On the survival of a class of subcritical branching processes in random environment

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Abstract

Let Z_n be the number of individuals in a subcritical BPRE evolving in the environment generated by iid probability distributions. Let X be the logarithm of the expected offspring size per individual given the environment. Assuming that the density of X has the form

$$p_X(x) = x^{-\beta-1} l_0(x) e^{-\rho x}$$

for some $\beta > 2$, a slowly varying function $l_0(x)$ and $\rho \in (0,1)$, we find the asymptotic survival probability $\mathbb{P}(Z_n > 0)$ as $n \to \infty$ and prove a Yaglom type conditional limit theorem for the process. The survival probability decreases exponentially with an additional polynomial term related to the tail of X. The proof relies on a fine study of a random walk (with negative drift and heavy tails) conditioned to stay positive until time n and to have a small positive value at time n, with $n \to \infty$.

1 Introduction and main results

We consider the model of branching processes in random environment introduced by Smith and Wilkinson [15]. The formal definition of these processes looks as follows. Let $\mathfrak{N}$ be the space of probability measures on $\mathbb{N}_0 = \{0, 1, 2, ...\}$. Equipped with the metric of total variation $\mathfrak{N}$ becomes a Polish space. Let $\mathfrak{e}$ be a random variable taking values in $\mathfrak{N}$. An infinite sequence $\mathcal{E} = (\mathfrak{e}_1, \mathfrak{e}_2, \ldots)$ of i.i.d. copies of $\mathfrak{e}$ is said to form a *random environment*. A sequence of $\mathbb{N}_0$-valued random variables $Z_0, Z_1, \ldots$ is called a *branching process in the random environment* $\mathcal{E}$, if Z_0 is independent of $\mathcal{E}$ and, given $\mathcal{E}$, the process $Z = (Z_0, Z_1, \ldots)$ is a Markov chain with

$$\mathcal{L}\left(Z_n \mid Z_{n-1} = z_{n-1}, \mathcal{E} = (e_1, e_2, \ldots)\right) \;=\; \mathcal{L}\left(\xi_{n1} + \cdots + \xi_{nz_{n-1}}\right) \qquad (1)$$

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for every $n \geq 1$, $z_{n-1} \in \mathbb{N}_0$ and $e_1, e_2, \ldots \in \mathfrak{N}$, where $\xi_{n1}, \xi_{n2}, \ldots$ are i.i.d. random variables with distribution $\mathfrak{e}_n$. Thus,

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_{ni} \tag{2}$$

and, given the environment, Z is an ordinary inhomogeneous Galton-Watson process. We will denote the corresponding probability measure and expectation on the underlying probability space by $\mathbb{P}$ and $\mathbb{E}$, respectively.

Let

$$X = \log\left(\sum_{k\geq 0} k\mathfrak{e}\left(\{k\}\right)\right), \qquad X_n = \log\left(\sum_{k\geq 0} k\mathfrak{e}_n\left(\{k\}\right)\right), \; n = 1, 2, ...,$$

be the logarithms of the expected offspring size per individual in the environments and

$$S_0 = 0, \; S_n = X_1 + \cdots + X_n, n \geq 1,$$

be their partial sums.

This paper deals with the subcritical branching processes in random environment, i.e., in the sequel we always assume that

$$\mathbb{E}\left[X\right] = -b < 0. \tag{3}$$

The subcritical branching processes in random environment admit an additional classification, which is based on the properties of the moment generating function

$$\varphi(t) = \mathbb{E}\left[e^{tX}\right] = \mathbb{E}\left[\left(\sum_{k\geq 0} k\mathfrak{e}\left(\{k\}\right)\right)^t\right], \quad t \geq 0.$$

Clearly, $\varphi'(0) = \mathbb{E}\left[X\right]$. Let

$$\rho_+ = \sup\left\{t \geq 0 : \varphi(t) < \infty\right\}$$

and ρ_{min} be the point where $\varphi(t)$ attains its minimal value on the interval $[0, \rho_+ \wedge 1]$. Then a subcritical branching process in random environment is called

weakly subcritical if $\rho_{min} \in (0, \rho_+ \wedge 1)$,
intermediately subcritical if $\rho_{min} = \rho_+ \wedge 1 > 0$ and $\varphi'(\rho_{min}) = 0$,
strongly subcritical if $\rho_{min} = \rho_+ \wedge 1$ and $\varphi'(\rho_{min}) < 0$.

Note that this classification is slightly different from that given in [8]. Weakly subcritical and intermediately subcritical branching processes have been studied in [13, 1, 2, 3] in detail. Let us recall that $\varphi'(\rho_+ \wedge 1) > 0$ for the weakly subcritical case.

The strongly subcritical case is also well studied for the case $\rho_+ \geq 1$, i.e., if $\rho_{min} = \rho_+ \wedge 1 = 1$ and $\varphi'(1) < 0$. In particular, it was shown in [13] and refined in [5] that if $\varphi'(1) = \mathbb{E}\left[Xe^X\right] < 0$ and $\mathbb{E}\left[Z_1 \log^+ Z_1\right] < \infty$ then, as $n \to \infty$

$$\mathbb{P}\left(Z_n > 0\right) \sim K\left(\mathbb{E}\left[\xi\right]\right)^n, \, K > 0, \tag{4}$$

and, in addition,

$$\lim_{n\to\infty} \mathbb{E}\left[s^{Z_n} | Z_n > 0\right] = \Psi(s), \tag{5}$$

where $\Psi(s)$ is the probability generating function of a proper nondegenerate random variable on $\mathbb{Z}_+$. This statement is actually an extension of the classical result for the ordinary subcritical Galton-Watson branching processes.

Our main concern in this paper is the strongly subcritical branching processes in random environment with $\rho_+ < 1$. More precisely, we assume that the following condition is valid:

Hypothesis A. The distribution of X has density

$$p_X\left(x\right) = \frac{l_0(x)}{x^{\beta+1}} e^{-\rho x}, \tag{6}$$

where $l_0(x)$ is a function slowly varying at infinity, $\beta > 2$, $\rho \in (0,1)$ and, in addition,

$$\varphi'(\rho) = \mathbb{E}\left[Xe^{\rho X}\right] < 0. \tag{7}$$

This assumption can be relaxed by assuming that $p_X(x)$ is the density of X for x large enough, or that the tail distribution $\mathbb{P}(X \in [x, x+h)) \sim \int_x^{x+h} p_X(y)dy$ for $x \to \infty$ (uniformly with respect to $h \leq 1$).
Clearly, $\rho = \rho_+ < 1$ under Hypothesis A. Observe that the case $\rho = \rho_+ = 0$ not included in Hypothesis A has been studied in [16] and yields a new type of the asymptotic behavior of subcritical branching processes in random environment. Namely, it was established that, as $n \to \infty$

$$\mathbb{P}\left(Z_n > 0\right) \sim K\mathbb{P}\left(X > nb\right) = K\frac{l_0(nb)}{\left(nb\right)^\beta}, \, K > 0, \tag{8}$$

so that the survival probability decays with a polynomial rate only. Moreover, for any $\varepsilon > 0$, some constant $\sigma > 0$ and any $x \in \mathbb{R}$

$$\mathbb{P}\left(\frac{\log Z_n - \log Z_{[n\varepsilon]} + n\left(1-\varepsilon\right)b}{\sigma\sqrt{n}} \leq x \,|\, Z_n > 0\right) = \mathbf{P}\left(B_1 - B_\varepsilon \leq x\right)$$

where B_t is a standard Brownian motion. Therefore, given the survival of the population up to time n, the number of individuals in the process at this moment tends to infinity as $n \to \infty$ that is not the case for other types of subcritical processes in random environment.

The goal of the paper is to investigate the asymptotic behavior of the survival probability of the process meeting Hypothesis A and to prove a Yaglom-type conditional limit theorem for the distribution of the number of individuals. To this aim we additionally assume that the sequence of conditional probability measures

$$\mathbb{P}^{[x]}(\cdot) = \mathbb{P}(\cdot \mid X = x)$$

is well defined for $x \to \infty$ under Hypothesis A. In Section 2, we provide natural examples when this assumption and Hypothesis B below are valid.

Denote by $\mathfrak{L} = \{\mathcal{L}\}$ the set of all proper probability measures $\mathcal{L}(\cdot)$ of non-negative random variables. Our next condition concerns the behavior of the measures $\mathbb{P}^{[x]}$ as $x \to \infty$:

Hypothesis B. There exists a probability measure $\mathbb{P}^*$ on $\mathfrak{L}$ such that, as $x \to \infty$,

$$\mathbb{P}^{[x]} \Longrightarrow \mathbb{P}^*$$

where the symbol $\Longrightarrow$ stands for the weak convergence of measures.

Now setting

$$a = -\frac{\varphi'(\rho)}{\varphi(\rho)} > 0,$$

we are ready to formulate the first main result of the paper.

Theorem 1 *If*

$$\mathbb{E}[-\log(1 - \mathfrak{e}(\{0\}))] < \infty, \quad \mathbb{E}\left[e^{-X} \sum_{k \geq 1} \mathfrak{e}(\{k\})\, k \log k\right] < \infty \tag{9}$$

and Hypotheses A and B are valid, then there exists a constant $C_0 > 0$ such that, as $n \to \infty$

$$\mathbb{P}(Z_n > 0) \sim C_0 \rho \varphi^{n-1}(\rho)\, e^{an\rho} \mathbb{P}(X > an) \sim C_0 \rho \varphi^{n-1}(\rho) \frac{l_0(n)}{(an)^{\beta+1}}. \tag{10}$$

The explicit form of C_0 can be found in Corollary 25. Let us now explain this asymptotic behavior and give at the same time an idea of the proof. Then we will provide some classes of processes which satisfy our assumptions.

For the proof, we introduce in the next section a new probability measure $\mathbf{P}$. Under this new probability measure, S has drift $-a < 0$ and the heavy tail distribution of its increments with polynomial decay β.Adding that $\mathbf{E}[\exp(\rho X)] = \varphi(\rho)$, we will get the survival probability as

$$\varphi^n(\rho)\, \mathbf{E}\left[e^{-\rho S_n} \mathbf{P}(Z_n > 0 | \mathfrak{e})\right]) \approx const \times \varphi^n(\rho)\, \mathbf{P}(L_n \geq 0, S_n \leq N)$$

where L_n is the minimum of the random walk up to time n and N is (large but) fixed. We then study this random walk with heavy tail to show that

$$\mathbf{P}(L_n \geq 0, S_n \leq N) \approx const \times \mathbf{P}(X_1 \in [an - M\sqrt{n}, an + M\sqrt{n}], S_n \in [0, 1])$$

for n large enough and conclude using the central limit theorem. The proof relies on a fine study of the random walk conditioned on $\{L_n \geq 0, S_n \leq N\}$, which results in the decomposition Theorem 9.

Our second main result is a Yaglom-type conditional limit theorem.

Theorem 2 *Under the conditions of Theorem 1,*

$$\lim_{n\to\infty} \mathbb{E}\left[s^{Z_n} | Z_n > 0\right] = \Omega(s),$$

where $\Omega(s)$ *is the probability generating function of a proper nondegenerate random variable on* $\mathbb{Z}_+$.

We see that, contrary to the case $\rho_{min} = \rho_+ \wedge 1 = 0$ this Yaglom-type limit theorem has the same form as for the ordinary Galton-Watson subcritical processes.

2 Examples

We provide here some examples meeting the conditions of Theorem 1. Thus, we assume that Hypothesis A is valid and we focus on the existence and convergence of $\mathbb{P}^{[x]}$. Let us first deal with the existence of random reproduction laws $\mathfrak{e}$ for which the conditional probability

$$\mathbb{P}^{[x]}(\cdot) = \mathbb{P}(\cdot \mid X = x)$$

is well defined.

Example 0. Assume that the environment $\mathfrak{e}$ takes its values in some set $\mathcal{M}$ of probability measures such that for all $\mu, \nu \in \mathcal{M}$

$$\sum_{k\geq 0} k\mu(k) < \sum_{k\geq 0} k\nu(k) \Rightarrow \mu \leq \nu,$$

where $\mu \leq \nu$ means that $\forall l \in \mathbb{N}, \mu[l,\infty) \leq \nu[l,\infty)$. We note that Hypothesis A ensures that $\mathbb{P}(.|X \in [x, x+\epsilon))$ is well defined. Then, for every $F : \mathcal{M} \to \mathbb{R}^+$ which is non decreasing in the sense that $\mu \leq \nu$ implies $F(\mu) \leq F(\nu)$, we get that the functional

$$\mathbb{E}(F(\mathfrak{e})|X \in [x, x+\epsilon))$$

decreases to some limit $p(F)$ as $\epsilon \to 0$. Thus, writing $F_{l,y}(\mu) = 1$ if $\mu[l,\infty) \geq y$ and 0 otherwise, we can define $\mathbb{P}^{[x]}$ via

$$\mathbb{P}^{[x]}(\mathfrak{e}[l,\infty) \geq y) = p(F_{l,y})$$

to get the expected conditional probability.

Let us now focus on Hypothesis B.

Example 1. Let $f(s;\mathfrak{e}) = \sum_{k\geq 0} \mathfrak{e}(\{k\})\, s^k$ be the (random) probability generating function corresponding to the random measure $\mathfrak{e} \in \mathfrak{N}$ and let (with a slight abuse of notation) $\xi = \xi(\mathfrak{e}) \geq 0$ be the integer-valued random variable with probability generating function $f(s;\mathfrak{e})$, i.e., $f(s;\mathfrak{e}) = E\left[s^{\xi(\mathfrak{e})}\right]$.

It is not difficult to understand that if $\mathbb{E}[\log f'(1;\mathfrak{e})] < 0$ and there exists a deterministic function $g(\lambda), \lambda \geq 0$, with $g(\lambda) < 1, \lambda > 0$, and $g(0) = 1$, such that, for every $\varepsilon > 0$

$$\lim_{y\to\infty} \mathbb{P}\left(\mathfrak{e} : \sup_{0\leq\lambda<\infty} \left| f\left(e^{-\lambda/y};\mathfrak{e}\right) - g(\lambda)\right| > \varepsilon \;\middle|\; f'(1;\mathfrak{e}) = y = e^x\right) = 0,$$

then Hypothesis B is satisfied for the respective subcritical branching process.

We now give two more natural examples for which Hypothesis B holds true and note that mixing two classes describing in these examples would provide a more general family which satisfies Hypothesis B.

