Averaging principle for diffusion processes via Dirichlet forms

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AVERAGING PRINCIPLE FOR DIFFUSION PROCESSES VIA DIRICHLET FORMS

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ABSTRACT. We study diffusion processes driven by a Brownian motion with regular drift in a finite dimension setting. The drift has two components on different time scales, a fast conservative component and a slow dissipative component. Using the theory of Dirichlet form and Mosco-convergence we obtain simpler proofs, interpretations and new results of the averaging principle for such processes when we speed up the conservative component. As a result, one obtains an effective process with values in the space of connected level sets of the conserved quantities. The use of Dirichlet forms provides a simple and nice way to characterize this process and its properties.

1. INTRODUCTION

Our aim in this article is to introduce a new method, to obtain new results and to prove averaging principles for stochastic diffusion equations. We consider standard Itô diffusions in a finite dimensional setting with drifts on different time scale.

We study the diffusion equation in $\mathbb{R}^2$, defined for $\alpha, \varepsilon > 0$ by

$$(1.1) \qquad \mathrm{d}X_t = A\nabla H(X_t)\mathrm{d}t - \alpha e(X_t)\mathrm{d}t + \sqrt{2\alpha\varepsilon}\mathrm{d}B_t$$

where $A\nabla$ is the symplectic gradient. H is the Hamiltonian and e is the drift and plays the role of a friction term (hence the minus sign in front of this term in Equation (1.1)). On the time scale $O\left(\frac{1}{\alpha}\right)$, $Y_t = X_{\frac{t}{\alpha}}$ is a solution to the following equation:

$$(1.2) \qquad \mathrm{d}Y_t = \frac{1}{\alpha}A\nabla H(X_t)\mathrm{d}t - e(X_t)\mathrm{d}t + \sqrt{2\varepsilon}\mathrm{d}B_t.$$

We want to consider the limit $\alpha \to 0$, for which we obtain an averaging along the orbits of the Hamiltonian system

$$(1.3) \qquad \dot{x}(t) = A\nabla H(x(t)).$$

In fact, one can make the observation that since H is preserved by the Hamiltonian vector field, the process $H(Y_t)$ satisfies

$$(1.4) \quad H(Y_t) = H(Y_0) + \int_0^t -\nabla H(Y_s) \cdot e(Y_s) + \varepsilon\Delta H(Y_s)\mathrm{d}s + \int_0^t \nabla H(Y_s) \cdot \mathrm{d}B_s$$

which does not depend on α. Then one could expect that the law of of the process $H(Y)$ will converge. However, we cannot obtain a convergence for $H(Y)$ itself if H is not injective but for a projection of Y on the "orbit space" Γ defined as the space of connected level sets of H. Previous convergence results have been obtained,

notably by Freildin and Wentzell in several papers (see the new edition of their book [6] for a review, and the previous articles [9, 8]).

The main novelty of this paper concerns the method we use to prove this results and that it leads to new, more general results. Freidlin and Wentzell prove their results using the martingale formulation of the diffusion $H(Y)$. As α goes to 0, they prove the convergence of the martingale problem and deduce the unicity of the limit and the equation satisfied by the solution. Existence is proven via tightness of the laws.

This paper is a nice application of the theory of convergence of Dirichlet form. We rely on the theory of non-symmetric Dirichlet forms (as exposed in the book [17] by Ma and Röckner). We prove convergence, in a suitable sense, of a sequence of Dirichlet forms associated to the process defined in Equation (1.1). As a consequence, we obtain convergence of the resolvents and therefore of the associated semigroups. In probabilistic terms, we prove convergence of the finite dimensional marginals of the process. Associated to the tightness of the sequence of laws, this proves the convergence in law.

We use results by Hino [10], and Tölle [21] about the convergence of non-symmetric Dirichlet forms. Note that in our case, the functional space also changes along the convergence, and we refer to Kolesnikov [13] who investigated such cases. We refer also to Kuwae and Shioya [16] for a quite complete exposition of spectral structures and their relations (semigroups, resolvents, Dirichlet forms and generators).

The use of Dirichlet forms is very quick and simple, despite giving an indirect description of the limiting process (or its infinitesimal generator). However, with the use of the coarea Formula, we can recover the infinitesimal generator and its domain, therefore achieve a complete description of the process.

We assume that we can define a nice Dirichlet form with a reference measure which is preserved under the accelerated flow. The other assumptions we make are quite general and could probably be greatly relaxed.

In this paper, we also prove convergence of the Dirichlet forms for higher dimensional diffusions on $\mathbb{R}^n$ defined by

$$(1.5) \qquad \mathrm{d}Y_t = \frac{1}{\alpha}v(Y_t)\mathrm{d}t + u(Y_t)\mathrm{d}t + \sqrt{2\varepsilon}\sigma(Y_t)\mathrm{d}B_t.$$

We consider the limit of the Dirichlet forms, as α goes to 0, for the process $G(Y)$ where G is a $\mathbb{R}^m$-valued function (with $m < n$) conserved along the flow of the vector field v. We only sketch the computation of the limiting infinitesimal generator since a complete computation would need a very detailed description of the space Γ (defined as the quotient space for the connected level sets of G) which is beyond the scope of this article.

Averaging principles for higher dimensional diffusions have been investigated in several papers. In [7], Freidlin and Wentzell show an averaging principle for a diffusion where the fast component concerns only the first two coordinates. In [4] by Freidlin and Weber, the fast component contains a Brownian term and there is only one first integral ($m = 1$ in our notation). In [5], the same authors investigate the perturbation of an Hamiltonian system with only one first integral but draws conclusions on the PDE counterpart of the averaging.

We give now more details for the Hamiltonian two-dimensional case. We choose a reference measure μ and define the (pre-)Dirichlet form, denoted E_α, associated to the infinitesimal generator, denoted L_α, of the diffusion given by Equation 1.2 in $L^2(\mu)$: for f, g two C^2 functions with compact support

$$(1.6) \qquad E_\alpha(f, g) = -\langle L_\alpha f, g \rangle_{L^2(\mu)} = -\int_{\mathbb{R}^2} L_\alpha f g \mathrm{d}\mu.$$

Under suitable conditions on μ, E_α is a Dirichlet form and characterizes completely the infinitesimal generator L_α.

The projected Dirichlet form, denoted $\mathcal{E}$, is constructed by restricting the set of test functions : we consider functions constant on connected level sets of H. We do it rigorously by considering the connected level sets of H as equivalence classes. We denote Γ the quotient set, and π the canonical projection onto Γ. Naturally, we can associate a $L^2(\Gamma)$ function space to the space Γ (containing functions f defined on Γ such that $f \circ \pi$ is in $L^2(\mu)$). The projected Dirichlet form $\mathcal{E}$ is defined as: for f, g

$$(1.7) \qquad \mathcal{E}(f, g) = E_\alpha(f \circ \pi, g \circ \pi).$$

Due to the choice of μ, we prove, and this is the most important remark, that $\mathcal{E}$ does not depend on α and is, in itself, a nice Dirichlet form on $L^2(\Gamma)$.

Moreover, we prove that, in a Mosco-convergence sense, the sequence of Dirichlet forms E_α and their domains, converges to $\mathcal{E}$. The convergence in law of the process follows by using the tightness.

The convergence of Dirichlet forms is quite abstract but powerful since it can be applied to very general cases (see Section 5). However, in order to have a more intuitive description of the limiting process, we have to write the infinitesimal operator $\mathcal{L}$ (and its domain) associated to the Dirichlet form as

$$(1.8) \qquad \mathcal{E}(f, g) = -\langle \mathcal{L}f, g \rangle_{L^2(\Gamma)} = -\int_{\mathbb{R}^2} (\mathcal{L}f) \circ \pi g \circ \pi \mathrm{d}\mu.$$

To this aim, we need a better understanding of the space Γ, it can be easily done in the Hamiltonian case on $\mathbb{R}^2$, but it is much more involved in higher dimensions.

In $\mathbb{R}^2$, the space Γ is a graph with vertices and edges, on each edge the averaging process is a classical diffusion whose drift and diffusion coefficients could be easily computed. However, at a vertex (a point gluing together several edges), we obtain a so-called gluing condition giving a restriction on the domain of the operator and therefore on the behavior of the process when (or if) it reaches this vertex.

The limiting diffusion is therefore a process on a graph and is described by

- an infinitesimal generator on each edge (a second order differential operator);
- a gluing condition at each vertex.

Analysis of such processes could be done based on one-dimensional diffusions (see Feller [2] or Mandl [18]), we also cite the work by Kant, Klauss, Voigt and Weber [12] which investigates such processes from a Dirichlet-form point of view and more recently the work of Kostrykin, Potthoff and Schrader [14]. Large deviations for diffusions process on graphs have been proven in [3].

The infinitesimal generator is solution of Equation (1.8). We compute the infinitesimal generator in two steps:

(1) using the coarea Formula, we compute the measure $\pi_*\mu$ on Γ (projection of μ on Γ) and we write the Dirichlet form as an integral on Γ;

(2) we use a integration by part on each edge to transfer the derivatives on g to f.

The identification of the two sides of Equation (1.8) is made separately on the edges and the vertices on Γ. On the edges, we obtain a second-order differential operator:

$$(1.9) \qquad\qquad \mathcal{L}u = au'' + bu'$$

whereas at a vertex O, we obtain a gluing relation:

$$(1.10) \qquad\qquad -\beta\mathcal{L}u(O) = \gamma u(O) + \sum_i \alpha_i D_i u(O)$$

where γ, β, α_i are constants, the sum is made on all edges incident to the vertex i and $D_i u$ is the one-sided derivative of u at O along the edge i.

The remaining part of this paper is organized as follows. In Section 2 we present the Hamiltonian case in $\mathbb{R}^2$, the main assumptions, we define the Dirichlet form E_α and the orbit space Γ. In Section 3, we prove the convergence in law of the projected process to a limiting process defined by the projected Dirichlet form $\mathcal{E}$. In Section 4, we compute the infinitesimal generator and draw some consequences about the behavior of the limiting process on Γ. Lastly, in the last section (Section 5), we generalize our method to diffusions in arbitrary dimension.

Acknowledgements. We thank an anonymous referee for pointing out the fact that condition (2.15) of Assumption 2.3 could probably be relaxed to $\nabla \cdot (hF) \leqslant c$ for some positive constant c. In this case, one should work with lower bounded semi-Dirichlet forms (see e.g. [19]). However, we ask for condition (2.15) in order to work with simple Dirichlet forms (and thus simplify the Mosco-convergence results).

2. Two dimensional case with additive noise

2.1. Properties of the process.
We consider the solution (Y_t) of equation (1.2).

Assumptions 2.1. We assume that e is a C^1 bounded vector field. We suppose that H is C^2, bounded from below and has compact level sets. We also assume that H has bounded second derivatives.

Remark 1. The assumption on e (and the fact that H must have bounded second derivatives) ensures the existence of a solution Y to Equation (1.2). The boundedness of the second derivative of H is a technical assumption (also made in [6]) which ensures an easy proof of the tightness of the process.

We follow Ma-Rockner [17]. The Dirichlet form is defined through the infinitesimal operator L_α, which is a closed, densely defined operator. We define this operator and its domain $\mathcal{D}(L_\alpha)$ via the transition semigroup of Equation (1.2):

$$(2.1) \quad \mathcal{D}(L_\alpha) = \left\{ f \in C_b(\mathbb{R}^2), t^{-1}(\mathbb{E}.[f(Y_t)] - f(\cdot)) \text{ converges uniformly as } t \to 0 \right\}$$

$$(2.2) \qquad L_\alpha f = \frac{1}{\alpha} A\nabla H \cdot \nabla f - e \cdot \nabla f + \varepsilon \Delta f \text{ for } f \in C_c^2(\mathbb{R}^2).$$

We consider a measure $\mathrm{d}\mu = h(x)\mathrm{d}x$, where h is C^2 and strictly positive. We define the Hilbert spaces $L^2(\mu)$ and $H^1(\mu)$ as the weighted L^2 and H^1 sets with their

scalar products $\langle \cdot, \cdot \rangle_{L^2(\mu)}$ and $\langle \cdot, \cdot \rangle_{H^1(\mu)}$:

$$(2.3) \qquad L^2(\mu) = \left\{ f, \langle f, f \rangle_{L^2(\mu)} = \int f^2 \mathrm{d}\mu < +\infty \right\}$$

$$(2.4) \qquad H^1(\mu) = \left\{ f, \langle f, f \rangle_{H^1(\mu)} = \int f^2 + |\nabla f|^2 \, \mathrm{d}\mu < +\infty \right\}.$$

Let us define the vector field F as

$$(2.5) \qquad F = e + \frac{\varepsilon}{h} \nabla h.$$

We have first the following Lemma

Lemma 2.2. *Let us consider the bilinear form, for $f, g \in \mathcal{D} \cap C_c(\mathbb{R}^2)$*

$$(2.6) \qquad E_\alpha(f, g) = - \langle L_\alpha f, g \rangle_{L^2(\mu)}$$

The bilinear form E_α can be uniquely decomposed in two parts: one symmetric, E_α^s, and one antisymmetric, E_α^a:

$$E_\alpha(f, g) = E_\alpha^s(f, g) + E_\alpha^a(f, g)$$

(2.7)

$$E_\alpha^s(f, g) = \varepsilon \int \nabla f \cdot \nabla g \mathrm{d}\mu - \frac{1}{2} \int \frac{1}{h} \nabla \cdot (hF) fg \mathrm{d}\mu + \frac{1}{2\alpha} \int \frac{1}{h} \nabla \cdot (hA\nabla H) fg \mathrm{d}\mu$$

