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► **To cite this version:**

Eric Menguy, Jean-Louis Boimond, Laurent Hardouin, Jean-Louis Ferrier. A First Step towards Adaptive Control for Linear Systems in Max Algebra. *Discrete Event Dynamic Systems*, 2000, 10 (4), pp.347-367. hal-00844937

**HAL Id: hal-00844937**

**<https://hal.science/hal-00844937>**

Submitted on 16 Jul 2013

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# A FIRST STEP TOWARDS ADAPTIVE CONTROL FOR LINEAR SYSTEMS IN MAX ALGEBRA

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**Keywords:** Discrete Event Systems, Timed Event Graphs, Max-Algebra, Identification, Adaptive Control.

## Abstract

A linear system theory has been developed for discrete event systems subject to synchronization. We are interested in the just in time control of such systems in presence of a possible mismatch between the system and its model. Taking such mismatch into account in controller synthesis prevents us from using the usual open loop control structure. The approach we propose to synthesize the controller is inspired by the principle of the indirect adaptive control well known in conventional system theory. Its design leads to solve an optimal tracking problem under an equality constraint. Preliminary results about the estimation of models are presented.

## 1 Introduction

Timed Event Graphs (TEG) are a subclass of Petri nets well adapted to modeling synchronization, moreover they can be considered as  $(max, +)$ -linear systems. A linear system theory has been developed [1, §6], [13] using the conventional linear system theory as a guideline. The concept of stationary, continuous and linear systems has been extended to the structures of moduloid in particular. For such systems, the output is the result of a "kind" of convolution product between the input and the system impulse response.

This paper deals with the just in time control, i.e., given desired output transition firing dates  $z = \{z(k)\}_{(k \in \mathbb{Z})}$  also called reference input, we search for the *latest* input transition firing dates  $u = \{u(k)\}_{(k \in \mathbb{Z})}$  such that output transition firing dates occur *at the latest before* the given ones. This tracking problem formally consists in finding the greatest solution to inequation  $G(u) \preceq z$  where  $G$  is a  $(max, +)$ -linear operator and  $G(u) = \{[G(u)](k)\}_{(k \in \mathbb{Z})}$  is the output associated with input  $u$ . In a production context, such a control minimizes the work in process (find the greatest input) while satisfying the customer demand (the output is less than or equal to the reference input).

An optimal control exists when the model is exact [1, §5.6], [7], its synthesis is based on an open loop control structure and principally uses the Residuation theory [2]. Extensions are proposed in the case of an update of reference input or in the presence of uncontrollable input [16], or on model matching [12], [11]. However these controller syntheses do not allow taking into account mismatch, always present in practice, between the system and its model (the latter being used to compute the

control). Some approaches have already been proposed like in [3] using the internal model control structure and in [15], [14, §6] using the notion of modified control. The approach we present here is based on the indirect adaptive control principle well known in conventional control theory. The term "indirect" means that the identification of system parameters is separated from the design of the controller, as shown in Fig. 1:

- A model is designed from the observed input-output data of the system in block "Estimation";
- The control law is designed in order to try to control the system in just in time from its model in block "Controller".

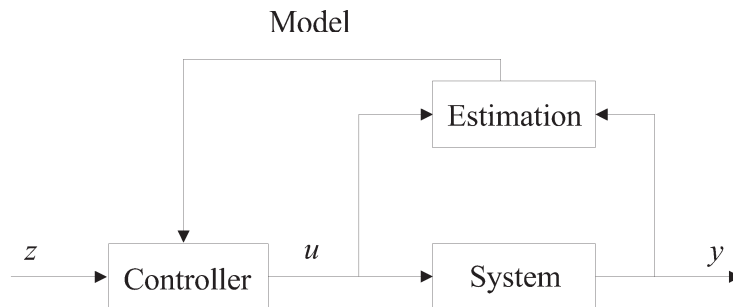


Figure 1: Block diagram of an indirect adaptive controller.

The paper is organized as follows. In section 2, we recall a few algebraic results about dioid, linear system and Residuation theories. The control problem is stated in section 3; for the sake of simplicity and without loss of generality, systems are assumed to be single-input, single-output. See [14] for the multi-input, multi-output case. We recall the expression of the optimal control when the model is exact, and we describe the adaptive control mechanism: the controller parameters are updated indirectly *via* the estimation of a model. We propose in section 4 an elementary estimation method which tries to fit a model (characterized by its periodic form) from the observed input-output data of the system. The controller design is considered in section 5 and leads to solve an optimal tracking problem under an equality constraint. It formally consists in finding the greatest solution to  $M(u) \preceq z$  which satisfies equality  $F(u) = c$ , where  $M$  and  $F$  are  $(max, +)$ -linear operators and  $c = \{c(k)\}_{(k \in \mathbb{Z})}$  is a fixed trajectory. The solution is also based on Residuation theory. To guarantee the existence of a solution, it may be necessary to modify the reference input which weakens the just in time objective: the firing dates of the output transition occur *as close as possible to* (eventually after) the desired ones. The control algorithm is presented in section 6. Before concluding we give a short example in section 7.

## 2 Preliminaries

### 2.1 Dioid

For an exhaustive presentation see [1], [8], [9]. A *dioid*  $(\mathcal{D}, \oplus, \otimes)$  is a semiring in which  $\oplus$  is idempotent ( $\forall a, a \oplus a = a$ ). Neutral elements of  $\oplus$  and  $\otimes$  are noted  $\varepsilon$  and  $e$  respectively. Due to idempotency of  $\oplus$ , a natural order relation is defined by  $a \preceq b \iff b = a \oplus b$  (the least upper bound of  $\{a, b\}$  is equal to  $a \oplus b$ ).

A dioid  $\mathcal{D}$  is *complete* if each subset  $A$  of  $\mathcal{D}$  admits a least upper bound denoted  $\bigoplus_{x \in A} x$ , and if  $\otimes$

distributes at left and at right over infinite sums. The greatest element noted  $T$  of a complete dioid  $\mathcal{D}$  is equal to  $\bigoplus_{x \in \mathcal{D}} x$ . The largest lower bound of every subset  $X$  of a complete dioid  $\mathcal{D}$  always exists and is noted  $\bigwedge_{x \in X} x$ .

A dioid  $\mathcal{D}$  is *distributive* if it is complete and if for all subsets  $C$  of  $\mathcal{D}$ ,  $(\bigwedge_{c \in C} c) \oplus a = \bigwedge_{c \in C} (c \oplus a)$  and  $(\bigoplus_{c \in C} c) \wedge a = \bigoplus_{c \in C} (c \wedge a)$ .

### Examples:

- The set  $\mathbb{R} \cup \{\mp\infty\}$ , endowed with *max* as  $\oplus$  and usual addition as  $\otimes$ , is a complete dioid noted  $\overline{\mathbb{R}}_{max}$  with neutral elements  $\varepsilon = -\infty$  for *max* and  $e = 0$  for  $+$ , moreover  $T = +\infty$ .
- Starting from a 'scalar' dioid  $\mathcal{D}$ , let us consider  $p \times p$  matrices with entries in  $\mathcal{D}$ . The sum and product of matrices are defined conventionally after the sum and product of scalars in  $\mathcal{D}$ . Let  $A = (A_{ij})$  and  $B = (B_{ij})$  two matrices in  $\mathcal{D}^{p \times p}$ , then

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij} \quad \text{and} \quad (A \otimes B)_{ij} = \bigoplus_{k=1}^p A_{ik} \otimes B_{kj}.$$

This set of matrices endowed with these two operations is also a dioid denoted  $\mathcal{D}^{p \times p}$ .

## 2.2 Linear System

See [1, §6], [10], [13] for an exhaustive presentation. With each transition  $X$  of a TEG, we associate  $x = \{x(k)\}_{(k \in \mathbb{Z})}$  where  $x(k)$  denotes the firing date numbered  $k$  of  $X$ . The map  $x$  is considered as a nondecreasing function from  $\mathbb{Z}$  into  $\overline{\mathbb{R}}_{max}$ , usually called "dater". We introduce the dioid  $\Sigma = Nondec(\overline{\mathbb{R}}_{max}^{\mathbb{Z}})$  of nondecreasing functions from  $\mathbb{Z}$  into  $\overline{\mathbb{R}}_{max}$  endowed with:

1. pointwise *max* as addition, noted  $\oplus$ :  $\forall k \in \mathbb{Z}, (u \oplus v)(k) = u(k) \oplus v(k) = \max(u(k), v(k))$ ;
2. the *sup-convolution* as multiplication, noted  $\otimes$ :

$$\forall k \in \mathbb{Z}, (u \otimes v)(k) = \bigoplus_{s \in \mathbb{Z}} [u(s) \otimes v(k-s)] = Sup_{s \in \mathbb{Z}} [u(s) + v(k-s)].$$

Let us note that in  $\overline{\mathbb{R}}_{max}$ ,  $\begin{cases} a + b = -\infty & \text{if } a \text{ or } b = -\infty \\ a + b = +\infty & \text{if } (a \text{ or } b = +\infty) \text{ and } (a \text{ and } b \neq -\infty) \end{cases}$ .