Let $\mathfrak{N}_f \subset \mathfrak{N}$ be the set of probability measures on $\mathbb{N}_0$ such that

$$e = e(t,y) \in \mathfrak{N}_f \iff f(s;e) = 1 - t + \frac{t}{1 + yt^{-1}(1-s)}$$

where $t \in (0,1]$ and $y \in (0,\infty)$, and let $\mathfrak{L}_g \subset \mathfrak{L}$ be the set of probability measures $L = L(t,y)$ on $[0,\infty)$ such that

$$L = L(t,y) \in \mathfrak{N}_f \iff g(t,\lambda) = \int e^{-\lambda y} L(t,dy) = 1 - t + \frac{t^2}{t+\lambda}.$$

Let, further, $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \subset (0,1] \times (0,\infty)$ be a Borel set. We write

$$e = e(t,y) \in T(\mathcal{B}) \subseteq \mathfrak{N}_f \text{ if } (t,y) \in \mathcal{B}$$

and

$$L = L(t,y) \in T(\mathcal{B}_1) \subseteq \mathfrak{L}_g \text{ if } t \in \mathcal{B}_1.$$

Let (θ,ζ) be a pair of random variables with values in $(0,1] \times (0,\infty)$ such that for a measure $\mathbf{P}^*(\cdot)$ with support on $(0,1]$ and any Borel set $\mathcal{B}_1 \subseteq (0,1]$

$$\lim_{x\to\infty} \mathbf{P}(\theta \in \mathcal{B}_1 | \zeta = x) = \mathbf{P}^*(\theta \in \mathcal{B}_1)$$

exists.

Example 2. Assume that the support of the probability measure $\mathbb{P}$ is concentrated on the set $\mathfrak{N}_f$ only and the random environment $\mathfrak{e}$ is specified by the relation

$$\mathfrak{e} = e(\theta,\zeta) \iff f(s;\mathfrak{e}) = 1 - \theta + \frac{\theta^2}{\theta + \zeta(1-s)}.$$

Clearly, $\log f'(1;\mathfrak{e}) = \log\zeta$. Thus,

$$\mathbb{P}\left(e\left(\theta,\zeta\right)\in T\left(\mathcal{B}\right)\right)=\mathbb{P}\left(f\left(s;\mathfrak{e}\right):\left(\theta,\zeta\right)\in\mathcal{B}\right)$$

and if $\mathcal{B}=\mathcal{B}_1\times\{x\}$ then

$$\begin{aligned}\lim_{x\to\infty}\mathbb{P}\left(f\left(e^{-\lambda\zeta^{-1}};\mathfrak{e}\right):\left(\theta,\zeta\right)\in\mathcal{B}|\zeta=e^x\right)&=\mathbf{P}^*\left(\theta\in\mathcal{B}_1\right)\\&=\mathbb{P}^*\left(g(\lambda;\theta):\theta\in\mathcal{B}_1\right)=\mathbb{P}^*\left(L(\theta;y)\in T\left(\mathcal{B}_1\right)\right).\end{aligned}$$

Note that if $\mathbf{P}\left(\theta=1|\zeta=x\right)=1$ for all sufficiently large x we get a particular case of Example 1.

Example 3. If the support of the environment is concentrated on such probability measures $\mathfrak{e}\in\mathfrak{N}$ that, for any $\varepsilon>0$

$$\lim_{y\to\infty}\mathbb{P}\left(\mathfrak{e}:\left|\frac{\xi(\mathfrak{e})}{f'(1;\mathfrak{e})}-1\right|>\varepsilon\;\Big|\;f'(1;\mathfrak{e})=e^X=y\right)=0 \tag{11}$$

and the density of the random variable $X=\log f'(1;\mathfrak{e})$ is positive for all sufficiently large x then $g(\lambda)=e^{-\lambda}$. Condition (11) is satisfied if, for instance,

$$\lim_{y\to\infty}\mathbb{P}\left(\mathfrak{e}:\frac{Var\xi(\mathfrak{e})}{\left(f'(1;\mathfrak{e})\right)^2}>\varepsilon\;\Big|\;f'(1;\mathfrak{e})=y\right)=0.$$

3 Preliminaries

3.1 Change of probability measure

A classical technique of studying subcritical branching processes in random environment is similar to that one used to investigate standard random walks satisfying the Cramer condition. Namely, denote by $\mathcal{F}_n$ the $\sigma-$algebra generated by the tuple $(\mathfrak{e}_1,\mathfrak{e}_2,...,\mathfrak{e}_n;Z_0,Z_1,...,Z_n)$ and let $\mathbb{P}^{(n)}$ be the restriction of $\mathbb{P}$ to $\mathcal{F}_n$. Setting

$$m=\varphi\left(\rho\right)=\mathbb{E}\left[e^{\rho X}\right],$$

we introduce another probability measure $\mathbf{P}$ by the following change of measure

$$d\mathbf{P}^{(n)}=m^{-n}e^{\rho S_n}d\mathbb{P}^{(n)},\ n=1,2,... \tag{12}$$

or, what is the same, for any random variable Y_n measurable with respect to $\mathcal{F}_n$ we let

$$\mathbf{E}\left[Y_n\right]=m^{-n}\mathbb{E}\left[Y_ne^{\rho S_n}\right]. \tag{13}$$

By (7),

$$\mathbf{E}\left[X\right]=m^{-1}\mathbb{E}\left[Xe^{\rho X}\right]=\varphi'\left(\rho\right)/\varphi\left(\rho\right)=-a<0. \tag{14}$$

Applying a Tauberian theorem we get

$$\begin{aligned} A(x) &= \mathbf{P}\left(X > x\right) = \frac{\mathbb{E}\left[I\left\{X > x\right\} e^{\rho X}\right]}{m} = \frac{1}{m}\int_x^\infty e^{\rho y} p_X(y) dy \\ &= \frac{1}{m}\int_x^\infty \frac{l_0(y)dy}{y^{\beta+1}} \sim \frac{1}{m\beta}\frac{l_0(x)}{x^\beta} = \frac{l(x)}{x^\beta}, \end{aligned} \tag{15}$$

where $l(x)$ is a function slowly varying at infinity. Thus, the random variable X under the measure $\mathbf{P}$ does not satisfy the Cramer condition and has finite variance.

Note that Hypothesis A implies the following property for the distribution function $A(x)$: for any fixed $\Delta > 0$

$$A(x+\Delta) - A(x) = -\frac{\Delta\beta A(x)}{x}(1 + o(1)) \tag{16}$$

as $x \to \infty$.

Let $\mathbf{\Phi} = \{\Phi\}$ be the metric space of the Laplace transforms $\Phi(\lambda) = \int_0^\infty e^{-\lambda u}\mathcal{L}(du)$, $\lambda \in [0, \infty)$, of the laws from $\mathfrak{L}$ endowed with the metric

$$d(\Phi_1, \Phi_2) = \sup_{2^{-1} \le \lambda \le 2} |\Phi_1(\lambda) - \Phi_2(\lambda)|.$$

Since the Laplace transform of the distribution of a nonnegative random variable is completely determined by its values on any interval of the positive half-line, convergence $\Phi_n \to \Phi$ as $n \to \infty$ in metric d is equivalent to weak convergence $\mathcal{L}_n \overset{w}{\to} \mathcal{L}$ of the respective probability measures.
From now on, to avoid confusions we agree to use P and E for the symbols of probability and expectation in the case when the respective distributions are not associated with the measures $\mathbb{P}$ or $\mathbf{P}$.
Let $\mathfrak{F}= \{f(s)\}$ be the set of all probability generating functions of integer-valued random variables $\eta \ge 0$, i.e. $f(s) = E\left[s^\eta\right]$ and let $\mathbf{\Phi}^{(f)} \subset \mathbf{\Phi}$ be the closure (in metric d) of the set of all Laplace transforms of the form

$$\Phi(\lambda; f) = f\left(\exp\left\{-\lambda/f'(1)\right\}\right), \quad f \in \mathfrak{F}.$$

The probability measure $\mathbf{P}$ on $\mathfrak{N}$ generates a natural probability measure on the metric space $\mathbf{\Phi}^{(f)}$ which we denote by the same symbol $\mathbf{P}$.
Introduce a sequence of probability measures on $\mathbf{\Phi}^{(f)}$ by the equality

$$\mathbf{P}^{[x]}(\cdot) = \mathbf{P}\left(\cdot \mid f'(1; \mathfrak{e}) = e^x\right).$$

With this new probability measure, Hypothesis B is now equivalent to

Hypothesis B'. There exists a measure $\mathbf{P}^*(\cdot)$ on $\mathbf{\Phi}^{(f)}$ (with the support on $\Phi(\lambda) : \Phi(0) = 1$, $\Phi(\lambda) < 1$, $\lambda > 0$) such that, as $x \to \infty$

$$\mathbf{P}^{[x]} \Longrightarrow \mathbf{P}^*.$$

In the other words, Hypothesis B' means that there exists a (random) a.s. continuous on $[0,\infty)$ function $g(\cdot)$ with values in $\mathbf{\Phi}^{(f)}$ such that, for every continuous bounded functional H on $\mathbf{\Phi}^{(f)}$

$$\lim_{x\to\infty}\mathbf{E}^{[x]}\left[H(\Phi)\right]=\mathbf{E}^{*}\left[H(g)\right].$$

Since, for any fixed $\lambda\geq 0$ the functional $H_{\lambda}(\Phi)=\Phi(\lambda)$ is continuous on $\mathbf{\Phi}^{(f)}$, we have for $y=e^{x}$

$$\lim_{y\to\infty}\mathbf{E}\left[f(e^{-\lambda/y};\mathfrak{e})\mid f'(1;\mathfrak{e})=y\right]=\mathbf{E}^{*}\left[g(\lambda)\right],\quad \lambda\in[0,\infty) \tag{17}$$

and $\mathbf{E}^{*}\left[g(0)\right]=1,\mathbf{E}^{*}\left[g(\lambda)\right]<1$ if $\lambda>0$. The prelimiting functions at the left-hand side of (17) have the form

$$\mathbf{E}\left[f(e^{-\lambda/y};\mathfrak{e})\mid f'(1;\mathfrak{e})=y\right]=\mathbf{E}\left[e^{-\lambda\xi(\mathfrak{e})/y}\mid f'(1;\mathfrak{e})=y\right]$$

and, therefore, are the Laplace transforms of the distributions of some random variables. Hence, by the continuity theorem for Laplace transforms there exists a proper nonnegative random variable θ such that

$$\lim_{y\to\infty}\mathbf{E}\left[f(e^{-\lambda/y};\mathfrak{e})\mid f'(1;\mathfrak{e})=y\right]=\mathbf{E}^{*}\left[e^{-\lambda\theta}\right],\quad \lambda\in[0,\infty).$$

Let now

$$h(s)=E\left[s^{\upsilon}\right]=\sum_{k=0}^{\infty}h_{k}s^{k},\ h(1)=1$$

be the (deterministic) probability generating function of the nonnegative integer-valued random variable υ. Since, for any fixed $\lambda\geq 0$ the functional $H_{\lambda,h}(\Phi)=h\left(\Phi(\lambda)\right)$ is continuous on $\mathbf{\Phi}^{(f)}$, we have

$$\lim_{y\to\infty}\mathbf{E}\left[h\left(f(e^{-\lambda/y};\mathfrak{e})\right)\mid f'(1;\mathfrak{e})=y\right]=\mathbf{E}^{*}\left[h\left(g(\lambda)\right)\right],\quad \lambda\in[0,\infty). \tag{18}$$

Further, denoting by $\xi_{i}(\mathfrak{e}),i=1,2,...$ independent copies of $\xi(\mathfrak{e})$ we get

$$\begin{aligned}\mathbf{E}\left[h\left(f(e^{-\lambda/y};\mathfrak{e})\right)\mid f'(1;\mathfrak{e})=y\right] &= \sum_{k=0}^{\infty}h_{k}\mathbf{E}\left[f^{k}(e^{-\lambda/y};\mathfrak{e})\mid f'(1;\mathfrak{e})=y\right]\\ &= \sum_{k=0}^{\infty}h_{k}\mathbf{E}\left[\exp\left\{-\frac{\lambda}{y}\sum_{i=1}^{k}\xi_{i}(\mathfrak{e})\right\}\ \Big|\ f'(1;\mathfrak{e})=y\right]\\ &= \mathbf{E}\left[\exp\left\{-\frac{\lambda}{y}\Xi\right\}\ \Big|\ f'(1;\mathfrak{e})=y\right]\end{aligned}$$

where

$$\Xi(\mathfrak{e})=\sum_{i=1}^{\upsilon}\xi_{i}(\mathfrak{e}).$$

Thus, similarly to the previous arguments there exists a proper random variable Θ such that

$$\lim_{y\to\infty} \mathbf{E}\left[\exp\left\{-\frac{\lambda}{y}\Xi(\mathfrak{e})\right\} \,\Big|\, f'(1;\mathfrak{e}) = y\right] = \mathbf{E}^*\left[e^{-\lambda\Theta}\right], \quad \lambda \in [0,\infty). \tag{19}$$

The prelimiting and limiting functions in (19) are monotone and continuous on $[0,\infty)$. Therefore, convergence in (19) (as well as in (18)) is uniform in $\lambda \in [0,\infty)$.

3.2 Some classical results on random walks

Our arguments essentially use a number of statements from the theory of random walks, that are included into this section.
In the sequel we shall meet the situations in which the random walk starts from any point $x \in \mathbb{R}$. In such cases we write for probabilities as usual $\mathbf{P}_x(\cdot)$. We use for brevity $\mathbf{P}$ instead of $\mathbf{P}_0$.
Let us introduce two important random variables

$$M_n \;=\; \max(S_1,\ldots,S_n)\,, \quad L_n \;=\; \min(S_1,\ldots,S_n)$$

and two right-continuous functions $U : \mathbb{R} \to \mathbb{R}_0 = \{x \ge 0\}$ and $V : \mathbb{R} \to \mathbb{R}_0$ given by

$$U(x) \;=\; 1 + \sum_{k=1}^{\infty} \mathbf{P}\left(-S_k \le x, M_k < 0\right)\,, \quad x \ge 0\,,$$
$$V(x) \;=\; 1 + \sum_{k=1}^{\infty} \mathbf{P}\left(-S_k > x, L_k \ge 0\right)\,, \quad x \le 0,$$

and 0 elsewhere. In particular $U(0) = V(0) = 1$. It is well-known that $U(x) = O(x)$ for $x \to \infty$. Moreover, $V(-x)$ is uniformly bounded in x in view of $\mathbf{E}X < 0$.

We define

$$\tau_n = \min\left\{0 \le k \le n : S_k = \min(0, L_n)\right\},\ \tau = \min\left\{k > 0 :\ S_k < 0\right\}$$

and let

$$D = \sum_{k=1}^{\infty} \frac{1}{k} \mathbf{P}\left(S_k \ge 0\right).$$

Now we list a number of known statements for convenience of references. The first lemma is directly taken from [9], Theorems 8.2.4, page 376 and 8.2.18, page 389.

Lemma 3 *Under conditions (14) and (15), as* $n \to \infty$

$$\mathbf{P}\left(L_n \geq 0\right) = \mathbf{P}\left(\tau > n\right) \sim e^D \mathbf{P}\left(X > an\right) \tag{20}$$

and for any fixed $x > 0$

$$\lim_{n\to\infty} \frac{\mathbf{P}\left(L_n \geq -x\right)}{\mathbf{P}\left(\tau > n\right)} = U(x). \tag{21}$$

The following statement is an easy corollary of Theorem 4.7.1 (page 218) of monograph [9].

