



**HAL**  
open science

# First time to exit of a continuous Itô process: general moment estimates and $L_1$ -convergence rate for discrete time approximations

Bruno Bouchard, Stefan Geiss, Emmanuel Gobet

## ► To cite this version:

Bruno Bouchard, Stefan Geiss, Emmanuel Gobet. First time to exit of a continuous Itô process: general moment estimates and  $L_1$ -convergence rate for discrete time approximations. 2013. hal-00844887v1

**HAL Id: hal-00844887**

**<https://hal.science/hal-00844887v1>**

Preprint submitted on 16 Jul 2013 (v1), last revised 9 Sep 2014 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# First time to exit of a continuous Itô process: general moment estimates and $L_1$ -convergence rate for discrete time approximations

Bruno Bouchard\*, Stefan Geiss† and Emmanuel Gobet‡

July 16, 2013

## Abstract

We establish general moment estimates for the discrete and continuous exit times of a general Itô process in terms of the distance to the boundary. These estimates serve as intermediate steps to obtain strong convergence results for the approximation of a continuous exit time by a discrete counterpart, computed on a grid. In particular, we prove that the discrete exit time of the Euler scheme of a diffusion converges in the  $L_1$  norm with an order  $1/2$  with respect to the mesh size.

**Key words:** Exit time; Strong approximation; Euler scheme.

**AMS 2000 subject classifications:** 65Cxx; 60F15.

## 1 Introduction

This paper is motivated by the study of the strong convergence rate of the discrete time approximation of the first exit time  $\theta$  of a process  $Z$  from a non-empty open subset  $\mathcal{O}$ .

---

\*CEREMADE, Université Paris Dauphine and CREST-ENSAE, place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France. Email: [bouchard@ceremade.dauphine.fr](mailto:bouchard@ceremade.dauphine.fr)

†Department of Mathematics, University of Innsbruck, A-6020 Innsbruck, Technikerstraße 13/7, Austria. Email: [stefan.geiss@uibk.ac.at](mailto:stefan.geiss@uibk.ac.at)

‡CMAP, Ecole Polytechnique and CNRS, Route de Saclay, 91128 Palaiseau cedex, France. Email: [emmanuel.gobet@polytechnique.edu](mailto:emmanuel.gobet@polytechnique.edu)

The interest for numerical discretization of diffusion processes dates back to the sixties, see [Mar55, Mul56] and [KP95] for general references. Different approaches can be used to approximate the first exit time of a diffusion. We briefly recall them for the sake of completeness and to make clear the contribution of this paper.

a. By the very nature of the problem, space discretization schemes naturally appear. The first version is based on the Walk On Sphere (WOS) schemes introduced in [Mul56]. In the Brownian motion case one simulates its position by the first hitting time of a ball contained in the domain and centered at the starting point: the position is uniformly distributed on the sphere and thus straightforward to sample. The sampled point is then used as a new starting point. One repeats the above procedure until one gets close enough to the boundary of  $\mathcal{O}$ . For a time-homogeneous diffusion process  $X$  the scheme is modified using small balls and an Euler-Maruyama approximation. In [Mil96, Mil98] strong error estimates on the *exit position*  $X_\theta$  are proved, assuming in particular that the domain  $\mathcal{O}$  is convex and that the diffusion coefficient satisfies a uniform ellipticity condition. These results do not include an approximation of the exit time  $\theta$ . Weak approximation results – i.e. for  $\mathbb{E}[\varphi(X_\theta)]$  with  $\varphi$  continuous and bounded – are established in [Mil97].

b. For polygonal domains moving from spheres to spheres may not be suitable because of the corners. One has to replace balls by parallelepipeds (tensor products of intervals). Exit times from parallelepipeds are easy to sample. Faure [Fau92] was probably the first one who developed these ideas. In [MT99] these ideas are further analyzed for diffusion processes with time-dependency by exploiting small parallelepipeds. Strong error estimates of the exit position and the exit time are established: the order of convergence of the exit time approximation is  $1 - \varepsilon$  with respect to the space step (for any  $0 < \varepsilon < 1$ ), i.e. equivalently  $\frac{1}{2} - \varepsilon$  (for any  $0 < \varepsilon < 1/2$ ) with respect to the time step, see [MT99, Theorem 8.2]. Here again, convexity of  $\mathcal{O}$  and strong ellipticity were assumed. Related experiments are discussed in [ZLD10]. Extensions to non-small parallelepipeds are investigated in [DL06].

c. To maintain a certain simplicity of the simulation, one can alternatively perform the usual Euler scheme on a grid  $\pi$  with deterministic time step  $|\pi|$  and stop when it exits  $\mathcal{O}$ . This is a crude approximation, nevertheless the simplest and quickest to use: this is why it has gained much interest in the applied stochastics community. It results in an order of weak convergence equal to  $\frac{1}{2}$  with respect to  $|\pi|$ , see [Gob00, GM10]. Interestingly, it is shown in [GM07] that this order of weak convergence remains valid for general Itô processes, far beyond the usual diffusion framework in which one can rely

on PDE tools to decompose the error. The strong convergence of the exit time is stated in [GM05, Theorem 4.2] but without speed. Finally, note that different techniques can be used to speed-up the convergence in the weak sense: sampling the continuous time exit using diffusion bridge techniques [Bal95, Gob00, BC02] (possibly with local modifications of the boundary [Gob01, BP06] or exponential-time stepping [JL05]) or using discrete exit times combined with an inward shifting of the boundary [GM10]. To our knowledge, no strong error estimates are available for these schemes.

As a matter of fact, until recently only little was known about the rate of  $L_1$  convergence of the discrete exit time of an Euler scheme of a diffusion towards the exit time of the exact diffusion, although there are important fields where the  $L_1$  criterion is the only relevant one. As examples let us mention the approximation of backward stochastic differential equations considered in a domain [BM09] and the multi level Monte Carlo methods [Hei01, Gil08]. In [BM09, Theorem 3.1] the authors prove that the convergence rate of the discrete exit time of the Euler scheme is of order  $\frac{1}{2} - \varepsilon$  with respect to  $|\pi|$  (for any  $0 < \varepsilon < 1/2$ ). Because of the aforementioned applications the question whether one can take  $\varepsilon = 0$  in the previous estimate has been raised. Also, their arguments are restricted to finite time horizons and the question whether they could be extended to an infinite time horizon was open.

In this paper we answer these questions to the positive: the discrete exit time of an Euler scheme converges at the rate  $1/2$  in the  $L_1$  norm, even if the time horizon is unbounded, see Theorem 3.9. In the same theorem we show that the stopped process converges at the rate  $1/4$  in  $L_2$ . Theorem 3.9 follows from an abstract version stated in Theorem 3.1, which we establish in a non-Markovian setting in the spirit of [GM07]. As a first step of our analysis we provide general controls on the expected time to exit in terms of the distance to the boundary, see Theorems 2.2 and 2.3 below. They are established both for continuous exit times and for discrete exit times, i.e. the latter are restricted to take values on a discrete grid. Essentially, we only use a mild non-characteristic boundary type condition and a uniform bound on the conditional expected times to exit. The fact that, as opposed to most of the papers quoted above, we analyze situations with unbounded time horizon in an  $L_\infty$  sense is delicate because the usual finite-time error estimates, e.g. on Euler schemes, blow up exponentially with respect to the time horizon.

The paper is organized as follows: In Section 2 we introduce a general set-up followed by the statement of our quantitative results on the moments of the first time to exit. The proof of the main results, Theorems 2.2 and 2.3, is split into several subsections. We first establish general Freidlin type

inequalities on moments of exit times, which will be controlled in terms of the probability of sub-harmonic paths in Section 2.4. Estimates on this probability yield to the proof of Theorem 2.2, that applies to continuous exit times. A final recursion argument is needed to pass from continuous exit times to discrete exit times, see Section 2.6. The application to the exit time approximation error is discussed in Section 3, first in an abstract setting, then for the solution of a stochastic differential equation whose exit time is estimated by the discrete exit time of its Euler scheme.

Throughout this paper, we let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space supporting a  $d$ -dimensional Brownian motion  $W$ . We denote by  $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$  the right-continuous completion of the natural filtration induced by  $W$ . The symbol  $\mathcal{T}$  denotes the set of stopping times that are finite a.s. We write  $\mathbb{E}_\tau$  and  $\mathbb{P}_\tau$  for the conditional expectation and probability, respectively, given  $\mathcal{F}_\tau$ . Finally, given a vector  $a \in \mathbb{R}^d$  or a matrix  $A \in \mathbb{R}^{m \times n}$ , the notation  $|a|$  and  $|A|$  stands for the euclidean and the Hilbert-Schmidt norm, respectively.

## 2 Moment estimates for continuous and discrete exit times

The main results of this section are Theorems 2.2 and 2.3. They are the basis to prove Theorem 3.1, which is the main result of the paper in its abstract form.

### 2.1 Assumptions

Let  $(\mathcal{Z}, d_{\mathcal{Z}})$  be a metric space equipped with the Borel  $\sigma$ -algebra generated by the open sets,  $\mathcal{O}$  be an open set of  $\mathcal{Z}$  with

$$\emptyset \neq \mathcal{O} \subsetneq \bar{\mathcal{O}} \subsetneq \mathcal{Z},$$

in which  $\bar{\mathcal{O}}$  denotes the closure of  $\mathcal{O}$ , and let  $(Z_t)_{t \geq 0}$  be a continuous  $\mathbb{F}$ -adapted  $\mathcal{Z}$ -valued process starting in  $Z_0 \equiv z_0 \in \mathcal{O}$ .

The two main results of this section concern estimates on the time taken by the process  $Z$  to reach the boundary of  $\mathcal{O}$ , either evaluated on points of  $\mathbb{R}_+$  or on some non-empty subset of time points  $\pi \subset \mathbb{R}_+$ . Therefore the standing assumption of this section is that either

- (a)  $\pi = \mathbb{R}_+$ ,
- (b) or  $0 \in \pi$  and, for all  $T > 0$ ,

$$0 < \inf_{[0, T] \setminus \pi} (\phi^+ - \phi) \leq \sup_{\mathbb{R}_+} (\phi^+ - \phi) =: |\pi| \leq 1,$$

where

$$\phi_t := \max\{s \in \pi : s \leq t\} \text{ and } \phi_t^+ := \min\{s \in \pi : s \geq t\}. \quad (1)$$

Note that these conditions ensure that the above max and min are actually attained. In other words, for  $\pi \neq \mathbb{R}_+$  the set  $\pi$  is composed by only finitely many points on each interval  $[0, T]$ ,  $T > 0$ . For  $t \geq 0$ ,  $\phi_t$  and  $\phi_t^+$  are the closest points in  $\pi$  to the left and to the right of  $t$ .

Our first assumption concerns the path regularity of the process  $Z$ . To simplify the notation, we set

$$\gamma(t, s) := d_Z(Z_t, Z_s), \quad t, s \geq 0. \quad (2)$$

**Assumption (Z) (Regularity of  $Z$  along  $\pi$ ).** *There exists a locally bounded map  $\kappa : \mathbb{R}_+ \times (0, \infty) \mapsto \mathbb{R}_+$  such that*

$$\mathbb{P}_\tau \left[ \sup_{\tau \leq t \leq \tau+T} \gamma(t, \phi_t \vee \tau) > \rho \right] \leq \kappa(T, \rho) |\pi|$$

for all  $\tau \in \mathcal{T}$ ,  $T \geq 0$ , and  $\rho > 0$ .

Our next set of assumptions concerns the regularity of the boundary  $\partial\mathcal{O}$  of  $\mathcal{O}$ .

**Assumption (P) (Distance process  $\delta(Z)$ ).** *There exist  $L \geq 1$  and an  $L$ -Lipschitz function  $\delta : \mathcal{Z} \mapsto \mathbb{R}$  such that  $\delta > 0$  on  $\mathcal{O}$ ,  $\delta = 0$  on  $\partial\mathcal{O}$ , and  $\delta < 0$  on  $\bar{\mathcal{O}}^c$ . In addition, the process  $P := \delta(Z)$  admits the Itô process decomposition*

$$P_t = P_0 + \int_0^t b_s ds + \int_0^t a_s^\top dW_s \quad (3)$$

for  $t \geq 0$ , where

- (i)  $(P, b, a)$  is a predictable process with values in  $[-L, L]^{d+2}$ ,
- (ii) there is a fixed  $r \in (0, L^{-3}/4)$  and a set  $\Omega_r \in \mathcal{F}$  of measure one, such that  $|P_t(\omega)| \vee \gamma(t, \phi_t)(\omega) \leq r$  implies that  $|a_t(\omega)| \geq 1/L$  whenever  $\omega \in \Omega_r$  and  $t \geq 0$ .

Before to go on, let us comment the latter assumptions.

**Remark 2.1.** (a) The process  $P = \delta(Z)$  measures the algebraic distance of  $Z$  to the boundary  $\partial\mathcal{O}$  in terms of the function  $\delta$ . The existence of a signed distance  $\delta$  that is 1-Lipschitz can be checked in various settings easily

(starting from the usual distance one can check whether for all segments  $[x, y] = \{z \in \mathcal{Z} : d_{\mathcal{Z}}(x, z) + d_{\mathcal{Z}}(z, y) = d_{\mathcal{Z}}(x, y)\}$  with  $x \in \mathcal{O}$  and  $y \in (\bar{\mathcal{O}})^c$  the intersection  $[x, y] \cap \partial\mathcal{O}$  is non-empty), and it can be modified outside a suitable neighborhood of  $\partial\mathcal{O}$  in order to be uniformly bounded.

(b) The Itô decomposition (3) may implicitly impose additional smoothness assumptions on  $\partial\mathcal{O}$ : for instance, if  $Z$  is an  $\mathbb{R}^d$ -valued Itô process, then  $P$  is also an Itô process provided that the domain is  $C^2$  with compact boundary, see [GM07, Proposition 2.1]. Hence, the condition (i) is not too restrictive.

(c) The coefficients  $b$  and  $a$  may depend on  $\pi$ . This will be the case in Section 3.2 when our abstract results will be applied to an Euler scheme.

(d) The condition (ii) is a uniform non-characteristic boundary condition. It ensures that the fluctuation of the paths of  $Z$  are not tangential to the boundary and is typically expressed for  $Z$  close to  $\partial\mathcal{O}$ , i.e.  $|P| \leq r$  in our context. The additional restriction  $\gamma(\cdot, \phi) = d_{\mathcal{Z}}(Z, Z_\phi) \leq r$  is imposed to account for situations where  $a$  depends on  $Z_\phi$  as in the case of the Euler scheme, see Section 3.2 below.