(2.8)

$$E_\alpha^a(f, g) = -\frac{1}{2\alpha} \int_{\mathbb{R}^2} A\nabla H \cdot [g\nabla f - f\nabla g] \, \mathrm{d}\mu + \frac{1}{2} \int F \cdot [\nabla fg - \nabla gf] \, \mathrm{d}\mu.$$

Proof. The lemma follows from an integration by parts. For $f, g \in C_c^2(\mathbb{R}^2)$, we get

$$(2.9) \qquad \int \Delta f g \mathrm{d}\mu = - \int \nabla f \cdot \nabla (gh) \mathrm{d}x = - \int \nabla f \cdot \nabla g \mathrm{d}\mu - \int \frac{1}{h} \nabla h \cdot \nabla f g \mathrm{d}\mu.$$

Also, for any regular vector field $G \in C^2(\mathbb{R}^2, \mathbb{R}^2)$, we have

$$(2.10) \qquad \int G \cdot \nabla f g \mathrm{d}\mu = \frac{1}{2} \int G \cdot (\nabla fg - \nabla gf) \mathrm{d}\mu - \frac{1}{2} \int \frac{1}{h} \nabla \cdot (hG) fg \mathrm{d}\mu.$$

Using Equations (2.9) and (2.10), we obtain Equations (2.7) and (2.8) for f, g sufficiently regular with compact support. By density of smooth functions in $\mathcal{D} \cap C_c(\mathbb{R}^2)$, we get the lemma. □

Using Ma-Rockner, we will extend the bilinear form E_α as a Dirichlet form. We denote

$$(2.11) \qquad E_\alpha^1(f, g) = E_\alpha(f, g) + \langle f, g \rangle_{L^2(\mu)}$$

$$(2.12) \qquad E_\alpha^{s,1}(f, g) = E_\alpha^s(f, g) + \langle f, g \rangle_{L^2(\mu)}$$

Let us recall ([17] Definition 4.5 p.34) that the bilinear form E_α with a domain $\mathcal{D}$ dense in $L^2(\mu)$ is a Dirichlet form if:

- E_α^s is positive definite on $\mathcal{D}$;
- $(E_\alpha, \mathcal{D})$ is closed (i.e. $\mathcal{D}$ is closed and complete w.r.t. $E_\alpha^{s,1}$, or equivalently $(\mathcal{D}, E_\alpha^{s,1})$ is a Hilbert space);
- $(E_\alpha, \mathcal{D})$ is coercive i.e. there exists $K > 0$ such that for all $f, g \in \mathcal{D}$

$$(2.13) \qquad \left| E_\alpha^1(f, g) \right| \leqslant K E_\alpha^{s,1}(f, f)^{1/2} E_\alpha^{s,1}(g, g)^{1/2};$$

- for all $u \in \mathscr{D}$, we have the contraction properties:

$$(2.14) \qquad \min(u_+, 1) \in \mathscr{D} \qquad E_\alpha(u + \min(u_+, 1), u - \min(u_+, 1)) \geqslant 0$$

$$E_\alpha(u - \min(u_+, 1), u + \min(u_+, 1)) \geqslant 0$$

where $u_+ = \max(u, 0)$ is the positive part of u.

A Dirichlet form is said to satisfy the local property if for any $f, g \in \mathscr{D}$ such that $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$, $E_\alpha(f, g) = 0$. The Dirichlet form is said to be regular if $C_c \cap \mathscr{D}$ is dense in $(\mathscr{D}, E_\alpha^{s,1})$ and dense in C_c with respect to the uniform norm.

We complete our set of assumptions.

Assumptions 2.3. We assume that there exists a function h constant on connected level sets of h such that

$$(2.15) \qquad\qquad\qquad \nabla \cdot (hF) \leqslant 0$$

i.e. the h-divergence of F is non positive (F is defined by Equation (2.5)). We also assume that the vector field F is bounded on $\mathbb{R}^2$, and that its divergence (with respect to h, i.e. $h^{-1}\nabla \cdot (hF)$) is also bounded.

We make several remarks on these assumptions.

Remark 2. These assumptions allow us to extend the bilinear form E_α as a Dirichlet form. We need the condition (2.15) to ensure that $E_\alpha^{s,1}$ is a positive bilinear form. The condition that h is constant on connected level sets ensures us that the symmetric part (2.7) does not depend on α. Lastly, the symmetric form $E_\alpha^{s,1}$ is a norm equivalent to the $H^1(\mu)$ norm thanks to the assumption that F and its h-divergence are bounded.

Remark 3. Note that Condition (2.15) simplifies in

$$(2.16) \qquad\qquad \varepsilon \Delta h + \nabla \cdot (he) = \varepsilon \Delta h + \nabla h \cdot e + h \nabla \cdot e \leqslant 0.$$

Therefore if $\nabla \cdot e \leqslant 0$, we can choose h to be constant and then μ can be the Lebesgue measure. Moreover, our assumption implies a necessary condition on e. In fact, we ask for h to be constant on connected level set, therefore, on the interior of $\{\nabla H = 0\}$, e must satisfies $\nabla \cdot e \leqslant 0$. Finally, note that if $e = \nabla f$ where f is a real-valued potential constant on connected level sets of H, one can choose the Gibbs measure i.e. $h = e^{-f/\varepsilon}$.

Remark 4. Let us also remark that the adjoint L_α^* (w.r.t. the usual scalar product in L^2) of L_α is, for $f \in C_c^2$

$$(2.17) \qquad\qquad L_\alpha^* f = \varepsilon \Delta f + \nabla \cdot (fe) - \frac{1}{\alpha} \nabla \cdot (fA\nabla H).$$

Thus the fact that $\nabla h \cdot A\nabla H = 0$ (since h is constant on connected level sets) and the condition (2.15) implies that

$$(2.18) \quad L_\alpha^* h = \varepsilon \Delta h + \nabla \cdot (he) - \frac{1}{\alpha} \nabla \cdot (hA\nabla H) = \varepsilon \Delta h + \nabla \cdot (he) = \nabla \cdot (hF) \leqslant 0.$$

Therefore h must be the density of a supermedian measure for the transitions probabilities of Equation (1.2) (cf. [17] p.62) i.e. for any $t \geqslant 0$, and $f \in C_b(\mathbb{R}^2)$

$$(2.19) \qquad\qquad \mathbb{E}_\mu[f(Y_t)] = \int \mathbb{E}_x[f(Y_t)]\mathrm{d}\mu(x) \leqslant \int f \mathrm{d}\mu.$$

Note that this is precisely the condition needed in order to define a Dirichlet form from a diffusion process ([17] Section IV.2 p.92).

Proposition 2.4. *Under the set of assumptions 2.1 and 2.3, E_α is coercive, closable and can be extended into a Dirichlet form, also denoted E_α, on $L^2(\mu)$ with domain $\mathscr{D}(E_\alpha) = H^1(\mu)$ defined as the completion in $L^2(\mu)$ of $\mathscr{D}(L_\alpha)$. Moreover E_α is regular and has the local property. The Dirichlet form is then $E_\alpha(f,g) = E_\alpha^s(f,g) + E_\alpha^a(f,g)$*

$$(2.20) \quad E_\alpha^s(f,g) = \varepsilon \int \nabla f \cdot \nabla g \, \mathrm{d}\mu - \frac{1}{2} \int \frac{1}{h} \nabla \cdot (hF) fg \, \mathrm{d}\mu$$

$$(2.21) \quad E_\alpha^a(f,g) = -\frac{1}{2\alpha} \int_{\mathbb{R}^2} A\nabla H \cdot [g\nabla f - f\nabla g] \, \mathrm{d}\mu + \frac{1}{2} \int F \cdot [\nabla fg - \nabla gf] \, \mathrm{d}\mu.$$

Proof. The domain $\mathscr{D}(E_\alpha)$ is defined as the completion of $\mathscr{D}(L_\alpha)$ with respect to the positive bilinear form $E_\alpha^{s,1}$ and it defines a coercive closed bilinear form ([17] Theorem 2.15 p.22). The contraction properties (2.14) are proved in [17] Section II.2.d. (p.48) under some conditions (Equations (2.16) p.48) which come as a consequence of our assumption (2.15).

The fact that the domain is $H^1(\mu)$ comes from the fact that, under the assumption 2.3, F and $\nabla \cdot (hF)$ are bounded, $E_\alpha^{s,1}$ and $\|\cdot\|_{H^1(\mu)}$ are equivalent norms. The regularity is obvious from the properties of the sets $H^1(\mu)$ and C_c. The local property is also obvious from Equations (2.7) and (2.8). □

2.2. **Orbit space.** We need to construct our new state space from the Hamiltonian H. We define a graph, with a set of vertices $\mathcal{V}$, and edges (with their length). In the rest of the section we denote $\mathcal{C}(x)$ the connected level set of H containing x.

We define a vertex O as a connected level set of H containing a stationary point. We denote $\mathcal{V}$ this set to which we add a artificial vertex $O = \infty$. Let us recall that we assume that H is bounded below (Assumption 2.1).

In order to define the edges, we prove a lemma.

Lemma 2.5. *Given any $x \in \mathbb{R}^2$ such that $\mathcal{C}(x)$ does not contain any stationary point (i.e. $\mathcal{C}(x)$ is not a vertex), there exists a unique closed interval I of $\mathbb{R}$ of the form $I = [m_-, m_+]$ or $I = [m_-, +\infty[$ such that $x \in H^{-1}(\mathring{I})$ and the connected domain A_I of $H^{-1}(I)$ containing x satisfies*

- *$H^{-1}(m_-) \cap A_I$ is a vertex, denoted O_I^- ;*
- *if $m_+ < +\infty$, $H^{-1}(m_+) \cap A_I$ is also a vertex denoted O_I^+ (with the convention $O_I^+ = \infty$ if $m_+ = +\infty$).*

Moreover for any $m \in I$, $A_I \cap H^{-1}(m)$ is a connected level set of H.

Proof. Let us consider $x \in \mathbb{R}^2$ such that $\mathcal{C}(x)$ does not contain a stationary point, and denote $m_0 = H(x)$. For $a < m_0$ and $b > m_0$, let us denote $C(a,b)$ the connected domain of $\{x, H(x) \in]a, b[\}$ containing $\mathcal{C}(x)$. Note that since H is C^1 and by assumption $\mathcal{C}(x)$ is compact, there exists $\eta > 0$ such that $C(m_0 - \eta, m_0 + \eta)$ does not contain any stationary point. Then let us denote,

$$(2.22) \qquad m_- = \min\{a < m_0, \forall z \in C(a, m_0 + \eta), \nabla H(z) \neq 0\}$$

$$(2.23) \qquad m_+ = \max\{b > m_0, \forall z \in C(m_0 - \eta, b), \nabla H(z) \neq 0\} \leqslant +\infty.$$

We have $m_- \leqslant m_0 - \eta$ and $m_+ \geqslant m_0 + \eta$. Then, we see that $I = [m_-, m_+]$ (or $I = [m_-, +\infty[$ if $m_+ = +\infty$) satisfies our properties. The facts that $x \in H^{-1}(\mathring{I})$ and that $H^{-1}(m_-) \cap A_I$ (resp. $H^{-1}(m_+) \cap A_I$) contains a vertex are obvious.

To prove that for all $m \in I$, $A_I \cap H^{-1}(m)$ is a connected level set, we consider the flow ϕ_t induced by the differential equation

$$(2.24) \qquad \dot{y}(t) = \frac{\nabla H(y(t))}{|\nabla H(y(t))|^2}.$$

Remark that this flow is well defined until the orbit reach a point for which $\nabla H(x) = 0$. Then it is well defined on $A_I \cap H^{-1}(\mathring{I})$ and that, for all y in this set, $H(\phi_t(y)) = t + H(y)$, for t such that $\phi_s(y)$ is not a stationnary point for any s. Then $\phi_t(\mathcal{C}(x))$ is a connected level set of H for all $t \in \mathring{I}$ and by definition of A_I, we get $A_I \cap H^{-1}(m) = \phi_{m-m_0}(\mathcal{C}(x))$. Therefore, it is a connected set for all $m \in \mathring{I}$. $\square$

Remark 5. Given a point x and the interval I associated to it by Lemma 2.5, for $y \in A_I \cap H^{-1}(\mathring{I})$, the interval associated by the lemma is also I. For all x, we denote $(I(x), O_I^-(x), O_I^+(x))$ the interval and the two vertices associated to x through Lemma 2.5. Under the Assumption 2.1, the set $S = (I_i, O_i^-, O_i^+)_i$ of distinct triplets given by Lemma 2.5 is countable, therefore the set of indices i is countable.

We are ready to define our graph.

Definition 2.6. Let consider the set $S = (I_i, O_i^-, O_i^+)_i$. Our graph Γ, is given by the collection of edges I_i, the collection of vertices $\mathcal{V}$. An edge I is related to the vertices O^-, O^+ such that $(I, O^-, O^+) \in S$.

We also define the projection π from $\mathbb{R}^2$ to Γ. For $x \in \mathbb{R}^2$, we define $\pi(x) = O$ if $\mathcal{C}(x)$ is the vertex O, otherwise $\pi(x) = (H(x), i(x))$ where $i(x)$ is defined as the index such that $I(x) = I_{i(x)}$.