Lemma 4 *Let* X *be a non-lattice random variable with* $\mathbf{E}\left[X\right] = -a < 0$ *whose distribution satisfies condition* (16). *If* $\tilde{S}_n = X_1 + \cdots + X_n + an$, *then for any* $\Delta > 0$ *uniformly in* $x \geq n^{2/3}$,

$$\mathbf{P}\left(\tilde{S}_n \in [x, x+\Delta)\right) = \frac{\Delta\beta n A(x)}{x}(1 + o(1)).$$

The next statement will be used several times in which i) does not require a proof and ii) is a special case of Theorem 1 in [10].

Lemma 5 *Let* (r_n) *be a regularly varying sequence with* $\sum_{k=0}^{\infty} r_k < \infty$.

i) If $\delta_n \sim dr_n$, $\eta_n \sim er_n$, *then* $\sum_{i=0}^{n} \delta_i \eta_{n-i} \sim cr_n$ *with* $c = d\sum_{k=0}^{\infty} \eta_k + e\sum_{k=0}^{\infty} \delta_k$ *as* $n \to \infty$.

ii) If $\sum_{k=0}^{\infty} \alpha_k t^k = \exp\left(\sum_{k=0}^{\infty} r_k t^k\right)$ *for* $|t| < 1$, *then* $\alpha_n \sim cr_n$ *with* $c = \sum_{k=0}^{\infty} \alpha_k$ *as* $n \to \infty$.

Introduce the notation

$$\begin{aligned} K_1(\lambda) &= \frac{1}{\lambda}\exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}\mathbf{E}\left[e^{\lambda S_n}; S_n < 0\right]\right\} \\ &= \frac{1}{\lambda}\left(1 + \sum_{n=1}^{\infty}\mathbf{E}\left[e^{\lambda S_n}; M_n < 0\right]\right) = \int_0^{\infty} e^{-\lambda x}U(x)dx, \end{aligned} \tag{22}$$

$$\begin{aligned} K_2(\lambda) &= \frac{1}{\lambda}\exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}\mathbf{E}\left[e^{-\lambda S_n}; S_n \geq 0\right]\right\} \\ &= \frac{1}{\lambda}\left(1 + \sum_{n=1}^{\infty}\mathbf{E}\left[e^{-\lambda S_n}; L_n \geq 0\right]\right) = \int_0^{\infty} e^{-\lambda z}V(-z)dz, \end{aligned} \tag{23}$$

and let

$$b_n = \beta\frac{\mathbf{P}\left(X > an\right)}{an}.$$

Note that the intermediate equalities in (22) and (23) are simply versions of the Baxter identities (see, for instance, Chapter XVIII.3 in [12] or Chapter 8.9 in [7]).

3.3 Asymptotic behavior of the distribution of (S_n, L_n)

Basing on the three previous lemmas, we prove the following statement.

Lemma 6 *Assume that* $\mathbf{E}[X] < 0$ *and that* $A(x)$ *meets condition* (16). *Then, for any* $\lambda > 0$ *as* $n \to \infty$

$$\mathbf{E}\left[e^{\lambda S_n}; \tau_n = n\right] = \mathbf{E}\left[e^{\lambda S_n}; M_n < 0\right] \sim K_1(\lambda)\, b_n \tag{24}$$

and

$$\mathbf{E}\left[e^{-\lambda S_n}; \tau > n\right] = \mathbf{E}\left[e^{-\lambda S_n}; L_n \geq 0\right] \sim K_2(\lambda)\, b_n. \tag{25}$$

Proof. We prove (25) only. Statement (24) (proved in [16] under a bit stronger conditions) may be checked in a similar way. First we evaluate the quantity

$$\mathbf{E}\left[e^{-\lambda S_n}; S_n \geq 0\right] = \mathbf{E}\left[e^{-\lambda S_n}; 0 \leq S_n < \lambda^{-1}(\beta+2)\log n\right] + O\left(n^{-\beta-2}\right). \tag{26}$$

Clearly, for any $\Delta > 0$

$$\begin{aligned}
&\sum_{0 \leq k \leq (\beta+2)\lambda^{-1}\Delta^{-1}\log n} e^{-\lambda(k+1)\Delta}\mathbf{P}\left(k\Delta + an \leq \tilde{S}_n \leq (k+1)\Delta + an\right) \\
&\quad \leq \mathbf{E}\left[e^{-\lambda S_n}; 0 \leq S_n < \lambda^{-1}(\beta+2)\log n\right] \\
&\quad \leq \sum_{0 \leq k \leq (\beta+2)\lambda^{-1}\Delta^{-1}\log n} e^{-\lambda k\Delta}\mathbf{P}\left(k\Delta + an \leq \tilde{S}_n \leq (k+1)\Delta + an\right).
\end{aligned}$$

Recall that by Lemma 4 in the range of k under consideration

$$\begin{aligned}
\mathbf{P}\left(k\Delta + an \leq \tilde{S}_n \leq (k+1)\Delta + an\right) &= \frac{\Delta\beta n}{(k\Delta + an)} A(k\Delta + an)(1 + o(1)) \\
&= \frac{\Delta\beta}{a} A(an)(1 + o(1)),
\end{aligned}$$

where $o(1)$ is uniform in $0 \leq k \leq (\beta+2)\lambda^{-1}\Delta^{-1}\log n$. Now passing to the limit as $n \to \infty$ we get

$$\begin{aligned}
\Delta\sum_{k=0}^{\infty} e^{-\lambda(k+1)\Delta} &\leq \liminf_{n\to\infty} \frac{a\mathbf{E}\left[e^{-\lambda S_n}; 0 \leq S_n < \lambda^{-1}(\beta+2)\log n\right]}{\beta A(an)} \\
&\leq \limsup_{n\to\infty} \frac{a\mathbf{E}\left[e^{-\lambda S_n}; 0 \leq S_n < \lambda^{-1}(\beta+2)\log n\right]}{\beta A(an)} \\
&\leq \Delta\sum_{k=0}^{\infty} e^{-\lambda k\Delta}.
\end{aligned}$$

Letting $\Delta \to 0+$, we see that

$$\lim_{n\to\infty} \frac{a\mathbf{E}\left[e^{-\lambda S_n}; 0 \leq S_n < \lambda^{-1}(\beta+2)\log n\right]}{\beta A(an)} = \lambda^{-1}.$$

Combining this with (26) we conclude that, as $n \to \infty$

$$\mathbf{E}\left[e^{-\lambda S_n}; S_n \geq 0\right] \sim \frac{\beta}{a\lambda} A\left(an\right)\left(1+o(1)\right) \sim \frac{\beta}{a\lambda}\mathbf{P}(X > an). \tag{27}$$

We know by the Baxter identity (see, for instance, Chapter 8.9 in [7]) that for $\lambda > 0$ and $t \in [0,1]$

$$1 + \sum_{n=1}^{\infty} t^n \mathbf{E}\left[e^{-\lambda S_n}; L_n \geq 0\right] = \exp\left\{\sum_{n=1}^{\infty} \frac{t^n}{n}\mathbf{E}\left[e^{-\lambda S_n}; S_n \geq 0\right]\right\}.$$

From (27) and Lemma 5 we get for $n \to \infty$,

$$\mathbf{E}\left[e^{-\lambda S_n}; L_n \geq 0\right] \sim K_2\left(\lambda\right)\frac{\beta \mathbf{P}(X > an)}{an},$$

where $K_2\left(\lambda\right)$ is specified by (23). This gives statement (25) of the lemma. ■

The important proposition below is a consequence of Lemmas 5 and 6.

Lemma 7 *For $x \geq 0, \lambda > 0$ we have as $n \to \infty$:*

$$\mathbf{E}_{-x}[e^{\lambda S_n}\,;\, M_n < 0] \;\sim\; b_n V(-x)\int_0^{\infty} e^{-\lambda z} U(z)\, dz, \tag{28}$$

$$\mathbf{E}_{x}[e^{-\lambda S_n}\,;\, L_n \geq 0] \;\sim\; b_n U(x)\int_0^{\infty} e^{-\lambda z} V(-z)\, dz. \tag{29}$$

Proof. This proof follows the line for proving Proposition 2.1 in [2]. By the continuity theorem for Laplace transforms the last two lemmas give for any $x \in [0,\infty)$ and $\lambda > 0$

$$b_n^{-1}\mathbf{E}[e^{\lambda S_n}; M_n < 0\,,\, S_n > -x] \;\to\; \int_0^{x} e^{-\lambda z} U(z)\, dz, \tag{30}$$

$$b_n^{-1}\mathbf{E}[e^{-\lambda S_n}; L_n \geq 0\,,\, S_n < x] \;\to\; \int_0^{x} e^{-\lambda z} V(-z)\, dz. \tag{31}$$

Further, using duality we have

$$\begin{aligned}
\mathbf{E}[e^{\lambda S_n}; M_n < x] \;&=\; \sum_{i=0}^{n-1} \mathbf{E}[e^{\lambda S_n}; S_0, \ldots, S_i \leq S_i < x\,,\, S_i > S_{i+1}, \ldots, S_n] \\
&\quad +\; \mathbf{E}[e^{\lambda S_n}; S_0, \ldots, S_n \leq S_n < x] \\
&=\; \sum_{i=0}^{n-1} \mathbf{E}[e^{\lambda S_i}; L_i \geq 0, S_i < x] \cdot \mathbf{E}[e^{\lambda S_{n-i}}; M_{n-i} < 0] \\
&\quad + \mathbf{E}[e^{\lambda S_n}; L_n \geq 0, S_n < x].
\end{aligned}$$

This formula together with (24), (31), the left continuity of $V(-z)$ for $z > 0$ implying $V(0) = V(0-) = 1$, and the equations

$$1 + \sum_{k=1}^{\infty} \mathbf{E}[e^{\lambda S_k}; L_k \geq 0, S_k < x]$$
$$= 1 + \int_{(0,x)} e^{\lambda z}\, dV(-z) \;=\; e^{\lambda x} V(-x) - \lambda \int_0^x e^{\lambda z} V(-z)\, dz\ ,$$
$$1 + \sum_{k=1}^{\infty} \mathbf{E}[e^{\lambda S_k}; M_k < 0] \;=\; \lambda \int_0^{\infty} e^{-\lambda z} U(z)\, dz$$

yield by Lemma 5 i) that for $\lambda > 0$ and $x > 0$

$$b_n^{-1} \mathbf{E}[e^{\lambda S_n}; M_n < x] \;\to\; V(-x) e^{\lambda x} \int_0^{\infty} e^{-\lambda z} U(z)\, dz,$$

which gives (28) by multiplying by $\exp(-\lambda x)$. Using similar arguments one can get (29). ■

The continuity theorem for Laplace transforms and (28) and (29) imply the next statement.

Corollary 8 *For any $x \geq 0$ and $T > -x$, as $n \to \infty$,*

$$\mathbf{P}\left(S_n < T, L_n \geq -x\right) \sim\; b_n U(x) \int_0^{x+T} V(-z)\, dz \tag{32}$$

and for any $x \geq 0$ and $T < x$, as $n \to \infty$,

$$\mathbf{P}\left(S_n > T, M_n < x\right) \;\sim\; b_n V(-x) \int_0^{x-T} U(z)\, dz\ . \tag{33}$$

Proof. By (29) and the continuity theorem for Laplace transforms for any $x \geq 0$ and $y > x$ we have

$$\mathbf{E}_x[e^{-\lambda S_n}\,;\, S_n < y, L_n \geq 0] \;\sim\; b_n U(x) \int_0^y e^{-\lambda z} V(-z)\, dz$$

giving

$$\mathbf{P}_x\left(S_n < y, L_n \geq 0\right) \sim\; b_n U(x) \int_0^y V(-z)\, dz$$

or

$$\mathbf{P}\left(S_n < y - x, L_n \geq -x\right) \sim\; b_n U(x) \int_0^y V(-z)\, dz$$

proving (32).
The asymptotic representation (33) may be proved by the same arguments. ■

4 Conditioning the random walk on its minimum and the final value

In this section it will be convenient to write $\mathbf{1}_n$ for the $n-$dimensional vector all whose coordinates are equal to 1 and set $\mathbf{S}_{j,n} = (S_j, S_{j+1}, \cdots, S_n)$ if $j \leq n$ with $\mathbf{S}_n = \mathbf{S}_{0,n}$ and $\mathbf{S}_{n,0} = (S_n, S_{n-1}, \cdots, S_0)$. Similar notation will be used for nonrandom vectors. Say, $\mathbf{s}_{n,0} = (s_n, s_{n-1}, \cdots, s_0)$.

We know from the previous results the asymptotic behavior of the probability of the event $\{S_n < T, L_n \geq -x\}$ as $n \to \infty$. In this section we would like to clarify the form of the trajectories giving the main contribution to the probability of this event. It is the key to understand the ingredients providing the survival of the branching process in random environment since it captures the associated environments.

Our aim is to demonstrate that if the event $\{S_n < T, L_n \geq -x\}$ occurs then the trajectory of the random walk on $[0, n]$ has a big jump of the order $an + O(\sqrt{n})$, such a jump is unique and happens at the beginning of the trajectory. Using this fact we also describe the full trajectory of the random walk. We note that Durrett [11] has obtained scaled limit results of such a negative random walk with heavy tails but conditionally on the minimum value only. Here the additional condition on the final value modifies the size of the big jump and we can provide a non scaled decomposition of the asymptotic conditional path.

More precisely, we prove the following result, which will come from a sequence of lemmas.

Theorem 9 *For all $x > 0$ and $T \in \mathbb{R}$ there exists a sequence of number $\pi_j = \pi_j(x) > 0$, $\sum_{j \geq 0} \pi_j = 1$, such that for each j the following properties hold :*

(i) $\lim_{n\to\infty} \mathbf{P}\left(X_j \geq an/2 | L_n \geq -x,\ S_n \leq T\right) = \pi_j$;

(ii) For each measurable and bounded function $F : \mathbb{R}^j \to \mathbb{R}$ and each family of measurable uniformly bounded functions $F_n : \mathbb{R}^{n+1} \to \mathbb{R}$ such that

$$\lim_{\varepsilon \to 0} \sup_{n \in \mathbb{N}, \mathbf{s}_n \in \mathbb{R}^{n+1}} |F_n(\mathbf{s}_n + \epsilon \mathbf{1}_{n+1}) - F_n(\mathbf{s}_n)| = 0, \tag{34}$$

we have as $n \to \infty$

$$\begin{aligned} &\mathbf{E}\left[F(\mathbf{S}_{j-1}) F_{n-j}(\mathbf{S}_{j,n}) | L_n \geq -x,\ S_n \leq T,\ X_j \geq an/2\right] \\ &\quad - \mathbf{E}\left[F(\mathbf{S}_{j-1}) | L_{j-1} \geq -x\right] \mathbf{E}_\mu\left[F_{n-j}(\mathbf{S}'_{n-j,0}) | L'_\infty \geq -x\right] \to 0, \end{aligned}$$

where S' is a random walk with step $-X$ and positive drift, L'_∞ is its global minimum and μ is a probability measure given by :

$$\mu(dy) = dy 1_{y \in [-x,T]} \mathbf{P}_y(L'_\infty \geq -x)\theta^{-1}, \quad \theta = \int_{-x}^{T} dy \mathbf{P}_y(L'_\infty \geq -x). \tag{35}$$

This theorem yields the decomposition of the trajectory of $(S_i : i \leq n)$ conditioned by its minimum L_n and final value S_n. It says that conditionally on

$L_n \geq -x$ and $S_n = s$, S jumps with probability π_j at some (finite) time j. Before this time, S is simply conditioned to be larger than $-x$. After this time, reversing the trajectory yields a random walk S' (with positive drift) conditioned to be larger than $-x$. The size of the jump at time j links the value S_{j-1} to $S'_{n-j-1} = s+a(n-j-1)+\sqrt{n}W_n$, where, thanks to the central limit theorem, W_n converges in distribution, as $n \to \infty$ to a Gaussian random variable Thus this big jumps is of order $an + W_n\sqrt{n}$, as stated below. The proof is defered to Section 4.2.