Now we can define the main objects of this section: given  $\ell \in \{0, r\}$ ,  $\tau \in \mathcal{T}$ , and an integer  $p \geq 1$ , we set

$$\begin{aligned}\theta_\ell(\tau) &:= \inf\{t \geq \tau : P_t \leq \ell\}, \\ \theta_\ell^\pi(\tau) &:= \inf\{t \geq \tau : t \in \pi, P_t \leq \ell\}, \\ \Phi_\ell^p(\tau) &:= \mathbb{E}_\tau [(\theta_\ell(\tau) - \tau)^p]^{\frac{1}{p}}, \\ \Phi_\ell^{p,\pi}(\tau) &:= \mathbb{E}_\tau [(\theta_\ell^\pi(\tau) - \tau)^p]^{\frac{1}{p}}.\end{aligned}$$

Observe that assumption **(P)** is not required to define  $\theta_0(\tau)$  and  $\theta_0^\pi(\tau)$  since they are simply the first exit times of  $Z$  from  $\mathcal{O}$  along  $\mathbb{R}_+$  or  $\pi$ .

Our aim is to provide pointwise estimates on  $\Phi_0^1(\tau)$  and  $\Phi_0^{1,\pi}(\tau)$ . Our arguments require an additional control on the first conditional moment of the times to exit. We impose a uniform control in terms of the constant  $L > 0$  of assumption **(P)**. Clearly, the fact that we use the same constant  $L$  in these assumptions is for notational convenience.

**Assumption (L) (Uniform bound on expectations of exit times).**  $\Phi_0^{1,\pi}(\tau) + \Phi_0^1(\tau) \leq L$  for all  $\tau \in \mathcal{T}$ .

We refer to [Fre85, Chapter III, Lemma 3.1] for sufficient conditions ensuring that the exit times of a stochastic differential equation have finite moments, that are bounded only in terms of the diameter of the domain, the bounds on the coefficients of the stochastic differential equation and a partial ellipticity condition.

In Lemma 2.7 below, we show that **(L)** implies that  $\theta_0^\pi(\tau) - \tau$  and  $\theta_0(\tau) - \tau$  have finite exponential moments, uniformly in  $\tau \in \mathcal{T}$ .

Up to a slight modification of the constant  $L$ , assumption **(L)** is equivalent to the following simpler

**Assumption (L')**.  $\Phi_0^{1,\pi}(\tau) + \Phi_0^1(\tau) \leq L$  a.s. on  $\{P_\tau > 0\}$  for all  $\tau \in \mathcal{T}$ .

For  $\Phi_0^1$  the equivalence is obvious because  $\Phi_0^1(\tau) = 0$  on  $\{P_\tau \leq 0\}$ . Regarding  $\Phi_0^{1,\pi}$ , **(L)** obviously implies **(L')**. Conversely, since  $\Phi_0^{1,\pi}(\tau) \leq \mathbb{E}_\tau [\Phi_0^{1,\pi}(\phi_\tau^+)] + |\pi|$ , where  $|\pi| \leq 1$ , and  $\Phi_0^{1,\pi}(\phi_\tau^+) = 0$  on  $\{P_{\phi_\tau^+} \leq 0\}$ , the assumption **(L')** implies that  $\Phi_0^{1,\pi}(\tau) \leq L + 1$  for all  $\tau \in \mathcal{T}$ .

We close this discussion with another equivalent condition.

**Assumption (L'')**. *There exist two constants  $c > 0$  and  $\alpha \in (0, 1)$  such that  $\max \left\{ \mathbb{P}_\tau[\theta_0^\pi(\tau) \geq \tau + c], \mathbb{P}_\tau[\theta_0(\tau) \geq \tau + c] \right\} \leq \alpha$  for all  $\tau \in \mathcal{T}$ .*

Indeed, **(L)** implies **(L'')** by Markov's inequality applied to the level  $c := L/\alpha$  for a given  $\alpha \in (0, 1)$ . Conversely, the fact that  $\theta_0^\pi(\tau + kc) = \theta_0^\pi(\tau)$  on  $\{\theta_0^\pi(\tau) \geq \tau + kc\}$  implies that

$$\mathbb{P}_\tau[\theta_0^\pi(\tau) \geq \tau + (k+1)c] = \mathbb{E}_\tau \left[ \mathbf{1}_{\{\theta_0^\pi(\tau) \geq \tau + kc\}} \mathbb{P}_{\tau+kc}[\theta_0^\pi(\tau + kc) \geq \tau + (k+1)c] \right].$$

Applying **(L'')** inductively, allows us to conclude that the left-hand side above is controlled by  $\alpha^{k+1}$ . It follows that

$$\begin{aligned} \mathbb{E}_\tau[\theta_0^\pi(\tau) - \tau] &\leq c + c \sum_{k \geq 0} \mathbb{P}_\tau[\theta_0^\pi(\tau) \geq \tau + (k+1)c] \\ &\leq c + c \sum_{k \geq 0} \alpha^{k+1} = c/(1 - \alpha) =: L/2. \end{aligned}$$

The same argument applies to  $\mathbb{E}_\tau[\theta_0(\tau) - \tau]$ . This proves that **(L'')** implies **(L)**.

## 2.2 First moment control near the boundary

Now we are in a position to state the main results of this section. We will denote by  $\mathcal{T}^\pi$  the set of stopping times with values in  $\pi$ . Remember that the following can be applied to situations where  $\pi = \mathbb{R}_+$ , in which case assumption **(Z)** is automatically satisfied and the extra term  $|\pi|^{\frac{1}{2}}$  below vanishes.



**Theorem 2.2.** *Let the assumptions **(Z)**, **(P)** and **(L)** be satisfied.*

(a) *If  $\tau \in \mathcal{T}^\pi$ , then*

$$\Phi_0^1(\tau) \leq c_{(2.2)} \left[ P_\tau + |\pi| \right] \mathbf{1}_{\{0 \leq P_\tau \leq r\}} + L \mathbf{1}_{\{r < P_\tau\}},$$

where  $c_{(2.2)} = c_{(2.2)}(r, L, d, \kappa) > 0$ .

(b) *If  $0 < \tilde{r} < r$  and  $\tau \in \mathcal{T}$ , then*

$$\Phi_0^1(\tau) \leq d_{(2.2)} \left[ P_\tau + |\pi|^{\frac{1}{2}} \right] \mathbf{1}_{\{0 \leq P_\tau \leq \tilde{r}\}} + L \mathbf{1}_{\{\tilde{r} < P_\tau\}},$$

where  $d_{(2.2)} = d_{(2.2)}(r, \tilde{r}, L, d, \kappa) > 0$ .

The proof of this theorem will be given in Section 2.5 below. Its counterpart for discrete exit times is proved in Section 2.6.

**Theorem 2.3.** *Let the assumptions **(Z)**, **(P)** and **(L)** be satisfied. Then there exists an  $\varepsilon_{(2.3)} = \varepsilon_{(2.3)}(r, L, d, \kappa) > 0$  such that for nets  $\pi \neq \mathbb{R}_+$  with  $|\pi| \leq \varepsilon_{(2.3)}$  one has the following:*

(a) *If  $\tau \in \mathcal{T}^\pi$ , then*

$$\Phi_0^{1,\pi}(\tau) \leq c_{(2.3)} \left[ P_\tau + |\pi|^{\frac{1}{2}} \right] \mathbf{1}_{\{0 \leq P_\tau \leq r\}} + L \mathbf{1}_{\{r < P_\tau\}},$$

where  $c_{(2.3)} = c_{(2.3)}(r, L, d, \kappa) > 0$ .

(b) *If  $0 < \tilde{r} < r$  and  $\tau \in \mathcal{T}$ , then*

$$\Phi_0^{1,\pi}(\tau) \leq d_{(2.3)} \left[ P_\tau + |\pi|^{\frac{1}{2}} \right] \mathbf{1}_{\{0 \leq P_\tau \leq \tilde{r}\}} + L \mathbf{1}_{\{\tilde{r} < P_\tau\}},$$

where  $d_{(2.3)} = d_{(2.3)}(r, \tilde{r}, L, d, \kappa) > 0$ .

Theorem 2.2 is similar to [GM07, Lemma 4.2], in which the time horizon is bounded and the counterpart of **(P-ii)** does not require  $\gamma(\cdot, \phi) \leq r$ . Our additional requirement yields to a weaker assumption and explains the presence of the additional  $|\pi|$ -terms in our result. We also refer to [Fre85, Chapter III, Section 3.3] who considers a Markovian setting for a uniformly fast exit of a diffusion from a domain.

Theorem 2.3 is of similar nature but is much more delicate to establish. An attempt to obtain such a result for the Euler scheme of stochastic differential equations on a finite time horizon can be found in [BM09] by a combination of their Lemmas 5.1, 5.2 and 5.3. However, they were only able

to achieve a bound in  $O_{|\pi| \rightarrow 0}(|\pi|^{\frac{1}{2}-\varepsilon})$  for all  $0 < \varepsilon < 1/2$ . We shall comment on this in Section 3 below.

The proofs of the above theorems are divided in several steps and provided in the next subsections (see Sections 2.5 and 2.6 for the final arguments). Both start with arguments inspired by [Fre85] and that were already exploited in [BM09]. One important novelty is our set of assumptions where we do not use any Markovian hypothesis and where we only assume that the delay to exit is uniformly bounded in expectation with respect to the initial time. Furthermore, we also refine many important estimates of [BM09] and use a new final recursion argument which is presented in Section 2.6. This recursion is crucial in order to recover the bound  $O_{|\pi| \rightarrow 0}(|\pi|^{\frac{1}{2}})$ , in contrast to the bound  $O_{|\pi| \rightarrow 0}(|\pi|^{\frac{1}{2}-\varepsilon})$  in [BM09].

**Remark 2.4.** Lemma 2.7 below implies the same estimates for  $(\Phi_0^{p,\pi}(\tau))^p$  and  $(\Phi_0^{p,\pi}(\tau))^p$ ,  $p \geq 2$ , as obtained for  $\Phi_0^{1,\pi}(\tau)$  and  $\Phi_0^1(\tau)$  in Theorems 2.2 and 2.3.

**Remark 2.5.** Theorems 2.2 and 2.3 extend to the case where  $\mathcal{O}$  is the intersection of countable many  $(\mathcal{O}_i)_{i \in I}$  satisfying the assumptions **(P)** and **(L)** for some family of processes  $(P^i)_{i \in I}$  with the same  $L \geq 1$  and  $r \in (0, L^{-3}/4)$ . Indeed, denote by  $\Phi_{0i}^{1,\pi}$  and  $\Phi_{0i}^1$  the counterparts of  $\Phi_0^{1,\pi}$  and  $\Phi_0^1$  associated to  $\mathcal{O}_i$ ,  $i \in I$ , then we have, a.s.,

$$\Phi_0^1(\tau) \leq \inf_{i \in I} \Phi_{0i}^1(\tau) \quad \text{and} \quad \Phi_0^{1,\pi}(\tau) \leq \inf_{i \in I} \Phi_{0i}^{1,\pi}(\tau)$$

whenever  $\mathcal{O} = \cap_{i \in I} \mathcal{O}_i$ .

**Remark 2.6.** Take  $d = 1$ ,  $\mathcal{O} = (-\infty, 1) \subset \mathcal{Z} = \mathbb{R}$ ,  $\pi = \mathbb{R}_+$ , and let  $Z = |W|^2 + z_0$  with  $1/2 < z_0 < 1$ . As distance function take an appropriate  $\delta \in C^\infty(\mathbb{R})$  with  $\delta$  constant outside  $(0, 2)$  and  $\delta(z) = 1 - z$  on  $[1/2, 3/2]$ . Then the conditions **(Z)**, **(P)** and **(L)** are satisfied and  $\Phi_0^1(0) = \mathbb{E}[\theta_0(0)] = \mathbb{E}[|W_{\theta_0(0)}|^2] = 1 - z_0 = P_0$ , which coincides with the upper-bound of Theorem 2.2 up to a multiplicative constant.

### 2.3 Freidlin type inequalities on moments of exit times

We start with a-priori estimates inspired by the proof of the exponential fast exit of Freidlin [Fre85, Lemma 3.3, Chapter 3]: a uniform bound on the conditional expected times to exit implies the existence of uniform conditional exponential moments for these exit times. We adapt Freidlin's arguments to our setting.

**Lemma 2.7.** *Let assumption (L) hold,  $p \geq 1$  be an integer,  $L^{(p)} := p!L^p$ , and  $\tau \in \mathcal{T}$ . Then we have*

$$(\Phi_0^p(\tau))^p \leq c_{p,(2.7)} \Phi_0^1(\tau) \quad \text{and} \quad (\Phi_0^{p,\pi}(\tau))^p \leq c_{p,(2.7)} \Phi_0^{1,\pi}(\tau)$$

with  $c_{p,(2.7)} := pL^{(p-1)}$ . Consequently,

$$\begin{aligned} \max \{ \Phi_0^p(\tau), \Phi_0^{p,\pi}(\tau) \}^p &\leq L^{(p)}, \\ \max \{ \mathbb{E}_\tau [e^{c(\theta_0(\tau)-\tau)}], \mathbb{E}_\tau [e^{c(\theta_0^\pi(\tau)-\tau)}] \} &\leq (1 - cL)^{-1}, \end{aligned}$$

where  $c \in [0, L^{-1})$ .