$(\Sigma, \oplus, \otimes)$  is a distributive dioid. The neutral elements of  $\oplus$  and  $\otimes$  are respectively defined by:

$$\varepsilon(k) = -\infty, \forall k \in \mathbb{Z} \quad \text{and} \quad e(k) = \begin{cases} 0 & \text{if } k \geq 0 \\ -\infty & \text{otherwise} \end{cases}.$$

The natural order over  $\Sigma$  is defined by:  $u \preceq v \iff u(k) \leq v(k), \forall k \in \mathbb{Z}$ , where  $\leq$  is the usual order in  $\mathbb{R}$ .

It can be shown (see [1, §6], [13]) that a stationary, continuous and linear system over the set of *daters*  $\Sigma$  is represented by a convolution operator, more precisely:

$$[S(u)](k) = (g \otimes u)(k) = \bigoplus_{s \in \mathbb{Z}} [g(s) \otimes u(k-s)] = Sup_{s \in \mathbb{Z}} [g(s) + u(k-s)],$$

$g$  is called the *impulse response* of system  $S$ . Let us consider some basic properties of map  $g$ .

**Proposition 1 :**

1.  $g(k) = (g \otimes e)(k)$ ,  $\forall k \in \mathbb{Z}$ . The response results from the impulse  $e$  (this input comes down to firing transition  $u$  an infinite number of times at time 0).
2.  $g$  is nondecreasing ( $g$  is a dater).
3.  $g$  is unique.

**Proof:**

1. This point is trivial because  $e$  is the neutral element of operator  $\otimes$ .
2. We have  $\forall k \in \mathbb{Z}$ ,  $g(k) = (g \otimes e)(k) = \text{Sup}_{s \in \mathbb{Z}} [g(s) + e(k - s)] = \text{Sup}_{s \leq k} g(s)$ , thus  $g$  is nondecreasing.
3. Let  $g_1, g_2$  two maps such that  $\forall u \in \Sigma$ ,  $[S(u)](k) = (g_1 \otimes u)(k) = (g_2 \otimes u)(k)$ . We clearly have  $[S(e)](k) = g_1(k) = g_2(k)$ ,  $\forall k \in \mathbb{Z}$ .  $\diamond$

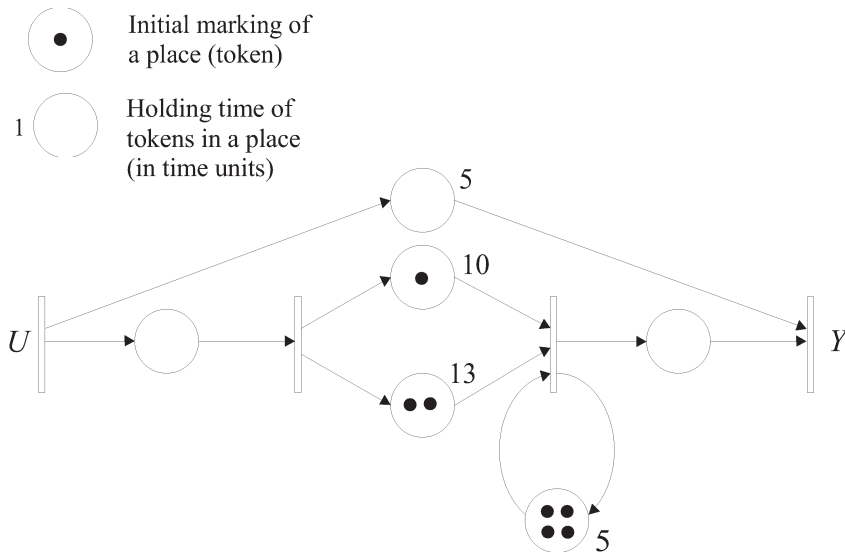


Figure 2: A timed event graph.

**Example:** The impulse response noted  $m_0$  of TEG in Fig. 2 is equal to

$$m_0(k) = \begin{cases} -\infty & \text{if } k < 0 \\ 5 & \text{if } k = 0 \\ \max(10 + 5 \lfloor \frac{k-1}{4} \rfloor, 13 + 5 \lfloor \frac{k-2}{4} \rfloor) & \text{if } k \geq 1 \end{cases} \quad (\lfloor x \rfloor \text{ is the truncated part of } x)$$

A fundamental result applying to TEG is the *periodicity* of these systems.

**Proposition 2** (see [5] for proof): The impulse response  $g$  of a TEG verifies the following conditions: it exists  $(\nu, \tau) \in \mathbb{N}^2$  such that

$$\begin{cases} g(k) = -\infty & \forall k < 0 \\ \exists N \in \mathbb{N}, \forall n \geq N & g(n + \nu) = \tau + g(n) \end{cases} \quad (1)$$

$(\nu, \tau)$  is called the *period* of  $g$  and  $N$  is the *transient length* ( $\nu$  firings occur every  $\tau$  units of time after a transient behavior of length  $N$ ).

Let us note that the impulse response  $m_0$  of the net in Fig. 2 is such that  $N = 1$ ,  $\nu = 4$  and  $\tau = 5$ .

**Remark 1** : We deduce from Eqs. (1) that the periodic impulse response  $g$  is entirely defined by parameters  $g(k)$ ,  $k \leq N + \nu - 1$ , since

$$g(k) = g(k - \lfloor \frac{k-N}{\nu} \rfloor \nu) + \lfloor \frac{k-N}{\nu} \rfloor \tau, \forall k \geq N + \nu.$$

**Definition 1 (Valuation)**: Let  $v \in \Sigma$ , we define the operator valuation, denoted  $val$ , by:

$$val : \Sigma \rightarrow \mathbb{Z} \cup \{\pm\infty\}; v \mapsto val v = Inf\{k \in \mathbb{Z} \mid v(k) > -\infty\}.$$

This operator satisfies the following properties.

**Proposition 3** :

1.  $val(u \oplus v) = Min(val u, val v)$ .  
In particular,  $val$  is an antitone map, i.e.,  $u \preceq v \Rightarrow val u \geq val v$ .
2.  $val(u \otimes v) = val u + val v$ .

**Proof:**

1.  $(u \oplus v)(k) = max(u(k), v(k)) > -\infty \iff u(k) > -\infty$  or  $v(k) > -\infty \iff k \geq val u$  or  $k \geq val v$ . We deduce that  $Inf\{k \in \mathbb{Z} \mid (u \oplus v)(k) > -\infty\} = Min(val u, val v)$ .
2.  $(u \otimes v)(k) = Sup_{s \in \mathbb{Z}} [u(s) + v(k-s)] > -\infty \iff \exists j \in \mathbb{Z}, u(j) + v(k-j) > -\infty \iff \exists j \in \mathbb{Z}, u(j) > -\infty$  and  $v(k-j) > -\infty \iff \exists j \in \mathbb{Z}, j \geq val u$  and  $k \geq j + val v$ . We deduce that  $Inf\{k \in \mathbb{Z} \mid (u \otimes v)(k) > -\infty\} = val u + val v$ .  $\diamond$

The following result shows that the firing numbered  $k$ ,  $k \geq val(g \otimes u)$ , of output transition  $Y$  depends only on firings numbered  $val u, \dots, k - val g$  of input transition  $U$ .

**Proposition 4** : If  $val u$  and  $val g$  are finite values, then  $\forall k \geq val(g \otimes u)$ ,

$$(g \otimes u)(k) = Max_{val g \leq j \leq k - val u} [g(j) + u(k-j)].$$

**Proof:** If  $(g \otimes u)(k) > -\infty$  then  $k \geq val(g \otimes u)$  and  $(g \otimes u)(k) = Sup_{\{j \mid g(j) + u(k-j) > -\infty\}} [g(j) + u(k-j)]$ .