Corollary 10 *Let $\varkappa = \inf\{n \geq 0 : X_j \geq an/2\}$. Under $\mathbf{P}$, conditionally on $L_n \geq -x$ and $S_n \leq T$, $\varkappa$ converges in distribution to a proper random variable whose distribution $(\pi_j : j \geq 0)$ is specified by*

$$\pi_j = \pi_j(x) = \frac{\mathbf{P}(L_j \geq x)}{\sum_{k\geq 0} \mathbf{P}(L_k \geq x)}$$

and

$$\frac{X_\varkappa - an}{\sqrt{n}}$$

converges in distribution to a centered Gaussian law with variance $\sigma^2 = Var(X)$.

Proof. The expression of q_j can be found in STEP 4 of the proof of the Theorem 9, see (38). The second part of the corollary is an application of the second part of this theorem with

$$F(s_0, \cdots, s_j) = 1, \qquad F_n(s_1, ..., s_{n+1}) = g((s_1 - an)/\sqrt{n})$$

for g uniformly continuous and bounded if one takes into account the positivity of the drift of S' allowing to neglect the condition $L'_\infty \geq -x$ and to use the central limit theorem. ■

4.1 Proof of the conditional description of the random walk

In this and subsequent sections we agree to denote by $C, C_1, C_2, ...$ positive constants which may be different in different formulas or even within one and the same complicated expression.

Our first result shows that the random walk may stay over a fixed level for a long time only if it has at least one big jump. Let

$$\mathcal{B}_j(y) = \{X_j + a \leq y\}, \qquad \mathcal{B}^{(n)}(y) = \cap_{j=1}^n \mathcal{B}_j(y).$$

Lemma 11 *If $\mathbf{E}[X] = -a < 0$ and condition (16) is valid then there exists $\delta_0 \in (0, 1/4)$ such that for all $\delta \in (0, \delta_0)$, $k \in \mathbb{Z}$, and $an/2 - u \geq M$*

$$\mathbf{P}_u(\max_{1\leq j\leq n} X_j \leq \delta an,\ S_n \geq k) \leq \varepsilon_M(k) n^{-\beta-1}, \quad \textit{where} \qquad \varepsilon_M(k) \downarrow_{M\to\infty} 0.$$

Proof. Set $Y_n = (S_n + an)/\sigma$ where $\sigma^2 = Var(X)$ and $S_0 = 0$. It follows from Theorem 4.1.2 and Corollary 4.1.3 (i) in [9] (see also estimate (4.7.7) in the mentioned book) that if $r > 2$ and $\delta > 0$ are fixed then for $x \geq n^{2/3}$ and all sufficiently large n

$$\mathbf{P}(\mathcal{B}^{(n)}\left(x\sigma r^{-1}\right), Y_n \geq x) \leq \left[n\mathbf{P}(X + a \geq \sigma x r^{-1})\right]^{r-\delta}.$$

Since $l(x)$ in (15) is slowly varying, $x^{-1/4}l(x) \to 0$ as $x \to \infty$. Hence we get for all sufficiently large n and $\beta > 2$

$$\left[n\mathbf{P}(X + a \geq \sigma x r^{-1})\right]^{r-\delta} \leq C\left(\frac{nl(x)}{x^{\beta}}\right)^{r-\delta} \leq C\left(\frac{n}{x^{\beta-1/4}}\right)^{r-\delta} \leq C\left(\frac{1}{n^{1/6}}\right)^{r-\delta}.$$

We fix now $r > 2, \delta_0 < 1/4$ with $r\delta_0 = 1/2$ so that $(r-\delta)/6 > \beta + 1$ for all $\delta \in (0, \delta_0)$. As a result we obtain that there exists $\gamma > 0$ such that

$$\mathbf{P}_u(\mathcal{B}^{(n)}\left(x\sigma r^{-1}\right),\ S_n \geq x\sigma - an + u) \leq Cn^{-\beta-1-\gamma}$$

for all $x \geq n^{2/3}$ where now $S_0 = u$. Setting $x\sigma = r\delta_0 an$ we get

$$\mathbf{P}_u(\mathcal{B}^{(n)}\left(\delta_0 an\right),\ S_n \geq -an/2 + u) \leq Cn^{-\beta-1-\gamma}.$$

Therefore, for every $k \in \mathbb{Z}$

$$\mathbf{P}_u(\max_{1\leq j\leq n} X_j \leq \delta_0 an;\ S_n \geq k) \leq Cn^{-\beta-1-\gamma} \tag{36}$$

for all $an/2 - u \geq M \to \infty$. Since the left-hand side is decreasing when $\delta_0 \downarrow 0$ the desired statement follows. ■

We know by (32) that for any fixed N and $l \geq -N$

$$\mathbf{P}\left(L_n \geq -N,\ S_n \in [l, l+1)\right) \sim\ b_n U(N) \int_{N+l}^{N+l+1} V(-z)\, dz,\ n \to \infty.$$

Hence, applying Lemma 11 with $u = 0$ we conclude that, as $n \to \infty$

$$\mathbf{P}\left(L_n \geq -N,\ S_n \in [l, l+1)\right) \sim \mathbf{P}\left(L_n \geq -N,\ S_n \in [l, l+1); \bar{\mathcal{B}}^{(n)}\left(\delta_0 an\right)\right),$$

meaning that for the event $\{L_n \geq -N,\ S_n \in [l, l+1)\}$ to occur it is necessary to have at least one jump exceeding $\delta_0 an$. The next statement shows that, in fact, there is exactly one such big jump on the interval $[0, n]$ that gives the contribution of order b_n to (32) and the jump occurs at the beginning of the interval.

Lemma 12 *Under conditions* $\mathbf{E}[X] = -a < 0$ *and (16) for any fixed* l *and* $\delta > 0$

$$\lim_{J\to\infty} \limsup_{n\to\infty} b_n^{-1}\mathbf{P}\left(L_n \geq -N,\ \max_{J\leq j\leq n} X_j \geq \delta an,\ S_n \in [l, l+1)\right) = 0.$$

Proof. Write for brevity $S_n \in [l)$ if $S_n \in [l, l+1)$. Then

$$\begin{aligned}
&\mathbf{P}\left(L_n \geq -N, X_j \geq \delta an, S_n \in [l)\right) \\
&\quad \leq \int_{-N}^{\infty} \mathbf{P}\left(S_{j-1} \in ds, L_{j-1} \geq -N\right) \times \\
&\qquad \int_{\delta an}^{\infty} \mathbf{P}\left(X_j \in dt\right) \mathbf{P}\left(S_{n-j} \in [l-t-s), L_{n-j} \geq -t-s-N\right) \\
&\quad \leq \int_{-N}^{\infty} \mathbf{P}\left(S_{j-1} \in ds, L_{j-1} \geq -N\right) \int_{\delta an}^{\infty} \mathbf{P}\left(X_j \in dt\right) \mathbf{P}\left(S_{n-j} \in [l-t-s)\right).
\end{aligned}$$

By condition (16),

$$\mathbf{P}\left(X_j \in [t)\right) \leq C \frac{\mathbf{P}\left(X > t\right)}{t}, \ t > 0.$$

This estimate and its monotonicity in t gives

$$\begin{aligned}
&\int_{-N}^{\infty} \mathbf{P}\left(S_{j-1} \in ds; L_{j-1} \geq -N\right) \int_{\delta an}^{\infty} \mathbf{P}\left(X_j \in dt\right) \mathbf{P}\left(S_{n-j} \in [l-t-s)\right) \\
\leq \quad & C_1 \frac{\mathbf{P}\left(X \geq \delta an\right)}{n} \int_{-N}^{\infty} \mathbf{P}\left(S_{j-1} \in ds; L_{j-1} \geq -N\right) \int_{\delta an}^{\infty} \mathbf{P}\left(S_{n-j} \in [l-t-s)\right) dt.
\end{aligned}$$

Now

$$\begin{aligned}
\int_{\delta an}^{\infty} \mathbf{P}\left(S_{n-j} \in [l-t-s)\right) dt \quad &\leq \quad \int_{-\infty}^{\infty} dt \int_{l-t-s}^{l-t-s+1} \mathbf{P}\left(S_{n-j} \in dw\right) \\
&= \int_{-\infty}^{\infty} \mathbf{P}\left(S_{n-j} \in dw\right) \int_{l-s-w}^{l-s-w+1} dt = 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbf{P}\left(L_n \geq -N, X_j \geq \delta an, S_n \in [l)\right) \\
&\quad \leq C \frac{\mathbf{P}\left(X > an\right)}{n} \int_{-N}^{\infty} \mathbf{P}\left(S_{j-1} \in ds, L_{j-1} \geq -N\right) \\
&\quad = C \frac{\mathbf{P}\left(X > an\right)}{n} \mathbf{P}\left(L_{j-1} \geq -N\right) = C_1 b_n \mathbf{P}\left(L_{j-1} \geq -N\right).
\end{aligned}$$

By (21) the series $\sum_{j \geq 1} \mathbf{P}\left(L_{j-1} \geq -N\right)$ converges meaning that a big jump may occur at the beginning only. Moreover, it is unique on account of the estimate

$$\mathbf{P}\left(X_i \geq \delta an, X_j \geq \delta an\right) = O\left(l^2(n) n^{-2\beta}\right) = o\left(b_n\right)$$

for all $i \neq j$ with $\max\left(i, j\right) \leq J$ and $\beta > 2$. ■

The next lemma gives an additional information about the properties of the random walk in the presence of a big jump. Let

$$\mathcal{R}_{\delta}\left(M, K\right) = \left\{\delta an \leq X_1 \leq an - M\sqrt{n}, \, |S_n| \leq K\right\}$$

and

$$\mathcal{R}\left(M, K\right) = \left\{X_1 \geq an + M\sqrt{n}, \, |S_n| \leq K\right\}.$$

Lemma 13 *Under conditions* $\mathbf{E}[X] = -a < 0$ *and (16) for any* $\delta \in (0,1)$ *and each fixed* K*,*

$$\lim_{M\to\infty} \limsup_{n\to\infty} b_n^{-1} \mathbf{P}\left(\mathcal{R}_\delta(M,K) \cup \mathcal{R}(M,K)\right) = 0.$$

Proof. Similarly to the previous lemma we have

$$\begin{aligned}
\mathbf{P}\left(\mathcal{R}_\delta(M,K)\right) &= \int_{\delta an}^{an-M\sqrt{n}} \mathbf{P}\left(S_{n-1} \in [-K-x, K-x]\right) \mathbf{P}\left(X_1 \in dx\right) \\
&\le C\frac{\mathbf{P}\left(X > \delta an\right)}{\delta an} \int_{\delta an}^{an-M\sqrt{n}} \mathbf{P}\left(S_{n-1} \in [-2K-x, 2K-x]\right) dx \\
&= C\frac{\mathbf{P}\left(X > \delta an\right)}{\delta an} \int_{\delta an}^{an-M\sqrt{n}} dx \int_{-2K-x}^{2K-x} \mathbf{P}\left(S_{n-1} \in dv\right) \\
&\le 4KC\frac{\mathbf{P}\left(X > \delta an\right)}{\delta an} \int_{-2K-an+M\sqrt{n}}^{2K-\delta an} \mathbf{P}\left(S_{n-1} \in dv\right) \\
&\le 4KC\frac{\mathbf{P}\left(X > \delta an\right)}{\delta an} \mathbf{P}\left(S_{n-1} \ge -2K - an + M\sqrt{n}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{P}\left(\mathcal{R}(M,K)\right) &= \int_{an+M\sqrt{n}}^{\infty} \mathbf{P}\left(S_{n-1} \in [-K-x, K-x]\right) \mathbf{P}\left(X_1 \in dx\right) \\
&\le C\frac{\mathbf{P}\left(X > an\right)}{an} \int_{an+M\sqrt{n}}^{\infty} \mathbf{P}\left(S_{n-1} \in [-2K-x, 2K-x]\right) dx \\
&\le 4KC\frac{\mathbf{P}\left(X > an\right)}{an} \mathbf{P}\left(S_{n-1} \le 2K - an - M\sqrt{n}\right).
\end{aligned}$$

Since $\limsup_{n\to\infty} \mathbf{P}\left(|S_{n-1} + an| \ge M\sqrt{n}\right)$ decreases to 0 as $M \to \infty$ by the central limit theorem, the desired statement follows. ■

4.2 Proof of Theorem 9

We start by the following important statement.