*Proof.* 1. The estimates for  $\Phi_0^p(\tau)$  and  $\Phi_0^{p,\pi}(\tau)$  are obtained in the same way, we only detail the second one by an induction over  $p$ . The case  $p = 1$  is an identity. Assume that the statement is proven for some  $p \geq 1$ . Observe that, on  $\{\theta_0^\pi(\tau) > t \geq \tau\} = \{\forall s \in [\tau, t] \cap \pi : Z_s \in \mathcal{O}\}$ , we have

$$\begin{aligned} \theta_0^\pi(\tau) &= \inf\{s \geq \tau : s \in \pi, Z_s \notin \mathcal{O}\} = \inf\{s \geq t \vee \tau : s \in \pi, Z_s \notin \mathcal{O}\} \\ &= \theta_0^\pi(t \vee \tau). \end{aligned}$$

Hence, for  $A \in \mathcal{F}_\tau$  we can write

$$\begin{aligned} &\frac{\mathbb{E} [(\Phi_0^{p+1,\pi}(\tau))^{p+1} \mathbf{1}_A]}{p+1} \\ &= \int_0^\infty \mathbb{E} [\mathbf{1}_A (\theta_0^\pi(\tau) - t)^p \mathbf{1}_{\{\theta_0^\pi(\tau) > t \geq \tau\}}] dt \\ &= \int_0^\infty \mathbb{E} [\mathbf{1}_A \mathbb{E}_{t \vee \tau} [(\theta_0^\pi(t \vee \tau) - t \vee \tau)^p] \mathbf{1}_{\{\theta_0^\pi(\tau) > t \geq \tau\}}] dt \\ &\leq p!L^{p-1} \int_0^\infty \mathbb{E} [\mathbf{1}_A \mathbb{E}_{t \vee \tau} [\theta_0^\pi(t \vee \tau) - t \vee \tau] \mathbf{1}_{\{\theta_0^\pi(\tau) > t \geq \tau\}}] dt \\ &\leq p!L^p \int_0^\infty \mathbb{E} [\mathbf{1}_A \mathbf{1}_{\{\theta_0^\pi(\tau) > t \geq \tau\}}] dt \\ &\leq L^{(p)} \mathbb{E} [\mathbf{1}_A \mathbb{E}_\tau [\theta_0^\pi(\tau) - \tau]], \end{aligned}$$

so that the proof is complete because  $A \in \mathcal{F}_\tau$  was arbitrary.

2. The consequently part is now obvious.  $\square$

## 2.4 An a-priori control in terms of the probability of strictly sub-harmonic paths

Now we provide a control on  $\Phi_0^1(\tau)$  in terms of the conditional probability of

$$\mathcal{A}_0^\tau := \{2Pb + |a|^2 \geq L^{-2}/2 \text{ on } [\tau, \theta_0(\tau)]\}^c.$$

Intuitively we can say, the more non-degenerate the process  $P_t^2$  from  $\tau$  to  $\theta_0(\tau)$  is, the smaller is the time of exit.

**Lemma 2.8.** *Let assumptions **(L)** and **(P-i)** be satisfied. Then there exists a constant  $c_{(2.8)} = c_{(2.8)}(L, d) > 0$  such that, for all  $\tau \in \mathcal{T}$ ,*

$$\Phi_0^1(\tau) \leq c_{(2.8)} \mathbb{P}_\tau[\mathcal{A}_0^\tau].$$

*Proof.* Let  $E := \{P_\tau \geq 0\} \in \mathcal{F}_\tau$  so that  $P_{\theta_0(\tau)} = 0$  on  $E$  and  $\Phi_0^1(\tau) = 0$  on  $E^c$ . Moreover, on  $E$  we obtain that

$$\begin{aligned} \theta_0(\tau) - \tau &\leq \mathbf{1}_{(\mathcal{A}_0^\tau)^c} 2L^2 \int_\tau^{\theta_0(\tau)} (2P_s b_s + |a_s|^2) ds + (\theta_0(\tau) - \tau) \mathbf{1}_{\mathcal{A}_0^\tau} \\ &= \mathbf{1}_{(\mathcal{A}_0^\tau)^c} 2L^2 (|P_{\theta_0(\tau)}|^2 - |P_\tau|^2) - \mathbf{1}_{(\mathcal{A}_0^\tau)^c} 2L^2 \int_\tau^{\theta_0(\tau)} 2P_s a_s^\top dW_s \\ &\quad + (\theta_0(\tau) - \tau) \mathbf{1}_{\mathcal{A}_0^\tau} \\ &\leq -\mathbf{1}_{(\mathcal{A}_0^\tau)^c} 4L^2 \int_\tau^{\theta_0(\tau)} P_s a_s^\top dW_s + (\theta_0(\tau) - \tau) \mathbf{1}_{\mathcal{A}_0^\tau}. \end{aligned}$$

Using the bound on  $\Phi_0^1(\tau)$  from assumption **(L)** and the bounds from assumption **(P-i)**, we obtain  $\mathbb{E} \int_0^\infty \mathbf{1}_{\{\tau < s \leq \theta_0(\tau)\}} P_s^2 |a_s|^2 ds < \infty$  and, on  $E$ ,

$$\begin{aligned} \mathbb{E}_\tau \left[ -\mathbf{1}_{(\mathcal{A}_0^\tau)^c} \int_\tau^{\theta_0(\tau)} P_s a_s^\top dW_s \right] &= \mathbb{E}_\tau \left[ \mathbf{1}_{\mathcal{A}_0^\tau} \int_\tau^{\theta_0(\tau)} P_s a_s^\top dW_s \right] \\ &\leq L^2 \sqrt{d} \mathbb{P}_\tau[\mathcal{A}_0^\tau]^{\frac{1}{2}} (\Phi_0^1(\tau))^{\frac{1}{2}}. \end{aligned}$$

On the other hand, Lemma 2.7 implies

$$\mathbb{E}_\tau [(\theta_0(\tau) - \tau) \mathbf{1}_{\mathcal{A}_0^\tau}] \leq \Phi_0^2(\tau) \mathbb{P}_\tau[\mathcal{A}_0^\tau]^{\frac{1}{2}} \leq \left[ c_{2,(2.7)} \Phi_0^1(\tau) \mathbb{P}_\tau[\mathcal{A}_0^\tau] \right]^{\frac{1}{2}}.$$

Combining the above estimates and using the inequality  $ab \leq a^2 + \frac{1}{4}b^2$  gives, on  $E$ ,

$$\begin{aligned} \Phi_0^1(\tau) &\leq 4L^4 \sqrt{d} \mathbb{P}_\tau[\mathcal{A}_0^\tau]^{\frac{1}{2}} (\Phi_0^1(\tau))^{\frac{1}{2}} + \left[ c_{2,(2.7)} \Phi_0^1(\tau) \mathbb{P}_\tau[\mathcal{A}_0^\tau] \right]^{\frac{1}{2}} \\ &\leq 16L^8 d \mathbb{P}_\tau[\mathcal{A}_0^\tau] + \frac{1}{4} \Phi_0^1(\tau) + c_{2,(2.7)} \mathbb{P}_\tau[\mathcal{A}_0^\tau] + \frac{1}{4} \Phi_0^1(\tau), \end{aligned}$$

which leads to the required result.  $\square$

## 2.5 Proof of Theorem 2.2

We start by two lemmas before we turn to the proof of Theorem 2.2.

**Lemma 2.9.** *Let  $\Psi \in \{\Phi_0^1, \Phi_0^{1,\pi}\}$  and assume that there is a constant  $c > 0$  such that for all  $\tau \in \mathcal{T}^\pi$  one has that*

$$\Psi(\tau) \leq c \left[ P_\tau + |\pi|^{\frac{1}{2}} \right] \mathbf{1}_{\{0 \leq P_\tau \leq r\}} + L \mathbf{1}_{\{r < P_\tau\}}.$$

Then for all  $0 < \tilde{r} < r$  there is a  $d_{(2.9)} = d_{(2.9)}(r - \tilde{r}, L, d, c) > 0$  such that for all  $\tau \in \mathcal{T}$  one has that

$$\Psi(\tau) \leq d_{(2.9)} \left[ P_\tau + |\pi|^{\frac{1}{2}} \right] \mathbf{1}_{\{0 \leq P_\tau \leq \tilde{r}\}} + L \mathbf{1}_{\{\tilde{r} < P_\tau\}}.$$

*Proof.* The case  $\pi = \mathbb{R}_+$  is trivial because  $\Psi(\tau) \leq L$  so that we can assume that  $\pi \neq \mathbb{R}^+$ . Using

$$\Psi(\tau) \leq \mathbb{E}_\tau \left[ \Psi(\phi_\tau^+) + |\pi| \right]$$

and

$$\mathbb{E}_\tau \left[ |P_{\phi_\tau^+} - P_\tau| \right] \leq L[1 + \sqrt{d}]|\pi|^{\frac{1}{2}} =: A|\pi|^{\frac{1}{2}},$$

which implies  $|P_\tau - \mathbb{E}_\tau [P_{\phi_\tau^+}]| \leq A|\pi|^{\frac{1}{2}}$ , we can conclude by

$$\begin{aligned} & \Psi(\tau) \\ & \leq \mathbb{E}_\tau \left[ \Psi(\phi_\tau^+) + |\pi| \right] \mathbf{1}_{\{0 \leq P_\tau \leq \tilde{r}\}} + L \mathbf{1}_{\{\tilde{r} < P_\tau\}} \\ & \leq \mathbb{E}_\tau \left[ c \left[ P_{\phi_\tau^+} + |\pi|^{\frac{1}{2}} \right] \mathbf{1}_{\{0 \leq P_{\phi_\tau^+} \leq r\}} + L \mathbf{1}_{\{r < P_{\phi_\tau^+}\}} + |\pi| \right] \mathbf{1}_{\{0 \leq P_\tau \leq \tilde{r}\}} \\ & \quad + L \mathbf{1}_{\{\tilde{r} < P_\tau\}} \\ & \leq \left[ cP_\tau + [c(1 + A) + 1]|\pi|^{\frac{1}{2}} \right] \mathbf{1}_{\{0 \leq P_\tau \leq \tilde{r}\}} \\ & \quad + L \mathbb{P}_\tau \left[ r < P_{\phi_\tau^+}, 0 \leq P_\tau \leq \tilde{r} \right] \mathbf{1}_{\{0 \leq P_\tau \leq \tilde{r}\}} + L \mathbf{1}_{\{\tilde{r} < P_\tau\}} \\ & \leq \left[ cP_\tau + [c(1 + A) + 1]|\pi|^{\frac{1}{2}} \right] \mathbf{1}_{\{0 \leq P_\tau \leq \tilde{r}\}} \\ & \quad + L \mathbb{P}_\tau \left[ |P_{\phi_\tau^+} - P_\tau| \geq r - \tilde{r} \right] \mathbf{1}_{\{0 \leq P_\tau \leq \tilde{r}\}} + L \mathbf{1}_{\{\tilde{r} < P_\tau\}} \\ & \leq \left[ cP_\tau + \left[ c(1 + A) + 1 + \frac{LA}{r - \tilde{r}} \right] |\pi|^{\frac{1}{2}} \right] \mathbf{1}_{\{0 \leq P_\tau \leq \tilde{r}\}} + L \mathbf{1}_{\{\tilde{r} < P_\tau\}}. \end{aligned}$$

□

Next we control the quantity  $\mathbb{P}_\tau[\mathcal{A}_0^\tau]$  to make Lemma 2.8 applicable:

**Lemma 2.10.** *Assume that **(Z)** and **(P)** hold. Then for all  $c > 0$  there exists an  $\eta(c) = \eta(c, r, L, d) > 0$  such that*

$$\mathbb{P}_\tau[\mathcal{A}_0^\tau] \leq \eta(c)P_\tau + c \Phi_0^1(\tau) + \kappa \left( \frac{2}{c}, r \right) |\pi| \quad \text{a.s. on } \{P_\tau \in [0, r]\}, \quad (4)$$

where  $\tau \in \mathcal{T}^\pi$  and  $\mathcal{A}_0^\tau := \{2Pb + |a|^2 \geq L^{-2}/2 \text{ on } [\tau, \theta_0(\tau)]\}^c$ .

*Proof.* Let  $\tilde{\theta}_r(\tau) := \inf\{t \geq \tau : P_t = r\} \in [0, \infty]$ . Assumption **(P-ii)** implies  $2Pb + |a|^2 \geq L^{-2}/2$   $\mathbb{P}$ -a.s. on  $\{|P| \vee \gamma(\cdot, \phi) \leq r\}$  for  $r \leq L^{-3}/4$ . It follows from the restriction  $\tau \in \mathcal{T}^\pi$  that on

$$E := \{P_\tau \in [0, r]\}$$

we have,  $\mathbb{P}$ -a.s., that

$$\begin{aligned} (\mathcal{A}_0^\tau)^c &\supseteq \left\{ \sup_{\tau \leq t \leq \theta_0(\tau)} |P_t| \leq r \right\} \cap \left\{ \sup_{\tau \leq t \leq \theta_0(\tau)} \gamma(t, \phi_t \vee \tau) \leq r \right\} \\ &\supseteq \{ \theta_0(\tau) \leq \tilde{\theta}_r(\tau) \} \cap \left\{ \sup_{\tau \leq t \leq \theta_0(\tau)} \gamma(t, \phi_t \vee \tau) \leq r \right\}. \end{aligned}$$

Setting  $\mathcal{B}_T := \{ \sup_{\tau \leq t \leq \tau+T} \gamma(t, \phi_t \vee \tau) \leq r \}$  for  $T := 2c^{-1}$ , we continue on  $E$  with

$$\begin{aligned} \mathbb{P}_\tau[\mathcal{A}_0^\tau] &\leq \mathbb{P}_\tau[\mathcal{A}_0^\tau, \theta_0(\tau) \leq \tau + T, \mathcal{B}_T] + \mathbb{P}_\tau[\theta_0(\tau) > \tau + T] + \mathbb{P}_\tau[\mathcal{B}_T^c] \\ &\leq \mathbb{P}_\tau[\tilde{\theta}_r(\tau) < \theta_0(\tau) \leq \tau + T, \mathcal{B}_T] + \mathbb{P}_\tau[\theta_0(\tau) > \tau + T] + \mathbb{P}_\tau[\mathcal{B}_T^c] \\ &\leq \mathbb{P}_\tau[\tilde{\theta}_r(\tau) < \theta_0(\tau) \leq \tau + T, \mathcal{B}_T] + \frac{c}{2} \Phi_0^1(\tau) + \kappa \left( \frac{2}{c}, r \right) |\pi|, \quad (5) \end{aligned}$$

where the last inequality follows from Chebyshev's inequality and assumption **(Z)**. To treat the first term in (5) we set, for  $t \geq 0$ ,

$$\theta_{0,r}^T := \theta_0(\tau) \wedge \tilde{\theta}_r(\tau) \wedge (\tau + T).$$