Moreover,  $g(j) + u(k-j) > -\infty \iff g(j) > -\infty$  and  $u(k-j) > -\infty \iff j \geq val g$  and  $k-j \geq val u \iff j \in [val g, k - val u]$ . This interval is non-empty since  $k \geq val(g \otimes u) = val g + val u$  using Prop. 3.2.  $\diamond$

## 2.3 Residuation Theory

For an exhaustive presentation see [2], [1, §4]. Laws  $\oplus$  and  $\otimes$  of a dioid are not reversible in general. Nevertheless *Residuation* is a general notion in lattice theory which allows defining "pseudo-inverses" of some isotone maps ( $f$  is isotone if  $a \preceq b \Rightarrow f(a) \preceq f(b)$ ). Let us present some basic results about Residuation theory.

**Definition 2 (Residual of map):** An isotone map  $f : \mathcal{D} \mapsto \mathcal{C}$  where  $\mathcal{D}$  and  $\mathcal{C}$  are ordered sets is said to be residuated if it exists an isotone map  $h : \mathcal{C} \mapsto \mathcal{D}$  such that  $f \circ h \preceq Id_{\mathcal{C}}$  and  $h \circ f \succeq Id_{\mathcal{D}}$  ( $Id_{\mathcal{D}}$  and  $Id_{\mathcal{C}}$  are the identity elements of  $\mathcal{D}$  and  $\mathcal{C}$  respectively). Map  $h$  also noted  $f^\sharp$  is unique and is called the residual of  $f$ .

If  $f$  is residuated then  $\forall y \in \mathcal{C}$ , the least upper bound of subset  $\{x \in \mathcal{D} \mid f(x) \preceq y\}$  exists and belongs to this subset. This *greatest* 'subsolution' is equal to  $f^\sharp(y)$ .

Let  $\mathcal{D}$  a complete dioid and consider the isotone map  $L_a : x \mapsto a \otimes x$ . The greatest 'subsolution' to inequation  $a \otimes x \preceq b$  exists and is equal to  $L_a^\sharp(b)$  also denoted  $\frac{b}{a}$ . Four basic results related to this quotient are given in the following proposition.

**Proposition 5 :** (see [1, §4.4.4 and §4.6] for proofs):

1.  $\forall a, b \in \mathcal{D}, \quad a \otimes \left(\frac{b}{a}\right) \preceq b$  and  $\frac{a \otimes b}{a} \succeq b$ .
2. Map  $x \mapsto \frac{x}{a}$  is isotone and map  $x \mapsto \frac{a}{x}$  is antitone (a map  $f$  is antitone if  $a \preceq b \Rightarrow f(a) \succeq f(b)$ ).
3.  $\frac{b \wedge c}{a} = \frac{b}{a} \wedge \frac{c}{a}$ .
4. Let  $A \in \mathcal{D}^{n \times p}$  and  $B \in \mathcal{D}^{n \times q}$ ,  $\frac{B}{A} \in \mathcal{D}^{p \times q}$  and  $\left(\frac{B}{A}\right)_{ij} = \bigwedge_{l=1}^n \frac{B_{lj}}{A_{li}}, \quad 1 \leq i \leq p, 1 \leq j \leq q$ .

**Remark 2 :**

1. If equation  $a \otimes x = b$  admits a solution, then  $\frac{b}{a}$  is the greatest solution to this equation. Indeed, let  $c$  a solution, by definition of  $\frac{b}{a}$  we have  $c \preceq \frac{b}{a}$ , and by isotony of  $x \mapsto a \otimes x$ , we have  $b = a \otimes c \preceq a \otimes \frac{b}{a} \preceq b$ . We deduce that  $a \otimes \frac{b}{a} = b$ .
2. Let  $u, v \in \Sigma$ , then the *dater*  $\frac{v}{u}$  exists and is defined by:

$$\frac{v}{u}(k) = \bigwedge_{s \in \mathbb{Z}} \frac{v(s)}{u(s-k)} = \text{Inf}_{s \in \mathbb{Z}} [v(s) - u(s-k)].$$

Let us note that in  $\overline{\mathbb{R}}_{max}$ ,  $\begin{cases} a - b = +\infty & \text{if } b = -\infty \text{ or } a = +\infty \\ a - b = -\infty & \text{if } (a = -\infty \text{ and } b \neq -\infty) \text{ or } (a \neq +\infty \text{ and } b = +\infty) \end{cases}$ .

### 3 Control Problem Statement

The just in time control consists in firing the input transition *at the latest* so that the firing dates of the output transition occur *at the latest before* the desired ones. Let us define reference input  $z = \{z(k)\}_{(k \in \mathbb{Z})}$  as the *dater* of the desired outputs:  $z(k) = t$  means that the firing numbered  $k$  of the output transition is desired at the latest at time  $t$ . Formally, we search for the *greatest solution* to equation:

$$y = g \otimes u \preceq z \quad (\text{i.e., } y(k) \leq z(k), \forall k \in \mathbb{Z}).$$

This optimal control noted  $u_{opt}$  exists and is given by Residuation theory (cf. section 2.3):

$$u_{opt} \stackrel{\text{def}}{=} \bigoplus_{\{u \in \Sigma \mid g \otimes u \preceq z\}} u = \frac{z}{g} \quad (\text{consider map } L_g : x \mapsto g \otimes x).$$

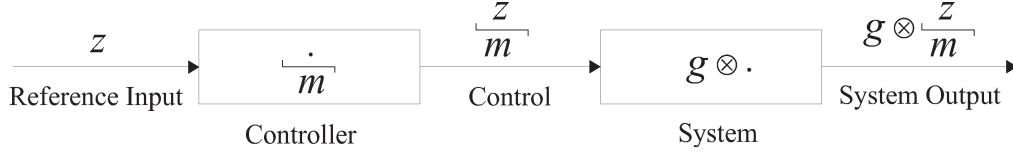


Figure 3: Open loop control.

Let us note  $g$  the system impulse response and  $m$  the model impulse response. In practice, the optimal model control  $\frac{z}{m}$  is applied to the system input, which corresponds to the open loop control represented in Fig. 3.

Such a control cannot take into account possible mismatch between the system and its model. To try to control the system in just in time, we propose using an open-loop adaptive control principle by proceeding as follows:

- Initially, the system is supposed to be known through a model noted  $m_0$ . Let  $u_0 = \frac{z}{m_0}$  the optimal model control. We apply control  $u_0$  to the system input until a time  $t_0$ , i.e.,

$$u_0(k), \text{ val } u_0 \leq k \leq K_0 \quad \text{with} \quad K_0 = \text{Sup} \{ k \in \mathbb{Z} \mid u_0(k) \leq t_0 \}.$$

- At time  $t_n$ ,  $n \geq 0$ , a model noted  $m_{n+1}$  is estimated from observed input-output data in block "Estimation" in Fig. 1:

$$\begin{cases} u_n(k), \text{ val } u_n \leq k \leq K_n & \text{with} \quad K_n = \text{Sup} \{ k \in \mathbb{Z} \mid u_n(k) \leq t_n \} \\ y(k) = (g \otimes u_n)(k), \text{ val } y \leq k \leq K'_n & \text{with} \quad K'_n = \text{Sup} \{ k \in \mathbb{Z} \mid (g \otimes u_n)(k) \leq t_n \} \end{cases}.$$

Control  $u_{n+1}$  is computed from this model in block "Controller" in Fig. 1. We apply control  $u_{n+1}$  until a time  $t_{n+1}$ , i.e.,

$$u_{n+1}(k), k \leq K_{n+1} \quad \text{with} \quad K_{n+1} = \text{Sup} \{ k \in \mathbb{Z} \mid u_{n+1}(k) \leq t_{n+1} \}.$$

**Remark 3 :** Due to controls already applied until time  $t_n$ , control  $u_{n+1}$  must verify

$$u_{n+1}(k) = u_n(k), \quad \forall k \leq K_n.$$

Control  $u_{n+1}$  computed at time  $t_n$  will be the greatest element of set

$$\{ u \in \Sigma \mid m_{n+1} \otimes u \preceq z \quad \text{and} \quad u(k) = u_n(k), \forall k \leq K_n \}.$$

The estimation of model parameters is tackled in next section. The resolution of this optimal tracking problem under an equality constraint is considered in section 5.