Lemma 14 *Let* F_n *be a bounded family of uniformly equicontinuous functionsas defined in Theorem 9 by (34). Then the family of functions*

$$g_n(s) = \sqrt{n}\mathbf{E}_s\left[F_n(\mathbf{S}_n); L_n \ge -x, S_n \le T\right], \quad n = 1, 2, ...,$$

is uniformly equicontinuous and uniformly bounded in $s \in \mathbb{R}$*.*

Proof. First, the fact that the family of functions F_n is bounded by C combined with the Stone local limit theorem for iid random variables having finite variance allows us to bound g_n by

$$C\sqrt{n}\mathbf{P}_s(S_n \in [-x, T]) = C\sqrt{n}\mathbf{P}(S_n \in [-x-s, T-s]) \le C_1 < \infty.$$

Second,

$$
\begin{aligned}
&|g_n(s+\epsilon)-g_n(s)| \\
&\quad=\sqrt{n}|\mathbf{E}_s\left[F_n(\mathbf{S}_n+\epsilon\mathbf{1}_{n+1});L_n+\epsilon\geq -x,S_n+\epsilon\leq T\right] \\
&\qquad-\mathbf{E}_s\left[F_n(\mathbf{S}_n);L_n\geq -x,S_n\leq T\right]| \\
&\qquad\quad\leq\sqrt{n}|\mathbf{E}_s\left[F_n(\mathbf{S}_n+\epsilon\mathbf{1}_{n+1})-F_n(\mathbf{S}_n);L_n+\epsilon\geq -x,S_n+\epsilon\leq T\right]| \\
&\qquad\qquad+\sqrt{n}|\mathbf{E}_s\left[F_n(\mathbf{S}_n);L_n+\epsilon\geq -x,S_n+\epsilon\leq T\right] \\
&\qquad\qquad-\mathbf{E}_s\left[F_n(\mathbf{S}_n);L_n\geq -x,S_n\leq T\right]| \\
&\quad\leq H_\epsilon C+\sqrt{n}|\mathbf{P}_s(L_n+\epsilon\geq -x,S_n+\epsilon\leq T)-\mathbf{P}_s(L_n\geq -x,S_n\leq T)|,
\end{aligned}
$$

where $H_\epsilon\to 0$ as $\epsilon\to 0$ again by the assumptions on F_n and the Stone local limit theorem. Let us prove now that the last term is small. Indeed,

$$
\begin{aligned}
&\sqrt{n}|\mathbf{P}_s(L_n+\epsilon\geq -x,S_n+\epsilon\leq T)-\mathbf{P}_s(L_n\geq -x,S_n\leq T)| \\
&\qquad\leq\sqrt{n}\left[\mathbf{P}_s(S_n\in[T-\epsilon,T])+\mathbf{P}_s(L_n\in[-x-\epsilon,-x[,S_n\leq T)\right]
\end{aligned}
$$

and only the second term raises a difficulty. By the total probability formula with respect to the (first) time k of the minimum we have

$$
\begin{aligned}
&\mathbf{P}_s(L_n\in[-x-\epsilon,-x[,S_n\leq T) \\
&\quad=\mathbf{P}(L_n+s+x\in[-\epsilon,0[,S_n\leq T-s) \\
&\quad=\sum_{k=0}^{n}\mathbf{P}(S_1>S_k,\cdots,S_{k-1}>S_k,S_k+s+x\in[-\epsilon,0), \\
&\qquad\qquad S_{k+1}\geq S_k,\cdots,S_n\geq S_k,S_n\leq T-s) \\
&\quad\leq\sum_{k=0}^{n-\lceil\sqrt{n}\rceil-1}\mathbf{P}(S_k+s+x\in[-\epsilon,0))\mathbf{P}(S_{k+1}\geq S_k,\cdots,S_n\geq S_k) \\
&\qquad+\sum_{k=n-\lceil\sqrt{n}\rceil}^{n}\mathbf{P}(S_1>S_k,\cdots,S_{k-1}>S_k,S_k+s+x\in[-\epsilon,0)) \\
&\qquad\qquad\times\mathbf{P}(S_{k+1}\geq S_k,\cdots,S_n\geq S_k).
\end{aligned}
$$

Now we use the representation

$$
\mathbf{P}(S_{k+1}\geq S_k,\cdots,S_n\geq S_k)=\mathbf{P}(L_{n-k}\geq 0)\sim C(n-k+1)^{-\beta}
$$

and the Stone local limit theorem according to which (see, for instance, [7], Section 8.4)

$$
\sqrt{2\pi n}\mathbf{P}(S_k+s+x\in[-\epsilon,0[)=\epsilon\exp\left\{-\frac{(s+x)^2}{2\sigma^2 n}\right\}+\delta_n,
$$

where $\delta_n \to 0$ as $n \to \infty$ uniformly in $k \in [n - \sqrt{n}, n]$ and $s + x \in \mathbb{R}$. Hence we conclude that

$$\begin{aligned}
&\sqrt{n}\mathbf{P}_s(L_n \in [-x-\epsilon, -x[, S_n \leq T) \\
&\quad \leq (\epsilon + \delta_n)\, C \sum_{k=0}^{n-\lceil\sqrt{n}\rceil-1} (n-k+1)^{-\beta} + C_1 \sum_{k=n-\lceil\sqrt{n}\rceil}^{n} (n-k+1)^{-\beta} \\
&\quad \leq C_2 \left(\epsilon + \delta_n + \sqrt{n}(\sqrt{n})^{-\beta}\right) \leq C_3 \left(\epsilon + \delta_n\right),
\end{aligned}$$

for n large enough, since $\beta > 1$. We end up the proof by noting that all these bounds are uniform with respect to s. ■

Proof of Theorem 9. We know by Lemmas 11, 12 and 13 that conditionally on the event $\{L_n \geq -x, S_n \leq T\}$, there is a (single) big jump, that it size is of order an with deviation $\sqrt{n}$ and that it happens at the beginning. Taking this into account and setting $\mathcal{A}_j^M = \{X_j - an \in [-M\sqrt{n}, M\sqrt{n}]\}$ we get

$$\mathbf{E}\left[F(\mathbf{S}_{j-1})F_{n-j}(\mathbf{S}_{j,n}); L_n \geq -x, S_n \leq T\right] = \varepsilon_{J,M,n} b_n + \sum_{j=0}^{J} A_{j,n}^M,$$

where

$$\lim_{J,M\to\infty} \sup_n |\varepsilon_{J,M,n}| = 0$$

and

$$A_{j,n}^M = \mathbf{E}\left[F(\mathbf{S}_{j-1})F_{n-j}(\mathbf{S}_{j,n}); L_n \geq -x, \mathcal{A}_j^M, S_n \leq T\right].$$

By the Markov property we have

$$A_{j,n}^M = \mathbf{E}\left[F(\mathbf{S}_{j-1})1_{\{L_{j-1}\geq -x\}} H_{j,n}^M(S_{j-1})\right],$$

where

$$H_{j,n}^M(s) = \mathbf{E}\left[1_{\mathcal{A}^M}\mathbf{E}_{s+X}\left[F_{n-j}(\mathbf{S}_{n-j}); L_{n-j} \geq -x, S_{n-j} \leq T\right]\right]$$

and $\mathcal{A}^M = \{X - an \in [-M\sqrt{n}, M\sqrt{n}]\}$.

STEP 1. We are proceeding by bounded convergence and show first the simple convergence. Thus, we consider

$$\begin{aligned}
b_n^{-1} H_{j,n}^M(s) &= \int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} b_n^{-1}\mathbf{P}(X \in dy)\mathbf{E}_{s+y}\left[F_{n-j}(\mathbf{S}_{n-j}); L_{n-j} \geq -x, S_{n-j} \leq T\right] \\
&= \frac{1}{\sqrt{n}} \int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(s+y)\mu_n(dy),
\end{aligned}$$

where

$$\mu_n(dy) = b_n^{-1}\mathbf{P}(X \in dy), \quad g_{j,n}(s) = \sqrt{n}\mathbf{E}_s\left[F_{n-j}(\mathbf{S}_{n-j}); L_{n-j} \geq -x, S_{n-j} \leq T\right].$$

We want to prove that

$$\frac{1}{\sqrt{n}}\int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(s+y)\mu_n(dy) - \frac{1}{\sqrt{n}}\int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(s+y)dy \to 0$$

as $n \to \infty$ by using the local converges of μ_n to the Lebesgue measure (with uniformity in $y \in [an - M\sqrt{n}, an + M\sqrt{n}]$ thanks to (15) and (16)) and the uniform equicontinuity of $g_{j,n}$ (compare with Lemma 14). Let us give the details. First, by Lemma 14 for any $\varepsilon > 0$ there exists $\eta > 0$ such that for all $n \geq n_0 = n_0(\varepsilon, \eta)$ we have

$$\sup_y \sup_{u\in[0,\eta]} |g_{j,n}(y) - g_{j,n}(y+u)| \leq \epsilon.$$

Let, further, $s_i (= s_i^n)$ be a subdivision of $[an - M\sqrt{n}, an + M\sqrt{n} - 1]$ with step η. Then, for sufficiently large $n \geq n_0$,

$$\left| b_n^{-1} H_{j,n}^M(s) - \frac{1}{\sqrt{n}} \sum_i g_{j,n}(s_i)\mu_n[s_i, s_{i+1}[\right| \leq 3M\epsilon.$$

Besides, $g_{j,n}(y)$ is bounded by C with respect to the pair n, y by Lemma 14. Recalling that by (16)

$$\sup_{y\in s+an+[-M\sqrt{n},M\sqrt{n}]} |\mu_n[y, y+\eta] - \eta| \leq \epsilon\eta$$

for n large enough, we get

$$\left| b_n^{-1} H_{j,n}^M(s) - \frac{1}{\sqrt{n}} \sum_i g_{j,n}(s_i)\eta \right| \leq 3M\epsilon + 2CM\epsilon\eta\frac{1}{\eta}.$$

Using again the uniform continuity of $g_{j,n}$ yields for n large enough

$$\left| \frac{1}{\sqrt{n}} \sum_i g_{j,n}(s_i)\eta - \frac{1}{\sqrt{n}}\int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(s+y)dy \right| \leq 3\epsilon M,$$

resulting in

$$\left| b_n^{-1} H_{j,n}^M(s) - \frac{1}{\sqrt{n}}\int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(s+y)dy \right| \leq M(6+2C)\epsilon$$

for n large enough.

Clearly, the sequence $b_n^{-1} H_{j,n}^M$ is bounded since both $g_{j,n}$ (see Lemma 14) and $\mu_n([an - M\sqrt{n}, an + M\sqrt{n}])/\sqrt{n}$ are bounded. This and the dominated convergence theorem lead to

$$b_n^{-1} A_{j,n}^M - \mathbf{E}\left[F(\mathbf{S}_{j-1}) 1_{\{L_{j-1}\geq -x\}} \frac{1}{\sqrt{n}}\int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(S_{j-1}+y)dy \right] \overset{n\to\infty}{\longrightarrow} 0. \tag{37}$$

STEP2. We can now complete the proof by reversing the random walk after time j. To this aim set $\mathbf{s}_k = (s_0, \cdots, s_k)$ and $\mathbf{s}_{n,0} = (s_n, \cdots, s_0)$ and recall that (see, for instance, Lemma 9 in [14])

$$ds_0 \mathbf{P}_{s_0}(\mathbf{S}_n \in d\mathbf{s}_n) = ds_n \mathbf{P}_{s_n}(\mathbf{S}'_{n,0} \in d\mathbf{s}_{n,0}).$$

Hence, letting

$$\mathcal{B}_n(\mathbf{s}_k) = \{|s_0 - an - s| \le M\sqrt{n},\, s_k \in [-x, T],\, \min_{0\le i\le k} s_i \ge -x\}$$

we get by integration

$$\begin{aligned}&\int 1_{\mathcal{B}_n(\mathbf{s}_{n-j})} F_{n-j}(\mathbf{s}_{n-j}) ds_0 \mathbf{P}_{s_0}(\mathbf{S}_{n-j} \in d\mathbf{s}_{n-j})\\ &\quad = \int 1_{\mathcal{B}_n(\mathbf{s}_{n-j})} F_{n-j}(\mathbf{s}_{n-j}) ds_{n-j} \mathbf{P}_{s_{n-j}}(\mathbf{S}'_{n-j,0} \in d\mathbf{s}_{n-j,0}).\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{1}{\sqrt{n}} \int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(s+y)dy\\ &= \int_{s'_0\in[-x,T]} \mathbf{E}_{s'_0}\left[F_{n-j}(\mathbf{S}'_{n-j,0}); L'_{n-j} \ge -x; \left|S'_{n-j} - an - s\right| \le M\sqrt{n}\right] ds'_0.\end{aligned}$$

Since, as $n \to \infty$

$$\mathbf{P}(\left|S'_{n-j} - an - s\right| \le M\sqrt{n}) \to \frac{1}{\sigma\sqrt{2\pi}} \int_{-M}^{M} \exp\left\{-\frac{y^2}{2\sigma^2}\right\} dy$$

for every $s \in \mathbb{R}$ and S' has a positive drift, we conclude that

$$\begin{aligned}K^M(s) = \limsup_{n\to\infty} \Bigg| &\frac{1}{\sqrt{n}} \int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(s+y)dy\\ &- \int_{s'_0\in[-x,T]} \mathbf{E}_{s'_0}\left[F_{n-j}(\mathbf{S}'_{n-j,0}); L'_\infty \ge -x\right] ds'_0\Bigg|\end{aligned}$$

goes to 0 as M becomes large. Further, by the bounded convergence and taking into account the boundness of $g_{j,n}$, we get (recall (35))

$$\begin{aligned}\limsup_{n\to\infty} \Bigg| &\mathbf{E}\left[F(\mathbf{S}_{j-1}) 1_{\{L_{j-1}\ge -x\}} \frac{1}{\sqrt{n}} \int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(S_{j-1}+y)dy\right]\\ &-\mathbf{E}\left[F(\mathbf{S}_{j-1}) 1_{\{L_{j-1}\ge -x\}} \theta \mathbf{E}_\mu\left[F_{n-j}(\mathbf{S}'_{n-j,0}) | L'_\infty \ge -x\right]\right]\Bigg|\\ &\le \left|\mathbf{E}\left[F(\mathbf{S}_{j-1}) 1_{\{L_{j-1}\ge -x\}} K^M(S_{j-1})\right]\right|\end{aligned}$$

where the right-hand side goes to 0 as $M \to \infty$. Using (35) once again we set

$$D_{j,n} = \mathbf{E}\left[F(\mathbf{S}_{j-1}) | L_{j-1} \ge -x\right] \mathbf{E}_\mu\left[F_{n-j}(\mathbf{S}'_{n-j}) | L'_\infty \ge -x\right]$$

and deduce from (37) that the function

$$R^M = \limsup_{n\to\infty} \left| b_n^{-1} A_{j,n}^M - \mathbf{P}(L_{j-1} \geq -x)\theta D_{j,n} \right|$$

goes to zero as $M \to \infty$. Writing

$$C_{j,n} = \mathbf{E}\left[F(\mathbf{S}_{j-1})F_{n-j}(\mathbf{S}_{j,n}); L_n \geq -x;\ S_n \leq T;\ X_j \geq an/2\right]$$

we have

$$\begin{aligned}\limsup_{n\to\infty} &\left| b_n^{-1} C_{j,n} - \mathbf{P}(L_{j-1} \geq -x)\theta D_{j,n} \right| \\ &\leq \limsup_{n\to\infty} b_n^{-1} \mathbf{P}(X_j \geq an/2, |X_j - an| > M\sqrt{n},\ S_n \leq T) + R^M.\end{aligned}$$

Combining the last limit and Lemma 13 ensures that the right-hand side of this inequality goes to 0 as $M \to \infty$. We conclude

$$\lim_{n\to\infty} \left(b_n^{-1} C_{j,n} - \mathbf{P}(L_{j-1} \geq -x)\theta D_{j,n} \right) = 0.$$

STEP 4. We apply the limit above to the family of functions $F = 1, F_{n-j} = 1$ and get

$$b_n^{-1} \mathbf{P}\left(L_n \geq -x,\ S_n \leq T,\ X_j \geq an/2\right) \overset{n\to\infty}{\longrightarrow} \mathbf{P}(L_{j-1} \geq -x)\theta. \tag{38}$$

Recalling (32) ensures that there exists $\pi_j(x) > 0$ such that

$$\mathbf{P}\left(X_j \geq an/2 \mid L_n \geq -x,\ S_n \leq T\right) \overset{n\to\infty}{\longrightarrow} \pi_j(x).$$

Using Lemmas 11, 12 and 13 shows that there is only one big jump at the beginning, and it has to be greater than $an/2$. Thus, $\sum_{j\geq 0} \pi_j(x) = 1$. Finally, the proof of the Theorem can be completed by using again the conclusion of STEP 3. ■

5 The asymptotic behavior of the survival probability

In this section we use the notation

$$\mathbf{E}_{\mathfrak{e}}\left[\cdot\right] = \mathbf{E}\left[\cdot \mid \mathcal{E}\right], \quad \mathbf{P}_{\mathfrak{e}}\left(\cdot\right) = \mathbf{P}\left(\cdot \mid \mathcal{E}\right)$$

i.e., consider the expectation and probability given the environment $\mathcal{E}$. Our aim is to prove the following statement.