We equip Γ with the minimal topology making π continuous.

Remark 6. The space Γ is therefore a disjoint countable union of interval $(I_i)_i$ of $\mathbb{R}$ glued together by one or two of their extremities (O_k) associated to stationnary points of H. $i(x)$ is a discrete first integral for the system but is defined only in the interior of the edges. At a vertex we can choose one the indices of the incident edges (e.g. the lowest integer). Note also that Γ is a tree (i.e. it does not have any cycle).

Let us now consider the equivalence relation $\sim$ on $\mathbb{R}^2$ defined by
$$(2.25)$$
$$x \sim y \Leftrightarrow x \text{ and } y \text{ are in the same connected component of a level set of } H.$$

Proposition 2.7. *We have*

$$(2.26) \qquad \Gamma = \mathbb{R}^2 / \sim .$$

Proof. We construct a bijection ϕ from $\mathbb{R}^2 / \sim$ to Γ. Given a connected level set in $\mathcal{C}$ in $\mathbb{R}^2 / \sim$, if there is a stationary point $x \in \mathcal{C}$, then by definition of Γ, $\mathcal{C}$ is a vertex and $\phi(\mathcal{C}) = \mathcal{C} \in \mathcal{V}$. If $\mathcal{C}$ does not contain any stationary point, then, according to Lemma 2.5, there exists a unique edge I_i and $\phi(\mathcal{C}) = (H(\mathcal{C}), i)$ where $H(\mathcal{C})$ is the common value of H along $\mathcal{C}$. ϕ^{-1} is also well defined since, according to 2.5, for any i and $h \in \mathring{I}$, the set $\pi^{-1}((h, i))$ is a connected level set of H, therefore a unique equivalence class in $\mathbb{R}^2 / \sim$. $\square$

We define $C_i(m)$ the connected level set

$$(2.27) \qquad C_i(m) = \{x, H(x) = m, i(x) = i\} = \{x, \pi(x) = (m, i)\} .$$

A_i is the domain of $\mathbb{R}^2$

$$(2.28) \qquad A_i = \left\{ x, i(x) = i, H(x) \in \mathring{I}_i \right\} = \bigcup_{m \in \mathring{I}_i} C_i(m).$$

Remark 7. Let h be a smooth function constant on connected level sets (such as the density of the measure μ satisfying Assumption 2.3). Note that on each A_i, there exists a function ψ_i such that $h = \psi_i(H)$. Then, we get, on each A_i,

$$(2.29) \qquad \nabla \cdot (hA\nabla H) = \nabla \cdot (\psi_i(H)A\nabla H) = 0.$$

Since for a point x which is not in any A_i, $\nabla H(x) = 0$, we have $\nabla \cdot (hA\nabla H) = 0$ on the whole space $\mathbb{R}^2$.

We introduce a decomposition of $\mathbb{R}^2$ into disjoints sets. Let us first introduce a partition of the vertices:

$$(2.30) \qquad \mathcal{V}_* = \left\{ O \in \mathcal{V}, \mu(\pi^{-1}(O)) > 0 \right\} \qquad \mathcal{V}_0 = \left\{ O \in \mathcal{V}, \mu(\pi^{-1}(O)) = 0 \right\}.$$

Lemma 2.8. *We have the following decompositions, and for $f \in L^1(\mu)$*

$$(2.31) \qquad \mathbb{R}^2 = \bigcup_i A_i \cup \bigcup_{O \in \mathcal{V}_*} \pi^{-1}(O) \cup \bigcup_{O \in \mathcal{V}_0} \pi^{-1}(O)$$

$$(2.32) \qquad \int_{\mathbb{R}^2} f \mathrm{d}\mu = \sum_i \int_{A_i} f \mathrm{d}\mu + \sum_{O \in \mathcal{V}_*} \int_{\pi^{-1}(O)} f \mathrm{d}\mu.$$

We also have $\nabla H = 0$, identically on $\pi^{-1}(O)$, for $O \in \mathcal{V}_*$.

2.2.1. *Examples.* We give in this section some examples of the space Γ obtain for some given H and some examples of vertices. In the following we denote, for $(x_1, x_2) \in \mathbb{R}^2$, $r = x_1^2 + x_2^2$.

Example 1. *The simplest example is maybe given by*

$$H_1(x_1, x_2) = \frac{r^2}{2}$$

The space Γ has a unique edge $I_1 = \mathbb{R}_+$ and one vertex $O = (0,0)$. The connected level set associated to $H_1 = m \in I_1$ is the circle centered on $(0,0)$ with radius $\sqrt{2m}$. The vertex O is a simple point.

Note that if we choose for H:

$$(2.33) \qquad H_1(x_1, x_2) = \begin{cases} 0 & \text{if } r \leqslant 1 \\ (r-1)^2 & \text{otherwise.} \end{cases}$$

The orbit space Γ is the same but the vertex O is the whole ball of radius 1. This is an example of a vertex having some positive mass.

Example 2. *Let us consider the function H_2 defined by $H_2(x_1, x_2) = \frac{x_1^4}{4} - \frac{x_1^2}{2} + \frac{x_2^2}{2}$.*

$$\infty$$

$$H_2 = 0 \qquad\qquad (0,0)$$

$$H_2 = -\frac{1}{4} \qquad (-1,0) \qquad\qquad (1,0)$$
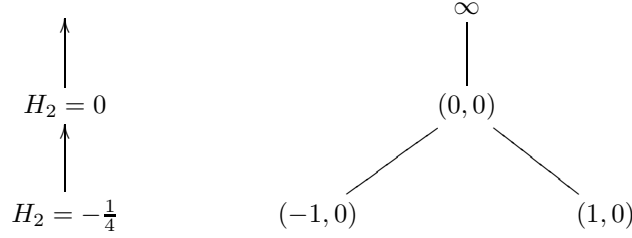
The space Γ is composed of three edges glued together at a point representing the connected level set of the saddle $(0,0)$. This connected level set is the internal vertex and does not have any mass. However, as in the first example, the vertex could have some mass (see Figure 1 and Figure 2).
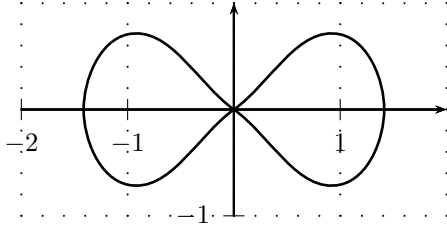

FIGURE 1. Connected level set associated to the internal vertex for H_2.
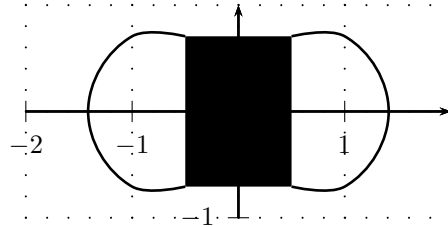
FIGURE 2. Connected level set associated to an internal vertex with some mass.

Example 3. *Let us consider the function H_3 defined by $H_3(x_1, x_2) = \frac{x_1^4}{4} - \frac{x_1^2}{2} + \frac{x_2^4}{4} - \frac{x_2^2}{2}$. The space Γ is composed of five branches glued together at a point O_1 representing the connected level set of the saddles $(\pm 1, 0), (0, \pm 1)$:*

$$\infty$$

$$H_3 = 0 \qquad\qquad (0,0)$$

$$H_3 = -\frac{1}{4} \qquad\qquad\qquad O_1$$

$$H_3 = -\frac{1}{2} \qquad (-1,-1) \qquad (-1,1) \qquad (1,-1) \qquad (1,1)$$
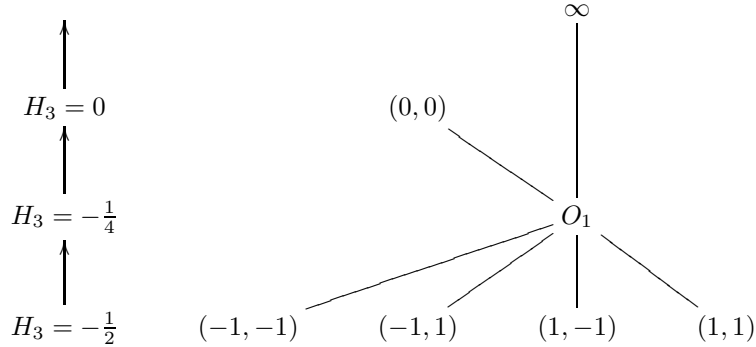
In this case the internal vertex at $H_3 = -1/4$ has a more complex structure.

3. Convergence of the Process

3.1. **Projected Dirichlet form.** We make the following remark: we see that if we evaluate E_α on functions which are constant on connected level sets of H, then
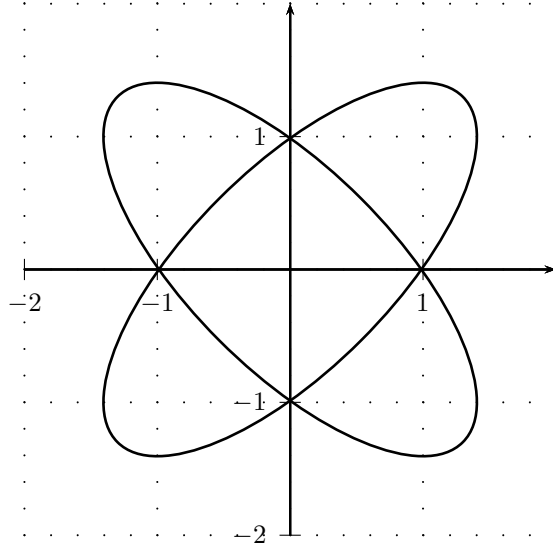
FIGURE 3. Connected level set associated to the internal vertex for H_3.

the α-dependence in the antisymmetric part (Equation (2.8)) of the Dirichlet form vanishes. More formally, the projection π defines a pull-back π_* on the functions on Γ :

$$\pi_* : \mathbb{R}^\Gamma \to \mathbb{R}^{\mathbb{R}^2}$$

(3.1)
$$u \mapsto \overline{u} = u \circ \pi = u(H, i)$$

The space Γ has a topological structure (induced by π). We define the L^2 and H^1 Hilbert space by using this pull-back:

(3.2)
$$L^2(\Gamma) = \pi_*^{-1}(L^2(\mu)) = \left\{ u, \overline{u} \in L^2(\mu) \right\}$$

(3.3)
$$\langle u, v \rangle_{L^2(\Gamma)} = \langle \overline{u}, \overline{v} \rangle_{L^2(\mu)} = \int_{\mathbb{R}^2} \overline{u}\,\overline{v} \mathrm{d}\mu$$

(3.4)
$$H^1(\Gamma) = \pi_*^{-1}(H^1(\mu)) = \left\{ u, \overline{u} \in H^1(\mu) \right\}$$

(3.5)
$$\langle u, v \rangle_{H^1(\Gamma)} = \langle \overline{u}, \overline{v} \rangle_{H^1(\mu)} = \int_{\mathbb{R}^2} \overline{u}\,\overline{v} + \nabla \overline{u} \cdot \nabla \overline{v} \mathrm{d}\mu.$$

Let us denote Π the subset of functions on $\mathbb{R}^2$ which are constant on connected level set: $\Pi = \pi_*(\mathbb{R}^\Gamma)$.

The pull-back naturally provides a identification (since it is an isometry by definition of $L^2(\Gamma)$ and $H^1(\Gamma)$):

(3.6)
$$L^2(\Gamma) \simeq L^2(\mu) \cap \Pi \qquad H^1(\Gamma) \simeq H^1(\mu) \cap \Pi = H^1(\mu) \cap L^2(\Gamma).$$

Lemma 3.1. *The spaces $L^2(\Gamma)$ and $H^1(\Gamma)$ are Hilbert spaces, and $H^1(\Gamma)$ is dense in $L^2(\Gamma)$.*

Proof. The pull-back π_* is continuous since it is an isometry by definition, then $L^2(\Gamma)$ and $H^1(\Gamma)$ are closed (relatively to $L^2(\mu)$ and $H^1(\mu)$). Therefore $L^2(\Gamma)$ and $H^1(\Gamma)$ are complete (for their respective norms induced by the norms on $L^2(\mu)$ and $H^1(\mu)$).