In view of assumption **(P)** we can define  $\mathbb{Q} \sim \mathbb{P}$  by the density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = H := \mathcal{E} \left( - \int_\tau^{\cdot} \lambda_s^\top dW_s \right)_{\theta_{0,r}^T},$$

where

$$\lambda := a [|a|^{-2} \wedge L^2] b \mathbf{1}_{[\tau, \theta_{0,r}^T]} \quad \text{so that} \quad |\lambda| \leq L^4 \sqrt{d} =: \lambda_\infty,$$

and deduce from Girsanov's Theorem (cf. [Bic10, p.163]) that

$$W^\mathbb{Q} := W + \mathbf{1}_{[\tau, \infty)} \int_\tau^{\theta_{0,r}^T \wedge \cdot} \lambda_s ds$$

is a Brownian motion associated to  $\mathbb{Q}$ . For any given  $\ell > 1$  we obtain

$$\begin{aligned} &\mathbb{P}_\tau[\tilde{\theta}_r(\tau) < \theta_0(\tau) \leq \tau + T, \mathcal{B}_T] \\ &\leq \mathbb{P}_\tau[H^{-1} > \ell] + \mathbb{E}_\tau^\mathbb{Q} \left[ H^{-1} \mathbf{1}_{\{H^{-1} \leq \ell\}} \mathbf{1}_{\{\tilde{\theta}_r(\tau) < \theta_0(\tau) \leq \tau + T\}} \mathbf{1}_{\mathcal{B}_T} \right] \\ &\leq \mathbb{P}_\tau[H^{-1} > \ell] + \ell \mathbb{Q}_\tau \left[ \tilde{\theta}_r(\tau) < \theta_0(\tau) \leq \tau + T, \mathcal{B}_T \right]. \quad (6) \end{aligned}$$

The first term above can be estimated, by using Chebyshev's inequality, the inequality  $\theta_{0,r}^T \leq \theta_0(\tau)$ , and Lemma 2.7:

$$\mathbb{P}_\tau [H^{-1} > \ell] \leq \frac{1}{|\log \ell|^2} 2\mathbb{E}_\tau \left[ \frac{1}{4} \lambda_\infty^4 |\theta_{0,r}^T - \tau|^2 + \lambda_\infty^2 (\theta_{0,r}^T - \tau) \right]$$

$$\leq \frac{(L\lambda_\infty^4 + 2\lambda_\infty^2)}{|\log \ell|^2} \Phi_0^1(\tau) \leq \frac{c}{2} \Phi_0^1(\tau), \quad (7)$$

where the last inequality holds by taking the constant  $\ell = \ell(c, L, d)$  large enough. To handle the second term in (6), set

$$M_t := \mathbb{E}_t^{\mathbb{Q}}[P_\tau] + \mathbf{1}_{[\tau, \infty)}(t) \int_\tau^t \mathbf{1}_{\{s < \theta_{0,r}^T\}} a_s^\top dW_s^{\mathbb{Q}} \quad \text{for } t \geq 0$$

so that  $M$  is a u.i.  $\mathbb{Q}$ -martingale. Let  $\theta_0^M(\tau)$  and  $\theta_r^M(\tau)$  be the first hitting times after  $\tau$  of levels 0 and  $r$  by  $M$ , and set  $\theta_{0,r}^{M,T} := \theta_0^M(\tau) \wedge \theta_r^M(\tau) \wedge (\tau + T)$ . Recalling assumption **(P)**, we see that

$$\mathbf{1}_{[\tau, \infty)}(t) \int_\tau^t \mathbf{1}_{\{s < \theta_{0,r}^T\}} a_s^\top \lambda_s ds = \mathbf{1}_{[\tau, \infty)}(t) \int_\tau^t \mathbf{1}_{\{s < \theta_{0,r}^T\}} b_s ds \quad \text{on } \mathcal{B}_T \cap E.$$

Hence, on  $\mathcal{B}_T \cap E$  the processes  $M$  and  $P$  coincide on  $[\tau, \theta_{0,r}^T]$ . By the optional sampling theorem and the non-negativity of  $(M_t)_{t \in [\tau, \theta_{0,r}^{M,T}]}$  a.s. on  $E$ , we then deduce

$$\begin{aligned} P_\tau \mathbf{1}_E &= M_\tau \mathbf{1}_E = \mathbb{E}_\tau^{\mathbb{Q}}(\mathbf{1}_E M_{\theta_{0,r}^{M,T}}) \geq \mathbf{1}_{E^c} r \mathbb{Q}_\tau(\theta_r^M(\tau) < \theta_0^M(\tau) \wedge (\tau + T)) \\ &\geq \mathbf{1}_{E^c} r \mathbb{Q}_\tau(\tilde{\theta}_r(\tau) < \theta_0(\tau) < \tau + T, \mathcal{B}_T). \end{aligned}$$

Plugging this inequality together with (7) into (6), gives on  $E$  that

$$\mathbb{P}_\tau \left[ \tilde{\theta}_r(\tau) < \theta_0(\tau) \leq \tau + T, \mathcal{B}_T \right] \leq \frac{c}{2} \Phi_0^1(\tau) + \frac{\ell(c, L, d)}{r} P_\tau.$$

□

**Proof of Theorem 2.2.** Part (a): For  $\tau \in \mathcal{T}^\pi$  and  $c > 0$  Lemmas 2.8 and 2.10 imply

$$\Phi_0^1(\tau) \leq c_{(2.8)} \mathbb{P}_\tau[\mathcal{A}_0^\tau] \leq c_{(2.8)} \left[ \eta(c) P_\tau + c \Phi_0^1(\tau) + \kappa \left( \frac{2}{c}, r \right) |\pi| \right]$$

a.s. on  $\{P_\tau \in [0, r]\}$ . Specializing to  $c = 1/(2c_{(2.8)})$  leads to

$$\Phi_0^1(\tau) \leq 2c_{(2.8)} \left[ \eta((2c_{(2.8)})^{-1}) P_\tau + \kappa(4c_{(2.8)}, r) |\pi| \right]$$

on  $\{P_\tau \in [0, r]\}$ . On  $\{P_\tau < 0\}$  we simply have  $\Phi_0^1(\tau) = 0$ , while  $\Phi_0^1(\tau) \leq L$  on  $\{P_\tau > r\}$  by assumption **(L)**.

Part (b) follows from Lemma 2.9. □

## 2.6 Proof of Theorem 2.3

Now we are in a position to conclude the proof of Theorem 2.3. It is based on a recursion argument. Namely, given  $\tau \in \mathcal{T}$  such that  $0 \leq P_\tau \leq r$ , we wait until the next time  $\vartheta$  in  $\mathbb{R}_+$  such that  $Z$  hits the boundary. The time it takes,  $\vartheta - \tau$ , is controlled by Theorem 2.2. If  $Z_{\phi_\vartheta^+} \notin \mathcal{O}$ , then we stop:  $\theta_0^\pi(\tau) - \tau \leq \vartheta - \tau + |\pi|$ . If not, then we know from standard estimates (Lemma 2.11 below) that  $P_{\phi_\vartheta^+} \in [0, r]$ , up to some event with a probability controlled by  $O(|\pi|^{\frac{1}{2}})$ . In this case one can restart the above procedure from  $\phi_\vartheta^+ \in \pi$ . Again, one waits for the next time in  $\mathbb{R}_+$  such that  $Z$  reaches the boundary and stops if  $Z \notin \mathcal{O}$  at the following time in  $\pi$ . One iterates this procedure. The key point is that the probability of the event set  $\{Z_{\phi_\vartheta^+} \in \mathcal{O}\}$  is uniformly controlled by some  $\alpha < 1$  (see Lemma 2.12 below).

Before we start with the proof of Theorem 2.3 we state two lemmas that are needed. The first one can be verified by Doob's maximal inequality and assumption **(P-i)**:

**Lemma 2.11.** *Under the assumption **(P-i)** one has, for all  $\tau \in \mathcal{T}$  and  $\lambda > 0$ ,*

$$\mathbb{P}_\tau \left[ \max_{\tau \leq t \leq \phi_\tau^+} |P_t - P_\tau| \geq \lambda \right] \leq \frac{1}{\lambda} \mathbb{E}_\tau \left[ \max_{\tau \leq t \leq \phi_\tau^+} |P_t - P_\tau|^2 \right]^{\frac{1}{2}} \leq \frac{c_{(2.11)}}{\lambda} |\pi|^{\frac{1}{2}},$$

where  $c_{(2.11)} := L + 2\sqrt{d}L$ .

**Lemma 2.12.** *Let assumptions **(Z)** and **(P)** hold. Then there exists an  $0 < \alpha_{(2.12)} = \alpha_{(2.12)}(r, L, d, \kappa) < 1$  such that, a.s.,*

$$\mathbb{P}_\tau [P_{\phi_\tau^+} > 0] \leq \alpha_{(2.12)} \quad \text{on} \quad \left\{ \gamma(\tau, \phi_\tau) \leq \frac{r}{2}, P_\tau = 0 \right\} \in \mathcal{F}_\tau$$

for all  $\tau \in \mathcal{T}$  and  $0 < |\pi| \leq \varepsilon_{(2.12)} = \varepsilon_{(2.12)}(r, L, d, \kappa)$ .

*Proof.* It is sufficient to check for  $B \in \mathcal{F}_\tau$  of positive measure with

$$B \subseteq \left\{ \gamma(\tau, \phi_\tau) \leq \frac{r}{2}, P_\tau = 0 \right\}$$

that

$$\mathbb{P} [P_{\phi_\tau^+} > 0, B] \leq \alpha_{(2.12)} \mathbb{P} [B].$$

Let

$$\mathcal{B} := \left\{ \max_{\tau \leq t \leq \phi_\tau^+} (|P_t| \vee \gamma(t, \phi_t \vee \tau)) \leq \frac{r}{2} \right\} \cap B$$



so that

$$\mathcal{B} \subseteq \left\{ \max_{\tau \leq t \leq \phi_\tau^+} (|P_t| \vee \gamma(t, \phi_t)) \leq r \right\}. \quad (8)$$

We use assumptions **(P)**, **(Z)** and Lemma 2.11 to continue with

$$\begin{aligned} & \mathbb{P} [P_{\phi_\tau^+} > 0, B] \\ &= \mathbb{P} \left[ \int_\tau^{\phi_\tau^+} b_s ds + \int_\tau^{\phi_\tau^+} a_s^\top dW_s > 0, B \right] \\ &\leq \mathbb{P} \left[ \int_\tau^{\phi_\tau^+} a_s^\top dW_s > -L(\phi_\tau^+ - \tau), \mathcal{B} \right] + \mathbb{P} [\mathcal{B}^c \cap B] \\ &\leq \mathbb{P} \left[ \int_\tau^{\phi_\tau^+} a_s^\top dW_s > -L(\phi_\tau^+ - \tau), \mathcal{B} \right] \\ &\quad + \left[ \kappa \left( 1, \frac{r}{2} \right) |\pi| + \frac{2c(2.11)}{r} |\pi|^{\frac{1}{2}} \right] \mathbb{P} [B]. \end{aligned}$$

Assuming that we are able to show that

$$\mathbb{P} \left[ \int_\tau^{\phi_\tau^+} a_s^\top dW_s > -L(\phi_\tau^+ - \tau), \mathcal{B} \right] \leq \theta \mathbb{P} [B] \quad (9)$$

for some  $\theta = \theta(L, d) \in (0, 1)$ , the proof would be complete as

$$\mathbb{P} [P_{\phi_\tau^+} > 0, B] \leq \left[ \theta + \kappa \left( 1, \frac{r}{2} \right) |\pi| + \frac{2c(2.11)}{r} |\pi|^{\frac{1}{2}} \right] \mathbb{P} [B]$$

and  $\varepsilon(2.12) = \varepsilon(2.12)(r, L, d, \kappa) > 0$  can be taken small enough to guaranty

$$\mathbb{P} [P_{\phi_\tau^+} > 0, B] \leq \alpha \mathbb{P} [B]$$

for some  $\alpha = \alpha(r, L, d, \kappa) \in (0, 1)$ . In order to check (9) we let

$$M_t := e^\top W_{t \wedge \tau} + \int_\tau^{\tau \vee t} \bar{a}_s^\top dW_s$$

where  $\bar{a}_s := a_s \mathbf{1}_{\{s \leq \phi_\tau^+\}} + e \mathbf{1}_{\{s > \phi_\tau^+\}}$  with  $e = d^{-\frac{1}{2}}(1, \dots, 1)^\top$ . Define  $\Lambda(s) := \inf\{t \geq 0 : \langle M \rangle_t > s\}$ . Applying the Dambis-Schwarz Theorem [RY05, p. 181] yields that  $B := M_\Lambda$  is a Brownian motion in the filtration  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$  defined by  $\mathcal{G} = \mathcal{F}_\Lambda$  and  $M = B_{\langle M \rangle}$ . One can also check that  $\mathcal{F}_\tau \subseteq \mathcal{G}_{\langle M \rangle_\tau}$ . Letting  $\eta := \phi_\tau^+ - \tau$  (which is  $\mathcal{F}_\tau$ -measurable), observing  $\eta \leq \sqrt{\eta}$  and  $\eta L^{-2} \leq \langle M \rangle_{\tau+\eta} - \langle M \rangle_\tau \leq \eta d L^2$  on  $\mathcal{B}$  by assumption **(P)** and (8), and taking an auxiliary one-dimensional Brownian motion  $\tilde{B}$  defined on some  $(\tilde{\Omega}, \tilde{\mathbb{P}})$ , we conclude by

$$\mathbb{P} \left[ \int_\tau^{\phi_\tau^+} a_s^\top dW_s > -L(\phi_\tau^+ - \tau), \mathcal{B} \right]$$

$$\begin{aligned}
&= \mathbb{P} \left[ B_{\langle M \rangle_{\phi_\tau^+}} - B_{\langle M \rangle_\tau} > -L(\phi_\tau^+ - \tau), \mathcal{B} \right] \\
&\leq \mathbb{P} \left[ \sup_{t \in [\langle M \rangle_\tau + \eta L^{-2}, \langle M \rangle_\tau + \eta d L^2]} B_t - B_{\langle M \rangle_\tau} > -L\eta, \mathcal{B} \right] \\
&\leq \mathbb{P} \left[ \sup_{t \in [\langle M \rangle_\tau + \eta L^{-2}, \langle M \rangle_\tau + \eta d L^2]} B_t - B_{\langle M \rangle_\tau} > -L\eta, B \right] \\
&\leq \tilde{\mathbb{P}} \times \mathbb{P} \left[ \sup_{\eta L^{-2} \leq u \leq \eta d L^2} \tilde{B}_u > -L\sqrt{\eta}, B \right] \\
&\leq \tilde{\mathbb{P}} \times \mathbb{P} \left[ \sup_{L^{-2} \leq u \leq d L^2} \tilde{B}_u > -L, B \right] \\
&= \tilde{\mathbb{P}} \left[ \sup_{L^{-2} \leq u \leq d L^2} \tilde{B}_u > -L \right] \mathbb{P}[B] \\
&=: \theta \mathbb{P}[B].
\end{aligned}$$