Lemma 15 *If Hypotheses A and B are valid then there exists a constant $C_0 > 0$ such that, as $n \to \infty$*

$$\mathbb{P}\left(Z_n > 0\right) \sim C_0 m^n \beta \frac{\mathbf{P}\left(X > an\right)}{an} = C_0 m^n b_n. \tag{39}$$

We recall from the discussion in Preliminaries that Hypotheses A and B (or B') ensure that there exists $g(\lambda)$ a.s. continuous on $[0,\infty)$ with $g(0)=1$ and with $\mathbf{E}[g(\lambda)]<1, \lambda>0$, such that for every continuous bounded function F on $[0,1]$

$$\lim_{y\to\infty}\sup_{\lambda\geq 0}\left|\mathbf{E}\left[F(f(e^{-\lambda/y}))\mid f'(1)=y\right]-\mathbf{E}^*\left[F(g(\lambda))\right]\right|=0. \tag{40}$$

Making the change of measure in accordance with (12) and (13) we see that it is necessary to show that, as $n\to\infty$

$$\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n}\right]\sim C_0 b_n. \tag{41}$$

The proof of this fact is conducted into several steps which we split into subsections.

5.1 Time of the minimum of S

First, we prove that the contribution to $\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n}\right]$ may be of order b_n only if the minimal value of S within the interval $[0,n]$ is attained at the beginning or at the end of this interval. To this aim we use, as earlier, the notation $\tau_n=\min\{0\leq k\leq n: S_k=L_n\}$ and show that the following statement is valid.

Lemma 16 *Given Hypotheses A and B we have*

$$\lim_{M\to\infty}\lim_{n\to\infty} b_n^{-1}\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n\in[M,n-M]\right]=0.$$

Proof. In view of the estimate

$$\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)\leq\min_{0\leq k\leq n}\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)\leq\exp\left\{\min_{0\leq k\leq n}S_k\right\}=e^{S_{\tau_n}}$$

we have

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n\in[M,n-M]\right]\\
&\quad\leq\mathbf{E}\left[e^{S_{\tau_n}-S_n};\tau_n\in[M,n-M]\right]\\
&\quad=\sum_{k=M}^{n-M}\mathbf{E}\left[e^{(1-\rho)S_k+\rho(S_k-S_n)};\tau_n=k\right]\\
&\quad=\sum_{k=M}^{n-M}\mathbf{E}\left[e^{(1-\rho)S_k};\tau_k=k\right]\mathbf{E}\left[e^{-\rho S_{n-k}};L_{n-k}\geq 0\right].
\end{aligned}\tag{42}$$

Hence, using Lemma 6 we get

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n\in[M,n-M]\right]\\
&\leq\left(\sum_{k=M}^{[n/2]}+\sum_{k=[n/2]+1}^{n-M}\right)\mathbf{E}\left[e^{(1-\rho)S_k};\tau_k=k\right]\mathbf{E}\left[e^{-\rho S_{n-k}};L_{n-k}\geq 0\right]\\
&\leq\frac{C_1}{n}\mathbf{P}\left(X>\frac{an}{2}\right)\sum_{k=M}^{[n/2]}\mathbf{E}\left[e^{(1-\rho)S_k};\tau_k=k\right]\\
&+\frac{C_2}{n}\mathbf{P}\left(X>\frac{an}{2}\right)\sum_{k=M}^{[n/2]}\mathbf{E}\left[e^{-\rho S_k};L_k\geq 0\right]\leq\varepsilon_M b_n \qquad (43)
\end{aligned}$$

where $\varepsilon_M\to 0$ as $M\to\infty$. ■

The following statement easily follows from (43) by taking $M=0$.

Corollary 17 *Given Hypotheses A and B there exists $C\in(0,\infty)$ such that, for all $n=1,2,...$*

$$\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n}\right]\leq\mathbf{E}\left[e^{S_{\tau_n}-\rho S_n}\right]\leq Cb_n.$$

5.2 Fluctuations of the random walk S

Introduce the event

$$\mathcal{C}_N=\left\{-N<S_{\tau_n}\leq S_n\leq N+S_{\tau_n}<N\right\}.$$

In particular, given $\mathcal{C}_N$

$$-N<S_n<N.$$

In what follows we agree to denote by $\varepsilon_N,\varepsilon_{N,n}$ or $\varepsilon_{N,K,n}$ functions of the low indices such that

$$\lim_{N\to\infty}\varepsilon_N=\lim_{N\to\infty}\limsup_{n\to\infty}|\varepsilon_{N,n}|=\lim_{N\to\infty}\limsup_{K\to\infty}\limsup_{n\to\infty}|\varepsilon_{N,K,n}|=0,$$

i.e., the lim sup (or *lim*) are sequentially taken with respect to the indices of $\varepsilon_{...}$ in the reverse order. Note that the functions are not necessarily the same in different formulas or even within one and the same complicated expression.

Lemma 18 *Given Hypotheses A and B for any fixed k*

$$\lim_{N\to\infty}\limsup_{n\to\infty}b_n^{-1}\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n=k,\bar{\mathcal{C}}_N\right]=0$$

and

$$\lim_{N\to\infty}\limsup_{n\to\infty}b_n^{-1}\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n=n-k,\bar{\mathcal{C}}_N\right]=0.$$

Proof. In view of (31)

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n=k,S_n-S_{\tau_n}\geq N\right]\\
&\quad\leq\mathbf{E}\left[e^{(1-\rho)S_{\tau_n}}e^{-\rho(S_n-S_{\tau_n})};\tau_n=k,S_n-S_{\tau_n}\geq N\right]\\
&\quad\leq\mathbf{E}\left[e^{-\rho S_{n-k}};L_{n-k}\geq 0,S_{n-k}\geq N\right]\leq\varepsilon_N b_n
\end{aligned}$$

where $\varepsilon_N\to 0$ as $N\to\infty$ since $\int_0^\infty\exp(-\rho z)V(-z)dz<\infty$. Further,

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n=k,S_{\tau_n}\leq -N\right]\\
&\quad\leq\mathbf{E}\left[e^{(1-\rho)S_{\tau_n}}e^{-\rho(S_n-S_{\tau_n})};\tau_n=k,S_{\tau_n}\leq -N\right]\\
&\quad\leq e^{-(1-\rho)N}\mathbf{E}\left[e^{-\rho S_{n-k}};L_{n-k}\geq 0\right]\leq\varepsilon_N b_n.
\end{aligned}\tag{44}$$

This, in particular, means that

$$\mathbf{E}\left[e^{(1-\rho)S_{\tau_n}}e^{-\rho(S_n-S_{\tau_n})};\tau_n=k,S_n\notin(-N,N)\right]=\varepsilon_{N,n}b_n\tag{45}$$

and

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n=k\right]\\
&\quad=\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n=k,S_{\tau_n}\geq -N,S_n-S_{\tau_n}\leq N\right]+\varepsilon_{N,n}b_n.
\end{aligned}$$

Similarly, by (30)

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n=n-k,S_{\tau_n}\leq -N\right]\\
&\quad\leq\mathbf{E}\left[e^{(1-\rho)S_{\tau_n}}e^{-\rho(S_n-S_{\tau_n})};\tau_n=n-k,S_{\tau_n}\leq -N\right]\\
&\quad\leq\mathbf{E}\left[e^{(1-\rho)S_{n-k}};\tau_{n-k}=n-k,S_{n-k}\leq -N\right]\\
&\quad=\mathbf{E}\left[e^{(1-\rho)S_{n-k}};M_{n-k}<0,S_{n-k}\leq -N\right]=\varepsilon_{N,n}b_n
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n=n-k,S_n-S_{\tau_n}\geq N\right]\\
&\quad\leq\mathbf{E}\left[e^{(1-\rho)S_{\tau_n}}e^{-\rho(S_n-S_{\tau_n})};\tau_n=n-k,S_n-S_{\tau_n}\geq N\right]\\
&\quad\leq e^{-\rho N}\mathbf{E}\left[e^{(1-\rho)S_{n-k}};\tau_{n-k}=n-k\right]\\
&\quad=e^{-\rho N}\mathbf{E}\left[e^{(1-\rho)S_{n-k}};M_{n-k}<0\right]=\varepsilon_{N,n}b_n.
\end{aligned}$$

As a result we get

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n=n-k\right]\\
&\quad=\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};\tau_n=n-k,S_{\tau_n}\geq -N,S_n-S_{\tau_n}\leq N\right]+\varepsilon_{N,n}b_n.
\end{aligned}$$

This completes the proof of the lemma. ■

Lemmas 16 and 18 easily imply the following statement.

Corollary 19 *Under Hypotheses A and B*

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n}\right]\\
&\quad=\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};|S_n|<N;\tau_n\in[0,M]\cup[n-M,n]\right]+\varepsilon_{N,M,n}b_n\\
&\qquad=\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};|S_n|<N\right]+\varepsilon_{N,n}b_n\\
&\qquad\quad=\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n};S_{\tau_n}\geq -N,S_n<N\right]+\tilde{\varepsilon}_{N,n}b_n \qquad (46)
\end{aligned}$$

where

$$\lim_{N\to\infty}\limsup_{M\to\infty}\limsup_{n\to\infty}|\varepsilon_{N,M,n}|=\lim_{N\to\infty}\limsup_{n\to\infty}\left(|\varepsilon_{N,n}|+|\tilde{\varepsilon}_{N,n}|\right)=0.$$

5.3 Asymptotic of the survival probability

In this section we investigate in detail the properties of the survival probability for the processes meeting Hypotheses A and B. As we know (see (13)) this probability is expressed as

$$\mathbb{P}\left(Z_n>0\right)=m^n\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n}\right].$$

We wish to show that $\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)e^{-\rho S_n}\right]$ is of order b_n as $n\to\infty$.

First we get rid of the trajectories giving the contribution of the order $o(b_n)$ to the quantity in question. Let

$$\mathcal{D}_N(j)=\left\{-N<S_{\tau_n}\leq S_n<N,\,X_j\geq\delta an\right\}.$$

Lemma 20 *If Hypotheses A and B are valid then there exists $\delta\in(0,1/4)$ such that*

$$\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)\exp(-\rho S_n)\right]=\sum_{j=1}^{J}\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)\exp(-\rho S_n);\mathcal{D}_N(j)\right]+\varepsilon_{N,J,n}b_n.$$

Proof. In view of the inequality

$$\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)\exp(-\rho S_n)\leq\exp(S_{\tau_n}-\rho S_n)=\exp((1-\rho)S_{\tau_n}-\rho(S_n-S_{\tau_n}))\leq 1$$

the statement of the lemma follows from Corollary 19, the estimate

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)\exp(-\rho S_n);S_{\tau_n}\geq -N,\,\max_{0\leq j\leq n}X_j<\delta an\right]\\
&\qquad\leq\mathbf{P}\left(S_{\tau_n}\geq -N,\,\max_{0\leq j\leq n}X_j<\delta an\right)=\varepsilon_{N,n}b_n,\,0<\delta\leq\delta_0\leq 1/4,
\end{aligned}$$

established in Lemma 11, Lemma 12, according to which

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)\exp(-\rho S_n);S_{\tau_n}\geq -N,S_n\leq N,\,\max_{J\leq j\leq n}X_j\geq\delta an\right]\\
&\qquad\leq\mathbf{P}\left(L_n\geq -N,S_n\leq N,\,\max_{J\leq j\leq n}X_j\geq\delta an\right)=\varepsilon_{N,J,n}b_n,
\end{aligned}$$

and the fact that the probability of the event for the random walk $\{S_j, 0 \le j \le J\}$ to have two increments exceeding δan is $o(b_n)$. ■

Now we fix $j \in [1, J]$ and investigate the quantity

$$\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right)\exp(-\rho S_n); \mathcal{D}_N(j)\right].$$

First, we check that S_{j-1} is bounded on the event we focus on.

Lemma 21 *If Hypotheses A and B are valid then, for every fixed j*

$$\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right)\exp(-\rho S_n); |S_{j-1}| \ge N, X_j \ge \delta an\right] = \varepsilon_{N,n} b_n.$$

Proof. First observe that

$$\begin{aligned}
& \mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n > 0\right)\exp(-\rho S_n); S_{j-1} \le -N, X_j \ge \delta an\right] \\
\le\ & \mathbf{E}\left[\exp((1-\rho)S_{\tau_n} - \rho(S_n - S_{\tau_n})); S_{j-1} \le -N, X_j \ge \delta an\right] \\
\le\ & \mathbf{E}\left[\exp((1-\rho)S_{\tau_n}); S_{j-1} \le -N, X_j \ge \delta an\right] \\
\le\ & \mathbf{E}\left[\exp(-(1-\rho)N); X_j \ge \delta an\right] \\
=\ & \exp(-(1-\rho)N)\mathbf{P}(X \ge \delta an) = \varepsilon_{N,n} b_n.
\end{aligned}$$

Further, taking $\gamma \in (0,1)$ such that $\gamma\beta > 1$, we get

$$\begin{aligned}
& \mathbf{E}\left[\exp(S_{\tau_n} - \rho S_n); S_{j-1} \ge n^\gamma, X_j \ge \delta an\right] \le \mathbf{P}(S_{j-1} \ge n^\gamma)\mathbf{P}(X \ge \delta an) \\
& \le j\mathbf{P}(X \ge n^\gamma/j)\mathbf{P}(X \ge \delta an) \sim \frac{j^{\beta+1}}{n^{\gamma\beta}} l(n^\gamma)\mathbf{P}(X \ge \delta an) = \varepsilon_n b_n. \quad (47)
\end{aligned}$$

Consider now the situation $S_{j-1} \in [N, n^\gamma]$, $j \ge 2$ and write

$$\begin{aligned}
& \mathbf{E}\left[\exp(S_{\tau_n} - \rho S_n); S_{j-1} \in [N, n^\gamma], X_j \ge \delta an\right] \\
& \qquad = \int_N^{n^\gamma} \int_{-\infty}^0 \mathbf{P}(S_{j-1} \in dy, L_{j-1} \in dz) H_{n,\delta}(y,z),
\end{aligned}$$

where

$$\begin{aligned}
H_{n,\delta}(y,z) &= \int_{\delta an}^\infty \mathbf{P}(X \in dt) \int_{-\infty}^0 \int_v^\infty \mathbf{P}_{y+t}(L_{n-j} \in dv, S_{n-j} \in dw) e^{z\wedge v} e^{-\rho w} \\
&= \int_{\delta an + y}^\infty \mathbf{P}(X \in dt - y) \int_{-\infty}^0 \int_v^\infty \mathbf{P}_t(L_{n-j} \in dv, S_{n-j} \in dw) e^{z\wedge v} e^{-\rho w}.
\end{aligned}$$