## 4 Identification

The input and output data from the system (represented by its *unknown* impulse response  $g$ ) are recorded and subjected to analysis in order to infer a model. The candidate models we consider have periodic impulse responses (cf. Prop. 2), in addition period  $(\nu, \tau)$  and transient length  $N$  are known. In the paper these values are constant, which sets the model structure, they are estimated beforehand



with an off-line method based on an analysis of the system impulse response [4]. Let  $\widehat{m}$  the estimated model. The identification method then consists in determining parameters  $\widehat{m}(k)$ ,  $k \leq N + \nu - 1$ , which allows reproducing the observed input-output data.

We consider the identification of a finite number of parameters of sequence  $\{g(k)\}_{(k \leq K)}$ ; we present two methods:

1. The first one provides an overestimation of system parameters compatible with observed input-output data. It is based on a direct application of Residuation theory.
2. The second one allows providing an estimation compatible with observed input-output data from an initial model. This method uses the previous overestimation.

## 4.1 Identification of a Finite Number of Parameters

We suppose that we have observed the firing dates  $y(\text{val } y), \dots, y(K)$  where  $K$ ,  $K \geq \text{val } y$ , is a fixed event. Using Prop.4, we easily verify that these outputs depend only on parameters  $g(k)$ ,  $\text{val } g \leq k \leq K - \text{val } u$ ; more precisely we have the following matricial representation:

$$\begin{bmatrix} u(\text{val } u) & \varepsilon & \dots & \varepsilon \\ u(\text{val } u + 1) & u(\text{val } u) & \dots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ u(K - \text{val } g) & \dots & \dots & u(\text{val } u) \end{bmatrix} \otimes \begin{bmatrix} g(\text{val } g) \\ \vdots \\ \vdots \\ g(K - \text{val } u) \end{bmatrix} = \begin{bmatrix} y(\text{val } y) \\ \vdots \\ \vdots \\ y(K) \end{bmatrix}$$

that we denote  $\mathcal{U}_K \otimes g_K = Y_K$ . The coefficients of matrice and the vectors are elements of  $\overline{\mathbb{R}}_{max}$  and matricial product  $\otimes$  is such that conventional sum and usual product are replaced by operator  $max$  and usual sum respectively (see section 2.1). Then we are interested in resolution of equation

$$\mathcal{U}_K \otimes x = Y_K. \quad (2)$$

to which vector  $g_K$  is solution.

**Remark 4 :**

- In the following we will suppose that  $\text{val } g$  is known. This parameter corresponds to the characteristic number introduced in [3]. A physical interpretation in TEG context is given in [14]:

$$\text{val } g = \text{Inf}_{c \in \mathcal{C}} [n(c)],$$

where  $\mathcal{C}$  is the set of paths from input transition to output transition and  $n(c)$  is the number of tokens in path  $c$ . For example, the value of  $\text{val } g$  of the TEG in Fig. 2 is equal to zero because the smallest number of tokens initially present in paths between transitions  $U$  and  $Y$  is equal to zero. Since the number of tokens in any path of a TEG is necessarily greater than or equal to zero, it is clear that  $\text{val } g \geq 0$ , i.e.,  $g(k) = -\infty$ ,  $\forall k < 0$  (such a system is said to be causal [1, §5.7]).

- As we will only consider vectors with coefficients in  $\mathbb{R}$ , map  $x \mapsto \mathcal{U}_K \otimes x$  is defined from  $\mathbb{R}^{(K - \text{val } y + 1) \times 1}$  into  $\mathbb{R}^{(K - \text{val } y + 1) \times 1}$  (each line of matrice  $\mathcal{U}_K$  has a finite coefficient at least).

#### 4.1.1 First solution: overestimation of $g_K$

Using Residuation theory (cf. section 2.3), the greatest solution to Eq. (2) is equal to

$$\tilde{g}_K = \frac{Y_K}{\mathcal{U}_K}.$$

This solution verifies the following properties.

**Proposition 6 :**

1.  $g_K \preceq \tilde{g}_K$ , i.e.,  $g_K(k) \leq \tilde{g}_K(k)$ ,  $\text{val } g \leq k \leq K - \text{val } u$  ( $\tilde{g}_K$  is an overestimation of  $g_K$ ).
2.  $\mathcal{U}_K \otimes \tilde{g}_K = Y_K$  ( $g_K$  and  $\tilde{g}_K$  lead to the same output vector  $Y_K$ ).
3.  $\tilde{g}_K(k) = \bigwedge_{j=k+\text{val } u, K} \frac{y(j)}{u(j-k)} = \text{Min}_{j=k+\text{val } u, K} [y(j) - u(j-k)]$ ,  $\text{val } g \leq k \leq K - \text{val } u$ .

**Proof:**

(1) is straightforward since  $\tilde{g}_K$  is by definition the greatest solution to Eq. (2).

(2) is a direct application of Remark 2.1.

(3) is a direct application of Prop. 5.4.  $\diamond$

**Remark 5 :** The additional observation of firing date  $y(K+1)$  allows overestimating vector  $g_{K+1} = [g(\text{val } g), \dots, g(K - \text{val } u), g(K+1 - \text{val } u)]^T$ . We easily deduce from Prop. 6.3. the following recursive expression of  $\tilde{g}_{K+1}$ :

$$\begin{cases} \tilde{g}_{K+1}(k) = \text{Min}(\tilde{g}_K(k), y(K+1) - u(K+1-k)), & \text{val } g \leq k \leq K - \text{val } u \\ \tilde{g}_{K+1}(K+1 - \text{val } u) = y(K+1) - u(\text{val } u) \end{cases}.$$

#### 4.1.2 Second method: estimation based on *a priori* knowledge of $g_K$

In this section we propose an estimation of  $g_K$  based on *a priori* estimation of this vector, let  $m_K = [m(\text{val } m), \dots, m(K - \text{val } u)]^T$  this initial estimation. As we suppose that  $\text{val } g$  is known, we have  $\text{val } m = \text{val } g$ . From vector  $m_K$ , we construct a vector  $m'_K$  defined by:

$$m'_K(k) = m_K(k) + \Delta_K(k), \text{val } m \leq k \leq K - \text{val } u,$$

also noted  $m'_K = m_K + \Delta_K$  where  $+$  is the conventional pointwise sum. Vector  $\Delta_K$  must be chosen in order to verify:

- **Condition (1):** If  $m_K$  is solution to Eq. (2) then  $m'_K = m_K$ .  
This first condition means that if the vector output of model  $m_K$  is equal to that of system ( $Y_K$ ), estimation  $m_K$  is considered exact and we have  $\Delta_K(k) = 0$ ,  $\text{val } m \leq k \leq K - \text{val } u$ .
- **Condition (2):**  $m'_K$  must be solution to Eq. (2), i.e.,  $\mathcal{U}_K \otimes m'_K = Y_K$ .

The following result gives an upper bound of  $\Delta_K$  under condition (2).

**Proposition 7 :** If  $m'_K$  is solution to Eq. (2) then  $\Delta_K \preceq \tilde{g}_K - m_K \stackrel{\text{def}}{=} \bar{\Delta}_K$  where operator minus is the conventional pointwise subtraction.

**Proof:** It is obvious since  $\tilde{g}_K$  is the greatest solution to Eq.(2).  $\diamond$

We have  $m'_K = \tilde{g}_K$  which corresponds to the first overestimation method if we choose  $\Delta_K = \bar{\Delta}_K$ . Since this choice ( $\Delta_K = \bar{\Delta}_K$ ) does not allow taking into account Condition (1), we propose

$$\Delta_K = \tilde{g}_K - \tilde{m}_K$$

where  $\tilde{m}_K$  is the greatest solution to equation  $\mathcal{U}_K \otimes x = \mathcal{U}_K \otimes m_K$  (see remark 2.1). We easily verify that:

1.  $\Delta_K \preceq \bar{\Delta}_K$  since  $m_K \preceq \tilde{m}_K$  by definition of  $\tilde{m}_K$  which is a necessary condition to satisfy Condition (2).
2. If  $m_K$  is solution to Eq. (2) then  $\tilde{m}_K = \tilde{g}_K$ ; hence  $\Delta_K(k) = 0$ ,  $\text{val } m \leq k \leq K - \text{val } u$  and Condition (1) is satisfied.

Vector  $m'_K = m_K + \Delta_K$  is not necessarily solution to Eq. (2). We then consider the following series:

$$\begin{cases} m_K^{(0)} = m_K \\ m_K^{(n+1)} = m_K^{(n)} + \Delta_K^{(n)} \end{cases} \quad \text{with } \Delta_K^{(n)} = \tilde{g}_K - \tilde{m}_K^{(n)}, n \geq 0$$

where  $\tilde{m}_K^{(n)}$  is the greatest solution to equation  $\mathcal{U}_K \otimes x = \mathcal{U}_K \otimes m_K^{(n)}$ .