□

**Proof of Theorem 2.3.** Part (a): For  $\tau \in \mathcal{T}^\pi$  and  $i \geq 0$  we define

$$\vartheta_0 := \theta_0(\tau), \quad \vartheta_{i+1} := \theta_0(\phi_{\vartheta_i}^+), \quad \vartheta_0^\pi := \theta_0^\pi(\tau), \quad \vartheta_{i+1}^\pi := \theta_0^\pi(\phi_{\vartheta_i}^+),$$

$$E_i := \{P_{\phi_{\vartheta_i}^+} > 0\}, \quad \text{and} \quad \mathcal{A}_i := \bigcap_{0 \leq j \leq i} E_j \in \mathcal{F}_{\phi_{\vartheta_i}^+}.$$

1. From the definitions we obtain for  $i \geq 0$ :

- a)  $\vartheta_{i+1} \leq \vartheta_{i+1}^\pi$  (by definitions of the stopping times);
- b)  $\phi_{\vartheta_{i+1}}^+ \leq \vartheta_{i+1} + |\pi|$  (by the definition of  $\phi^+$ );
- c)  $\vartheta_{i+1}^\pi = \vartheta_{i+2}^\pi$  on  $E_{i+1} = \{Z_{\phi_{\vartheta_{i+1}}^+} \in \mathcal{O}\}$  (since  $\phi_{\vartheta_{i+1}}^+ < \vartheta_{i+1}^\pi$  on  $E_{i+1}$ );
- d)  $\vartheta_{i+1}^\pi \leq \vartheta_{i+1} + |\pi|$  on  $(E_{i+1})^c = \{Z_{\phi_{\vartheta_{i+1}}^+} \notin \mathcal{O}\}$  (by definition of the stopping time  $\vartheta_{i+1}^\pi$ ).

Item c) leads to

$$\vartheta_{i+1}^\pi = \vartheta_{i+2}^\pi \mathbf{1}_{E_{i+1}} + \vartheta_{i+1}^\pi \mathbf{1}_{[E_{i+1}]^c},$$

$$\vartheta_{i+1}^\pi - \phi_{\vartheta_i}^+ = (\vartheta_{i+2}^\pi - \phi_{\vartheta_{i+1}}^+) \mathbf{1}_{E_{i+1}} + (\phi_{\vartheta_{i+1}}^+ - \phi_{\vartheta_i}^+) \mathbf{1}_{E_{i+1}} + (\vartheta_{i+1}^\pi - \phi_{\vartheta_i}^+) \mathbf{1}_{[E_{i+1}]^c}.$$

With b) and d) we continue to

$$\begin{aligned}
\vartheta_{i+1}^\pi - \phi_{\vartheta_i}^+ &\leq (\vartheta_{i+2}^\pi - \phi_{\vartheta_{i+1}}^+) \mathbf{1}_{E_{i+1}} + (\vartheta_{i+1} + |\pi| - \phi_{\vartheta_i}^+) \mathbf{1}_{E_{i+1}} \\
&\quad + (\vartheta_{i+1} + |\pi| - \phi_{\vartheta_i}^+) \mathbf{1}_{[E_{i+1}]^c} \\
&= (\vartheta_{i+2}^\pi - \phi_{\vartheta_{i+1}}^+) \mathbf{1}_{E_{i+1}} + |\pi| + (\vartheta_{i+1} - \phi_{\vartheta_i}^+)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_\tau [(\vartheta_{i+1}^\pi - \phi_{\vartheta_i}^+) \mathbf{1}_{\mathcal{A}_i}] &\leq \mathbb{E}_\tau \left[ (\vartheta_{i+2}^\pi - \phi_{\vartheta_{i+1}}^+) \mathbf{1}_{\mathcal{A}_i} \mathbf{1}_{E_{i+1}} \right] \\
&\quad + |\pi| \mathbb{P}_\tau [\mathcal{A}_i] + \mathbb{E}_\tau [(\vartheta_{i+1} - \phi_{\vartheta_i}^+) \mathbf{1}_{\mathcal{A}_i}] \\
&= \mathbb{E}_\tau \left[ (\vartheta_{i+2}^\pi - \phi_{\vartheta_{i+1}}^+) \mathbf{1}_{\mathcal{A}_{i+1}} \right] \\
&\quad + |\pi| \mathbb{P}_\tau [\mathcal{A}_i] + \mathbb{E}_\tau [(\vartheta_{i+1} - \phi_{\vartheta_i}^+) \mathbf{1}_{\mathcal{A}_i}].
\end{aligned}$$

Summing up the above inequalities from  $i = 0$  to  $i = n - 1$  yields

$$\begin{aligned}
\mathbb{E}_\tau [(\vartheta_1^\pi - \phi_{\vartheta_0}^+) \mathbf{1}_{\mathcal{A}_0}] &\leq \mathbb{E}_\tau [(\vartheta_{n+1}^\pi - \phi_{\vartheta_n}^+) \mathbf{1}_{\mathcal{A}_n}] \\
&\quad + \sum_{i=0}^{n-1} (|\pi| \mathbb{P}_\tau [\mathcal{A}_i] + \mathbb{E}_\tau [(\vartheta_{i+1} - \phi_{\vartheta_i}^+) \mathbf{1}_{\mathcal{A}_i}]). \quad (10)
\end{aligned}$$

2. For  $\sigma \in \mathcal{T}$  set  $\mathcal{A}^\sigma := \{\gamma(\sigma, \phi_\sigma) \leq r/2\} \in \mathcal{F}_\sigma$  so that, for  $i \geq 1$ ,

$$\begin{aligned}
\mathbb{P}_\tau [\mathcal{A}_i] &= \mathbb{E}_\tau [\mathbf{1}_{\mathcal{A}_{i-1}} \mathbf{1}_{\mathcal{A}^{\vartheta_i}} \mathbb{P}_{\vartheta_i} [E_i]] + \mathbb{E}_\tau \left[ \mathbf{1}_{\mathcal{A}_{i-1}} \mathbb{P}_{\phi_{\vartheta_{i-1}}^+} [[\mathcal{A}^{\vartheta_i}]^c \cap E_i] \right] \\
&\leq \alpha_{(2.12)} \mathbb{P}_\tau [\mathcal{A}_{i-1}] + \mathbb{E}_\tau \left[ \mathbf{1}_{\mathcal{A}_{i-1}} \mathbb{P}_{\phi_{\vartheta_{i-1}}^+} [[\mathcal{A}^{\vartheta_i}]^c \cap E_i] \right],
\end{aligned}$$

because of  $\mathcal{F}_{\phi_{\vartheta_{i-1}}^+} \subseteq \mathcal{F}_{\vartheta_i}$ , Lemma 2.12, and  $P_{\vartheta_i} = 0$  on  $\mathcal{A}_{i-1}$ . To treat the second term we take a fixed  $T > 0$  and use **(Z)** and **(L)** to get

$$\begin{aligned}
\mathbb{P}_{\phi_{\vartheta_{i-1}}^+} [[\mathcal{A}^{\vartheta_i}]^c \cap E_i] &\leq \mathbb{P}_{\phi_{\vartheta_{i-1}}^+} \left[ \{\gamma(\vartheta_i, \phi_{\vartheta_i}) > r/2\} \cap \{\vartheta_i \leq \phi_{\vartheta_{i-1}}^+ + T\} \right] \\
&\quad + \mathbb{P}_{\phi_{\vartheta_{i-1}}^+} \left[ \vartheta_i > \phi_{\vartheta_{i-1}}^+ + T \right] \\
&\leq \mathbb{P}_{\phi_{\vartheta_{i-1}}^+} \left[ \sup_{\phi_{\vartheta_{i-1}}^+ \leq t \leq \phi_{\vartheta_{i-1}}^+ + T} \gamma(t, \phi_t \vee \phi_{\vartheta_{i-1}}^+) > r/2 \right] \\
&\quad + \mathbb{P}_{\phi_{\vartheta_{i-1}}^+} \left[ \vartheta_i > \phi_{\vartheta_{i-1}}^+ + T \right] \\
&\leq \kappa(T, r/2) |\pi| + \mathbb{E}_{\phi_{\vartheta_{i-1}}^+} [\theta_0(\phi_{\vartheta_{i-1}}^+)] / T \\
&\leq \kappa(T, r/2) |\pi| + L/T.
\end{aligned}$$

By taking  $T > 0$  large enough and then  $\varepsilon_{(11)} \in (0, \varepsilon_{(2.12)})$  small enough such that

$$\alpha_{(2.12)} + \kappa(T, r/2) \varepsilon_{(11)} + \frac{L}{T} =: \alpha < 1$$

and assuming that  $|\pi| \leq \varepsilon_{(11)}$ , we obtain  $\mathbb{P}_\tau [\mathcal{A}_i] \leq \alpha \mathbb{P}_\tau [\mathcal{A}_{i-1}]$  and, by induction,

$$\mathbb{P}_\tau [\mathcal{A}_j] \leq \alpha^j \quad \text{for all } j \geq 0. \quad (11)$$

3. Let us set  $F_i := \{P_{\phi_{\vartheta_i}^+} > r\}$  for  $i \geq 0$ . Because of  $\phi_{\vartheta_i}^+ \in \mathcal{T}^\pi$ , applying Theorem 2.2 and using the fact that  $\mathcal{A}_i \in \mathcal{F}_{\phi_{\vartheta_i}^+}$  and assumption **(L)**, lead to

$$\begin{aligned} & \mathbb{E}_\tau [(\vartheta_{i+1} - \phi_{\vartheta_i}^+) \mathbf{1}_{\mathcal{A}_i}] \\ & \leq \mathbb{E}_\tau \left[ \mathbb{E}_{\phi_{\vartheta_i}^+} [(\vartheta_{i+1} - \phi_{\vartheta_i}^+) \mathbf{1}_{\mathcal{A}_i}] \right] \\ & \leq \mathbb{E}_\tau \left[ c(2.2) \left( (P_{\phi_{\vartheta_i}^+})^+ + |\pi| \right) \mathbf{1}_{\mathcal{A}_{i-1}} \mathbf{1}_{E_i \cap [F_i]^c} \right] + L \mathbb{P}_\tau [\mathcal{A}_i \cap F_i] \\ & \leq \mathbb{E}_\tau \left[ c(2.2) \left( (P_{\phi_{\vartheta_i}^+})^+ + |\pi| \right) \mathbf{1}_{\mathcal{A}_{i-1}} \right] + L \mathbb{E}_\tau [\mathbf{1}_{\mathcal{A}_{i-1}} \mathbb{P}_{\vartheta_i} [F_i]], \end{aligned}$$

where  $\mathcal{A}_{-1} := \Omega$ . Because  $P_{\vartheta_i} \leq 0$ , Lemma 2.11 implies

$$\mathbb{E}_{\vartheta_i} \left[ (P_{\phi_{\vartheta_i}^+})^+ \right] \leq c(2.11) |\pi|^{\frac{1}{2}} \quad \text{and} \quad \mathbb{P}_{\vartheta_i} [F_i] \leq \frac{c(2.11)}{r} |\pi|^{\frac{1}{2}},$$

and (11) yields

$$\begin{aligned} & \mathbb{E}_\tau [(\vartheta_{i+1} - \phi_{\vartheta_i}^+) \mathbf{1}_{\mathcal{A}_i}] \\ & \leq \mathbb{E}_\tau \left[ c(2.2) \left( c(2.11) |\pi|^{\frac{1}{2}} + |\pi|^{\frac{1}{2}} \right) \mathbf{1}_{\mathcal{A}_{i-1}} \right] + L \mathbb{P}_\tau [\mathcal{A}_{i-1}] \frac{c(2.11)}{r} |\pi|^{\frac{1}{2}} \\ & \leq D |\pi|^{\frac{1}{2}} \alpha^{(i-1)+} \end{aligned}$$

with  $D := c(2.2)c(2.11) + c(2.2) + Lc(2.11)/r$ . If we insert the last estimate into (10) and let  $n \rightarrow +\infty$ , then we get

$$\mathbb{E}_\tau [(\vartheta_1^\pi - \phi_{\vartheta_0}^+) \mathbf{1}_{\mathcal{A}_0}] \leq |\pi|^{\frac{1}{2}} \frac{|\pi|^{\frac{1}{2}} + (2 - \alpha)D}{1 - \alpha},$$

where we exploit Lemma 2.7 to check

$$\mathbb{E}_\tau [|\vartheta_{n+1}^\pi - \phi_{\vartheta_n}^+| \mathbf{1}_{\mathcal{A}_n}] \leq \sqrt{L^{(2)}} \mathbb{P}_\tau [\mathcal{A}_n]^{\frac{1}{2}}.$$

Observe now that

$$\theta_0^\pi(\tau) = [\phi_{\vartheta_0}^+ + (\vartheta_1^\pi - \phi_{\vartheta_0}^+)] \mathbf{1}_{\mathcal{A}_0} + \phi_{\vartheta_0}^+ \mathbf{1}_{[\mathcal{A}_0]^c} \leq |\pi| + \vartheta_0 + (\vartheta_1^\pi - \phi_{\vartheta_0}^+) \mathbf{1}_{\mathcal{A}_0},$$

so that by an application of the previous estimate and Theorem 2.2(a) we obtain Theorem 2.3(a).