By our conditions $\mathbf{P}(X \in dt - y) = \mathbf{P}(X \in dt)\left(1 + o(1)\right)$ uniformly in $y \in [0, n^\gamma]$ and $t \ge \delta an$. Thus, for all sufficiently large n

$$\begin{aligned}
H_{n,\delta}(y,z) &\le 2\int_{\delta an}^\infty \mathbf{P}(X \in dt) \int_{-\infty}^0 \int_v^\infty \mathbf{P}_t(L_{n-j} \in dv, S_{n-j} \in dw) e^{z\wedge v} e^{-\rho w} \\
&\le 2\int_{\delta an}^\infty \mathbf{P}(X \in dt) \int_{-\infty}^0 \int_v^\infty \mathbf{P}_t(L_{n-j} \in dv, S_{n-j} \in dw) e^{v} e^{-\rho w} \\
&= 2\int_{\delta an}^\infty \mathbf{P}(X \in dt) \mathbf{E}_t\left[e^{S_{\tau_{n-j}} - \rho S_{n-j}}\right] \\
&\le 2\mathbf{E}_0\left[e^{S_{\tau_{n-j+1}} - \rho S_{n-j+1}}; X_1 \ge \delta an\right] = 2H_{n,\delta}(0,0).
\end{aligned}$$

By integrating this inequality we get for sufficiently large n

$$\begin{aligned}\int_N^{n^\gamma}\int_{-\infty}^{0}&\mathbf{P}(S_{j-1}\in dy, L_{j-1}\in dz)H_{n,\delta}(y,z)\\ &\le 2\int_N^{n^\gamma}\int_{-\infty}^{0}\mathbf{P}(S_{j-1}\in dy, L_{j-1}\in dz)H_{n,\delta}(0,0)\\ &\le 2\mathbf{P}(S_{j-1}\ge N)\mathbf{E}_0\left[e^{S_{\tau_{n-j+1}}-\rho S_{n-j+1}};X_1\ge\delta an\right].\end{aligned}$$

Since

$$\begin{aligned}b_n^{-1}H_{n,\delta}(0,0) &= b_n\mathbf{E}\left[e^{S_{\tau_{n-j+1}}-\rho S_{n-j+1}};X_1\ge\delta an\right]\\ &\le b_n^{-1}\mathbf{E}\left[e^{S_{\tau_{n-j+1}}-\rho S_{n-j+1}}\right]=O(1)\end{aligned}$$

as $n\to\infty$ (see Corollary 17) and $\mathbf{P}(S_{j-1}\ge N)\to 0$ as $N\to\infty$, we obtain

$$\mathbf{E}\left[\exp(S_{\tau_n}-\rho S_n);\, S_{j-1}\in[N,n^\gamma], X_j\ge\delta n\right]=\varepsilon_{N,n}b_n. \tag{48}$$

Combining (47) and (48) proves the lemma. ■

Lemma 22 *Given Hypotheses A and B we have for each fixed j*

$$\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}\left(Z_n>0\right)\exp(-\rho S_n);|S_n-S_{j-1}|>K, X_j\ge\delta an\right]=\varepsilon_{K,n}(j)b_n.$$

Proof. We know from Lemma 21 that only the values $S_{j-1}\le N$ for sufficiently large but fixed N are of importance. Thus, we just need to prove that for fixed N

$$\mathbf{E}\left[e^{S_{\tau_n}-\rho S_n};S_{j-1}\le N,|S_n-S_{j-1}|>K,X_j\ge\delta an\right]=\varepsilon_{N,K,n}(j)b_n$$

where $\lim_{K\to\infty}\limsup_{n\to\infty}|\varepsilon_{N,K,n}(j)|=0$.To this aim we set $L_{j,n}=\min\{S_k-S_{j-1}: j-1\le k\le n\}$ and, using the inequality $S_{\tau_n}\le S_{j-1}+L_{j,n}$, deduce the estimate

$$\begin{aligned}&\mathbf{E}\left[e^{S_{\tau_n}-\rho S_n};S_{j-1}\le N,|S_n-S_{j-1}|>K,X_j\ge\delta an\right]\\ &\quad\le\mathbf{E}\left[e^{S_{j-1}+L_{j,n}-\rho(S_n-S_{j-1})-\rho S_{j-1}};S_{j-1}\le N,|S_n-S_{j-1}|>K\right]\\ &\quad=\mathbf{E}\left[e^{(1-\rho)S_{j-1}};S_{j-1}\le N\right]\mathbf{E}\left[e^{L_{j,n}-\rho(S_n-S_{j-1})};|S_n-S_{j-1}|>K\right].\end{aligned}$$

We conclude with $\mathbf{E}\left[e^{(1-\rho)S_{j-1}};S_{j-1}\le N\right]<\infty$ and we can now control the term

$$\mathbf{E}\left[e^{L_{j,n}-\rho(S_n-S_{j-1})};|S_n-S_{j-1}|>K\right]=\mathbf{E}\left[e^{S_{\tau_{n-j+1}}-\rho S_{n-j+1}};|S_{n-j+1}|>K\right]$$

by $\varepsilon_{K,n}b_n$. Indeed it is now exactly the term controlled in a similar situation in (45). ■

We give the last technical lemma.

Lemma 23 *Assume that g is a random function which satisfies (40). Then for every (deterministic) probability generating function $h(s)$ and every $\varepsilon > 0$ there exists $\kappa > 0$ such that*

$$\left|\mathbf{E}\left[1 - h(g(e^v w))\right] - \mathbf{E}\left[1 - h(g(e^{v'} w))\right]\right| \leq h'(1)\varepsilon$$

for $|v - v'| \leq \kappa, w \in [0, 2]$.

Proof. Clearly,

$$\left|\mathbf{E}\left[1 - h(g(e^v w))\right] - \mathbf{E}\left[1 - h(g(e^{v'} w))\right]\right| \leq h'(1)\mathbf{E}\left[|g(e^{v'} w) - g(e^v w)|\right].$$

We know that $0 \leq g(\lambda) \leq 1$ for all $\lambda \in [0, \infty)$, $g(\lambda)$ is nonincreasing with respect to λ a.s. and has a finite limit as $\lambda \to \infty$. Therefore, $g(\lambda)$ is a.s. uniformly continuous on $[0, \infty)$ implying that a.s.

$$\lim_{\kappa \to 0} \sup_{|v - v'| \leq \kappa, w \in [0,2]} |g(e^{v'} w) - g(e^v w)| = 0.$$

Hence, by the bounded convergence theorem

$$\sup_{|v - v'| \leq \kappa, w \in [0,2]} \mathbf{E}\left[|g(e^{v'} w) - g(e^v w)|\right] \leq \mathbf{E}\left[\sup_{|v - v'| \leq \kappa, w \in [0,2]} |g(e^{v'} w) - g(e^v w)|\right]$$

goes to zero as $\kappa \to 0$, which ends up the proof. ■

Introduce a sequence of generating functions

$$f_n(s) = f(s; \mathfrak{e}_n) \ = \ \sum_{k=0}^{\infty} \mathfrak{e}_n(\{k\}) s^k, \qquad 0 \leq s \leq 1,$$

specified by the environmental sequence $(\mathfrak{e}_1, \mathfrak{e}_2, ..., \mathfrak{e}_n, ...)$ and denote

$$f_{i,k} = f_{i+1} \circ \cdots \circ f_k \quad (i < k), \qquad f_{i,k} = f_i \circ \cdots \circ f_{k+1} \qquad (i > k), \quad f_{i,i} = Id.$$

Clearly,

$$\mathbf{P}_{\mathfrak{e}}(Z_n > 0) = 1 - f_{0,n}(0).$$

Let

$$W_{n,j} = \frac{1 - f_{n,j}(0)}{e^{S_n - S_j}}, \qquad G_{n,j} = -\frac{S_n - S_j + an}{\sigma\sqrt{n}}.$$

Observe that by monotonicity

$$\lim_{n \to \infty} W_{n,j} = W_j$$

and $W_j \stackrel{d}{=} W$, $j = 1, 2, ...$ where $\mathbf{P}(W \in (0, 1]) = 1$ in view of conditions (9) and Theorem 5 in [6] II, while by the central limit theorem

$$\lim_{n \to \infty} G_{n,j} = G$$

where G is a random variable having the standard normal distribution.

Since $f_{j,n}$ is distributed as $f_{n,j}$, we write

$$\begin{aligned}
&\mathbf{E}[\mathbf{P}_{\mathfrak{c}}\left(Z_n>0\right)e^{-\rho S_n};X_j\geq\delta an]=\mathbf{E}[(1-f_{0,n}(0))\,e^{-\rho S_n};X_j\geq\delta an]\\
&\quad=\mathbf{E}[(1-f_{0,j-1}(f_j(f_{n,j}(0))))e^{-\rho S_n};X_j\geq\delta an]\\
&\quad=\mathbf{E}[e^{-\rho S_j}\left(1-f_{0,j-1}(f_j(f_{n,j}(0)))\right)e^{-\rho(S_n-S_j)};X_j\geq\delta an]\\
&\quad=\mathbf{E}[e^{-\rho S_j}\left(1-f_{0,j-1}(f_j(1-e^{S_n-S_j}W_{n,j}))\right)e^{-\rho(S_n-S_j)};X_j\geq\delta an].
\end{aligned}$$

We state now the key result:

Lemma 24 *Assume that Hypotheses A and B are valid and let g be the function satisfying (40). Then, as $n\to\infty$*

$$b_n^{-1}\mathbf{E}[(1-f_{0,n}(0))\,e^{-\rho S_n};X_j\geq\delta an]\to\mathbf{E}\left[e^{-\rho S_{j-1}}\int_{-\infty}^{\infty}(1-f_{0,j-1}(g(e^vW)))\,e^{-\rho v}dv\right]$$

where S_{j-1} and W are independent and the right-hand side term is finite.

Proof. Fix M and K and for

$$-M\leq x\leq M,\ -K\leq v\leq K,\ 0\leq w\leq 1$$

consider the probability

$$T_{K,M,n}(x,v,w)=b_n^{-1}\mathbf{P}(an-M\sqrt{n}\leq X_1\leq an+x\sqrt{n},-K\leq S_n\leq v,W_{n,1}\leq w).$$

Denote $S_{n,m}=S_n-S_m,\ 0\leq m\leq n$. Clearly,

$$\begin{aligned}
T_{K,M,n}(x,v,w) &= b_n^{-1}\int_{-\infty}^{\infty}\mathbf{P}\left(S_{n,1}\in dy,W_{n,1}\leq w\right)\\
&\quad\times\mathbf{P}\left(X_1\in\left[an-M\sqrt{n},an+x\sqrt{n}\right]\cap\left[-K-y,v-y\right]\right).
\end{aligned}$$

Since $p_X\left(an+t\sqrt{n}\right)\sim b_n$ for $t\in[-M,M]$, we have

$$\begin{aligned}
T_{K,M,n}(x,v,w) &= (1+\varepsilon_n)\int_{-\infty}^{\infty}\mathbf{P}\left(S_{n,1}\in dy,W_{n,1}\leq w\right)\times\\
&\quad\times\left|\left[an-M\sqrt{n},an+x\sqrt{n}\right]\cap\left[-K-y,v-y\right]\right|.
\end{aligned}$$

Further,

$$\left[an-M\sqrt{n},an+x\sqrt{n}\right]\cap\left[-K-y,v-y\right]=\emptyset$$

if

$$an-M\sqrt{n}>v-y\text{ or }an+x\sqrt{n}<-K-y,$$

and

$$\left|\left[an-M\sqrt{n},an+x\sqrt{n}\right]\cap\left[-K-y,v-y\right]\right|=v+K$$

if

$$an - M\sqrt{n} \leq -K - y \leq v - y \leq an + x\sqrt{n}$$

or

$$v - an - x\sqrt{n} \leq y \leq -K - an + M\sqrt{n}.$$

Hence it follows that

$$\begin{aligned} T_{K,M,n}(x,v,w) = (1+\varepsilon_n)(v+K)\,\mathbf{P}\left((S_{n,1}+an) \in \left[v - x\sqrt{n}, -K + M\sqrt{n}\right], W_{n,1} \leq w\right) \\ +(1+\varepsilon_n)\Delta_{M,K,n} \end{aligned}$$

where

$$\begin{aligned} 0 \quad \leq \quad & \Delta_{M,K,n} \leq 2K\mathbf{P}\left(S_{n,1}+an \in \left[M\sqrt{n}-2K, M\sqrt{n}+2K\right], W_{n,1} \leq w\right) \\ & +2K\mathbf{P}\left(S_{n,1}+an \in \left[-x\sqrt{n}-2K, -x\sqrt{n}+2K\right], W_{n,1} \leq w\right) \\ \leq \quad & 2K\mathbf{P}\left(S_{n,1}+an \in \left[M\sqrt{n}-2K, M\sqrt{n}+2K\right]\right) \\ & +2K\mathbf{P}\left(S_{n,1}+an \in \left[-x\sqrt{n}-2K, -x\sqrt{n}+2K\right]\right) = \tilde{\varepsilon}_n. \end{aligned}$$

Passing to the limit as $n \to \infty$ and using the central limit theorem and monotonicity of $W_{n,1}$ in n we get

$$\lim_{n\to\infty} T_{K,M,n}(x,v,w) = (v+K)\,\mathbf{P}\left(-x \leq \sigma G \leq M, W \leq w\right) \tag{49}$$

where, as earlier, G is a random variable having the standard normal distribution.

