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IDENTIFICATION OF LINEAR SYSTEMS USING MA AND ARMA MODEL IN DIOID

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Abstract: The behavior of timed event graphs can linearly be described by max algebra. The impulse response of such systems can be decomposed into a finite sum of simple elements (elementary systems which present some analogies with the classical first order systems in conventionnal linear system theory). This paper deals with two identification methods of several simple elements. The identification methods performances and the complexity of their algorithms are compared.
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Résumé : Le comportement des graphes d'événements temporisé peut se décrire de manière linéaire dans l'algèbre (max, +). La réponse impulsionnelle de ces systèmes peut se décomposer en une somme finie d'éléments simples (systèmes élémentaires présentant des similitudes avec les systèmes linéaires conventionnels du premier ordre). Ce papier s'intresse à l'identification de plusieurs éléments simples en employant deux méthodes diffrentes. Les performances de ces méthodes ainsi que la complexité de leurs algorithmes sont comparées.

Keywords: Discrete-event systems, Identification, Dioid, Linear systems, Timed event graphs.

1. INTRODUCTION

The time behavior of Timed Event Graphs (TEG) (a subclass of Petri nets involving synchronisation) is described by linear equations in dioids which allows analogies with the conventionnal linear system theory. This algebraic approach has been used to study performance evaluation (Baccelli *et al.*, 1992), (Cohen *et al.*, 1989) or control design (Boimond, 1993), (Boimond and Ferrier, 1996), (Hardouin *et al.*, 1997) which need to have a mathematical model.

Most works on modelling deal with the model minimal realisation (Olsder, 1987), (Schutter, 1996). This paper does not care about this problem. We are interested in developing a simple method to compute the parameters of the transfer relation, i.e., the coefficients of its impulse response, using its input-output behavior. The modelling method is based on the decomposition property of a linear system into a finite sum of simple elements (under the form $t\gamma^n(\tau\gamma^\nu)^*$) which is reminiscent with decomposition of a linear system into a sum of first order systems in conventional linear system theory.

Identification methods of one simple element expressed as an ARMA model have been proposed in (Gallot *et al.*, 1997*b*) and (Gallot *et al.*, 1997*a*). In this paper, we propose an approach based on a MA model. Estimation performances of both approches are compared. Moreover,

we extend our methods to the simultaneous identification of several simple elements.

The paper is organised as follows: in section 2, we remind elementary notions on dioid and particularly on the max-plus algebra. We show in section 3 the possible representation of a TEG by linear equations over $\overline{\mathbb{R}}_{max} = \{\mathbb{R} \cup \pm\infty,\ max,\ +\}$. Notions on transfer relation will be defined as well. The fourth section deals with the two identification methods of one simple element and their extensions to the simultaneous identification of several simple elements. A short example is given in section 5.

2. DIOIDS

2.1 *Definitions*

A *dioid* $(\mathcal{D}, \oplus, \otimes)$ is a semiring in which $\oplus$ is idempotent ($\forall a.\ a \oplus a = a$), neutral elements of $\oplus$, $\otimes$ are noted ε and e respectively. Moreover, thanks to the idempotency of $\oplus$, $\mathcal{D}$ is endowed with an order relation defined by $a \preceq b \Leftrightarrow a \oplus b = b$.

If $\mathcal{D}$ is closed for infinite sums and if multiplication distributes over infinite sums too then $\mathcal{D}$ is said *complete*. Then the sum of all elements of $\mathcal{D}$, denoted T, is absorbing for the addition.
If $\mathcal{D}$ is complete then a lower bound ($\succeq \varepsilon$) can be defined for any subset $\mathcal{C}$ of $\mathcal{D}$. Let us define $a \wedge b$ the lower bound of $\mathcal{C} = \{a, b\}$. We have $a \succeq b \Leftrightarrow a = a \oplus b \Leftrightarrow b = a \wedge b$.

A mapping f from an ordered set $\mathcal{D}$ into an ordered set $\mathcal{C}$ is *isotone* (resp. *antitone*) if $\forall a, b \in \mathcal{D},\ a \succeq b \Rightarrow f(a) \succeq f(b)$ (resp. $f(a) \preceq f(b)$).

2.2 *The linear equation* $x = ax \oplus b$

Theorem 1. ((Cohen *et al.*, 1989)). The least solution of $x = ax \oplus b$ is given by $x = a^*b$ with $a^* = \bigoplus_{n \in \mathbb{N}} a^n$.