To prove that $H^1(\Gamma)$ is dense in $L^2(\Gamma)$, we want to show that

$$(3.7) \qquad\qquad H^1(\Gamma)^{\perp_{L^2(\Gamma)}} = \{0\}$$

where the orthogonal is taken in $L^2(\Gamma)$. Then, by definition

$$H^1(\Gamma)^{\perp_{L^2(\Gamma)}} = H^1(\Gamma)^\perp \cap L^2(\Gamma) = \overline{(H^1(\mu)^\perp \cup L^2(\Gamma)^\perp)} \cap L^2(\Gamma)$$

$$(3.8) \qquad\qquad = L^2(\Gamma)^\perp \cap L^2(\Gamma) = \{0\}.$$

We used that $H^1(\mu)$ (as a subspace of $L^2(\mu)$) is dense in $L^2(\mu)$ i.e. $H^1(\mu)^\perp = \{0\}$. $\qquad\square$

The limiting form is defined by projection (Equation (1.7)) but may, a priori, depend on α. We denote $\mathcal{E}_\alpha$ the projection of E_α and define its domain $\mathscr{D}(\mathcal{E}_\alpha)$: for u, v

$$(3.9) \quad \mathcal{E}_\alpha(u,v) = E_\alpha(u \circ \pi, v \circ \pi)$$

$$(3.10) \quad \mathscr{D}(\mathcal{E}_\alpha) = \left\{ u \in L^2(\Gamma), E_\alpha^s(u \circ \pi, u \circ \pi) < +\infty \right\} = \pi_*^{-1}(H^1(\mu)) = H^1(\Gamma).$$

Theorem 3.2. *The form $\mathcal{E}_\alpha$ does not depend on α and defines a Dirichlet form on $L^2(\Gamma)$ with domain $H^1(\Gamma)$, denoted $\mathcal{E}$. Moreover, $\mathcal{E}$ is regular and has the local property. We decompose $\mathcal{E} = \mathcal{E}^s + \mathcal{E}^a$ into its symmetric and antisymmetric parts, for all $u, v \in H^1(\Gamma)$*

$$(3.11) \quad \mathcal{E}^s(u,v) = \varepsilon \sum_{i \in I} \int_{A_i} |\nabla H|^2 \, \overline{\partial_1 u \partial_1 v} \mathrm{d}\mu$$

$$- \frac{1}{2} \sum_{i \in I} \int_{A_i} \frac{1}{h} \nabla \cdot (hF) \overline{uv} \mathrm{d}\mu - \frac{1}{2} \sum_{O \in \mathcal{V}_*} \int_{\pi^{-1}(O)} \frac{1}{h} \nabla \cdot (hF) \overline{uv} \mathrm{d}\mu$$

$$(3.12) \quad \mathcal{E}^a(u,v) = \frac{1}{2} \sum_{i \in I} \int_{A_i} F \cdot \nabla H \left[\overline{v \partial_1 u} - \overline{u \partial_1 v} \right] \mathrm{d}\mu.$$

Using this theorem, we can associate to $\mathcal{E}_\alpha$ a continuous Hunt process ([17] Theorem IV 3.5, section IV 4. a), Theorem V 1.5). Let us denote by $(\Omega, \mathcal{F}, (Z_t)_t)$ this process.

Proof of Theorem 3.2. We compute the projection of the antisymmetric part of $\mathcal{E}_\alpha$. Let us recall that for a function u on Γ, $\overline{u} = u \circ \pi$ denotes its pull-back on $\mathbb{R}^2$. For $u, v \in H^1(\Gamma)$, we have, from Equation (2.21) and Lemma 2.8

$$(3.13)$$

$$\int_{\mathbb{R}^2} A\nabla H \cdot \left[\overline{v} \nabla \overline{u} - \overline{u} \nabla \overline{v} \right] \mathrm{d}\mu = \sum_{i \in I} \int_{A_i} A\nabla H \cdot \left[v(H,i) \nabla u(H,i) - u(H,i) \nabla v(H,i) \right] \mathrm{d}\mu$$

$$+ \sum_{O \in \mathcal{V}_*} \int_{\pi^{-1}(O)} A\nabla H \cdot \left[v(H,i) \nabla u(H,i) - u(H,i) \nabla v(H,i) \right] \mathrm{d}\mu$$

Then, from Lemma 2.8, $\nabla H = 0$ on $\pi^{-1}(O)$ for $O \in \mathcal{V}_*$. We obtain that

$$(3.14) \quad \int_{\mathbb{R}^2} A\nabla H \cdot \left[\overline{v} \nabla \overline{u} - \overline{u} \nabla \overline{v} \right] \mathrm{d}\mu = \sum_{i \in I} \int_{A_i} A\nabla H \cdot \nabla H \left[\overline{v \partial_1 u} - \overline{u \partial_1 v} \right] \mathrm{d}\mu = 0.$$

We also have

$$(3.15) \quad \int F \cdot \left[\overline{v} \nabla \overline{u} - \overline{u} \nabla \overline{v} \right] \mathrm{d}\mu = \sum_{i \in I} \int_{A_i} F \cdot \nabla H \left[\overline{v \partial_1 u} - \overline{u \partial_1 v} \right] \mathrm{d}\mu.$$

Therefore, we have

$$(3.16) \qquad \mathcal{E}_\alpha^a(u,v) = \frac{1}{2} \sum_{i \in I} \int_{A_i} F \cdot \nabla H \left[\overline{v \partial_1 u} - \overline{u \partial_1 v} \right] d\mu.$$

Note that for the symmetric part $\mathcal{E}_\alpha^s$ from (2.20), the same calculation holds

$$(3.17) \quad \mathcal{E}_\alpha^s(u,v) = \varepsilon \sum_{i \in I} \int_{A_i} |\nabla H|^2 \overline{\partial_1 u \partial_1 v} d\mu$$
$$- \frac{1}{2} \sum_{i \in I} \int_{A_i} \frac{1}{h} \nabla \cdot (hF) \overline{uv} d\mu - \frac{1}{2} \sum_{O \in \mathcal{V}_*} \int_{\pi^{-1}(O)} \frac{1}{h} \nabla \cdot (hF) \overline{uv} d\mu.$$

Thus $\mathcal{E}_\alpha$ is a bilinear form defined on $H^1(\Gamma)$ and does not depend on α (thus denoted $\mathcal{E}$). We have to prove that $(\mathcal{E}, H^1(\Gamma))$ is a Dirichlet form following the definition recalled before Assumption 2.3. Let us denote $\mathcal{E}^1$ and $\mathcal{E}^{s,1}$ the forms E_α^1 and $E_\alpha^{s,1}$ (defined by Equations (2.11) and (2.12)) projected on $H^1(\Gamma)$.

Since E_α^s is positive definite on $H^1(\mu)$, $\mathcal{E}^s$ is also positive definite on $H^1(\Gamma)$. From Lemma 3.1, we know that $H^1(\Gamma)$ is dense in $L^2(\Gamma)$. $(\mathcal{E}, H^1(\Gamma))$ is closed on $L^2(\Gamma)$ since $(\mathcal{E}^{s,1})^{1/2}$ is a norm equivalent to $\|\cdot\|_{H^1(\Gamma)}$. The coercivity of $\mathcal{E}$ is also inherited from the coercivity of E_α. The contraction property is satisfied since it is satisfied by E_α and that, for $u \in H^1(\Gamma)$

$$(3.18) \qquad \min(u_+, 1) \circ \pi = \min((u \circ \pi)_+, 1).$$

The local property is also trivially satisfied since E_α has the local property and if $u, v \in H^1(\Gamma)$ are such that $\operatorname{supp} u \cap \operatorname{supp} v = \varnothing$, then

$$(3.19) \qquad \operatorname{supp}(u \circ \pi) \cap \operatorname{supp}(v \circ \pi) \subset \pi^{-1}(\operatorname{supp} u \cap \operatorname{supp} v) = \varnothing.$$

It now remains to show the regularity of the Dirichlet form. Since $C_c \cap H^1(\mu)$ is dense in $H^1(\mu)$, we have $(C_c \cap H^1(\mu))^\perp = 0$ where the orthogonal is taken with respect to the H^1-scalar product. Then we have:

$$(C_c(\Gamma) \cap H^1(\Gamma))^{\perp_{H^1(\Gamma)}} = (C_c \cap H^1(\mu) \cap H^1(\Gamma))^\perp \cap H^1(\Gamma)$$
$$= \overline{((C_c \cap H^1(\mu))^\perp \cup H^1(\Gamma)^\perp)} \cap H^1(\Gamma)$$
$$(3.20) \qquad = H^1(\Gamma)^\perp \cap H^1(\Gamma) = 0$$

which prove the density of $C_c(\Gamma) \cap H^1(\Gamma)$ in $H^1(\Gamma)$. The fact that $C_c(\Gamma) \cap H^1(\Gamma)$ is dense in $C_c(\Gamma)$ (for the uniform norm) is a consequence of the Stone-Weierstrass theorem. $\qquad \square$

3.2. Mosco-convergence.
We would want to prove the averaging principle by the Mosco convergence of the Dirichlet form $(E_\alpha, H^1(\mu))$ on $L^2(\mu)$ to $(\mathcal{E}, H^1(\Gamma))$ defined on $L^2(\Gamma)$ as $\alpha \to 0$. We then define (following [10]) weak and strong convergence for elements in $L^2(\mu)$ to elements in $L^2(\Gamma)$.

Definition 3.3 (Convergence). The sequence (u_n) in $L^2(\mu)$ strongly converge to $u \in L^2(\Gamma)$, noted $u_n \xrightarrow[n \to +\infty]{L^2(\Gamma)} u$, if

$$(3.21) \qquad \|u_n - u \circ \pi\|_{L^2(\mu)} \xrightarrow[n \to +\infty]{} 0.$$

The sequence (u_n) in $L^2(\mu)$ weakly converge to $u \in L^2(\Gamma)$, noted $u_n \xrightarrow[n\to+\infty]{L^2(\Gamma)} u$, if

$$(3.22) \qquad \begin{cases} \sup_n \|u_n\|_{L^2(\mu)} < +\infty \\ \langle u_n, v \circ \pi \rangle_{L^2(\mu)} \xrightarrow[n\to+\infty]{} \langle u \circ \pi, v \circ \pi \rangle_{L^2(\mu)} = \langle u, v \rangle_{L^2(\Gamma)}, \forall v \in L^2(\Gamma). \end{cases}$$

We have the following proposition.

Proposition 3.4. *We have the equivalence:*

$$(3.23) \qquad u_n \xrightarrow[n\to+\infty]{L^2(\Gamma)} u \iff \begin{cases} u_n \xrightarrow[n\to+\infty]{L^2(\Gamma)} u \\ \|u_n\|_{L^2(\mu)} \xrightarrow[n\to+\infty]{} \|u \circ \pi\|_{L^2(\mu)}. \end{cases}$$

Proof. This proposition is the transposition of the usual equivalence in $L^2(\mu)$. $\square$

Definition 3.5 (Mosco-convergence [10])**.** For a sequence (α_n) converging to 0, E_{α_n} Mosco-converges to $\mathcal{E}$ if the two following conditions hold:

(1) for any sequence (u_n) in $L^2(\mu)$ weakly converging to $u \in L^2(\Gamma)$ and such that

$$(3.24) \qquad \sup_n E_{\alpha_n}^{s,1}(u_n) < +\infty$$

then u is in $H^1(\Gamma)$.

(2) for all $v \in H^1(\Gamma)$, and all sequences $(u_n) \in H^1(\mu)$ weakly converging in $L^2(\mu)$ to $u \in H^1(\Gamma)$, there exists a sequence (v_n) strongly converging to v, such that

$$(3.25) \qquad E_{\alpha_n}(u_n, v_n) \xrightarrow[n\to+\infty]{} \mathcal{E}(u, v)$$

The second condition could however be replaced by a weaker condition (see [10] Section 3, Condition (F'2)): we ask that for all subsequences (α_{n_k}), all sequences (u_k) weakly convergent in $L^2(\mu)$ to $u \in H^1(\Gamma)$ such that

$$(3.26) \qquad \sup_n E_{\alpha_{n_k}}^{s,1}(u_k) < +\infty$$

and for all $v \in H^1(\Gamma)$, there exists a sequence (v_k) strongly converging to v such that

$$(3.27) \qquad \liminf_k E_{\alpha_{n_k}}(u_k, v_k) \leqslant \mathcal{E}(u, v)$$

Theorem 3.6. *For all sequence (α_n) converging to 0, E_{α_n} Mosco-converges to $\mathcal{E}$.*

Proof. We have to check that the sequence E_{α_n} satisfies the condition of the definition 3.5.

Let us check the first condition. Let (u_n) be a sequence in $L^2(\mu)$ weakly converging to u in $L^2(\Gamma)$ and such that $E_{\alpha_n}^{s,1}(u_n)$ is bounded. Then since $E_\alpha^{s,1}$ does not depend on α (Equation (2.20)) and is equivalent to the norm in H^1, the sequence (u_n) is bounded in $H^1(\mu)$. Therefore there exists a subsequence converging weakly in $H^1(\mu)$, thus converging weakly in $L^2(\mu)$. By uniqueness of the limit, we deduce that $u \circ \pi$ is in $H^1(\mu)$.

For the second condition, we prove the alternate form (Equation (3.27)). We fix a sequence (u_n) weakly convergent in $L^2(\mathbb{R}^2)$ to $u \in H^1(\Gamma)$ and bounded in

$H^1(\mathbb{R}^2)$. Let us consider the constant sequence $v_n = v \circ \pi$, then the Condition (3.27) becomes

$$\liminf_k E_{\alpha_{n_k}}(u_k, v \circ \pi) \leqslant \mathcal{E}(u, v). \tag{3.28}$$

Since (u_k) is bounded in $H^1(\mu)$, it is weakly precompact in $H^1(\mu)$. Let $w \in H^1(\mu)$ be a weak accumulation point of (u_k) in $H^1(\mu)$, then it is also a weak accumulation point in $L^2(\mu)$, thus $u \circ \pi = w$. Therefore u_k weakly converge in $H^1(\mu)$ to $u \circ \pi$.