The following proposition shows that this series converges on a solution to Eq. (2).

**Proposition 8 :**

1. The series  $(m_K^{(n)})_{(n \in \mathbb{N}^*)}$  is nondecreasing and bounded by  $\tilde{g}_K$ .
2. Let  $\hat{m}_K = \lim_{n \rightarrow +\infty} m_K^{(n)}$ ,  $\hat{m}_K$  is solution to Eq. (2).

**Proof:**

1. As  $\tilde{m}_K^{(n)}$  ( $\forall n \geq 0$ ) is the greatest solution to equation  $\mathcal{U}_K \otimes x = \mathcal{U}_K \otimes m_K^{(n)}$ , we have  $m_K^{(n)} \preceq \tilde{m}_K^{(n)}$  and

$$m_K^{(n+1)} = m_K^{(n)} + [\tilde{g}_K - \tilde{m}_K^{(n)}] = [m_K^{(n)} - \tilde{m}_K^{(n)}] + \tilde{g}_K \preceq \tilde{g}_K.$$

We deduce that  $\forall n \geq 1$ ,  $m_K^{(n)}$  is bounded by  $\tilde{g}_K$ .

By isotony of map  $x \mapsto \mathcal{U}_K \otimes x$ , we have  $m_K^{(n)} \preceq \tilde{g}_K \Rightarrow \mathcal{U}_K \otimes m_K^{(n)} = \mathcal{U}_K \otimes \tilde{m}_K^{(n)} \preceq \mathcal{U}_K \otimes \tilde{g}_K = Y_K$ . As  $\tilde{g}_K$  is the greatest solution to Eq. (2), we deduce that  $\tilde{m}_K^{(n)} \preceq \tilde{g}_K$ . Then  $\forall k \in \{\text{val } g, \dots, K - \text{val } u\}$ ,  $\tilde{g}_K(k) - \tilde{m}_K^{(n)}(k) \geq 0$  and  $m_K^{(n+1)}(k) \geq m_K^{(n)}(k)$ . The series  $(m_K^{(n)})_{(n \in \mathbb{N}^*)}$  is therefore nondecreasing.

2.  $(m_K^{(n)})_{(n \in \mathbb{N}^*)}$  converges since it is a nondecreasing and bounded series in  $\mathbb{R}^{(K - \text{val } y + 1) \times 1}$ . Let  $\hat{m}_K = \lim_{n \rightarrow +\infty} m_K^{(n)}$ , then

$$\hat{m}_K = \hat{m}_K + [\tilde{g}_K - \hat{m}_K].$$

We deduce that  $\hat{m}_K = \tilde{g}_K$  and  $\hat{m}_K \otimes \mathcal{U}_K = \tilde{g}_K \otimes \mathcal{U}_K = Y_K$ , i.e.,  $\hat{m}_K$  satisfies Eq. (2).  $\diamond$

**Remark 6 :** Additional results on this iterative estimation method, such as the influence of initial estimation of  $g_K$ , the speed of the convergence, or the effect of growth of dimension  $K$ , remain to be stated. They are directly linked to the concept of distance which should be compatible with the considered algebra structure.

## 4.2 Identification of Periodic Impulse Response

The model structure is characterized by its period  $(\nu, \tau)$  and its transient length  $N$ . In the context of the adaptive control, these values are supposed to be equal to those of the initial model  $m_0$  which are estimated with an off-line method. Also, the system impulse response  $g$  will be entirely identified by the identification of parameters  $g(k)$ ,  $\text{val } g \leq k \leq N + \nu - 1$ , only (cf. Remark 1). These unknown parameters will be identified using the previous estimation method. For their identifications we must observe system outputs  $y(k)$ ,  $k \leq N + \nu - 1 + \text{val } u$ , at least. For  $K \geq N + \nu - 1 + \text{val } u$ , we construct a model noted  $\hat{m}$  from estimated vector  $\hat{m}_K$  as follows:

$$\hat{m}(k) = \begin{cases} \hat{m}_K(k) & \text{if } k \leq N + \nu - 1 \\ \hat{m}_K(k - \lfloor \frac{k-N}{\nu} \rfloor \nu) + \lfloor \frac{k-N}{\nu} \rfloor \tau & \text{if } k \geq N + \nu \end{cases} \quad (3)$$

**Remark 7 :**

1.  $\hat{m}$  is not necessarily a *dater*, i.e., a nondecreasing map. We then consider its "nondecreasing" version:  $\hat{m} \otimes e$ . We indeed prove that  $\forall u \in \overline{\mathbb{R}}_{max}^{\mathbb{Z}}$ ,  $u \otimes e$  is in  $Nondec(\overline{\mathbb{R}}_{max}^{\mathbb{Z}})$  and is the least *dater* greater than  $u$  [1, §6.3.3.] ( $\forall k \in \mathbb{Z}$ ,  $(u \otimes e)(k) = \sup_{s \leq k} u(s)$ ).

In fact the product of  $\hat{m}$  with a *dater* is equivalent to the product of its non decreasing version with this same *dater* since  $\forall u \in Nondec(\overline{\mathbb{R}}_{max}^{\mathbb{Z}})$ ,  $u = u \otimes e$  and  $\hat{m} \otimes u = \hat{m} \otimes (u \otimes e) = (\hat{m} \otimes e) \otimes u$  using associativity and commutativity of  $\otimes$ . Hence it is equivalent to consider  $\hat{m}$  or  $\hat{m} \otimes e$  in  $Nondec(\overline{\mathbb{R}}_{max}^{\mathbb{Z}})$ .

2. The notion of the input persistent excitation is not discussed here, even if it has an important part in estimation. Let us note that some results have been stated about particular inputs analogous to *sine* functions in the conventional linear system theory [1, §6], [6].

## 5 Controller Design Problem for Known System

We recall that control  $u_{n+1}$  computed at time  $t_n$  (cf. Remark 3) is the greatest solution to set

$$\{ u \in \Sigma \mid m_{n+1} \otimes u \preceq z \quad \text{and} \quad u(k) = u_n(k), \forall k \leq K_n \}.$$

So, in this section we are looking for the *greatest* element of set

$$S_1 = \{ u \in \Sigma \mid h \otimes u \preceq z \quad \text{and} \quad u(k) = v(k), \forall k \leq K \},$$

$h \in \Sigma$ ,  $v$  is a fixed *dater* and  $K$  a fixed event. In relation to the computation of control  $u_{n+1}$  at time  $t_n$ , *daters*  $v$ ,  $h$  and event  $K$  correspond to  $u_n$ ,  $m_{n+1}$  and  $K_n$  respectively.

Let us define map  $r_K^-$  from  $\Sigma$  into  $\Sigma$ :

$$[r_K^-(u)](k) = \begin{cases} u(k) & \text{if } k \leq K \\ u(K) & \text{if } k > K \end{cases},$$

which allows expressing constraint  $(u(k) = v(k), \forall k \leq K)$  by  $r_K^-(u) = r_K^-(v)$  and set  $S_1$  by

$$S_1 = \{ u \in \Sigma \mid h \otimes u \preceq z \quad \text{and} \quad r_K^-(u) = r_K^-(v) \}.$$

We give some properties of operator  $r_K^-$ .

**Proposition 9 :**

1.  $r_{\bar{K}}$  is a  $\oplus$ -morphism:  $\forall u, u' \in \Sigma, r_{\bar{K}}(u \oplus u') = r_{\bar{K}}(u) \oplus r_{\bar{K}}(u')$  (in particular, it is isotone).
2.  $r_{\bar{K}}$  is a projector:  $r_{\bar{K}} \circ r_{\bar{K}} = r_{\bar{K}}$ .
3.  $r_{\bar{K}} \preceq Id_{\Sigma}$ .
4.  $r_{\bar{K}}$  is a residuated map and  $(r_{\bar{K}})^{\sharp}$  is map  $x \mapsto r_{\bar{K}}(x) \oplus d_K$  where dater  $d_a, a \in \mathbb{Z}$ , is defined by:

$$d_a(k) = \begin{cases} -\infty & \text{if } k \leq a \\ +\infty & \text{if } k > a \end{cases}.$$

5.  $r_{\bar{K}}(v) = Inf \{ u \in \Sigma \mid r_{\bar{K}}(u) = r_{\bar{K}}(v) \}$ .

