{-1}$, the periodic sampling implementation (4) will not guarantee, in general, the stated LQ performance. Moreover, asymptotic stability of the origin may be lost too if T is too large. Here we suitably select the flow and jump sets $\mathcal{F}_E$ and $\mathcal{J}_E$ in (5) to preserve those properties, as clarified in the theorem below.

Before stating the theorem, it is useful to recall that, according to the hybrid framework in (Goebel et al. [2009, 2012]), each solution x to a hybrid system is defined on *hybrid time domains* $E = \mathrm{dom}(x)$ corresponding to a suitable subset of $\mathbb{R} \times \mathbb{N}$ satisfying the following property for all pairs $(T, J) \in \mathbb{R}^+ \times \mathbb{N}$:

$$E \cap ([0, T] \times [0, J]) = \bigcup_{j=0}^{J} [t_j, t_{j+1}] \times \{j\},$$

for the nondecreasing sequence of times $0 = t_0 \leq \cdots \leq t_{J+1} = T$, where $t_1, \ldots, t_J$ are called "jump times". We also call $\mathrm{dom}_j(x) = \{j \in \mathbb{N} : \exists t, \text{ s.t. } (t, j) \in \mathrm{dom}(x)\}$. Then $\mathrm{dom}(x)$ can be seen as the union of finitely many or infinitely many intervals of the type $[t_j, t_{j+1}] \times \{j\}$, $j \in \mathrm{dom}_j(x)$ with the last interval possibly satisfying $t_{j+1} = \infty$ and being open. When rewriting the LQ cost (3) in this context, one needs to take special care of the hybrid nature of solutions. A possible way to generalize (3) is the following one:

$$J(\alpha) = \max_{x(0,0) \in \mathcal{B}(\alpha)} \sum_{j \in \mathrm{dom}_j(x)} \int_{t_j}^{t_{j+1}} \psi(x(t, j), s(t, j)) \mathrm{d}t \tag{11}$$

where $Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times m}$ are symmetric positive definite matrices and $\psi(x, s) = x^T Q x + s^T R s$.

The following result relies upon Theorem 1.

Theorem 2. Consider any feasible solution to the constraints (7) and the corresponding values $P = W^{-1}$, $K = YW^{-1}$, α, μ. Given any scalar $\hat{\mu} \geq \mu$, define the following flow and jump sets for (5):

$$\mathcal{F}_E = \left\{ (x, s) : \quad \begin{bmatrix} x \\ s \end{bmatrix}^T \Pi_{\hat{\mu}} \begin{bmatrix} x \\ s \end{bmatrix} \leq 0 \right\}, \tag{12a}$$

$$\mathcal{J}_E = \left\{ (x, s) : \quad \begin{bmatrix} x \\ s \end{bmatrix}^T \Pi_{\hat{\mu}} \begin{bmatrix} x \\ s \end{bmatrix} \geq 0 \right\}. \tag{12b}$$

$$\Pi_{\hat{\mu}} = \begin{bmatrix} PA + A^T P + Q/\hat{\mu} & PB \\ B^T P & R/\hat{\mu} \end{bmatrix}. \tag{12c}$$

Then, denoting by $\mathcal{U}_0 = \{u \in \mathbb{R}^m : |\operatorname{diag}(u_0)^{-1} u|_\infty \leq 1\}$ the range of the saturation function, the event-triggered closed-loop system (5), (12) is such that the set

$$\mathcal{A} = \{0\} \times \mathcal{U}_0, \tag{13}$$

is locally exponentially stable with region of attraction including the set $\mathcal{E}(P) \times \mathcal{U}_0$.

Moreover, for each initial condition in $\mathcal{E}(P) \times \mathcal{U}_0$ there exists at least one solution having unbounded time domain in the ordinary time direction t. Finally, for all $x(0,0) \in \mathcal{E}(P) \supset \mathcal{B}(\alpha)$, the worst case LQ cost given by (11) satisfies $J(\alpha) \leq \hat{\mu}$.

Remark 2. Theorem 2 guarantees local stability of the origin while the results proposed in Kiener et al. [2013] only guarantees the convergence of the closed-loop trajectories to a bounded set around the origin.

Remark 3. The jump and flow sets in (12) can be suitably modified by selecting the desired LQ performance level $\hat{\mu}$. In particular, note that for $\hat{\mu} = \mu$ one recovers the same LQ performance as the continuous-time solution of Theorem 1. However, one may increase $\hat{\mu}$ and give up some performance because this leads to strictly larger flow sets and strictly smaller jump sets (the strict nature of this property comes from positive definiteness of ψ). As confirmed by the simulation results of Section 4, it is expected that enlarging the flow set and reducing the jump set leads to less jumps in solutions starting from the same initial conditions, namely smaller average sample rate. This is desirable from an event-triggered viewpoint. While this observation is only qualitative, its advantages are readily appreciated by inspecting the numerical results of Section 4.