Since by Assumption 2.3, $h^{-1}\nabla \cdot (hF)$ is bounded, we have the convergence of the symmetric part (2.20). The convergence of the antisymmetric part (2.21) comes, for the term with F, from the fact that F is assumed to be bounded (Assumption 2.3). For the other term, with $A\nabla H$, using (2.10), we have that

$$\frac{1}{2\alpha_{n_k}} \int A\nabla H \cdot (u_k \nabla \overline{v} - \overline{v} \nabla u_k) \mathrm{d}\mu = \frac{1}{\alpha_{n_k}} \int (A\nabla H \cdot \nabla \overline{v}) u_k \mathrm{d}\mu$$
$$+ \frac{1}{2\alpha_{n_k}} \int \frac{1}{h} \nabla \cdot (hA\nabla H) u_k \overline{v} \mathrm{d}\mu. \tag{3.29}$$

The first term vanishes because $\nabla \overline{v} = \nabla H \overline{\partial_1 v}$. The second vanishes since $\nabla h \cdot A\nabla H = 0$ by Assumption 2.3. $\qquad\square$

3.3. Convergence of finite dimensional marginals.
Let us denote Y^α the process defined by Equation (1.2). It is associated to the Dirichlet forms E_α.

An important consequence of the Mosco convergence of the Dirichlet form is the strong convergence of the semigroup, resolvent and generator associated to the form.

Definition 3.7. Let $(B_n)_n$ be a sequence of bounded operators on $L^2(\mu)$ and B a bounded operator on $L^2(\Gamma)$, then:

- (B_n) strongly converge to B if for every sequence (u_n) strongly converging to u, $(B_n u_n)$ strongly converge to Bu,
- (B_n) weakly converge to B if for every sequence (u_n) weakly converging to u, $(B_n u_n)$ weakly converge to Bu.

The following Theorem ([10], Theorem 3.5, [21], Theorem 2.53) gives us the convergence of the C_0-contraction semigroups and resolvants associated to E_{α_n} and $\mathcal{E}$.

Theorem 3.8. *Let T^n, T be the C_0-semigroups and G_λ^n, G_λ be the C_0-resolvents associated to E_{α_n} and $\mathcal{E}$. We have the following equivalence:*

(1) *T_t^n strongly converges to T_t for all $t \geqslant 0$;*
(2) *G_λ^n strongly converges to G_λ for all $\lambda \geqslant 0$;*
(3) *E_{α_n} Mosco-converges to $\mathcal{E}$.*

Remark 8. Let us consider f_n strongly converging to $\overline{f}$, the strong convergence of $T_t^n f_n$ implies that

$$\left\| T_t^n f_n - \overline{T_t f} \right\|_{L^2(\mu)} \xrightarrow[n \to +\infty]{} 0. \tag{3.30}$$

We now consider the convergence of finite dimensional distributions of the processes $Z^{\alpha_n} = (\pi(Y^{\alpha_n}))_n$. We suppose that the law of the initial condition $\nu_n = \mathscr{L}(Y_0^{\alpha_n})$ has a density g_n with respect to μ and g_n converges weakly in $L^2(\mu)$ to

$g \in L^2(\Gamma)$ with $\langle \overline{g}, 1 \rangle_{L^2(\mu)} = 1$. Then ν_n converges weakly to the probability measure $\overline{g}\mathrm{d}\mu$ which defines a probability measure $\widetilde{\nu}$ on Γ. In particular, it means that for all $f_n \in C_b(\mathbb{R}^2)$ converging strongly uniformly to $\overline{f}$

$$(3.31) \qquad \int_{\mathbb{R}^2} f_n \mathrm{d}\nu_n \xrightarrow[n \to +\infty]{} \int_{\mathbb{R}^2} \overline{f}\,\overline{g}\mathrm{d}\mu = \int_{\Gamma} f \mathrm{d}\widetilde{\nu}.$$

We consider Z the Markov process on Γ associated to the Dirichlet form $\mathcal{E}$ with initial law ν.

Proposition 3.9. *We consider (Y^{α_n}) the processes with initial distribution ν_n and the process Z with initial distribution ν. Under the condition that ν_n converges weakly to ν, the finite dimensional distributions of $(Z^{\alpha_n}) = (\pi(Y^{\alpha_n}))$ converge as n goes to infinity to the finite dimensional distributions of Z.*

By this proposition, we mean the following: for all $N > 0$, $0 < t_1 < t_2 < \cdots < t_N$, and $f_1, \cdots f_N \in C_b(\Gamma)$ then

$$(3.32) \quad \mathbb{E}_{\nu_n}[f_1(Z_{t_1}^{\alpha_n})f_2(Z_{t_2}^{\alpha_n}) \cdots f_N(Z_{t_N}^{\alpha_n})] \xrightarrow[n \to +\infty]{} \mathbb{E}_{\nu}[f_1(Z_{t_1})f_2(Z_{t_2}) \cdots f_N(Z_{t_N})].$$

Proof. Let us prove the proposition for $N = 2$, for the sake of simplicity. We use the Markov property of the process Y^{α_n} for all n. Then

$$\mathbb{E}_{\nu_n}[f_1(Z_{t_1}^{\alpha_n})f_2(Z_{t_2}^{\alpha_n})] = \mathbb{E}_{\nu_n}\left[\overline{f_1}(Y_{t_1}^{\alpha_n})\overline{f_2}(Y_{t_2}^{\alpha_n})\right] = \mathbb{E}_{\nu_n}\left[\overline{f_1}(Y_{t_1}^{\alpha_n})\mathbb{E}_{Y_{t_1}^{\alpha_n}}\left[\overline{f_2}(Y_{t_2}^{\alpha_n})\right]\right]$$

$$= \mathbb{E}_{\nu_n}\left[\overline{f_1}(Y_{t_1}^{\alpha_n})T_{t_2-t_1}^n\overline{f_2}(Y_{t_1}^{\alpha_n})\right] = \int_{\mathbb{R}^2} T_{t_1}^n(\overline{f_1}T_{t_2-t_1}^n\overline{f_2})g_n\mathrm{d}\mu$$

$$\xrightarrow[n \to +\infty]{} \int_{\mathbb{R}^2} \overline{T_{t_1}(f_1 T_{t_2-t_1}f_2)}\,\overline{g}\mathrm{d}\mu = \int_{\Gamma} T_{t_1}(f_1 T_{t_2-t_1}f_2)\mathrm{d}\nu$$

$$(3.33) \qquad\qquad = \mathbb{E}_{\nu}[f_1(Z_{t_1})f_2(Z_{t_2})].$$

The convergence comes from the fact that g_n weakly converge to $\overline{g}$ and the fact that

$$(3.34) \qquad T_{t_1}^n(\overline{f_1}T_{t_2-t_1}^n\overline{f_2}) \xrightarrow[n \to +\infty]{} \overline{T_{t_1}(f_1 T_{t_2-t_1}f_2)}$$

strongly in $L^2(\mu)$ by a repeated application of Equation (3.30). $\qquad\square$

3.4. Convergence in law.

The tightness of the law of $(Z^{\alpha_n})_n$ follows from the Lemma 3.2 of Chapter 8 of [6] for the case $e = 0$. With Proposition 3.9 it gives the weak convergence of the law of the processes as α_n goes to 0.

Proposition 3.10 (Tightness, Freidlin Wentzell Lemma 3.2 Chapter 8). *The family of distributions of $(Z^{\alpha_n})_n$ in the space $C(\mathbb{R}^+; \Gamma)$ is tight.*

The proof follows usual ideas to prove the tightness, and comes ultimately from Ascoli-Arzéla Theorem. The first argument is given by: for all $T > 0$ and $\delta > 0$, there are H_0 and n_0, such that:

$$(3.35) \qquad n \geqslant n_0 \implies \mathbb{P}_x[\max_{0 \leqslant t \leqslant T} H(Y_t^{\alpha_n, \varepsilon}) \geqslant H_0] \leqslant \delta.$$

The equicontinuity condition comes from Stroock-Varadhan [20]. Let us first precise the metric used on Γ: the distance $\rho(y, y')$ is the minimum distance of the

paths on Γ. If $y = (H_0, i_0)$ and $y' = (H_0', i_0')$, if γ is a path from y to y' passing through the vertices $O_1, O_2 \ldots O_l$, we denote $\widetilde{\rho}$ the length of this path:

$$(3.36) \qquad \widetilde{\rho}(\gamma) = |H_0 - H(O_1)| + \sum_{i=1}^{l-1} |H(O_i) - H(O_{i+1})| + |H_0' - H(O_l)|.$$

Then $\rho(y, y')$ is simply the minimum of the lengths of all such paths.

The equicontinuity then follows from this: for all compact K of Γ and all δ sufficiently small there exists a constant A_δ such that for every $a \in K$ there exists a function f_δ^a with $f_\delta^a(a) = 1, f_\delta^a(z) = 0$ for $\rho(z, a) > \delta$, and $0 \leqslant f_\delta^a \leqslant 1$, such that, for all n, $f_\delta^a(Z_t^{\alpha_n, \varepsilon}) + A_\delta t$ is a submartingale.

Note that their proof use the fact that H has bounded second derivatives. Using our Assumption 2.1 on e, the generalization is straightforward.

We can then conclude by the convergence of the law of the processes $(Z^{\alpha_n})_n$ to the law of the process Z.

Theorem 3.11. *For all g_n weakly converging in L^2 to $g \circ \pi$, and such that g_n and $g \circ \pi$ are densities (with respect to μ) of probability laws in $\mathbb{R}^2$, then the processes $(\pi(Y^{\alpha_n, \varepsilon}))$ where $Y_0^{\alpha_n, \varepsilon}$ is distributed as $g_n \mathrm{d}\mu$, converge in law to the process Z^ε with initial law given by $g \mathrm{d}(\pi_* \mu)$.*

Proof. The theorem follows from Proposition 3.9 and Proposition 3.10. $\qquad\qquad\square$

4. IDENTIFICATION OF THE LIMITING PROCESS

We want to identify the limiting process on the space Γ by identifying the infinitesimal operator and its domain associated to the Dirichlet form $\mathcal{E}$. The domain of the operator $\mathcal{L}$ associated to the Dirichlet form could be defined as ([17] Proposition I 2.16 p.23):

$$(4.1) \quad \mathscr{D}(\mathcal{L}) = \left\{ u \in H^1(\Gamma), v \mapsto \mathcal{E}(u, v) \text{ is continuous w.r.t. } \|\cdot\|_{L^2(\Gamma)} \text{ on } H^1(\Gamma) \right\}.$$

Accordingly, the infinitesimal generator applied to a function $u \in \mathscr{D}(\mathcal{L})$ is given as $\mathcal{L}u = w$ where $w \in L^2(\Gamma)$ is the unique function such that, for all $v \in H^1(\mu)$

$$(4.2) \qquad\qquad\qquad \mathcal{E}(u, v) = - \langle w, v \rangle_{L^2(\Gamma)}.$$

In fact, we need to write the Dirichlet form as an integral with respect to the projected measure.

We denote θ the function defined on Γ such that $h = \theta \circ \pi$.

For all i, for all $m \in \mathring{I}_i$, we define

$$(4.3) \qquad T(m, i) = \oint_{C_i(m)} \frac{\mathrm{d}l}{|\nabla H|} \qquad\qquad \mathrm{d}\omega_{(m,i)} = \frac{1}{T(m, i)} \frac{\mathrm{d}l}{|\nabla H|}.$$

Note that $T(m, i)$ is the period of the orbit of the Hamiltonian flow along the orbit $C_i(m)$ and $\omega_{(m,i)}$ is the invariant measure of mass 1 for this flow on $C_i(m)$, we

would call it the normalized Liouville measure. Let us also define

$$(4.4) \qquad S^2(m,i) = \oint_{C_i(m)} |\nabla H|^2 \, \mathrm{d}\omega_{(m,i)} = \frac{1}{T(m,i)} \oint_{C_i(m)} |\nabla H| \, \mathrm{d}l$$

$$(4.5) \qquad B^0(m,i) = -\oint_{C_i(m)} e \cdot \nabla H \mathrm{d}\omega_{(m,i)} = -\frac{1}{T(m,i)} \oint_{C_i(m)} e \cdot \nabla H \frac{\mathrm{d}l}{|\nabla H|}$$

$$(4.6) \qquad B^1(m,i) = \oint_{C_i(m)} \Delta H \mathrm{d}\omega_{(m,i)} = \frac{1}{T(m,i)} \oint_{C_i(m)} \Delta H \frac{\mathrm{d}l}{|\nabla H|}.$$

We define, for all vertices O and all edges I_i incident to O,

$$(4.7) \qquad \alpha_i(O) = \lim_{m \to H(O), m \in I_i} \int_{C_i(m)} |\nabla H| \, \mathrm{d}l, \qquad \gamma(O) = -\int_{\pi^{-1}(O)} \nabla \cdot e \mathrm{d}x.$$

Note that due to Assumption 2.3, $\gamma(O)$ is positive. We consider $J(O)$ the set of edges incident to O. We define the partition of $J(O)$ into two sets

$$(4.8) \qquad J_+(O) = \{i | i \in J(O), m \geqslant H(O), \forall m \in I_i\},$$

$$(4.9) \qquad J_-(O) = \{i | i \in J(O), m \leqslant H(O), \forall m \in I_i\}.$$

For a function u on Γ, we denote $u_i = u_{|I_i}$ its restriction on the edge I_i (which is a real interval). We have the main theorem of this section.