Part (b) follows again from Lemma 2.9. □

### 3 $L_1$ -error for exit time approximations

The main motivation of this paper is to study the error made by approximating the exit time  $\theta$  of a diffusion  $X$  from a domain  $\mathcal{O}$  by the discrete exit time  $\bar{\theta}$  of an Euler scheme  $\bar{X}$  associated to  $X$ , where  $\bar{\theta}$  is computed on the

grid  $\bar{\pi}$  of the Euler scheme. If  $X$  exits before  $\bar{X}$ , then standard estimates based on mild regularity assumptions imply that  $\bar{X}$  should be close to the boundary, the distance being of order of the square root of the mesh-size of the time grid, i.e.  $|\bar{\pi}|^{\frac{1}{2}}$ . If we also know that the expectation of the time it takes the Euler scheme  $\bar{X}$  to exit the domain is proportional to its distance to the boundary up to an additional term  $|\bar{\pi}|^{\frac{1}{2}}$ , then we can conclude that  $\mathbb{E}[|\bar{\theta} - \theta| \mathbf{1}_{\{\theta \leq \bar{\theta}\}}]$  is controlled in  $|\bar{\pi}|^{\frac{1}{2}}$ . The same idea applies if  $\bar{X}$  exits before  $X$ . In this section, we show how Theorems 2.2 and 2.3 are used to follow this idea. We start with an abstract statement and then specialize it to the case where  $X$  solves a stochastic differential equation and  $\bar{X}$  is its Euler scheme.

### 3.1 Upper-bound in an abstract setting

We fix an open non-empty subset  $\mathcal{O}$  of a metric space  $(\mathcal{Z}, d_{\mathcal{Z}})$ , satisfying the assumptions of Section 2.1, and two  $\mathcal{Z}$ -valued processes  $X$  and  $\bar{X}$ . We consider the first exit time  $\theta_0 := \theta_0(0)$  of  $X$  on  $\pi := \mathbb{R}_+$  and  $\bar{\theta}_0^{\bar{\pi}} := \bar{\theta}_0^{\bar{\pi}}(0)$  of  $\bar{X}$  on  $\bar{\pi} \subsetneq \mathbb{R}_+$  (with  $0 \in \bar{\pi}$ ), i.e.

$$\theta_0 := \inf\{t \geq 0 : X_t \notin \mathcal{O}\} \quad \text{and} \quad \bar{\theta}_0^{\bar{\pi}} := \inf\{t \geq 0 : t \in \bar{\pi} \text{ and } \bar{X}_t \notin \mathcal{O}\}.$$

We let  $\bar{\phi}$  and  $\bar{\phi}^+$  be the functions defined in (1) associated to  $\bar{\pi}$ , and such that

$$0 < \inf_{[0, T] \setminus \bar{\pi}} (\bar{\phi}^+ - \bar{\phi}) \leq \sup_{\mathbb{R}_+} (\bar{\phi}^+ - \bar{\phi}) =: |\bar{\pi}| \leq 1$$

for all  $T > 0$ , while  $\phi = \phi^+$  is just the identity.

We also fix a *distance function*  $\delta : \mathcal{Z} \mapsto \mathbb{R}$  such that  $\delta > 0$  on  $\mathcal{O}$ ,  $\delta = 0$  on  $\partial\mathcal{O}$ , and  $\delta < 0$  on  $\bar{\mathcal{O}}^c$ , and set  $P := \delta(X)$  and  $\bar{P} := \delta(\bar{X})$ .

Throughout this section we assume that the assumptions **(Z)**, **(P)** and **(L)** of Section 2.1 hold for  $(X, \pi, P)$ ,  $(\bar{X}, \bar{\pi}, \bar{P})$ , and  $\delta$  with the same  $(r, L, \kappa)$ . Obviously, the estimate contained in **(Z)** is trivial for  $(X, \pi, P)$  since  $\pi = \mathbb{R}_+$  and  $\phi$  is the identity.

**Theorem 3.1.** *Assume that there exists  $\rho > 0$  such that*

$$\mathbb{E} [|P_{\vartheta} - \bar{P}_{\vartheta}|^2] \leq \rho |\bar{\pi}| \quad \text{for all } \vartheta \in \mathcal{T} \text{ with } \vartheta \leq \theta_0. \quad (12)$$

*Then for all integers  $p \geq 1$  there exists  $c_{(3.1)} = c_{(3.1)}(r, L, d, \kappa, \rho, p) > 0$  and  $\varepsilon_{(3.1)} = \varepsilon_{(3.1)}(r, L, d, \kappa) > 0$  such that*

$$\mathbb{E} [|\theta_0 - \bar{\theta}_0^{\bar{\pi}}|^p] \leq \mathbb{E} \left[ \left[ \mathbb{E}_{\vartheta^{\bar{\pi}}} [|\theta_0 - \bar{\theta}_0^{\bar{\pi}}|^p] \right]^2 \right]^{\frac{1}{2}} \leq c_{(3.1)} |\bar{\pi}|^{\frac{1}{2}}$$

*for  $|\bar{\pi}| \leq \varepsilon_{(3.1)}$  where  $\vartheta^{\bar{\pi}} := \bar{\phi}_{\theta_0}^+ \wedge \bar{\theta}_0^{\bar{\pi}}$ .*

*Proof.* First we observe that

$$\begin{aligned}\mathbb{E}_{\vartheta^{\bar{\pi}}} [|\theta_0 - \bar{\theta}_0^{\bar{\pi}}|^p] &\leq (\Phi_0^p(\bar{\theta}_0^{\bar{\pi}}))^p \quad \text{on } \{\theta_0 \geq \bar{\theta}_0^{\bar{\pi}}\}, \\ \mathbb{E}_{\vartheta^{\bar{\pi}}} [|\bar{\theta}_0^{\bar{\pi}} - \bar{\phi}_{\theta_0}^+\|^p] &\leq (\Phi_0^{p,\pi}(\bar{\phi}_{\theta_0}^+))^p \quad \text{on } \{\theta_0 < \bar{\theta}_0^{\bar{\pi}}\},\end{aligned}$$

where we used the fact that  $\{\theta_0 < \bar{\theta}_0^{\bar{\pi}}\} \subseteq \{\bar{\phi}_{\theta_0}^+ \leq \bar{\theta}_0^{\bar{\pi}}\}$ . We continue with Lemma 2.7 to get

$$\begin{aligned}\mathbb{E}_{\vartheta^{\bar{\pi}}} [|\theta_0 - \bar{\theta}_0^{\bar{\pi}}|^p] &\leq p!L^{p-1}\Phi_0^1(\bar{\theta}_0^{\bar{\pi}}) \quad \text{on } \{\theta_0 \geq \bar{\theta}_0^{\bar{\pi}}\}, \\ \mathbb{E}_{\vartheta^{\bar{\pi}}} [|\bar{\theta}_0^{\bar{\pi}} - \bar{\phi}_{\theta_0}^+\|^p] &\leq p!L^{p-1}\Phi_0^{1,\pi}(\bar{\phi}_{\theta_0}^+) \quad \text{on } \{\theta_0 < \bar{\theta}_0^{\bar{\pi}}\}.\end{aligned}$$

Applying Theorem 2.2 to  $(X, \pi, P)$  and  $\tau = \bar{\theta}_0^{\bar{\pi}}$ , and Theorem 2.3 to  $(\bar{X}, \bar{\pi}, \bar{P})$  and  $\tau = \bar{\phi}_{\theta_0}^+$ , enables us to deduce

$$\mathbb{E}_{\vartheta^{\bar{\pi}}} [|\theta_0 - \bar{\theta}_0^{\bar{\pi}}|^p] \leq p!L^{p-1} \left[ c_{(2.2)}(P_{\bar{\theta}_0^{\bar{\pi}}})_+ + L\mathbf{1}_{\{P_{\bar{\theta}_0^{\bar{\pi}}} > r\}} \right] \quad \text{on } \{\theta_0 \geq \bar{\theta}_0^{\bar{\pi}}\}$$

and

$$\mathbb{E}_{\vartheta^{\bar{\pi}}} [|\bar{\theta}_0^{\bar{\pi}} - \bar{\phi}_{\theta_0}^+\|^p] \leq p!L^{p-1} \left[ \bar{c}_{(2.3)}[(\bar{P}_{\bar{\phi}_{\theta_0}^+})_+ + |\bar{\pi}|^{\frac{1}{2}}] + L\mathbf{1}_{\{\bar{P}_{\bar{\phi}_{\theta_0}^+} > r\}} \right]$$

on  $\{\theta_0 < \bar{\theta}_0^{\bar{\pi}}\}$ . By the above inequalities, and the fact that  $P_{\theta_0} \vee \bar{P}_{\bar{\theta}_0^{\bar{\pi}}} \leq 0$ , we obtain

$$\begin{aligned}&\mathbb{E}_{\vartheta^{\bar{\pi}}} [|\theta_0 - \bar{\theta}_0^{\bar{\pi}}|^p] \\ &\leq p!L^{p-1} \left( c_{(2.2)}|P_{\bar{\theta}_0^{\bar{\pi}}} - \bar{P}_{\bar{\theta}_0^{\bar{\pi}}}| + L\mathbf{1}_{\{P_{\bar{\theta}_0^{\bar{\pi}}} - \bar{P}_{\bar{\theta}_0^{\bar{\pi}}} \geq r\}} \right) \mathbf{1}_{\{\theta_0 \geq \bar{\theta}_0^{\bar{\pi}}\}} \\ &\quad + p!L^{p-1} \left( \bar{c}_{(2.3)} \left[ |\bar{P}_{\bar{\phi}_{\theta_0}^+} - P_{\theta_0}| + |\bar{\pi}|^{\frac{1}{2}} \right] + L\mathbf{1}_{\{\bar{P}_{\bar{\phi}_{\theta_0}^+} - P_{\theta_0} \geq r\}} \right) \mathbf{1}_{\{\theta_0 < \bar{\theta}_0^{\bar{\pi}}\}},\end{aligned}$$

and Chebyshev's inequality implies that

$$\begin{aligned}\mathbb{E} \left[ \left| \mathbb{E}_{\vartheta^{\bar{\pi}}} [|\theta_0 - \bar{\theta}_0^{\bar{\pi}}|^p] \right|^2 \right] &\leq c \left( \mathbb{E} \left[ |P_{\bar{\theta}_0^{\bar{\pi}} \wedge \theta_0} - \bar{P}_{\bar{\theta}_0^{\bar{\pi}} \wedge \theta_0}|^2 + |\bar{P}_{\bar{\phi}_{\theta_0}^+} - P_{\theta_0}|^2 \right] + |\bar{\pi}| \right) \\ &\leq c \mathbb{E} \left[ |P_{\bar{\theta}_0^{\bar{\pi}} \wedge \theta_0} - \bar{P}_{\bar{\theta}_0^{\bar{\pi}} \wedge \theta_0}|^2 + 2|P_{\theta_0} - \bar{P}_{\theta_0}|^2 \right] \\ &\quad + c \mathbb{E} \left[ 2|\bar{P}_{\bar{\phi}_{\theta_0}^+} - \bar{P}_{\theta_0}|^2 + |\bar{\pi}| \right]\end{aligned}$$

for some  $c = c(r, L, d, \kappa, p) > 0$ . In view of the assumption (12) and Lemma 2.11 applied to  $\bar{P}$ , this leads to the desired result.  $\square$

We conclude this section with sufficient conditions ensuring that (12) holds. In the following,  $\|\cdot\|_q$  denotes the  $L^q$ -norm for  $q \geq 1$ .

**Lemma 3.2.** *Assume  $\vartheta, \tau \in \mathcal{T}$  such that  $0 \leq \vartheta \leq \tau$ .*

(a) We have that

$$\|d_{\mathcal{Z}}(X_{\vartheta}, \bar{X}_{\vartheta})\|_2 \leq \inf_{2 < q < \infty} \left( \sum_{k=0}^{\infty} \mathbb{P}[\tau \geq k]^{\frac{q-2}{q}} \left\| \sup_{t \in [k, k+1)} d_{\mathcal{Z}}(X_t, \bar{X}_t) \right\|_q^2 \right)^{\frac{1}{2}}.$$

(b) Assume  $2 < q < \infty$ ,  $\rho > 0$ ,  $0 \leq \alpha < \beta < \infty$ , and  $Q : \{0, 1, 2, \dots\} \rightarrow \mathbb{R}_+$  such that

$$(i) \quad \delta := \frac{q-2}{2q}\beta - \alpha > 0,$$

$$(ii) \quad c^2 := \rho^{\frac{q-2}{q}} \sum_{k=0}^{\infty} [e^{-k\delta} Q(k)]^2 < \infty,$$

$$(iii) \quad \mathbb{P}[\tau \geq k] \leq \rho e^{-\beta k} \text{ for } k = 0, 1, 2, \dots,$$

$$(iv) \quad \left\| \sup_{t \in [k, k+1)} d_{\mathcal{Z}}(X_t, \bar{X}_t) \right\|_q \leq Q(k) e^{\alpha k} |\bar{\pi}|^{\frac{1}{2}} \text{ for } k = 0, 1, 2, \dots$$

Then one has

$$\|d_{\mathcal{Z}}(X_{\vartheta}, \bar{X}_{\vartheta})\|_2 \leq c |\bar{\pi}|^{\frac{1}{2}}.$$

*Proof.* (a) We simply observe that

$$\begin{aligned} \|d_{\mathcal{Z}}(X_{\vartheta}, \bar{X}_{\vartheta})\|_2 &\leq \left( \sum_{k=0}^{\infty} \mathbb{E} \left[ \sup_{t \in [k, k+1)} d_{\mathcal{Z}}(X_t, \bar{X}_t)^2 \mathbf{1}_{\vartheta \in [k, k+1)} \right] \right)^{\frac{1}{2}} \\ &\leq \left( \sum_{k=0}^{\infty} \mathbb{P}[\vartheta \in [k, k+1)]^{\frac{q-2}{q}} \left\| \sup_{t \in [k, k+1)} d_{\mathcal{Z}}(X_t, \bar{X}_t) \right\|_q^2 \right)^{\frac{1}{2}} \\ &\leq \left( \sum_{k=0}^{\infty} \mathbb{P}[\tau \geq k]^{\frac{q-2}{q}} \left\| \sup_{t \in [k, k+1)} d_{\mathcal{Z}}(X_t, \bar{X}_t) \right\|_q^2 \right)^{\frac{1}{2}}. \end{aligned}$$

(b) follows immediately.  $\square$

Combining Theorem 3.1 and Lemma 3.2 gives

**Corollary 3.3.** *Let the conditions of Lemma 3.2(b) be satisfied with  $\tau = \theta_0$  and let  $p \geq 1$  be an integer. Then there exists  $c > 0$ , depending at most on  $(r, L, d, \kappa, p, \alpha, \beta, q, Q, \rho)$ , such that*

$$\mathbb{E} [|\theta_0 - \bar{\theta}_0^{\bar{\pi}}|^p] \leq c |\bar{\pi}|^{\frac{1}{2}} \quad \text{whenever } |\bar{\pi}| \leq \varepsilon \quad (3.1)$$

with  $\varepsilon(3.1) > 0$  taken from Theorem 3.1.