**Proof:** Points 1., 2. and 3. are trivial.

Let us prove point 4. Let us note  $f : x \mapsto r_{\bar{K}}(x) \oplus d_K$ , we have

1.  $f$  is isotone: if  $x \preceq y$  then  $r_{\bar{K}}(x) \preceq r_{\bar{K}}(y)$  since  $r_{\bar{K}}$  is isotone and  $r_{\bar{K}}(x) \oplus d_K \preceq r_{\bar{K}}(y) \oplus d_K$ .
2.  $(r_{\bar{K}} \circ f)(x) = r_{\bar{K}}(r_{\bar{K}}(x) \oplus d_K) = r_{\bar{K}}(r_{\bar{K}}(x)) \oplus r_{\bar{K}}(d_K) = r_{\bar{K}}(r_{\bar{K}}(x)) \oplus \varepsilon = r_{\bar{K}}(x) \preceq x$ .
3.  $(f \circ r_{\bar{K}})(x) = f(r_{\bar{K}}(x)) = r_{\bar{K}}(r_{\bar{K}}(x)) \oplus d_K = r_{\bar{K}}(x) \oplus d_K \succeq x$ .

From definition 2, we deduce that  $f = (r_{\bar{K}})^{\sharp}$ .

Let us prove point 5. Let us note  $A = \{ u \in \Sigma \mid r_{\bar{K}}(u) = r_{\bar{K}}(v) \}$ .  $r_{\bar{K}}(v) \in A$  since  $r_{\bar{K}}(r_{\bar{K}}(v)) = r_{\bar{K}}(v)$  ( $r_{\bar{K}}$  is a projection) and  $\forall u \in A$ , we have  $r_{\bar{K}}(v) = r_{\bar{K}}(u) \preceq u$  ( $r_{\bar{K}} \preceq Id_{\Sigma}$ ). We deduce point 5.  $\diamond$ .

From theorem 1 (cf. Annexe), the greatest element of  $S_1$ , if  $S_1$  is not empty, is equal to

$$\frac{z}{h} \wedge (r_{\bar{K}})^{\sharp}(r_{\bar{K}}(v)).$$

The equality constraint (defined by map  $r_{\bar{K}}$  and dater  $r_{\bar{K}}(v)$ ) cannot be modified and can lead to an empty set  $S_1$ . To always have a solution and to have output transition firing dates *as close as possible* to the desired ones defined by  $z$ , we consider the *least modification* to bring additively (in sense of  $\oplus$ ) to  $z$ . This approach leads to consider a new set  $S_2$  defined from Prop. 12 (cf. Annexe) by:

$$S_2 = \{ u \in \Sigma \mid h \otimes u \preceq z \oplus h \otimes \underline{u} \text{ and } r_{\bar{K}}(u) = r_{\bar{K}}(v) \} \text{ with } \underline{u} = Inf \{ u \in \Sigma \mid r_{\bar{K}}(u) = r_{\bar{K}}(v) \}$$

(we easily deduce from Prop. 9.5 that  $\underline{u} = r_{\bar{K}}(v)$  belongs to subset  $\{ u \in \Sigma \mid r_{\bar{K}}(u) = r_{\bar{K}}(v) \}$ ).

From theorem 1 (cf. Annexe), its greatest element, noted  $u_{opt,K}$ , is equal to

$$u_{opt,K} = \frac{z \oplus h \otimes \underline{u}}{h} \wedge (r_{\bar{K}})^{\sharp}(r_{\bar{K}}(v)).$$

The next proposition gives the formal expression of  $u_{opt,K}$ .

**Proposition 10 :** Let  $r_{\bar{K}}^+$  the map from  $\Sigma$  into  $\Sigma$  defined by  $r_{\bar{K}}^+(x) = x \wedge d_K$ . We have

$$u_{opt,K} = r_{\bar{K}}(v) \oplus r_{\bar{K}}^+\left(\frac{z \oplus h \otimes r_{\bar{K}}(v)}{h}\right).$$

**Proof:** We recall that  $u_{opt,K} = \frac{z \oplus h \otimes \underline{u}}{h} \wedge (r_K^-)^\sharp(r_K^-(v))$ .

From Prop. 9.5,  $\underline{u} = r_K^-(v)$ . From the expression of  $(r_K^-)^\sharp$  and since  $r_K^-$  is a projection,  $(r_K^-)^\sharp(r_K^-(v)) = r_K^-(r_K^-(v)) \oplus d_K = r_K^-(v) \oplus d_K$ . Then

$$\begin{aligned} u_{opt,K} &= \frac{z \oplus h \otimes r_K^-(v)}{h} \wedge (r_K^-(v) \oplus d_K) \\ &= \left( \frac{z \oplus h \otimes r_K^-(v)}{h} \wedge r_K^-(v) \right) \oplus \left( \frac{z \oplus h \otimes r_K^-(v)}{h} \wedge d_K \right) \quad ((\Sigma, \oplus, \otimes) \text{ is a distributive dioid}) \\ &= \left( \frac{z \oplus h \otimes r_K^-(v)}{h} \wedge r_K^-(v) \right) \oplus r_K^+ \left( \frac{z \oplus h \otimes r_K^-(v)}{h} \right). \end{aligned}$$

As  $z \oplus h \otimes r_K^-(v) \succeq h \otimes r_K^-(v)$  and using Props. 5.2 and 5.1, we have  $\frac{z \oplus h \otimes r_K^-(v)}{h} \succeq \frac{h \otimes r_K^-(v)}{h} \succeq r_K^-(v)$ . We deduce that  $\frac{z \oplus h \otimes r_K^-(v)}{h} \wedge r_K^-(v) = r_K^-(v)$ .  $\diamond$

Let us give another expression of  $u_{opt,K}$ .

**Proposition 11 :**

$$u_{opt,K} = r_K^-(v) \oplus \frac{r_{K+val h}^+(z \oplus h \otimes r_K^-(v))}{h}. \quad (4)$$

**Proof:** We must prove the following result:

**Lemma 1 :** Let  $u, v \in \Sigma$ ,  $val u$  is supposed finite. Then  $\frac{r_K^+(v)}{u} = r_{K-val u}^+ \left( \frac{v}{u} \right)$ .

**Proof of lemma:**  $\frac{r_K^+(v)}{u} = \frac{v \wedge d_K}{u} = \frac{v}{u} \wedge \frac{d_K}{u}$  using Prop. 5.3. We then must prove that  $\frac{d_K}{u} = d_{K-val u}$ .  $\frac{d_K}{u}$  is by definition the greatest solution to inequation

$$u \otimes x \preceq d_K. \quad (5)$$

- As  $val(u \otimes d_{K-val u}) = val u + K + 1 - val u = K + 1$  using Prop. 3.2., we deduce that  $u \otimes d_{K-val u} \preceq d_K$  (we easily verify that  $d_K = Sup \{ w \in \Sigma \mid val w \geq K + 1 \}$ ) and  $d_{K-val u}$  is solution to Eq. (5).
- For all  $x$  solution to Eq. (5),

$$\begin{aligned} val(u \otimes x) &= val u + val x \quad \text{using Prop. 3.2.} \\ &\geq val d_K = K \quad \text{using Prop. 3.1.} \end{aligned}$$

Then  $val x \geq K + 1 - val u$  and  $x \preceq d_{K-val u}$  ( $d_{K-val u} = Sup \{ w \in \Sigma \mid val w \geq K + 1 - val u \}$ ).

- We deduce that  $d_{K-val u}$  is the greatest solution to Eq. (5) and  $d_{K-val u} = \frac{d_K}{u}$ .  $\diamond$

From this lemma, we immediately deduce that  $r_K^+ \left( \frac{z \oplus h \otimes r_K^-(v)}{h} \right) = \frac{r_{K+val h}^+(z \oplus h \otimes r_K^-(v))}{h}$ .  $\diamond$

Control  $u_{opt,K}$  allows firing the system input *at the latest* dates so that the output firing dates occur *as close as possible* to those defined by  $z$ . More precisely, we see from Eq. (4) that  $u_{opt,K}$  is such that firing date  $y(k)$  occurs exactly at  $(g \otimes v)(k)$  for  $k \leq K + val h$  and *at the latest* before  $(z \oplus h \otimes r_K^-(v))(k)$  for  $k > K + val h$ .