Theorem 4.1. *The domain $\mathscr{D}(\mathcal{L}) \subset H^1(\Gamma)$ is the set of real-valued functions u on Γ such that:*

(1) *u is continuous on Γ, for all i, u_i is in $H^2(I_i)$,*
(2) *for all (m,i) in the interior of an edge (i.e. $m \in \mathring{I}_i$), the differential operator*

$$(4.10) \qquad \mathcal{L}^i u_i(m) = \varepsilon S_i^2 u_i'' + (B_i^0 + \varepsilon B_i^1) u_i'$$

   defines a continuous function on $\mathring{I}_i$,
(3) *these functions $\mathcal{L}_i u_i$ have a common limit at a common vertex O, denoted $\mathcal{L}u(O)$,*
(4) *at a vertex O, we have the relation*

$$(4.11) \quad \gamma(O)u(O) + \varepsilon \left(\sum_{i \in J_+(O)} \alpha_i(O) D_i u(O) - \sum_{i \in J_-(O)} \alpha_i(O) D_i u(O) \right)$$
$$= -\left| \pi^{-1}(O) \right| \mathcal{L}u(O),$$

   where $\left| \pi^{-1}(O) \right|$ is the Lebesgue measure (area) of $\pi^{-1}(O)$
(5) *$\mathcal{L}u$ is in $L^2(\Gamma)$.*

Remark 9. This theorem shows that the underlying process is specified on two different domains : on the edges, we have a diffusion given by the restriction of the infinitesimal generator (Equation (4.10)); at each vertex, we have a gluing condition (Equation (4.11)), deduced from the domain, defining the properties of the vertex.

- Let us notice that the process Z does not depend on the choice of the measure μ. This confirms that μ is only a convenient tool we use to define the Dirichlet form.
- The coefficients of the diffusion on each edge are averages, with respect to the measure $\mathrm{d}\omega$, of, respectively, $|\nabla H|^2$, $-e \cdot \nabla H$ and ΔH on each connected level set $C_i(m)$.

- This Theorem also gives a way to construct the reference measure μ. In fact on each edge, we have a one dimensional diffusion therefore a natural candidate for a invariant measure which has a density with respect to the Lebesgue measure. Due to the tree-like structure of Γ, we obtain a invariant measure on G by adding suitable constants to these densities on each edge, then the lift on $\mathbb{R}^2$ give us a suitable candidate for μ.

First we derive from the Dirichlet form, the infinitesimal generator and its domain. Then we deduce the diffusion process on the edges and its behavior when it reaches a vertex.

4.1. **Proof of Theorem 4.1.**
In order to compute the different coefficients of the generator, we need a lemma.

Lemma 4.2. *For a vector field G of class C^1 on A_i, we have*

$$(4.12) \qquad \frac{\mathrm{d}}{\mathrm{d}m} \oint_{C_i(m)} G \cdot \frac{\nabla H}{|\nabla H|} \mathrm{d}l = \oint_{C_i(m)} \nabla \cdot G \frac{\mathrm{d}l}{|\nabla H|}.$$

For a function g, of class C^1 on A_i, we have

$$(4.13) \qquad \frac{\mathrm{d}}{\mathrm{d}m} \oint_{C_i(m)} g \, |\nabla H| \, \mathrm{d}l = \oint_{C_i(m)} [\nabla g \cdot \nabla H + g \Delta H] \frac{\mathrm{d}l}{|\nabla H|}.$$

The second property (4.13) is given in [6] (Lemma 1.1 p.265). We formulate a short proof.

Proof. This lemma can be proved by using the co-area formula and the divergence theorem on a domain $D_i(m_0, m) = \{x \in C_i(m'), m' \in [(m_0, m)]\}$. Let us suppose that $m_0 < m$. First, let us remark that the unit vector $n = \frac{\nabla H}{|\nabla H|}$ is a normal vector to the curve $C_i(m')$ which points towards the exterior of the domain along $C_i(m)$ and inwards along $C_i(m_0)$. Therefore, using the divergence Theorem on the domain $D_i(m_0, m)$ for the integrand $\nabla \cdot G$, we obtain:

$$\int_{D_i(m_0,m)} \nabla \cdot G \mathrm{d}x = \int_{\partial D_i(m_0,m)} G \cdot \mathrm{d}n_{ext}$$

$$(4.14) \qquad = \oint_{C_i(m)} G \cdot \frac{\nabla H}{|\nabla H|} \mathrm{d}l - \oint_{C_i(m_0)} G \cdot \frac{\nabla H}{|\nabla H|} \mathrm{d}l.$$

Using the co-area formula on the same integral, we have:

$$(4.15) \qquad \int_{D_i(m_0,m)} \nabla \cdot G \mathrm{d}x = \int_{m_0}^{m} \oint_{C_i(m')} \frac{\nabla \cdot G}{|\nabla H|} \mathrm{d}l \mathrm{d}m'.$$

The same holds if $m < m_0$. Thus the integral $\oint_{C_i(m_0)} G \cdot \frac{\nabla H}{|\nabla H|} \mathrm{d}l$ is differentiable at m_0 and we obtain the result.

For (4.13), we remark that

$$(4.16) \qquad g \, |\nabla H| = g \nabla H \cdot \frac{\nabla H}{|\nabla H|}$$

and apply (4.12) to $G = g \nabla H$. We obtain the result since $\nabla \cdot [g \nabla H] = \nabla g \cdot \nabla H + g \Delta H$. $\qquad \square$

Proof of Theorem 4.1. First, we compute the generator for functions which are in $C^2(\Gamma)$. Let us consider functions u, v in $C^2(\Gamma)$. For the symmetric part, Equation (3.11), we get

$$\mathcal{E}^s(u,v) = \sum_{i \in I} \int_{A_i} \varepsilon |\nabla H|^2 u_i'(H)v_i'(H) - \frac{1}{2h} \nabla \cdot (hF)u_i(H)v_i(H)\mathrm{d}\mu$$

$$(4.17) \qquad\qquad - \frac{1}{2} \sum_{O \in \mathcal{V}_*} \int_{\pi^{-1}(O)} \frac{1}{h} \nabla \cdot (hF)\overline{uv}\mathrm{d}\mu.$$

We use the coarea formula (since $|\nabla H| \neq 0$ on each A_i), to obtain

$$\mathcal{E}^s(u,v) = \sum_{i \in I} \int_{I_i} \varepsilon u_i'(m)v_i'(m) \oint_{C_i(m)} |\nabla H| \, h \mathrm{d}l$$

$$- u_i(m)v_i(m)\frac{1}{2} \oint_{C_i(m)} \frac{\nabla \cdot (hF)}{|\nabla H|}\mathrm{d}l\mathrm{d}m$$

$$- \frac{1}{2} \sum_{O \in \mathcal{V}_*} u(O)v(O) \int_{\pi^{-1}(O)} \nabla \cdot (hF)(x)\mathrm{d}x$$

$$(4.18) \qquad = \sum_{i \in I} \int_{I_i} \varepsilon a_i u_i'v_i'\mathrm{d}m - \frac{1}{2}\int_{I_i} c_i u_i v_i\mathrm{d}m - \frac{1}{2}\sum_{O \in \mathcal{V}_*} \gamma(O)u(O)v(O).$$

We have denoted a and c the quantities

$$(4.19) \qquad a(m,i) = \oint_{C_i(m)} |\nabla H| \, h\mathrm{d}l, \qquad c(m,i) = \oint_{C_i(m)} \frac{\nabla \cdot (hF)}{|\nabla H|}\mathrm{d}l.$$

For the antisymmetric part, Equation (3.12), we get, using also the coarea formula

$$(4.20) \qquad \mathcal{E}^a(u,v) = \frac{1}{2}\sum_{i \in I}\int_{I_i} b_i \left[v_i u_i' - u_i v_i'\right]\mathrm{d}m$$

where b denotes the quantity

$$(4.21) \qquad b(m,i) = \oint_{C_i(m)} hF \cdot \nabla H \frac{\mathrm{d}l}{|\nabla H|}.$$

Then, since we assume that $u \in C^2(\Gamma)$, we have for all i

$$(4.22) \quad \int_{I_i} a_i u_i'v_i'\mathrm{d}m = \lim_{m \to m_i^+}(a_i u_i'v_i)(m) - \lim_{m \to m_i^-}(a_i u_i'v_i)(m) - \int_{I_i}(a_i u_i')' v_i\mathrm{d}m$$

where we denote $I_i = \left[m_i^+, m_i^-\right]$ (from Proposition 2.7). Then, we get that

$$(4.23) \qquad \lim_{m \to m_i^+} a_i(m) = \theta(O)\alpha_i(O)$$

where O is the vertex incident to I_i at m_+^i. The same holds at m_i^-.

By summing the integrals (4.22) over the edges of Γ, we rewrite the first part as a sum over the vertices. Since v has a unique value at each vertex we get

$$\sum_i \int_{I_i} a_i u_i' v_i' \mathrm{d}m = -\sum_i \int_{I_i} (a_i u_i')' v_i \mathrm{d}m$$

$$(4.24) \quad + \sum_{O \in \mathcal{V}} v(O) \left(\sum_{i \in J_+(O)} \theta(O) \alpha_i(O) D_i u(O) - \sum_{i \in J_-(O)} \theta(O) \alpha_i(O) D_i u(O) \right).$$

We do the same calculation for the integral (4.20) and get

$$\sum_{i \in I} \int_{I_i} b_i u_i v_i' \mathrm{d}m = -\sum_i \int_{I_i} (b_i u_i)' v_i \mathrm{d}m$$

$$(4.25) \qquad + \sum_{O \in \mathcal{V}} v(O) u(O) \left(\sum_{i \in J_+(O)} \beta_i(O) - \sum_{i \in J_-(O)} \beta_i(O) \right)$$

where $\beta_i(O)$ is defined as

$$(4.26) \qquad\qquad \beta_i(O) = \lim_{m \to H(O)} b(m, i).$$

Without loss of generality, we suppose for a moment that $H(O) = 0$. Let us choose $\delta_0 > 0$ such that, for all $\delta < \delta_0$, the connected domain, denoted Ω_δ, of $H^{-1}(]-\delta, \delta[)$ containing $\pi^{-1}(O)$, satisfies

$$(4.27) \qquad\qquad \nabla H(x) \neq 0 \text{ for all } x \in \Omega_\delta \setminus \pi^{-1}(O).$$

Let us apply the divergence formula on Ω_δ to the vector field hF:

$$(4.28) \qquad\qquad \int_{\Omega_\delta} \nabla \cdot (hF) \mathrm{d}x = \int_{\partial \Omega_\delta} hF \cdot \mathrm{d}n_{ext}.$$

Using the same method as in the proof of Lemma 4.2, we have

$$\int_{\Omega_\delta} \nabla \cdot (hF) \mathrm{d}x = \sum_{i \in J_+(O)} \oint_{C_i(\delta)} hF \cdot \frac{\nabla H}{|\nabla H|} \mathrm{d}l - \sum_{i \in J_-(O)} \oint_{C_i(-\delta)} hF \cdot \frac{\nabla H}{|\nabla H|} \mathrm{d}l$$

$$(4.29) \qquad = \sum_{i \in J_+(O)} b(\delta, i) - \sum_{i \in J_-(O)} b(-\delta, i).$$

Since H is C^1, we get that

$$(4.30) \qquad \int_{\Omega_\delta} \nabla \cdot (hF) \mathrm{d}x \xrightarrow[\delta \to 0]{} \int_{\pi^{-1}(O)} \nabla \cdot (hF) \mathrm{d}x = -\theta(O)\gamma(O).$$

We deduce from (4.29)

$$(4.31) \qquad\qquad \theta(O)\gamma(O) = \sum_{i \in J_+(O)} \beta_i(O) - \sum_{i \in J_-(O)} \beta_i(O).$$

Therefore, after simplification, the whole Dirichlet form is

(4.32)

$$\mathcal{E}(u,v) = -\sum_i \int_{I_i} v_i \left[\varepsilon a_i u_i'' - (b_i - \varepsilon a_i')u_i' + \frac{1}{2}(c_i - b_i')u_i \right] \mathrm{d}m$$

$$+ \sum_{O \in \mathcal{V}} \theta(O)v(O) \left[\varepsilon \left(\sum_{i \in J_+(O)} \alpha_i(O)D_i u(O) - \sum_{i \in J_-(O)} \alpha_i(O)D_i u(O) \right) \right.$$

$$\left. + u(O)\gamma(O) \right].$$

We need to identify the projected measure $\mathrm{d}\pi_*\mu$: for $v \in C_c(\Gamma)$, by definition, $\overline{v} = v \circ \pi$ is continuous with compact support, thus integrable. We get, using also the coarea formula

$$\int_\Gamma v \mathrm{d}\pi_*\mu = \int_{\mathbb{R}^2} \overline{v}\mathrm{d}\mu = \sum_i \int_{A_i} \overline{v}h\mathrm{d}x + \sum_{O \in \mathcal{V}_*} \int_{\pi^{-1}(O)} \overline{v}h\mathrm{d}x$$

$$= \sum_i \int_{I_i} v_i(m) \oint_{C_i(m)} \frac{h}{|\nabla H|}\mathrm{d}l\mathrm{d}m + \sum_{O \in \mathcal{V}_*} v(O)\mu(\pi^{-1}(O))$$

(4.33)

$$= \sum_i \int_{I_i} v_i d_i \mathrm{d}m + \sum_{O \in \mathcal{V}_*} v(O)\theta(O)\left|\pi^{-1}(O)\right|$$

where d denotes

(4.34)
$$d(m,i) = \oint_{C_i(m)} \frac{h}{|\nabla H|}\mathrm{d}l = \theta_i(m)T_i(m).$$

To identify the operator $\mathcal{L}$, we solve (4.2) for any $v \in C^2(\Gamma)$. Using Equations (4.32) and (4.33), which are valid for any test functions v, therefore we see that w must satisfy

(4.35)
$$w_i d_i = \varepsilon a_i u_i'' - (b_i - \varepsilon a_i')u_i' + \frac{1}{2}(c_i - b_i')u_i$$

(4.36)
$$w(O)\theta(O)\left|\pi^{-1}(O)\right| = \theta(O)\gamma(O)u(O)$$

$$- \varepsilon\theta(O)\left(\sum_{i \in J_+(O)} \alpha_i(O)D_i u(O) - \sum_{i \in J_-(O)} \alpha_i(O)D_i u(O) \right).$$

Thus by definition, the domain of the operator is the set of functions u such that, on each edge I_i, we can define $w_i = \mathcal{L}_i u_i$, using Equation (4.35)

We deduce that $u \in H^2(I_i)$ for every edge I_i, therefore u has at the boundary of each edge limits of the first derivatives. The value of w on each vertex is given by continuity. Therefore, these limits must coincide and satisfy Condition (4.36). Then we can define $w = \mathcal{L}u$. The restriction is that w must be in $L^2(\Gamma)$

(4.37)
$$\mathscr{D}(\mathcal{L}) = \left\{ u \in H^1(\Gamma), \mathcal{L}u \in L^2(\Gamma) \right\}.$$

To finish the proof we need to compute the coefficients of the generator in Equation (4.35). Using Equation (4.13) for a (given by (4.19)) and Equation (4.12) for

b (given by (4.21)) from Lemma 4.2, we have

$$(4.38) \quad a_i'(m) = \oint_{C_i(m)} [\nabla h \cdot \nabla H + h \Delta H] \frac{\mathrm{d}l}{|\nabla H|}, \quad b_i'(m) = \oint_{C_i(m)} \nabla \cdot (hF) \frac{\mathrm{d}l}{|\nabla H|}.$$

Then we see that $b_i' = c_i$ (given by Equation (4.19)) on each edge I_i.