### 3.2 Application to the Euler scheme approximation of the first exit time of a SDE

Now we specialize the discussion to the case where  $\mathcal{Z} = \mathbb{R}^d$  endowed with the usual Euclidean norm  $|\cdot|$  and where  $X$  is the strong solution of the stochastic differential equation

$$X_t = x_0 + \int_0^t \mu(X_s) ds + \int_0^t \sigma(X_s) dW_s$$

for some fixed  $x_0 \in \mathcal{O}$ , where  $(\mu, \sigma) : \mathbb{R}^d \rightarrow (\mathbb{R}^d, \mathbb{R}^{d \times d})$  satisfy

**Assumption 3.4.** *There exists  $0 < L_\mu, L_\sigma \leq L$  such that, for all  $x, y \in \mathbb{R}^d$ ,*

$$|\mu(x) - \mu(y)| \leq L_\mu |x - y|, \quad |\sigma(x) - \sigma(y)| \leq L_\sigma |x - y|,$$

and  $|\mu(x)| + |\sigma(x)| \leq L$ .

Only in Lemma 3.8 and Theorem 3.9 we use the possibly smaller constants  $L_\mu$  and  $L_\sigma$  from the assumption above. We let  $\bar{X}$  be the Euler scheme based on the grid  $\bar{\pi}$ , i.e.

$$\bar{X}_t = x_0 + \int_0^t \mu(\bar{X}_{\bar{\phi}_s}) ds + \int_0^t \sigma(\bar{X}_{\bar{\phi}_s}) dW_s.$$

**Lemma 3.5.** *Under the Assumption 3.4 the process  $\bar{X}$  shares condition **(Z)**.*

*Proof.* Let us fix  $\tau \in \mathcal{T}$ ,  $A \in \mathcal{F}_\tau$  of positive measure,  $T > 0$  and set

$$Y_t := \bar{X}_{\tau+tT} \mathbf{1}_A \quad \text{for } t \in [0, 1].$$

For  $2 < p < \infty$  Assumption 3.4 implies that

$$\mathbb{E}[|Y_t - Y_s|^p] \leq c[T^p + T^{\frac{p}{2}}]|t - s|^{\frac{p}{2}} \mathbb{P}[A]$$

for some  $c = c(L, d, p) > 0$  independent from the choice of  $A \in \mathcal{F}_\tau$ . Fix  $\alpha \in (0, \frac{1}{2} - \frac{1}{p})$ . Then it follows from the continuity of  $Y$  and (the proof of) Kolmogorov's theorem in [RY05, Theorem 2.1, p.26] that

$$\begin{aligned} \mathbb{E} \left[ \mathbf{1}_A \sup_{\tau \leq t \leq \tau+T} |\bar{X}_t - \bar{X}_{\bar{\phi}_t \vee \tau}|^p \right] &\leq \mathbb{E} \left[ \mathbf{1}_A \sup_{|t-s| \leq |\bar{\pi}|/T} |Y_t - Y_s|^p \right] \\ &\leq \frac{|\bar{\pi}|^{p\alpha}}{T^{p\alpha}} \mathbb{E} \left[ \mathbf{1}_A \sup_{|t-s| \leq |\bar{\pi}|/T; s, t \in D} \frac{|Y_t - Y_s|^p}{(|\bar{\pi}|/T)^{p\alpha}} \right] \\ &\leq \frac{|\bar{\pi}|^{p\alpha}}{T^{p\alpha}} \mathbb{E} \left[ \mathbf{1}_A \sup_{0 \leq s < t \leq 1; s, t \in D} \frac{|Y_t - Y_s|^p}{|t - s|^{p\alpha}} \right] \end{aligned}$$



$$\leq \frac{|\bar{\pi}|^{p\alpha}}{T^{p\alpha}} c' [T^p + T^{\frac{p}{2}}] \mathbb{P}[A],$$

where  $c' = c'(c, p, \alpha) > 0$  and  $D \subseteq [0, 1]$  are the dyadic points. Choosing  $p = 6$  and  $\alpha = 1/6 \in (0, \frac{1}{2} - \frac{1}{p}) = (0, \frac{2}{6})$  gives

$$\mathbb{E}_\tau \left[ \sup_{\tau \leq t \leq \tau+T} |\bar{X}_t - \bar{X}_{\bar{\phi}_t \vee \tau}|^p \right] \leq c' [T^5 + T^2] |\bar{\pi}|$$

and

$$\mathbb{P}_\tau \left[ \sup_{\tau \leq t \leq \tau+T} |\bar{X}_t - \bar{X}_{\bar{\phi}_t \vee \tau}| \geq \rho \right] \leq \frac{c' [T^5 + T^2]}{\rho^p} |\bar{\pi}| \quad \text{for } \rho > 0.$$

□

We also impose a uniform non-characteristic boundary condition. We denote by  $D\delta$  and  $D^2\delta$  the gradient (considered as row vector) and the Hessian matrix of  $\delta$ , respectively.

**Assumption 3.6.** *There exists a bounded  $C_b^2$  function  $\delta : \mathbb{R}^d \mapsto \mathbb{R}$  such that  $\delta > 0$  on  $\mathcal{O}$ ,  $\delta = 0$  on  $\partial\mathcal{O}$  and  $\delta < 0$  on  $\bar{\mathcal{O}}^c$ , which satisfies  $|D\delta| \leq 1$  and the non-characteristic boundary condition*

$$|(D\delta)\sigma| \geq 2L^{-1} \quad \text{on } \{|\delta| \leq r\}. \quad (13)$$

Under the above conditions we can verify assumption **(P)** after a possible change of  $r > 0$  and  $L > 0$ :

**Lemma 3.7.** *Let the Assumptions 3.4 and 3.6 hold. Then  $P$  and  $\bar{P}$  satisfy the condition **(P)** for  $r > 0$  small enough and  $L \geq 1$  large enough, independently of  $|\bar{\pi}|$ .*

*Proof.* First we apply Itô's Lemma to obtain that  $d\bar{P}_t = \bar{b}_t dt + \bar{a}_t dW_t$  with

$$\bar{b}_t := D\delta(\bar{X}_t)\mu(\bar{X}_{\bar{\phi}_t}) + \frac{1}{2} \text{Tr}[(\sigma\sigma^\top)(\bar{X}_{\bar{\phi}_t})D^2\delta(\bar{X}_t)] \quad \text{and} \quad \bar{a}_t := D\delta(\bar{X}_t)\sigma(\bar{X}_{\bar{\phi}_t}).$$

Up to an increase of  $L$  in Assumption **(P)** (which potentially leads to a decrease of  $r$  to satisfy  $0 < r < 1/(4L^3)$ ), condition (i) is satisfied because  $\delta, \mu, \sigma, D\delta, D^2\delta$  are bounded. Since  $D\delta$  is bounded by  $L$  and  $\sigma$  is  $L$ -Lipschitz,

$$|D\delta(\bar{X}_t)\sigma(\bar{X}_{\bar{\phi}_t}) - D\delta(\bar{X}_t)\sigma(\bar{X}_t)| \leq L^2 |\bar{X}_{\bar{\phi}_t} - \bar{X}_t|.$$

Consequently,

$$\begin{aligned} |\bar{a}_t| &\geq |D\delta(\bar{X}_t)\sigma(\bar{X}_t)| - |D\delta(\bar{X}_t)\sigma(\bar{X}_{\bar{\phi}_t}) - D\delta(\bar{X}_t)\sigma(\bar{X}_t)| \\ &\geq |D\delta(\bar{X}_t)\sigma(\bar{X}_t)| - L^2 |\bar{X}_t - \bar{X}_{\bar{\phi}_t}|. \end{aligned}$$

For  $|\bar{P}_t| \leq r$ ,  $|\bar{X}_t - \bar{X}_{\bar{\phi}_t}| \leq r$  and  $0 < r < 1/(4L^3)$  this finally gives  $|\bar{a}_t| \geq 1/L$  so that  $\bar{P}$  satisfies **(P-ii)**. The argument for  $P$  is analog.  $\square$

In order to apply Corollary 3.3 it remains to check the conditions of Lemma 3.2. We shall not discuss here the exponential moment condition in Lemma 3.2 (b-iii) for  $\tau = \theta_0$  in full detail, instead of we refer to Remark 3.10 below, Lemma 2.7 and [Fre85, Chapter III, Lemma 3.3] for sufficient conditions: for instance,  $\mathcal{O}$  is bounded in one direction and a uniform ellipticity condition holds in this direction.

Now we show how one can simply establish a bound as in Lemma 3.2 (b-iv) <sup>1</sup>.

**Lemma 3.8.** *Let Assumption 3.4 hold. Then, for all  $4 \leq q < \infty$  there is a  $c > 0$  depending at most on  $(q, L, L_\mu, L_\sigma, d)$  and a non-negative polynomial  $Q_q$  such that*

$$\left\| \sup_{t \in [0, T]} |X_t - \bar{X}_t| \right\|_q \leq c Q_q(T) e^{\alpha T} |\bar{\pi}|^{\frac{1}{2}} \quad \text{with} \quad \alpha := d(6L_\mu + 3qL_\sigma^2).$$

*Proof.* 1. Let  $2 \leq v < \infty$  and set  $\Delta := X - \bar{X}$ . It follows from Itô's Lemma that

$$\begin{aligned} |\Delta_s|^{2v} &= \int_0^s 2v |\Delta_u|^{2v-2} \Delta_u^\top d\Delta_u + \sum_{i=1}^d \int_0^s v |\Delta_u|^{2v-2} d\langle \Delta^i \rangle_u \\ &\quad + \sum_{i,j=1}^d 2v(v-1) \int_0^s \Delta_u^i \Delta_u^j |\Delta_u|^{2v-4} d\langle \Delta^i, \Delta^j \rangle_u. \end{aligned}$$

Under Assumption 3.4 we obtain

$$\begin{aligned} &\mathbb{E} [|\Delta_s|^{2v}] \\ &\leq \int_0^s 2v \mathbb{E} [|\Delta_u|^{2v-2} |\mu(X_u) - \mu(\bar{X}_{\bar{\phi}_u})|] du \\ &\quad + \int_0^s v(1 + 2d(v-1)) \mathbb{E} [|\Delta_u|^{2v-2} |\sigma(X_u) - \sigma(\bar{X}_{\bar{\phi}_u})|^2] du \\ &\leq A \int_0^s \mathbb{E} [|\Delta_u|^{2v-2} (|\Delta_u| + |\bar{X}_u - \bar{X}_{\bar{\phi}_u}|)] du \\ &\quad + B \int_0^s \mathbb{E} [|\Delta_u|^{2v-2} (|\Delta_u|^2 + |\bar{X}_u - \bar{X}_{\bar{\phi}_u}|^2)] du \\ &= [A + B] \int_0^s \mathbb{E} [|\Delta_u|^{2v}] du + A \int_0^s \mathbb{E} [|\Delta_u|^{2v-2} |\bar{X}_u - \bar{X}_{\bar{\phi}_u}|] du \end{aligned}$$

---

<sup>1</sup>The article [Avi07, Theorem A.1] provides a similar result, which (however) involves a  $T^2$  term in the exponential, while we need a linear term.

$$+B \int_0^s \mathbb{E} [|\Delta_u|^{2v-2} |\bar{X}_u - \bar{X}_{\bar{\phi}_u}|^2] du$$

for  $A := 2vL_\mu$  and  $B := 2v(1 + 2d(v-1))L_\sigma^2 \leq 6dv^2L_\sigma^2$ . Exploiting

$$|\Delta_u|^{2v-1} |\bar{X}_u - \bar{X}_{\bar{\phi}_u}| \leq \frac{2v-1}{2v} |\Delta_u|^{2v} + \frac{1}{2v} |\bar{X}_u - \bar{X}_{\bar{\phi}_u}|^{2v}$$

and

$$|\Delta_u|^{2v-2} |\bar{X}_u - \bar{X}_{\bar{\phi}_u}|^2 \leq \frac{v-1}{v} |\Delta_u|^{2v} + \frac{1}{v} |\bar{X}_u - \bar{X}_{\bar{\phi}_u}|^{2v}$$

we arrive at

$$\begin{aligned} & \mathbb{E} [|\Delta_s|^{2v}] \\ & \leq \left[ A + B + A \frac{2v-1}{2v} + B \frac{v-1}{v} \right] \int_0^s \mathbb{E} [|\Delta_u|^{2v}] du \\ & \quad + \left[ \frac{A}{2v} + \frac{B}{v} \right] \int_0^s \mathbb{E} [|\bar{X}_u - \bar{X}_{\bar{\phi}_u}|^{2v}] du \\ & \leq 2[A + B] \int_0^s \mathbb{E} [|\Delta_u|^{2v}] du + \left[ \frac{A}{2v} + \frac{B}{v} \right] \int_0^s \mathbb{E} [|\bar{X}_u - \bar{X}_{\bar{\phi}_u}|^{2v}] du \\ & \leq 12d[vL_\mu + v^2L_\sigma^2] \int_0^s \mathbb{E} [|\Delta_u|^{2v}] du \\ & \quad + 6d[L_\mu + vL_\sigma^2] \int_0^s \mathbb{E} [|\bar{X}_u - \bar{X}_{\bar{\phi}_u}|^{2v}] du. \end{aligned}$$

Exploiting

$$\sup_{u \geq 0} \mathbb{E} [|\bar{X}_u - \bar{X}_{\bar{\phi}_u}|^{2v}] \leq c_v |\bar{\pi}|^v$$

for some constant  $c_v = c(v, L) > 0$ , where we use the boundedness part of Assumption 3.4, we derive

$$\mathbb{E} [|\Delta_s|^{2v}] \leq 12d[vL_\mu + v^2L_\sigma^2] \int_0^s \mathbb{E} [|\Delta_u|^{2v}] du + 6d[L_\mu + vL_\sigma^2] s c_v |\bar{\pi}|^v$$

and, by Gronwall's Lemma,

$$\mathbb{E} [|\Delta_s|^{2v}] \leq 6d[L_\mu + vL_\sigma^2] c_v s e^{s12d[vL_\mu + v^2L_\sigma^2]} |\bar{\pi}|^v.$$