We denote

$$\begin{aligned} t_{n,j}(dx,dv,dw) \quad &= \quad b_n^{-1}\mathbf{P}(X_j - na \in \sqrt{n}dx, S_{n,j} \in dv, W_{n,j} \in dw)e^{-\rho v} \\ &= \quad b_n^{-1}\mathbf{P}(X_1 - na \in \sqrt{n}dx, S_{n-j+1,1} \in dv, W_{n-j,1} \in dw)e^{-\rho v} \end{aligned}$$

and let

$$\mathcal{D}(K,M) = \{(x,v,w) : |x| \leq M, |v| \leq K, 0 \leq w \leq 1\}.$$

We know from the above that for each fixed j as $n \to \infty$

$$t_{n,j}(dx,dv,dw)I\left\{\mathcal{D}(K,M)\right\} \to e^{-\rho v}dv\mathbf{P}\left(\sigma G \in dx, W \in dw\right)I\left\{\mathcal{D}(K,M)\right\}.$$

For a deterministic probability generating function $h(s)$ set

$$H_n(h,x,v,w) = \mathbf{E}\left[1 - h\left(f\left(\exp\left\{-e^v we^{-X}\right\}\right)\right) \mid X = na + x\sqrt{n}\right]$$

and introduce the event

$$\mathcal{C}_{n,j}(x,v,w) = \left\{X_j = na + x\sqrt{n}, S_{n,j} = v, W_{n,j} = w\right\}.$$

Clearly, for $(x,v,w) \in \mathcal{D}(K,M)$

$$H_n(h,x,v,w) = \mathbf{E}\left[1 - h\left(f_j\left(\exp\left\{-e^{S_{n,j}}W_{n,j}e^{-X_j}\right\}\right)\right) \mid \mathcal{C}_{n,j}(x,v,w)\right].$$

The uniform convergence in (18) with respect to any compact set of λ from $[0,\infty)$ ensures that, uniformly for $|x| \le M$, $w \in [0,2]$ and $|v| \le K$ we have

$$|H_n(h,x,v,w) - \mathbf{E}\,[1 - h(g(e^v w)]\,| \le \varepsilon_n.$$

Let $w_\varepsilon = (1+\varepsilon)\,e^v w$ and

$$F_{n,j}(h,K,M;\varepsilon) = \int_{D(K,M)} t_{n,j}(dx,dv,dw) H_n(h,x,v,w_\varepsilon).$$

It follows from the estimates above that

$$\left| F_{n,j}(h,K,M;\varepsilon) - \int_{D(K,M)} t_{n,j}(dx,dv,dw)\mathbf{E}\,[1-h(g(e^v w_\varepsilon)] \right| \le \varepsilon_n \int_{D(K,M)} t_{n,j}(dx,dv,dw) \le C\varepsilon_n.$$

Besides, in view of (49)

$$\begin{aligned}
&\lim_{n\to\infty} \int_{D(K,M)} t_{n,j}(dx,dv,dw)\mathbf{E}\,[1-h(g(e^v w_\varepsilon)] \\
&\quad = \int_{D(K,M)} e^{-\rho v}\mathbf{E}\,[1-h(g(e^v w_\varepsilon)]\,dv\mathbf{P}\,(W\in dw, \sigma G \in dx) \\
&\quad = \int_{-K}^{K} e^{-\rho v}dv \int_0^1 \mathbf{E}\,[1-h(g(e^v w_\varepsilon)]\,\mathbf{P}\,(W\in dw, \sigma G \in [-M,M]) \\
&\qquad =: F(h,K,M;\varepsilon).
\end{aligned}$$

To proceed further we introduce the event

$$\mathcal{T}_{N,K,M}(j) = \left\{ |S_{j-1}| \le N,\ |S_n - S_{j-1}| \le K,\ |X_j - an| \le M\sqrt{n} \right\}.$$

By Corollary 19 and Lemmas 13, 21, and 22

$$\begin{aligned}
&\mathbf{E}\left[(1-f_{0,n}(0))\,e^{-\rho S_n}; X_j \ge \delta an\right] \\
= \;& \mathbf{E}[(1-f_{0,n}(0))\,e^{-\rho S_n}; |S_{j-1}| \le N, X_j \ge \delta an] + \varepsilon_{N,n} b_n \\
= \;& \mathbf{E}[(1-f_{0,n}(0))\,e^{-\rho S_n}; |S_{j-1}| \le N, |S_n - S_{j-1}| \le K, X_j \ge \delta an] + \varepsilon_{N,K,n} b_n \\
= \;& \mathbf{E}[(1-f_{0,n}(0))\,e^{-\rho S_n}; \mathcal{T}_{N,K,M}(j)] + \varepsilon_{N,K,M,n} b_n \qquad (50)
\end{aligned}$$

where

$$\lim_{N\to\infty} \limsup_{K\to\infty} \limsup_{M\to\infty} \limsup_{n\to\infty} |\varepsilon_{N,K,M,n}(j)| = 0. \qquad (51)$$

Thus, to prove the lemma it is sufficient to study only the quantity

$$\begin{aligned}
&\mathbf{E}\left[(1-f_{0,n}(0))\,e^{-\rho S_n}; \mathcal{T}_{N,K,M}(j)\right] \\
&\quad = \mathbf{E}[[1-f_{0,j-1}(f_j(f_{n,j}(0)))]e^{-\rho S_j} e^{-\rho(S_n - S_j)}; \mathcal{T}_{N,K,M}(j)].
\end{aligned}$$

Clearly,

$$\left\{X_j \geq an - M\sqrt{n}, |S_n - S_{j-1}| \leq K\right\} \Longrightarrow \left\{S_n - S_j \leq K - an + M\sqrt{n}\right\}.$$

This, in view of the inequality

$$e^{S_n - S_j} W_{n,j} = 1 - f_{n,j}(0) \leq e^{S_n - S_j}$$

and the representation $e^{-x} = 1 - x + o(x)$, $x \to 0$, means that if the event $\mathcal{T}_{N,K,M}(j)$ occurs then, for any $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that for all $n \geq n_0$

$$e^{-(1+\varepsilon)(1-f_{n,j}(0))} \leq f_{n,j}(0) \leq e^{-(1-f_{n,j}(0))}.$$

Thus,

$$\begin{aligned}
&\mathbf{E}[[1 - f_{0,j-1}(f_j(e^{-(1-f_{n,j}(0))}))]e^{-\rho S_j} e^{-\rho(S_n - S_j)}; \mathcal{T}_{N,K,M}(j)] \\
&\quad \leq b_n^{-1} \mathbf{E}\left[(1 - f_{0,n}(0))\, e^{-\rho S_n}; \mathcal{T}_{N,K,M}(j)\right] \\
&\quad \leq \mathbf{E}[\Big(1 - f_{0,j-1}(f_j(e^{-(1+\varepsilon)(1-f_{n,j}(0))}))\Big)\, e^{-\rho S_j} e^{-\rho(S_n - S_j)}; \mathcal{T}_{N,K,M}(j)].
\end{aligned}$$

Hence, denoting by $\mathcal{F}_{j-1}$ the σ-algebra generated by the sequence

$$(f_1, ..., f_{j-1}; S_1, ..., S_{j-1})$$

and setting

$$\hat{F}_{n,j}(f_{0,j-1}, K, M; \varepsilon) = \mathbf{E}\left[F_{n,j}(f_{0,j-1}, K, M; \varepsilon) | \mathcal{F}_{j-1}\right]$$

and

$$\hat{F}(f_{0,j-1}, K, M; \varepsilon) = \mathbf{E}\left[F(f_{0,j-1}, K, M; \varepsilon) | \mathcal{F}_{j-1}\right],$$

we get

$$\begin{aligned}
&\mathbf{E}\left[e^{-\rho S_{j-1}} \hat{F}_{n,j}(f_{0,j-1}, K, M; 0); |S_{j-1}| \leq N\right] \\
&\quad \leq b_n^{-1} \mathbf{E}\left[(1 - f_{0,n}(0))\, e^{-\rho S_n}; \mathcal{T}_{N,K,M}(j)\right] \\
&\quad \leq \mathbf{E}[e^{-\rho S_{j-1}} \hat{F}_{n,j}(f_{0,j-1}, K, M; \varepsilon); |S_{j-1}| \leq N].
\end{aligned}$$

Now the dominated convergence theorem gives for any fixed $\varepsilon \in [0, 1)$

$$\begin{aligned}
&\lim_{n\to\infty} \mathbf{E}[e^{-\rho S_{j-1}} \hat{F}_{n,j}(f_{0,j-1}, K, M; \varepsilon); |S_{j-1}| \leq N] \\
&\quad = \mathbf{E}[e^{-\rho S_{j-1}} \lim_{n\to\infty} \hat{F}_{n,j}(f_{0,j-1}, K, M; \varepsilon); |S_{j-1}| \leq N] \\
&\quad = \mathbf{E}[e^{-\rho S_{j-1}} \hat{F}(f_{0,j-1}, K, M; \varepsilon); |S_{j-1}| \leq N].
\end{aligned}$$

Hence we deduce that

$$\begin{aligned}
&\mathbf{E}\left[e^{-\rho S_{j-1}} \hat{F}(f_{0,j-1}, K, M; 0); |S_{j-1}| \leq N\right] \\
&\quad \leq \liminf_{n\to\infty} b_n^{-1} \mathbf{E}\left[(1 - f_{0,n}(0))\, e^{-\rho S_n}; \mathcal{T}_{N,K,M}(j)\right] \\
&\quad \leq \limsup_{n\to\infty} b_n^{-1} \mathbf{E}\left[(1 - f_{0,n}(0))\, e^{-\rho S_n}; \mathcal{T}_{N,K,M}(j)\right] \\
&\quad \leq \mathbf{E}[e^{-\rho S_{j-1}} \hat{F}(f_{0,j-1}, K, M; \varepsilon); |S_{j-1}| \leq N].
\end{aligned}$$

Letting $\varepsilon \downarrow 0$ we see that

$$\lim_{n\to\infty} b_n^{-1}\mathbf{E}\left[(1-f_{0,n}(0))\,e^{-\rho S_n};\mathcal{T}_{N,K,M}(j)\right] = \mathbf{E}\left[e^{-\rho S_{j-1}}\hat{F}(f_{0,j-1},K,M;0);|S_{j-1}|\le N\right].$$

Since the right-hand side of this equality is monotone in K, M and N, we conclude by letting sequentially M, K and N to infinity and taking into account the estimates $0 \le W \le 1$ that

$$\begin{aligned}&\lim_{M\to\infty}\lim_{n\to\infty} b_n^{-1}\mathbf{E}\left[(1-f_{0,n}(0))\,e^{-\rho S_n};\mathcal{T}_{N,K,M}(j)\right]\\ &\quad=\mathbf{E}\left[e^{-\rho S_{j-1}}\int_{-K}^{K}e^{-\rho v}dv\int_0^1\mathbf{E}\left[1-f_{0,j-1}(g(e^v w)|\mathcal{F}_{j-1}\right]\mathbf{P}\left(W\in dw\right);|S_{j-1}|\le N\right]\end{aligned}$$

and

$$\begin{aligned}&\lim_{N\to\infty}\lim_{K\to\infty}\lim_{M\to\infty}\lim_{n\to\infty} b_n^{-1}\mathbf{E}\left[(1-f_{0,n}(0))\,e^{-\rho S_n};\mathcal{T}_{N,K,M}(j)\right]\\ =\;&\mathbf{E}\left[e^{-\rho S_{j-1}}\int_{-\infty}^{\infty}e^{-\rho v}dv\int_0^1\mathbf{E}\left[1-f_{0,j-1}(g(e^v w)|\mathcal{F}_{j-1}\right]\mathbf{P}\left(W\in dw\right)\right]\\ =\;&\mathbf{E}\left[e^{-\rho S_{j-1}}\int_{-\infty}^{\infty}e^{-\rho v}\left(1-f_{0,j-1}(g(e^v W)\right)dv\right].\end{aligned}$$

It remains to check that the right-hand side of this equality is positive and finite.

Positivity follows from conditions (9), since under these conditions $W > 0$ with probability 1 according to Theorem 5 [6], II. Finiteness is evident, since $\mathbf{E}\left[(1-f_{0,n}(0))\,e^{-\rho S_n}\right] = O\left(b_n\right)$ by Corollary 17. This gives the whole result. ■

Now we have an important corollary, which, in fact, proves Theorem 1 with the explicit form of the constant C_0 mentioned in the statement of the theorem.

Corollary 25 *Given Hypotheses A and B,*

$$\mathbb{P}\left(Z_n>0\right)\sim C_0 m^n b_n \sim C_0\rho m^{n-1}\mathbb{P}\left(X>an\right)e^{an\rho}$$

where

$$\begin{aligned}C_0 &= \sum_{j=1}^{\infty}\mathbf{E}\left[e^{-\rho S_{j-1}}\int_{-\infty}^{\infty}\left(1-f_{0,j-1}(g(e^v W))\right)e^{-\rho v}dv\right]\\ &= \sum_{j=0}^{\infty}\left(\mathbb{E}\left[e^{\rho X}\right]\right)^j\int_{-\infty}^{\infty}\mathbb{E}\left[1-f_{0,j}(g(e^v W))\right]e^{-\rho v}dv.\end{aligned}$$

Proof. It follows from (46) that for each fixed j

$$\mathbf{E}\left[(1-f_{0,n}(0))\exp(-\rho S_n);\mathcal{D}_N(j)\right] = \mathbf{E}[(1-f_{0,n}(0))\,e^{-\rho S_n};X_j\ge\delta an]+\varepsilon_{N,n}b_n.$$

Using this fact and Lemmas 24 and 20 we get

$$\lim_{n\to\infty} m^{-n} b_n^{-1} \mathbb{P}\left(Z_n > 0\right) = \lim_{n\to\infty} m^{-n} b_n^{-1} \mathbb{E}\left[(1 - f_{0,n}(0))\right]$$
$$= \lim_{n\to\infty} b_n^{-1} \mathbf{E}\left[(1 - f_{0,n}(0)) \exp(-\rho S_n)\right] = C_0.$$

To complete the proof it remains to observe that in view of (15)

$$b_n = \beta \frac{\mathbf{P}\left(X > an\right)}{an} \sim \frac{1}{m} \frac{l_0(an)}{(an)^{\beta+1}} = \frac{1}{m} \frac{l_0(an)}{(an)^{\beta+1}} e^{-\rho an} e^{\rho an}$$
$$\sim \frac{\rho}{m} e^{\rho an} \int_{an}^{\infty} p_X(x) dx = \frac{\rho}{m} e^{\rho an} \mathbb{P}\left(X > an\right).$$

5.4 Conditional limit theorem

In this section we prove Theorem 2. Let

$$W_{n,j}(s) = \frac{1 - f_{n,j}(s)}{e^{S_n - S_j}},\ s \in [0, 1).$$

By monotonicity

$$W_{n,j}(s) \overset{n\to\infty}{\longrightarrow} W_j(s)$$

and $W_j(s) \overset{d}{=} W(s)$, $j = 1, 2, ...$ where $\mathbf{P}(W(s) \in (0, 1]) = 1$ thanks to [6] II Theorem 5.

Similarly to Lemma 24 one can show that, as $n \to \infty$

$$\lim_{n\to\infty} b_n^{-1} \mathbf{E}[(1 - f_{0,n}(s))\, e^{-\rho S_n}]$$
$$= \lim_{n\to\infty} b_n^{-1} \sum_{j=1}^{\infty} \mathbf{E}[(1 - f_{0,n}(s))\, e^{-\rho S_n}; X_j \geq \delta an]$$
$$= \mathbf{E}\left[e^{-\rho S_{j-1}} \int_{-\infty}^{\infty} (1 - f_{0,j-1}(g(e^v W(s))))\, e^{-\rho v} dv\right] = \Omega_0(s).$$

Hence we get

$$\lim_{n\to\infty} \mathbb{E}\left[s^{Z_n} | Z_n > 0\right] = 1 - \lim_{n\to\infty} \frac{\mathbf{E}[(1 - f_{0,n}(s))\, e^{-\rho S_n}]}{\mathbf{E}[(1 - f_{0,n}(0))\, e^{-\rho S_n}]}$$
$$= 1 - C_0^{-1} \Omega_0(s) =: \Omega(s).$$

Theorem 2 is proved.

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