Since $h = \overline{\theta}$, the vector field F is

$$(4.39) \qquad\qquad\qquad F = e + \varepsilon \frac{\overline{\partial_1 \theta}}{\overline{\theta}} \nabla H.$$

Therefore, we have, from Equations (4.19) and (4.21)

$$(4.40) \qquad\qquad a_i(m) = \theta_i(m) \oint_{C_i(m)} |\nabla H| \, \mathrm{d}l$$

$$(4.41) \qquad\qquad a_i'(m) = \theta_i(m) \oint_{C_i(m)} \Delta H \frac{\mathrm{d}l}{|\nabla H|} + \theta_i'(m) \oint_{C_i(m)} |\nabla H| \, \mathrm{d}l$$

$$(4.42) \qquad\qquad b_i(m) = \theta_i(m) \oint_{C_i(m)} e \cdot \nabla H \frac{\mathrm{d}l}{|\nabla H|} + \varepsilon \theta_i'(m) \oint_{C_i(m)} |\nabla H| \, \mathrm{d}l.$$

The coefficients of the generator (Equation (4.35)) could then be written

$$(4.43) \qquad\qquad \frac{a_i(m)}{d_i(m)} = \frac{1}{T_i(m)} \oint_{C_i(m)} |\nabla H| \, \mathrm{d}l = S_i^2(m)$$

$$(4.44) \qquad -\frac{1}{d_i(m)} (b_i(m) - \varepsilon a_i'(m)) = \frac{1}{T_i(m)} \oint_{C_i(m)} [\varepsilon \Delta H - e \cdot \nabla H] \frac{\mathrm{d}l}{|\nabla H|}$$
$$= \varepsilon B_i^1(m) + B_i^0(m)$$

which give us the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.2. Local behavior.

Theorem 4.1 is sufficient to define and describe the process Z^ε. However we would like to give a more intuitive description of the process. Note that for any vertex O, we have:

(1) if $\left| \pi^{-1}(O) \right| = 0$ then $\gamma(O) = 0$;
(2) the coefficients $\alpha_i(O)$ are such that

$$(4.45) \qquad\qquad \sum_{i \in J_+(O)} \alpha_i(O) = \sum_{i \in J_-(O)} \alpha_i(O).$$

From Theorem 4.1, we see that on each edge, the process is a continuous diffusion whose characteristics are explicitly given as averaging along the connected level sets of the Hamiltonian H. However at the edges, the gluing conditions are not so clear could give several different behavior.

Note that at the vertices two issues must be addressed in order to successfully describe the behavior of the process :

- is the vertex accessible and from which edges ?
- what happend when the process reach the vertex ?

We can determine the different gluing conditions we can have.

- For an exterior vertex O (e.g. a vertex with only one incident edge), we get two types of boundary conditions
  (1) no gluing conditions if $\pi^{-1}(O)$ is a null measure set (i.e. a single point),

(2) or the boundary condition

$$(4.46) \qquad \left|\pi^{-1}(O)\right| \mathcal{L}u(0) = \gamma(O)u(O).$$

- For an interior vertex O, we could have again two types of boundary conditions (with the relation (4.45))

    (1) purely first order gluing condition if $\pi^{-1}(O)$ is a null measure set

$$(4.47) \qquad \sum_{i \in J_+(O)} \alpha_i(O)D_iu(O) - \sum_{i \in J_-(O)} \alpha_i(O)D_iu(O) = 0,$$

    (2) mixed gluing conditions

$$(4.48) \quad \left|\pi^{-1}(O)\right| \mathcal{L}u(0) = \gamma(O)u(O) - \sum_{i \in J_+(O)} \alpha_i(O)D_iu(O) + \sum_{i \in J_-(O)} \alpha_i(O)D_iu(O).$$

Detailed analysis of such process has been conducted by several authors as Feller [2] or Mandl [18]. We also refer to [12] and [14].

5. Generalization

In this last section, we present a generalization of our previous results for more general diffusions. However, we only sketch the computation of the generator of the diffusion from the limiting Dirichlet form.

Consider the diffusion defined on $\mathbb{R}^n$ by

$$(5.1) \qquad \mathrm{d}Y_t = \frac{1}{\alpha}v(Y_t)\mathrm{d}t + u(Y_t)\mathrm{d}t + \sqrt{2\varepsilon}\sigma(Y_t)\mathrm{d}B_t.$$

The vector field v plays the role of the $A\nabla H$ for the 2-dimensional case and u plays the role of a friction term. Since we do not suppose that v is given by some Hamiltonian, we assume instead existence of some m first integrals $G = (G_1, \cdots, G_m)$ for the flow defined by v. We wish to derive the convergence in law of the process $G(Y)$ as α goes to 0.

In this part, for a vector Y or a matrix a, we denote by Y^* or a^* the transposition of these elements. In particular a scalar product between two vectors X and Y could be written

$$(5.2) \qquad X^*Y = X \cdot Y = \sum_i X_iY_i$$

where the usual matrix product takes place in the left-hand side expression.

Let us denote $a = \sigma\sigma^*$ the diffusion matrix which is a symmetric matrix. For the function $G : \mathbb{R}^n \to \mathbb{R}^m$, we define the matrix DG by

$$(5.3) \qquad DG = \begin{pmatrix} \partial_1 G_1 & \cdots & \partial_n G_1 \\ \vdots & & \vdots \\ \partial_1 G_m & \cdots & \partial_n G_m \end{pmatrix}.$$

We make the following assumptions.

Assumptions 5.1. We assume that v, u and σ are Lipschitz bounded functions and that a is uniformly elliptic.. The function G satisfies

$$(5.4) \qquad DGv = 0.$$

We suppose also that G has compact level sets. We assume that there exists a C^1 function h strictly positive, which satisfies

$$(5.5) \qquad\qquad\qquad\qquad \nabla^*[vh] = 0$$

$$(5.6) \qquad\qquad\qquad -\nabla^*[uh] + \varepsilon \sum_{i,j} \partial_{ij}(a_{ij}h) \leqslant 0.$$

We define the measure $\mathrm{d}\mu = h\mathrm{d}x$.

Remark 10. The assumptions on v, u and σ ensure the existence of a strong solution to the stochastic differential equation (1.2). Equation (5.4) ensures that G is conserved along orbits of the flow of v. The function h is, as in the first part, the density of a measure μ. Equation (5.5) ensures that μ is also preserved by the flow generated by v. To define a proper Dirichlet form, μ must be supermedian (for the infinitesimal generator of the diffusion (5.1) see [17] pp.62 and 98), Equation (5.6) ensures that.

5.1. **Dirichlet form.** This first proposition gives the Dirichlet form for the diffusion (5.1).

Let us denote the vector fields Φ, Φ_α:

$$(5.7) \qquad\qquad\qquad \Phi = u - \frac{\varepsilon}{h}(\nabla^*(ah))^*$$

$$(5.8) \qquad\qquad\qquad \Phi_\alpha = \frac{1}{\alpha}v + \Phi = \frac{1}{\alpha}v + u - \frac{\varepsilon}{h}(\nabla^*(ah))^*.$$

The infinitesimal generator, L_α, could be written, for $f \in C_c^2(\mathbb{R}^n)$

$$(5.9) \qquad L_\alpha f = \frac{1}{\alpha}v^*\nabla f + u^*\nabla f + \varepsilon \sum_{i,j} a_{ij}\partial_{ij}f = \frac{1}{\alpha}v^*\nabla f + u^*\nabla f + \varepsilon a : \nabla^2 f$$

where $a : b$ denotes the Frobenius product of the two matrices and $\nabla^2 f$ is the matrix of the second derivatives of f.

We consider the Dirichlet form E_α associated to L_α in $L^2(\mu)$. For $f, g \in C_c^2$, we have

$$(5.10)$$
$$E_\alpha(f,g) = -\langle L_\alpha f, g\rangle_\mu = -\frac{1}{\alpha}\int(v^*\nabla f)g\mathrm{d}\mu - \int(u^*\nabla f)g\mathrm{d}\mu - \varepsilon\int(a : \nabla^2 f)g\mathrm{d}\mu.$$

Proposition 5.2. *Under the set of assumptions 5.1, E_α is coercive, closable and can be extended to a Dirichlet form E_α on $L^2(\mu)$ with domain $\mathscr{D}(E_\alpha) = H^1(\mu)$ defined as the completion in $L^2(\mu)$ of $\mathscr{D}(L_\alpha)$. Moreover E_α is regular and has the local property. The Dirichlet form is then $E_\alpha(f,g) = E_\alpha^s(f,g) + E_\alpha^a(f,g)$*

$$(5.11) \quad E_\alpha^s(f,g) = \varepsilon\langle\sigma^*\nabla f, \sigma^*\nabla g\rangle_\mu + \frac{1}{2}\int\frac{1}{h}\nabla^*(\Phi h)fg\mathrm{d}\mu$$

$$(5.12) \quad E_\alpha^a(f,g) = \frac{1}{2}\int\Phi_\alpha^*[f\nabla g - g\nabla f]\mathrm{d}\mu = \frac{1}{2}\int\left(\frac{1}{\alpha}v + \Phi\right)^*[f\nabla g - g\nabla f]\mathrm{d}\mu.$$

Proof. Let us first consider $f, g \in C_c^2(\mathbb{R}^n)$. The first and second integrals in Equation (5.10) are treated as in (2.10). For the third integral, we get

$$-\int (a : \nabla^2 f) g \mathrm{d}\mu = -\sum_{ij} \int (a_{ij} \partial_{ij} f) gh = \sum_{ij} \int \partial_j f \partial_i (a_{ij} gh)$$

$$= \sum_{ij} \int a_{ij} (\partial_j f \partial_i g) h + \sum_{ij} \int \partial_i(h) a_{ij} \partial_j f g + \sum_{ij} \int \partial_j f \partial_i(a_{ij}) gh$$

$$(5.13) \qquad = \int (\sigma^* \nabla f)^* (\sigma^* \nabla g) h + \int [(a\nabla h)^* \nabla f] g + \int [(\nabla^* a) \nabla f] gh.$$

Then we can decompose this bilinear form in (f, g) in a symmetric and antisymmetric part using similar computations as in (2.10). Thus, Equation (5.13) becomes

$$-\int (a : \nabla^2 f) gh = \langle \sigma^* \nabla f, \sigma^* \nabla g \rangle_\mu + \frac{1}{2} \int (a\nabla h + (\nabla^* a)^* h)^* [g\nabla f - f\nabla g]$$

$$- \frac{1}{2} \int [\nabla^*(a\nabla h) + h\nabla^*((\nabla^* a)^*) + (\nabla h)^*(\nabla^* a)^*] fg$$

$$(5.14) \qquad = \langle \sigma^* \nabla f, \sigma^* \nabla g \rangle_\mu + \frac{1}{2} \int [\nabla^*(ah)] [g\nabla f - f\nabla g] - \frac{1}{2} \int [\nabla^2 : (ah)] fg.$$

Putting all the terms together we obtain the announced Dirichlet form.

The domain $\mathscr{D}(E_\alpha)$ is defined as the completion of $\mathscr{D}(L_\alpha)$ with respect to the positive bilinear form $E_\alpha^{s,1}$ and it defines a coercive closed bilinear form ([17] Theorem 2.15 p.22). The contraction properties (2.14) are proved in [17] Section II.2.d. (p.48) under suitable conditions (Equations (2.16) p.48) which come as a consequence of our assumption (5.6).