Remark 4. The shape of the jump and flow sets in (12) heavily relies on the properties of the Lyapunov function that are established in Theorem 1. For example, an advantageous feature of these sets stands in the fact that whenever the continuous-time feedback of Theorem 1 would lead to a control input that remains saturated for some time interval (this is the case, for example, during the initial transient of a trajectory which starts far from the attractor), the event-triggered solution (5), (12) does not jump (namely it does not sample) for all that period of time. This fact can be appreciated by noticing that the flow set is defined as the set where the desirable Lyapunov decrease established in Theorem 1 is preserved (possibly an even larger version of it if $\hat{\mu}$ is strictly larger than μ). Since for all such responses the plant input remains constant also for the continuous-time solution (where the Lyapunov decrease is established by Theorem 1), then the event-triggered solution remains in the flow set without performing any sample until it is necessary to modify the plant input. This aspect is well illustrated by some of the simulations reported in Section 4.

Remark 5. Note that due to the definition of jump and flow sets in (12), the hybrid system exhibits Zeno solutions at the origin (even though it also admits solutions that never jump at the origin). This is caused by the fact that both jump and flow sets are closed, which causes a nonempty intersection (including the origin) and could be avoided picking a jump set which is not closed. Selecting closed jump and closed sets however ensures well posedness of the hybrid dynamics, as illustrated in [Goebel et al., 2012, Ch. 4 and 6] and allows to capture in the set of the (non-necessarily unique) solutions to the well-posed dynamics any possible limiting solution produced by arbitrarily small perturbations of the dynamics. For our specific case, the presence of a Zeno solution at the origin reveals that there might be defective trajectories which exhibit many jumps (many samples) close to the origin. This fact is confirmed to a certain extent by the simulation results reported in Section 4 and motivates future research where the proposed policy is preserved far from the origin (this is very effective also in light of Remark 4) and is suitably modified close to the origin where the state is small so that a small error might be perhaps tolerated.

3.3 Self-triggered control design

The event-triggered control design of the previous section has the drawback of requiring a continuous monitoring of the plant state to verify whether the state $\left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right]$ belongs to the flow or jump set and possibly trigger a sample if it belongs to the jump set. A more convenient implementation could be that of a self-triggered approach where it is not necessary to continuously monitor the plant state and it is possible to decide when to sample again based on the only knowledge of the past sampled value of the state.

Due to the linear nature of dynamics (1) this is possible using the architecture in (6) and explicitly computing the current value of the state $x(t,j)$, $t \geq t_j$ based on the last sample $x(t_j,j)$ which is held in the additional state variable ξ. In particular, it is evident that for $j \geq 1$, any solution to (6) satisfies $\xi(t,j) = \xi(t_j,j) = x(t_j,j)$ which means that the following holds:

$$\begin{aligned}\begin{bmatrix} x(t,j) \\ s(t,j) \end{bmatrix} &= \exp\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\tau\right)\begin{bmatrix} \xi(t,j) \\ s(t,j) \end{bmatrix} \\ &:= M(\tau)\begin{bmatrix} \xi(t,j) \\ s(t,j) \end{bmatrix}.\end{aligned}$$

The following flow and jump sets for (6) are then selected:

$$\begin{aligned}\mathcal{F}_S &= \left\{(\xi,s,\tau): \; \left(M(\tau)\begin{bmatrix} \xi \\ s \end{bmatrix}\right)^T \Pi_{\hat\mu}\left(M(\tau)\begin{bmatrix} \xi \\ s \end{bmatrix}\right) \leq 0\right\}, \\ \mathcal{J}_S &= \left\{(x,s): \; \begin{bmatrix} x \\ s \end{bmatrix}^T \Pi_{\hat\mu}\begin{bmatrix} x \\ s \end{bmatrix} \geq 0\right\}.\end{aligned} \tag{14}$$

where $\Pi_{\hat\mu}$ is defined in (12c), and the trajectories of the arising self-triggered closed loop (6), (14) coincide with those of the event-triggered one in the previous section. In particular, one can prove the next result following exactly the same steps as those in the proof of Theorem 2.

Theorem 3. Consider any feasible solution to the constraints (7) and the corresponding values $P = W^{-1}$, $K = YW^{-1}$, α, μ. Given any scalar $\hat\mu \geq \mu$, the event-triggered system (6), (14), (12c) is such that the set

$$\mathcal{A} = \{0\} \times \mathcal{E}(P) \times \mathcal{U}_0 \times [0,T], \tag{15}$$

is locally exponentially stable with region of attraction including the set $\mathcal{E}(P) \times \mathcal{U}_0$.

Moreover, for each initial condition in $\mathcal{E}(P) \times \mathcal{E}(P) \times \mathcal{U}_0 \times [0,T]$ there exists at least one solution having unbounded time domain in the ordinary time direction t. Finally, for all $x(0,0) \in \mathcal{E}(P) \supset \mathcal{B}(\alpha)$, the worst case LQ cost given by (11) satisfies $J(\alpha) \leq \hat\mu$.