2. Using the Itô decomposition of  $|\Delta|^2$  and the Burkholder-Davis-Gundy and Hölder inequalities, we obtain (for another constant  $c'_v = c'(v, L_\mu, L_\sigma) > 0$ ) that

$$\mathbb{E} \left[ \sup_{0 \leq s \leq T} |\Delta_s|^{2v} \right] \leq c'_v [T^{v-1} + T^{v/2-1}] \int_0^T \mathbb{E} [|\Delta_u|^{2v} + |\bar{X}_u - \bar{X}_{\bar{\phi}_u}|^{2v}] du$$

$$\begin{aligned}
&\leq c'_v [T^v + T^{v/2}] |\bar{\pi}|^v [c_v + 6d[L_\mu + vL_\sigma^2] c_v e^{12dT[vL_\mu + v^2L_\sigma^2]}] \\
&\leq c'_v c_v [T^p + T^{p/2}] [1 + 6d[L_\mu + vL_\sigma^2]] e^{12dT[vL_\mu + v^2L_\sigma^2]} |\bar{\pi}|^v.
\end{aligned}$$

Consequently, for  $q \geq 4$ ,

$$\left\| \sup_{0 \leq s \leq T} |\Delta_s| \right\|_q \leq C(q, L, L_\mu, L_\sigma, d) Q_q(T) e^{Td[6L_\mu + 3qL_\sigma^2]} |\bar{\pi}|^{\frac{1}{2}}.$$

□

Using Lemmas 3.2, 3.5 and 3.7 and Corollary 3.3, we can summarize the results of this section as follows:

**Theorem 3.9.** *Let the Assumptions 3.4 and 3.6 hold and assume that*

$$\mathbb{E}_\tau [|\bar{\theta}_0^\pi(\tau) - \tau| + |\theta_0(\tau) - \tau|] \leq L \quad \text{for all } \tau \in \mathcal{T}.$$

Assume that there are  $\rho > 0$ ,  $4 \leq q < \infty$ , and  $\beta > \frac{2qd}{q-2}(6L_\mu + 3qL_\sigma^2)$  such that

$$\mathbb{P}[\theta_0 \geq k] \leq \rho e^{-\beta k} \quad \text{for all } k = 0, 1, 2, \dots$$

Then there exist  $c, \varepsilon > 0$  and, for any integer  $p \geq 1$ , a constant  $c_p > 0$  such that, for  $|\bar{\pi}| \leq \varepsilon$ ,

$$\mathbb{E} [|\theta_0 - \bar{\theta}_0^\pi|^p] \leq c_p |\bar{\pi}|^{\frac{1}{2}} \quad \text{and} \quad \left( \mathbb{E} [ |X_{\theta_0} - \bar{X}_{\bar{\theta}_0^\pi}|^2 ] \right)^{\frac{1}{2}} \leq c |\bar{\pi}|^{\frac{1}{4}}.$$

**Remark 3.10.** (a) One key assumption of the above theorem is that  $\beta$  is sufficiently large. For example, in the case  $\mu = 0$  this can be realized by shrinking an appropriate open set  $\mathcal{O}$  so that the modulus of the first eigenvalue of the corresponding elliptic operator (under Dirichlet boundary conditions) is large enough (see [Fre85, p. 203]).

(b) We give an example how to check  $\mathbb{E}_\tau [|\bar{\theta}_0^\pi(\tau) - \tau|] \leq L$  for all  $\tau \in \mathcal{T}$ . By **(L'')** this can be reduced to find  $\alpha \in (0, 1)$  and  $c > 0$  with

$$\mathbb{P}_\tau [\bar{\theta}_0^\pi(\tau) - \tau \geq c] \leq \alpha \quad \text{for all } \tau \in \mathcal{T}. \tag{14}$$

Because for  $c > 1$  one has

$$\mathbb{P}_\tau [\bar{\theta}_0^\pi(\tau) - \tau \geq c] \leq \mathbb{E}_\tau [\mathbb{P}_{\bar{\phi}_\tau^+} [\bar{\theta}_0^\pi(\bar{\phi}_\tau^+) - \bar{\phi}_\tau^+ \geq c - 1]],$$

it is sufficient to check (14) for  $\tau \in \mathcal{T}^\pi$ . Let  $B_R$  be the open ball of radius  $R > 0$  centered at zero and  $\mathcal{O}(R) := \mathcal{O} + B_R$ . The set  $\mathcal{O}(R)$  is open and we assume that for all  $R > 0$  we have  $\mathcal{O}(R) \subsetneq \overline{\mathcal{O}(R)} \subsetneq \mathbb{R}^d$ . If  $(X_t^x)_{t \geq 0}$  is the diffusion started in  $x \in \mathbb{R}^d$ , then we let  $\theta_0^x(R) := \inf\{t \geq$

$0 : X_t^x \notin \mathcal{O}(R)\}$  for  $x \in \mathcal{O}$ . We assume the existence of  $R, \rho, \beta > 0$  and  $4 \leq q < \infty$  such that

$$\sup_{x \in \mathcal{O}} \mathbb{P} [\theta_0^x(R) \geq T] \leq \rho e^{-\beta T} \quad \text{for all } T \geq 0,$$

cf. [Fre85, Chapter 3] and  $\beta > \frac{2qd}{q-2}(6L_\mu + 3qL_\sigma^2)$ . Given  $\tau \in \mathcal{T}^{\bar{\pi}}$ , we let

$$\check{X}_t = x_0 + \int_0^t \check{\mu}_s ds + \int_0^t \check{\sigma}_s dW_s$$

with  $\check{\mu}_t := \mathbf{1}_{(0,\tau]}(t)\mu(\check{X}_{\check{\phi}_t}) + \mathbf{1}_{(\tau,\infty)}(t)\mu(\check{X}_t)$  and with the corresponding definition for  $\check{\sigma}$ . Let  $\check{\theta}_0(\tau, R) := \inf\{t \geq \tau : \check{X}_t \notin \mathcal{O}(R)\}$ . For  $c \geq 2$  and  $\tau \in \mathcal{T}^{\bar{\pi}}$  with  $|\bar{\pi}| \leq \bar{\varepsilon}$ , where  $\bar{\varepsilon} \in (0, 1]$  is chosen at the end of the computation, we get from Lemma 3.2 for a set  $A \in \mathcal{F}_\tau$  of positive measure with  $A \subseteq \{\tau = t\} \cap \{\bar{X}_t^{\bar{\pi}} \in \mathcal{O}\}$  (note that  $\tau$  takes only countable many values) that

$$\begin{aligned} & \mathbb{P} [A \cap \{\bar{\theta}_0^{\bar{\pi}}(t) - \tau \geq c\}] \\ &= \mathbb{P} [A \cap \{\bar{\theta}_0^{\bar{\pi}}(t) - t \geq c\}] \\ &\leq \mathbb{P} [A \cap \{\check{\theta}_0(t, R) - t \geq c/2\}] \\ &\quad + \mathbb{P} [A \cap \{|\check{X}_{\check{\theta}_0(t, R)} - \bar{X}_{\check{\theta}_0(t, R)}| \geq R/2\}] \\ &\quad + \mathbb{P} [A \cap \{|\bar{X}_{\check{\phi}_{\check{\theta}_0(t, R)}^+} - \bar{X}_{\check{\theta}_0(t, R)}| \geq R/2\}] \\ &\leq \mathbb{P} [A] \left[ \sup_{x \in \mathcal{O}} \mathbb{P} [\theta_0^x(R) \geq c/2] \right. \\ &\quad \left. + \sup_{x \in \mathcal{O}} \sup_{|\bar{\pi}| \leq \bar{\varepsilon}} \mathbb{P} [ |X_{\theta_0^x(R)}^x - \bar{X}_{\theta_0^x(R)}^{x, \bar{\pi}}| \geq R/2 ] + \frac{c'(L)}{R^2} |\bar{\pi}| \right] \\ &\leq \mathbb{P} [A] \left[ \frac{c'(L)}{R^2} |\bar{\pi}| + \rho e^{-\beta c/2} \right. \\ &\quad \left. + \sup_{x \in \mathcal{O}} \sup_{|\bar{\pi}| \leq \bar{\varepsilon}} \frac{4}{R^2} \rho^{\frac{q-2}{q}} \sum_{k=0}^{\infty} e^{-\beta k \frac{q-2}{q}} \left\| \sup_{t \in [k, k+1)} |X_t^x - \bar{X}_t^{x, \bar{\pi}}| \right\|_q^2 \right], \end{aligned}$$

where  $\bar{X}^{x, \bar{\pi}}$  is the Euler scheme for  $X^x$  based on the net  $\bar{\pi}$ . First we arrange the second term to be small by taking an appropriate  $c \geq 2$ . The remaining terms can be controlled by choosing  $\bar{\varepsilon} > 0$  small enough, where we use Lemma 3.8.

**Remark 3.11.** The main aim of [BM09] was to study the strong error made when approximating the solution of a BSDE whose terminal condition is of the form  $g(X_{\theta_0 \wedge T})$ , for some Lipschitz map  $g$  and  $T > 0$ , by a backward Euler

scheme; see [BM09] for the corresponding definitions and references. Because of  $|\theta_0 \wedge T - \bar{\theta}_0^\pi \wedge T| \leq |\theta_0 - \bar{\theta}_0^\pi|$  the result of Theorem 3.9 complements [BM09, Theorem 3.1] in which the upper-bound takes the form  $O_{|\bar{\pi}| \rightarrow 0}(|\bar{\pi}|^{\frac{1}{2}-\varepsilon})$  for all  $0 < \varepsilon < 1/2$ . Moreover, the upper-bound of the second inequality of [BM09, Theorem 3.3] is of the form  $O_{|\bar{\pi}| \rightarrow 0}(|\bar{\pi}|^{\frac{1}{4}-\varepsilon})$  for all  $0 < \varepsilon < 1/4$ . This comes from the control they obtained on the exit time of their Theorem 3.1. With Theorem 3.9 of this paper it can be reduced to  $O_{|\bar{\pi}| \rightarrow 0}(|\bar{\pi}|^{\frac{1}{4}})$ . Our results open the door to the study of backward Euler type approximations of BSDEs with a terminal condition of the form  $g(X_{\theta_0})$ , i.e. there is no finite time horizon  $T > 0$ . This will however require to study at first the regularity of the solution of the BSDE, which is beyond the scope of this paper.

## References

- [Avi07] R. Avikainen. Convergence rates for approximations of functionals of sdes. *arXiv preprint arXiv:0712.3635*, 2007.
- [Bal95] P. Baldi. Exact asymptotics for the probability of exit from a domain and applications to simulation. *The Annals of Probability*, 23(4):1644–1670, 1995.
- [BC02] P. Baldi and L. Caramellino. Asymptotics of hitting probabilities for general one-dimensional diffusions. *Annals of Applied Probability*, 12:1071–1095, 2002.
- [Bic10] K. Bichteler. *Stochastic Integration with Jumps*. Cambridge University Press, 2010.
- [BM09] B. Bouchard and S. Menozzi. Strong approximations of BSDEs in a domain. *Bernoulli*, 15(4):1117–1147, 2009.
- [BP06] F.M. Buchmann and W.P. Petersen. An exit probability approach to solving high dimensional Dirichlet problems. *SIAM J. Sci. Comput.*, 28(3):1153–1166, 2006.
- [DL06] M. Deaconu and A. Lejay. A random walk on rectangles algorithm. *Methodology And Computing In Applied Probability*, 8(1):135–151, 2006.
- [Fau92] O. Faure. *Simulation du mouvement brownien et des diffusions*. PhD thesis, 1992.

- [Fre85] M. Freidlin. *Functional integration and partial differential equations*. Annals of Mathematics Studies - Princeton University Press, 1985.
- [Gil08] M.B. Giles. Multilevel Monte Carlo path simulation. *Operation Research*, 56:607–617, 2008.
- [GM05] E. Gobet and S. Maire. Sequential control variates for functionals of Markov processes. *SIAM Journal on Numerical Analysis*, 43(3):1256–1275, 2005.
- [GM07] E. Gobet and S. Menozzi. Discrete sampling of functionals of Itô processes. *Séminaire de probabilités XL - Lecture Notes in Mathematics 1899 Springer Verlag*, pages 355–374, 2007.
- [GM10] E. Gobet and S. Menozzi. Stopped diffusion processes: boundary corrections and overshoot. *Stochastic Processes and Their Applications*, 120:130–162, 2010.
- [Gob00] E. Gobet. Euler schemes for the weak approximation of killed diffusion. *Stochastic Processes and their Applications*, 87:167–197, 2000.
- [Gob01] E. Gobet. Euler schemes and half-space approximation for the simulation of diffusions in a domain. *ESAIM: Probability and Statistics*, 5:261–297, 2001.
- [Hei01] S. Heinrich. Multilevel monte carlo methods. In *Large-scale scientific computing*, pages 58–67. Springer, 2001.
- [JL05] K.M. Jansons and G.D. Lythe. Multidimensional exponential timestepping with boundary test. *SIAM Journal on Scientific Computing*, 27(3):793–808, 2005.
- [KP95] P.E. Kloeden and E. Platen. *Numerical solution of stochastic differential equations*. Springer Verlag, 1995.
- [Mar55] G. Maruyama. Continuous Markov processes and stochastic equations. *Rendiconti del Circolo Matematico di Palermo, Serie II*, 4:48–90, 1955.
- [Mil96] G.N. Milstein. The simulation of phase trajectories of a diffusion process in a bounded domain. *Stochastics Stochastics Rep.*, 56(1-2):103–125, 1996.

- [Mil97] G.N. Milstein. Weak approximation of a diffusion process in a bounded domain. *Stochastics and Stochastics Reports*, 62:147–200, 1997.
- [Mil98] G.N. Milstein. On the mean-square approximation of a diffusion process in a bounded domain. *Stochastics Stochastics Rep.*, 64(3-4):211–233, 1998.
- [MT99] G.N. Milstein and M.V. Tretyakov. Simulation of a space-time bounded diffusion. *Ann. Appl. Probab.*, 9(3):732–779, 1999.
- [Mul56] M.E. Muller. Some continuous Monte Carlo methods for the Dirichlet problem. *Ann. Math. Statist.*, 27(569–589), 1956.
- [RY05] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*. Springer, 2005.
- [ZLD10] S. Zein, A. Lejay, and M. Deaconu. An efficient algorithm to simulate a Brownian motion over irregular domains. *Commun. Comput. Phys.*, 8(4):901–916, 2010.