The fact that the domain is $H^1(\mu)$ comes from the fact that, under the assumption 5.1, Φ_α and $\nabla \cdot (h\Phi)$ are bounded and a is uniformly elliptic. Therefore $E_\alpha^{s,1}$ and $\|\cdot\|_{H^1(\mu)}$ are equivalent norms. The regularity is obvious from the properties of the sets $H^1(\mu)$ and C_c. The local property is also obvious from Equations (5.11) and (5.12). $\qquad \square$

5.2. **Projected Dirichlet form.** We consider the projection π associated to the equivalent classes defined by

$$x \sim y \iff \text{ and } x, y \text{ are in the same connected component of a level set of } G.$$

The space $\Gamma = \mathbb{R}^n / \sim$ is the new state space. However, a complete description of this state space is quite complicated and not really necessary for the Mosco-convergence. As in the first part, for all f defined on Γ, we define $\overline{f} = f \circ \pi$, its lift on $\mathbb{R}^n$.

In order to have a Mosco-convergence of the Dirichlet form E_α as $\alpha \to 0$, we have to make the terms in Equation (5.12) depending on α vanishing, i.e. we want

$$(5.15) \qquad\qquad v^* [f\nabla g - g\nabla f] = 0.$$

This is achieved by the choice of test functions and is equivalent to a projection of the Dirichlet form.

Lemma 5.3. *For all $f \in C^1(\Gamma)$, we have*

$$(5.16) \qquad\qquad v^* \nabla \overline{f} = 0.$$

Proof. Let us consider the flow ϕ_t generated by the vector field v. We define, for all x, t

$$(5.17) \qquad F(x,t) = \overline{f}(\phi_t(x)).$$

By definition, we have $G(\phi_t(x)) = G(x)$ for all t, therefore $\phi_t(x)$ stays in a unique equivalence class for all $t \in \mathbb{R}$. Then $F(x,t)$ is a constant function in t, thus

$$(5.18) \qquad \partial_t F(x,t) = v^* \nabla \overline{f}(\phi_t(x)) = 0, \text{ for all } t.$$

Putting $t = 0$, we get the lemma. $\qquad \square$

We test the Dirichlet form E_α on $\overline{f_1} = f_1 \circ \pi, \overline{f_2} = f_2 \circ \pi$. Using the previous lemma, we see that $E_\alpha(\overline{f_1}, \overline{f_2})$ defines a bilinear form, for functions in $C_c^1(\Gamma)$ which does not depend on α. We denote it $\mathcal{E}$:

$$(5.19) \qquad \mathcal{E}(f_1, f_2) = \varepsilon \int_{\mathbb{R}^n} (\nabla \overline{f_1})^* a \nabla \overline{f_2} \mathrm{d}\mu$$
$$+ \frac{1}{2} \int \Phi^* \left[\overline{f_1} \nabla \overline{f_2} - \overline{f_2} \nabla \overline{f_1} \right] \mathrm{d}\mu + \frac{1}{2} \int \frac{1}{h} \nabla^* (\Phi h) \overline{f_1 f_2} \mathrm{d}\mu.$$

We have the following Proposition.

Proposition 5.4. *The form $\mathcal{E}$ define a Dirichlet form on $L^2(\Gamma)$ with domain $H^1(\Gamma)$. Moreover, $\mathcal{E}$ is regular and has the local property.*

Remark 11. This proposition allows us to define a process Z on Γ associated to the Dirichlet form, $\mathcal{E}$.

Proof. The proof follows exactly the proof of Theorem 3.2. $\qquad \square$

5.3. **Mosco-convergence and convergence in law.** Let us denote Y^α the process solution of the diffusion equation (5.1) with $\alpha > 0$. We denote Z the process associated to the Dirichlet form $\mathcal{E}$.

We have the following Theorem.

Theorem 5.5. *For all sequence α_n converging to 0, for all g_n weakly converging in L^2 to $g \circ \pi$, and such that g_n and $g \circ \pi$ are densities (with respect to μ) of probability laws in $\mathbb{R}^n$, the sequence of processes $Z^{\alpha_n} = \pi(Y^{\alpha_n})$ where $Y_0^{\alpha_n}$ is distributed as $g_n \mathrm{d}\mu$, converges in law to the process Z with initial law given by $g \mathrm{d}(\pi_* \mu)$.*

The proof follows exactly as in the first part (Theorem 3.11) by proving two facts: the convergence of the finite dimensional marginals of the processes (Z^{α_n}) (which follows itself form the convergence of the associated Dirichlet forms), and the tightness.

Proposition 5.6. *For all sequence (α_n) converging to 0, E_{α_n} Mosco-converges to $\mathcal{E}$.*

Proof. The proof follows exactly the proof of Theorem 3.2. $\qquad \square$

We consider Z the Markov process on Γ associated to the Dirichlet form $\mathcal{E}$ with initial law ν.

Proposition 5.7 (Tightness). *The family of distributions of $(Z^{\alpha_n})_n$ is tight in the space $C(\mathbb{R}^+; \Gamma)$.*

Proof. By the same proof of Proposition 3.10, we prove that each family of processes $G_i(Y^{\alpha_n})$ is tight, thus, using Corollary 3.33 p.317 in [11] the family of processes $G(Y^{\alpha_n})$ is tight. Therefore Z^{α_n} is also tight since all bounded continuous function on Γ could be decomposed as a countable sum of bounded continuous functions depending only on G. □

Proof of Theorem 5.5. From Proposition 5.6, we deduce as in the proof of Proposition 3.9 the convergence of the finite dimensional marginals. The theorem follows from the tightness proved in Proposition 5.7. □

5.4. **Identification of the limiting process.** The process Z is only defined through the Dirichlet form $\mathcal{E}$. In order to obtain a more intuitive representation of this process, through a stochastic diffusion equation for example or its infinitesimal generator, $\mathcal{L}_\varepsilon$, one should write the Dirichlet form as a scalar product in $L^2(\Gamma)$. In this last section, we would like to expose what one should expect and how computations could be made.

The equation is therefore the following, we wish to find $w \in L^2(\Gamma)$ such that, for all $v \in H^1(\Gamma)$

$$\mathcal{E}(u,v) = \varepsilon \int_{\mathbb{R}^n} (\nabla \overline{u})^* a \nabla \overline{v} \mathrm{d}\mu + \frac{1}{2}\int \Phi^* \left[\overline{u}\nabla \overline{v} - \overline{v}\nabla \overline{u}\right] \mathrm{d}\mu + \frac{1}{2}\int \frac{1}{h}\nabla^*(\Phi h)\overline{uv}\mathrm{d}\mu$$

$$(5.20) \qquad = -\int_\Gamma wv \mathrm{d}\pi_*\mu.$$

In order to do this identification, one has to do several things:

- write the Dirichlet form $\mathcal{E}$ as an integral on Γ;
- write the image measure of μ via the projection π on Γ.

5.4.1. *Orbit Space.* As in the two dimensional case, Γ is decomposed into several connected submanifolds of $\mathbb{R}^m$. This decomposition is done by using the minimal rank of the Jacobian JG along a connected level set of G. This rank is the dimension of the submanifold. Each submanifold of dimension higher than one has a boundary made of a union of submanifolds of dimension strictly lower. We describe Γ by this collection of manifolds ordered along their dimensions and by the relations describing their boundaries.

We denote I_i^k the collection of submanifolds of dimension k and R_i^k the preimage associated to the submanifold I_i^k.

5.4.2. *Image measure.* On each submanifold, we need to do a change of variable to transfer an integral on the preimage of a submanifold R_i^k (which is a domain in $\mathbb{R}^n$) to a integral on this submanifold I_i^k (a domain in $\mathbb{R}^m$).

We denote $\mathrm{d}\mathcal{H}_k$ for $k \geqslant 0$, the k-dimensional Hausdorff measure.

For $k = m$, this change of variable is just the coarea formula. The integrals on subsets of rank strictly lower than m must be treated carefully. In these cases, we need a coarea formula from a domain R_i^k to I_i^k, a submanifold in $\mathbb{R}^m$ of dimension k.

We introduce a k-dimensional Jacobian $J_k G$ ([15])
(5.21)
$$J_k G(y) = \sup \left\{ \frac{\mathcal{H}_k(DG(y)P)}{\mathcal{H}_k(P)}, \text{ for } P \text{ a } k\text{-dimensional parallelepiped in } \mathbb{R}^n \right\}.$$

Remark that for the usual Jacobian we have $JG = J_m G$. We obtain

$$(5.22) \qquad \int_{R_i^k} f(x)\mathrm{d}x = \int_{I_i^k} \mathrm{d}\mathcal{H}_k(g) \int_{G^{-1}(g)\cap R_i^k} \frac{f(y)}{J_k G(y)}\mathrm{d}\mathcal{H}_{n-k}(y).$$

This allows us to make the change of variable (and also identify the image measure $\pi_*\mu$ on Γ). Let us choose a function f integrable (w.r.t. μ) on $\mathbb{R}^n$ and then, for $p = (g, k, i)$ we have

$$(5.23) \qquad \int_{\mathbb{R}^n} f(x)\mathrm{d}x = \sum_{k=0}^m \sum_i \int_{I_i^k} \bar{f}(p)\mathrm{d}\mathcal{H}_k(p).$$

The average function $\bar{f}$ is defined on Γ, for $g \in R_i^k$, by

$$(5.24) \qquad \bar{f}(g, k, i) = \int_{G^{-1}(g)\cap R_i^k} \frac{f(x)}{J_k G(x)}\mathrm{d}\mathcal{H}_{n-k}(x).$$

5.4.3. *Projected Dirichlet form.* We have to decompose all the integrals of the left-hand side of Equation (5.20) and apply the coarea formula, in order to obtain integrals on the components of Γ. For the first integral of Equation (5.20), we obtain

$$(5.25)$$
$$\int_{\mathbb{R}^n} \nabla\overline{u}^* a\nabla\overline{v}\mathrm{d}\mu = \sum_{k\leqslant m} \sum_i \int_{I_i^k} \mathrm{d}\mathcal{H}_k(g) \int_{G^{-1}(g)\cap R_i^k} \nabla\overline{u}(x)^* a(x)\nabla\overline{v}(x)h(x)\frac{\mathrm{d}\mathcal{H}_{n-k}(x)}{J_k G(x)}.$$

On the set R_i^k, we have $\overline{u}(x) = u(G(x), k, i)$, then let us denote $D_g u$ the derivative of u with respect to the first variable. Note that this derivative has value in the respective tangent space of the submanifold i.e. $\mathbb{R}^k$. This is a derivative along the submanifold I_i^k in $\mathbb{R}^m$. If we omit the constant quantities (k, i), we get

$$(5.26) \qquad (\nabla\overline{u})^* a\nabla\overline{v} = D_g u^*(DGaDG^*)D_g v$$

Note that since $a = \sigma\sigma^*$, the matrix $DGaDG^* = (DG\sigma)(DG\sigma)^*$ is also symmetric and non-negative. Let us also define the average quantity a_G defined on Γ

$$(5.27) \qquad a_G((g, k, i)) = \int_{G^{-1}(g)\cap R_i^k} \frac{(DGaDG^*)h}{J_k G}\mathrm{d}\mathcal{H}_{n-k}.$$

The integral (5.25) becomes

$$(5.28) \qquad \int_{\mathbb{R}^n} (\nabla\overline{u})^* a\nabla\overline{v}\mathrm{d}\mu = \sum_{k\leqslant m} \sum_i \int_{I_i^k} (D_g u)^* a_G D_g v\mathrm{d}\mathcal{H}_k = \int_\Gamma (D_g u)^* a_G D_g v.$$

The second and third integrals in (5.20) are treated accordingly. We denote, for $p = (g, k, i)$

$$(5.29) \qquad \Phi_G(p) = \int_{G^{-1}(g)\cap R_i^k} \frac{DG\Phi h}{J_k G}\mathrm{d}\mathcal{H}_{n-k}$$

$$(5.30) \qquad F_G(p) = \int_{G^{-1}(g)\cap R_i^k} \frac{\nabla^*(\Phi h)}{J_k G}\mathrm{d}\mathcal{H}_{n-k}.$$

Therefore, we get the following formulation for the projected Dirichlet form

$$(5.31) \qquad \mathcal{E}(u, v) = \varepsilon \int_\Gamma (D_g u)^* a_G D_g v + \frac{1}{2}\int_\Gamma \Phi_G^* [uD_g v - vD_g u] + \frac{1}{2}\int_\Gamma F_G uv.$$

5.4.4. *Identification of the infinitesimal generator.* In order to solve Equation (5.20), we need to write the integrals in Equation (5.31) where v does not have any derivative. This can be obtained, at least formally, via some integrations by parts .

We need to identify the integrals for each I_i^k. We only have boundary terms if $k < m$, therefore let us first give the generator on the sets I_j^m. From Equation (5.31), we obtain on each I_j^m,

$$(5.32) \qquad \mathcal{L}_\varepsilon u = \frac{1}{d} \left[\varepsilon D_g^*(a_G D_g u) + (\Phi_G)^* D_g u + \frac{1}{2} \left(D_g^* \Phi_G - F_G \right) u \right].$$

Note that since all sets I_i^k, for $k < m$, are contained in the boundary of a domain I_j^m.

Collecting the boundary term along every submanifold gives gluing conditions, this is the same process as the 2-dimensional case. Heuristically, these gluing conditions connect the value of the generator restricted on a submanifold I_i^k to the limiting values of the generator at the boundary of the submanifolds I_j^{k+1} such that $\partial I_j^{k+1} \subset I_i^k$.

Remark 12. For an explicit computation of the generator, defined by Equation (5.32), one would need a lemma analogous to the Lemma 4.2 to compute the derivatives of the averaging a_G and Φ_G.

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