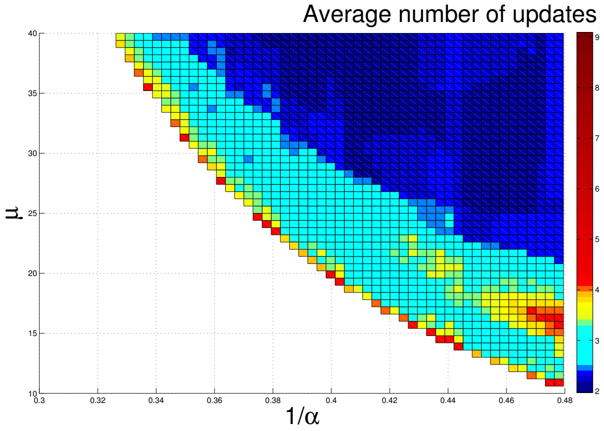

Fig. 1. Graph representing the average number of control updates with respect to the performance index $\hat\mu$ and the size of attraction set $1/\alpha$.

4. SIMULATION EXAMPLES

Consider the system (1) studied in Zaccarian and Teel [2011] with

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -f/m \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}. \tag{16}$$

where $k = 1$, $m = 0.1$ and $f = 0.01$ and $u_0 = 1$. The cost function (3) is taken with $Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $R = 0.5$. The simulation results are described below.

First, Figure 1 shows the evolution of the average number of control updates with respect to $1/\alpha \in [0.32\ 0.48]$, the inverse of the size of the attraction set and $\hat\mu \in [10\ 40]$, the performance index. To obtain this figure, a gridding of $1/\alpha$ over the interval $[0.32\ 0.48]$ is performed. For each value of $1/\alpha$, the matrices P and K are obtained by solving the conditions of Theorem 1 and the optimization problem (9). Then the performance index $\hat\mu$ is gridded from its minimal value until 40. Finally, for these parameters, several simulations are run using 12 different initial conditions which satisfies $V(x_0) = 1$. The simulation stops when V becomes lower than 0.1. This allows focussing on the transient response of the system. The average number of control updates is computed over the simulation time and the initial conditions.

In Figure 1, the boundary between the regions for which there exists a solution to Theorem 1 and the region where such a solution does not exist, represents the optimal solutions to the saturated LQ control design. This curve shows the trade-off between the size of the domain of attraction and the performance index. In Figure 1, the greatest number of updates is achieved when the parameters α and $\hat\mu$ are taken close to the optimal solutions to the saturated LQ control design. As stated in Remark 3, Figure 1 also depicts that selecting a larger $\hat\mu$ reduces the average number of control updates since it enlarges the flow set and reduces the jump set.

Solving problem (9) with $\alpha = 2.5$ delivers the Lyapunov matrix and the controller gain given by

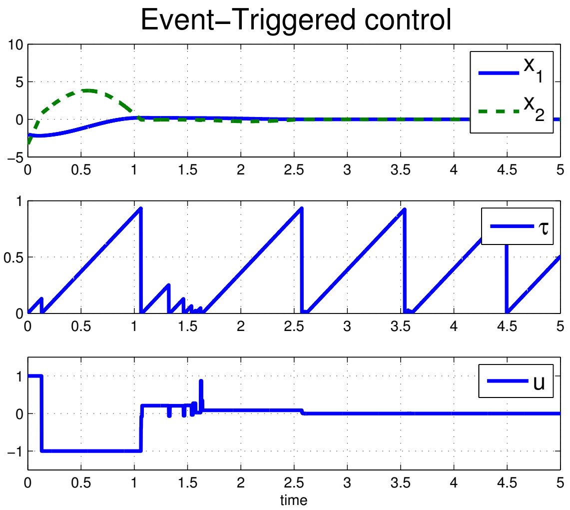

Fig. 2. Simulation results representing the state, the timer and the control input for the initial conditions $x_0 = [0.375 \quad -6.1005]^T$.

$$P = \begin{bmatrix} 0.1597 & 0.0062 \\ 0.0062 & 0.0240 \end{bmatrix}, \quad K = [-3.2182 \quad -12.1459].$$

Using this controller gain and $\hat{\mu} = 30$, simulations are provided in Figure 2 with the initial conditions $x_0 = [0.375 \quad -6.1005]^T$.Using the condition provided Seuret and Gomes da Silva Jr. [2012] on stability of saturated sampled-data systems, this control gain ensures stability of the closed-loop system if the timer τ is lower than $0.015s$. However, in Figure 2, the time interval is greater than 0.9. This shows the benefits of the event-triggered control provided in Theorem 2.

5. CONCLUSION

Focusing on linear plants with input saturation we proposed a stabilizing linear static state feedback law with LQ performance guarantees and guaranteed domain of attraction. Then we proposed different algorithms to implement the saturated control law in a sampled-data context. More precisely periodic, event-triggered and self-triggered sampling architectures have been considered. It has been shown that a suitable event-triggered algorithm may recover the same LQ performance as the continuous-time saturated controller and the same domain of attraction. The event-triggered algorithm is then modified to derive a self-triggered algorithm ensuring the same properties for the closed-loop system.

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