## 6 Algorithm

An algorithm is presented here to provide controls  $u_n$ ,  $n \geq 0$  and models  $m_n$ ,  $n \geq 1$ . The period  $(\nu, \tau)$  and the transient length  $N$  of the models we consider are supposed to be known.

- Until time  $t_0 \geq y(N + \nu - 1 + \text{val } u_0)$ , the system is known through a model  $m_0$ . Let  $u_0 = \frac{z}{m_0}$  the optimal model control. Controls

$$u_0(k), \text{val } u_0 \leq k \leq K_0 \quad \text{with} \quad K_0 = \text{Sup} \{ k \in \mathbb{Z} \mid u_0(k) \leq t_0 \}$$

are applied to the system input until a time  $t_0$ .

- At time  $t_n$ ,  $n \geq 0$ ,
  - we have applied controls  $u_n(k)$ ,  $k \leq K_n$  with  $K_n = \text{Sup} \{ k \in \mathbb{Z} \mid u_n(k) \leq t_n \}$ ;
  - we dispose of observed outputs  $y(k)$ ,  $k \leq K'_n$  with  $K'_n = \text{Sup} \{ k \in \mathbb{Z} \mid y(k) \leq t_n \}$ .

We compute (cf. section 4.1.2, Prop. 8) the vector  $(m_{n+1})_{K'_n} = (\widehat{m}_n)_{K'_n}$  from initial vector  $(m_n)_{K'_n}$  and model  $m_{n+1}$  as described in section 4.2 (cf. Eq. (3)):

$$m_{n+1}(k) = \begin{cases} (m_{n+1})_{K'_n}(k) & \text{if } k \leq N + (\nu - 1) \\ (m_{n+1})_{K'_n}(k - \lfloor \frac{k-N}{\nu} \rfloor \nu) + \lfloor \frac{k-N}{\nu} \rfloor \tau & \text{if } k \geq N + \nu \end{cases}$$

The expression of control  $u_{n+1}$  is given by Eq. (4) with  $h = m_{n+1}$ ,  $v = u_n$  and  $K = K_n$ , i.e.,

$$u_{n+1} = r_{K_n}^-(u_n) \oplus \frac{r_{K_n + \text{val } m_{n+1}}^+[z \oplus m_{n+1} \otimes r_{K_n}^-(u_n)]}{m_{n+1}}$$

Controls

$$u_{n+1}(k), k \leq K_{n+1} \quad \text{with} \quad K_{n+1} = \text{Sup} \{ k \in \mathbb{Z} \mid u_{n+1}(k) \leq t_{n+1} \}$$

are applied to the system input until a time  $t_{n+1}$ .

## 7 Example

We present an application of the previous algorithm. Let  $(\nu, \tau)$  and  $N$  respectively the period and the transient length of the models. They will be considered equal to those of the initial model  $m_0$ .

Samples  $K'_n$ ,  $n \geq 0$  are defined by:

$$\begin{cases} K'_0 = N + \nu - 1 + \text{val } u_0 \text{ where } u_0 = \frac{z}{m_0} \text{ is the optimal model control} \\ K'_n = K'_{n-1} + 1, \forall n \geq 1 \end{cases}$$

We denote  $u_a$  the control computed at the last step of the algorithm. It is the control effectively applied to the system input from the beginning, since

$$\forall n \geq 0, u_a(k) = u_n(k), \forall k \leq K_n \quad (r_{K_n}^-(u_a) = r_{K_n}^-(u_n))$$

by construction of controls  $u_n$ . Let  $y_a = g \otimes u_a$  the corresponding system output. We denote  $m_a$  the impulse response of the model computed at the last step of the algorithm.

We consider a linear system whose model is described by the TEG in Fig. 2. Impulse responses of the initial model  $m_0$  and the system  $g$  are represented in Fig. 4. The system and the model have the

same period ( $\nu = 4, \tau = 5$ ) but different transient lengths ( $N = 1$  and  $N_g = 20$ ). Moreover  $g \preceq m_0$ ; roughly speaking we can say that the system is "faster" than its model (for a same input, system output occurs later than model output).

Reference input  $z$  is represented in Fig. 5 (dotted and thick line). It is such that  $z(k) = +\infty, \forall k > k_f = 30$  and means that production must be controlled until event  $k_f$ , since beyond this event the production is desired at the latest before  $+\infty$  (we fix no time constraint).

Let  $u_0 = \frac{z}{m_0}$  the optimal model control and  $u_{opt} = \frac{z}{g}$  the optimal system control. Controls  $u_0$  and  $u_{opt}$  are represented in Fig. 6. Inequality  $g \preceq m_0$  implies that  $u_{opt} \succeq u_0$  by antitony of  $x \mapsto \frac{z}{x}$  (cf. Prop. 5.2.) and the isotony of  $x \mapsto g \otimes x$  implies that  $g \otimes u_0 \preceq g \otimes u_{opt}$  as we can see in Fig. 5. Controls  $u_a$  and  $u_0$  are initially equal until event  $K_0 = 5$  since controls initially applied to the system input are  $u_0(k), k \leq K_0$  (step 0 of the algorithm). Beyond this event, controls  $u_a$  and  $u_{opt}$  are nearly similar.

We "reach" from model  $m_0$  a model  $m_a$  close but no equal to  $g$  (see Fig. 4). It is equal to

$$m_a(k) = \begin{cases} -\infty & \text{if } k < 0 \\ 1 & \text{if } k = 0 \\ \max(1 + 5 \lfloor \frac{k-1}{4} \rfloor, 3 + 5 \lfloor \frac{k-2}{4} \rfloor, 4 + 5 \lfloor \frac{k-3}{4} \rfloor) & \text{if } k \geq 1 \end{cases} .$$

**Remark 8 :**

- The position of  $g$  with respect to  $m_0$  ( $g \preceq m_0$ ) suggests that a modification from  $u_0$  to "reach"  $u_{opt}$  will perform in the positive sense along the time axis.
- For this example we initially have  $g \otimes u_0 \preceq z$  and  $z \oplus g \otimes r_K^-(u_n) = z, \forall n \geq 0$ , i.e., past action controls have no effect at each step of the algorithm. The reference input effectively tracked through model  $m_{n+1}$  is therefore  $z$  and  $u_{n+1} = r_{K_n}^-(u_n) \oplus r_{K_n}^+(\frac{z}{m_{n+1}})$ .



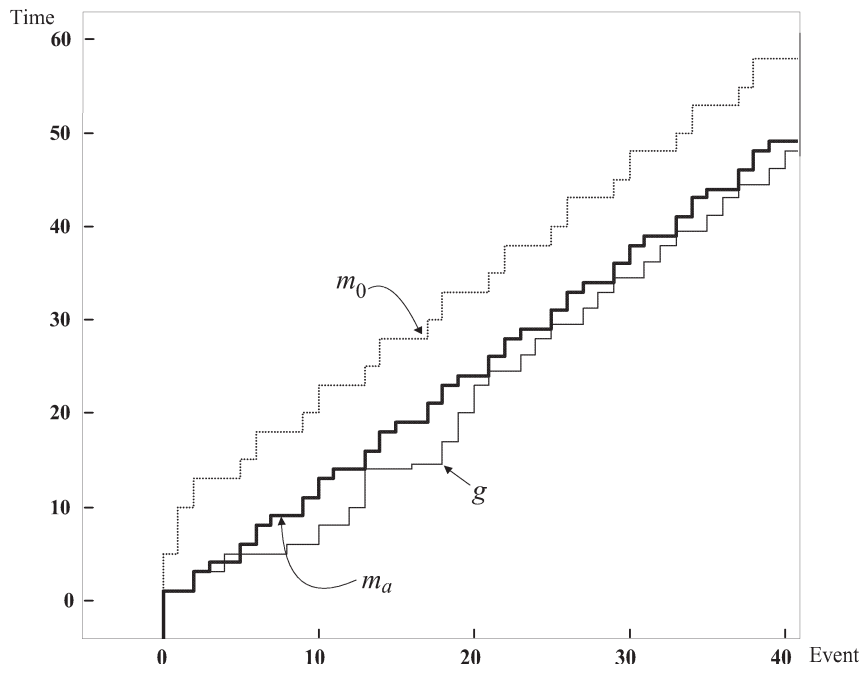


Figure 4: Impulse responses  $g$  (thin line),  $m_0$  (dotted line) and  $m_a$  (thick line).

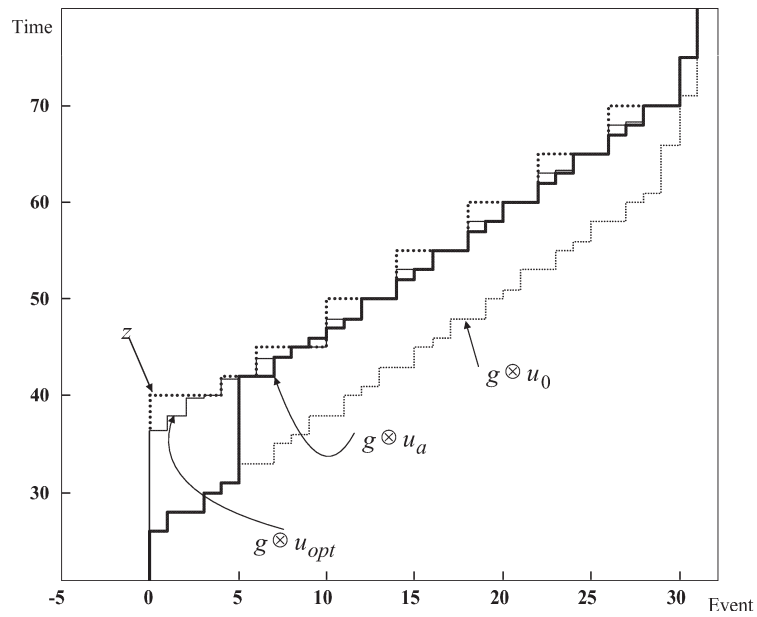


Figure 5: Outputs  $g \otimes u_{opt}$  (thin line),  $g \otimes u_0$  (dotted line),  $g \otimes u_a$  (thick line) and reference input  $z$  (dotted and thick line).

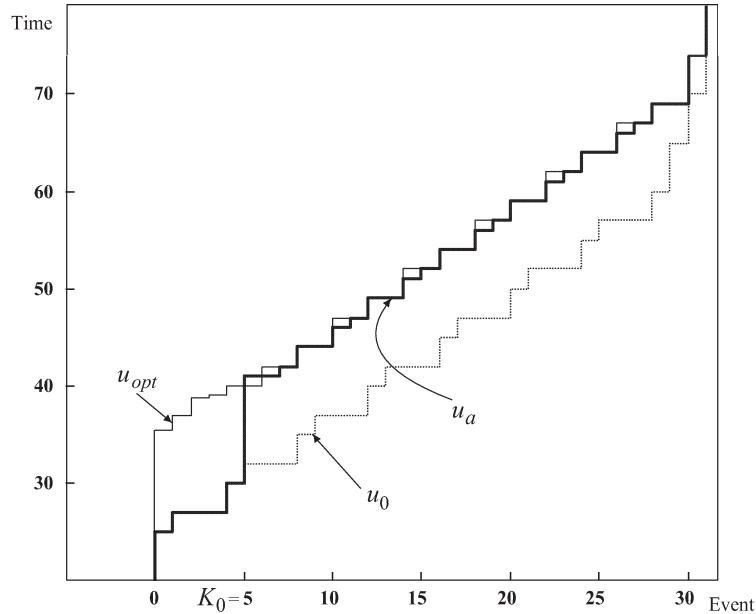


Figure 6: Controls  $u_{opt}$  (thin line),  $u_0$  (dotted line) and  $u_a$  (thick line).

## 8 Conclusion

The proposed control structure is applied to systems (TEG) modeled in Max-Algebra. This control structure allows taking into account mismatch between the system and its model. It is based on adaptive control principle and an algorithm with recursive structure is established. The control law is based on:

- "inversion" of dynamic system,
- system identification which led us to consider the estimation of models.

Concerning identification of linear systems, we have just established preliminary results. Two notions remain to be analyzed.

- Concerning the model structure: The valuation, the length of the transient behavior and the periodicity have been supposed to be known (structure hypothesis); these parameters are fixed beforehand with an off-line estimation method. The notion of persistent signal will certainly have an important part in an on-line estimation (cf. remark 7), the effect of input signal and indirectly of matrix  $\mathcal{U}_K$  (cf. section 4.1) has not been taken into account in the analysis of estimation methods.
- The notion of performance has not really been considered, its study implies the use of adapted tools in the considered algebra, like for example the concept of distance and indirectly of norm. This fact clearly appears in section 4.1.2 in which we propose an estimation method based on an *a priori* knowledge of the system impulse response without giving clear results on the obtained model.

Like in conventional theory, this field of study offers some interesting applications like in control, for example, to get appropriate models for the synthesis of a controller, the design of a prediction algorithm, or simulation.

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## Annexe

**Theorem 1** : Let  $\mathcal{D}$  and  $\mathcal{C}$  two complete dioids,  $F, G : \mathcal{D} \rightarrow \mathcal{C}$  two residuated maps and  $c_1, c_2 \in \mathcal{C}$ .  
Let us consider sets

$$S_a = \{ x \in \mathcal{D} \mid G(x) \preceq c_1 \text{ and } F(x) = c_2 \} \quad \text{and} \quad S_b = \{ x \in \mathcal{D} \mid G(x) \preceq c_1 \text{ and } F(x) \preceq c_2 \}.$$

We suppose that  $S_a$  is not empty and we note  $x_a = \bigoplus_{x \in S_a} x$ ,  $x_b = \bigoplus_{x \in S_b} x$ , then:

1.  $x_a = x_b$ .
2.  $x_b = G^\sharp(c_1) \wedge F^\sharp(c_2)$ .

**Proof:**

1.  $S_a$  is supposed not empty, therefore  $x_a$  exists since  $\mathcal{D}$  is complete. As  $S_a \subset S_b$ , we obviously have  $x_a \preceq x_b$ . Let  $x \in S_a$ , we have  $x \preceq x_b$  and  $c_2 = F(x) \preceq F(x_b) \preceq c_2$  by isotony of  $F$ . Then  $F(x_b) = c_2$  and  $x_b \in S_a$ , we deduce that  $x_b \preceq x_a$ . We conclude that  $x_a = x_b$ .
2. (a)  $G^\sharp(c_1) \wedge F^\sharp(c_2) \preceq F^\sharp(c_2) \Rightarrow F(G^\sharp(c_1) \wedge F^\sharp(c_2)) \preceq F(F^\sharp(c_2)) \preceq c_2$  by isotony of  $F$  and definition of  $F^\sharp$ . We equivalently have  $G(G^\sharp(c_1) \wedge F^\sharp(c_2)) \preceq G(G^\sharp(c_1)) \preceq c_1$ , hence  $G^\sharp(c_1) \wedge F^\sharp(c_2) \in S_b$ .  
(b)  $\forall x \in S_b$ ,  $x \preceq F^\sharp(c_2)$  since  $F^\sharp(c_2)$  is the greatest solution to  $F(x) \preceq c_2$ . We equivalently have  $x \preceq G^\sharp(c_1)$ . We deduce that  $x \preceq G^\sharp(c_1) \wedge F^\sharp(c_2)$ .  
(c) We conclude from (a) and (b) that  $G^\sharp(c_1) \wedge F^\sharp(c_2)$  is the greatest element of  $S_b$ .  $\diamond$

**Proposition 12** : Let  $\mathcal{D}$  and  $\mathcal{C}$  two complete dioids,  $F, G : \mathcal{D} \rightarrow \mathcal{C}$  two residuated maps and  $c_1, c_2 \in \mathcal{C}$ . We assume that the largest lower bound of the subset  $\{ x \in \mathcal{D} \mid F(x) = c_2 \}$ , denoted  $\underline{x}$ , belongs to this subset. Then  $c_1 \oplus G(\underline{x})$  is the least dater of the form  $c_1 \oplus \Delta c_1$  such that set  $S(\Delta c_1) = \{ x \in \mathcal{D} \mid G(x) \preceq c_1 \oplus \Delta c_1 \text{ and } F(x) = c_2 \}$  is not empty.

**Proof:**

- $S(G(\underline{x}))$  is not empty since it contains  $\underline{x}$ .
- If  $S(\Delta c_1)$  is not empty then  $c_1 \oplus G(\underline{x}) \preceq c_1 \oplus \Delta c_1$ . Indeed, let  $y \in S(\Delta c_1)$ , as  $\underline{x} \preceq y$  and  $G$  is isotone, we have  $G(\underline{x}) \preceq G(y) \preceq c_1 \oplus \Delta c_1$ . We deduce that  $c_1 \oplus G(\underline{x}) \preceq c_1 \oplus \Delta c_1$ .  $\diamond$