2.3 *Residuation*

A complete description of the Residuation theory can be found in (Blyth and Janowitz, 1972).

Theorem 2. ((Baccelli *et al.*, 1992), § 4.4.2). Let f be an isotone mapping from the complete dioid $\mathcal{D}$ into the complete dioid $\mathcal{C}$. The following three statments are equivalent:

- For all $b \in \mathcal{C}$, there exists a greatest subsolution to the equation $f(x) = b$.
- $f(\varepsilon) = \varepsilon$ and f is lower semi-continuous, i.e.,

$$f(\bigoplus_{x \in \mathcal{X} \subset \mathcal{D}} x) = \bigoplus_{x \in \mathcal{X} \subset \mathcal{D}} f(x).$$

- There exists an isotone mapping $f^\sharp$ from $\mathcal{C}$ into $\mathcal{D}$ such that: $f \circ f^\sharp \preceq I_{\mathcal{C}}$ (identity of $\mathcal{C}$); $f^\sharp \circ f \succeq I_{\mathcal{D}}$ (identity of $\mathcal{D}$).

When f satisfies these properties, it is said to be residuated and $f^\sharp$ is called its residual.

This theorem can be applied to the mapping $x \mapsto a \otimes x$ in a complete dioid. The residual mapping will be denoted $y \mapsto a \backslash y = \frac{y}{a}$.

Starting from a "scalar" dioid $\mathcal{D}$, consider $m \times n$ matrices with entries in $\mathcal{D}$. The set of $m \times n$ matrices is also a dioid denoted $\mathcal{D}^{m \times n}$.

Theorem 3. ((Baccelli *et al.*, 1992), § 4.6.2). Let $A \in \mathcal{D}^{m \times n}$ and $B \in \mathcal{D}^{m \times p}$, the greatest subsolution of equation $AX = B$ is noted $C = A \backslash B$:

$$C_{ij} = \bigwedge_{k=1}^{m} (A_{ki} \backslash B_{kj}),\ i = 1 \text{ to } n,\ j = 1 \text{ to } p.$$

2.4 *An example: the dioid* $\overline{\mathbb{R}}_{max}$

The algebraic structure $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ with max and $-$ as $\oplus$ and $\otimes$ respectively, is called the *max-plus algebra* (noted $\mathbb{R}_{max}$). The set $(\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}, \oplus, \otimes)$ is the complete dioid denoted $\overline{\mathbb{R}}_{max}$. By convention, we have $(-\infty) - (-\infty) = -\infty$. The element $+\infty$ is also denoted T.

Let $r \in \mathbb{R}$. The r^{th} power of $x \in \overline{\mathbb{R}}_{max}$ is noted $x^{\otimes r}$ or more simply x^r and corresponds to $r \times x$ in conventionnal algebra. If a, b are finite values, then $b \backslash a = \frac{a}{b} = a - b$. If $a, b, c, d \in \overline{\mathbb{R}}_{max}$, we have the following basic properties:

$$\frac{a \wedge b}{c} = \frac{a}{c} \wedge \frac{b}{c} \tag{1}$$

$$\frac{\frac{c}{a}}{b} = \frac{c}{ab} \tag{2}$$

$$(a \wedge b)c \preceq (ac) \wedge (bc) \tag{3}$$

$$(\bigwedge_l a_l) \otimes (\bigwedge_k b_k) \preceq \bigwedge_l (a_l \otimes b_l) \tag{4}$$

where $\wedge$ represents the *min* operator.

The proofs of eqs. (1), (2) and (3) are given in (Baccelli *et al.*, 1992), § 4.3 and 4.4. A direct consequence of eq. (3) is: $(\bigwedge_l a_l) \otimes (\bigwedge_k b_k) \preceq \bigwedge_l (a_l \otimes (\bigwedge_k b_k))$. Moreover $\forall l.\ \bigwedge_l b_k \preceq b_l$, hence we have, by isotony of the law $\otimes$, $\bigwedge_l (a_l \otimes (\bigwedge_k b_k)) \preceq \bigwedge_l (a_l \otimes b_l)$, which leads to eq. (4).

3. LINEAR DESCRIPTION OF TEG

Event graphs are a particular class of Petri nets in which each place has exactly one upstream and one downstream

transition. In a TEG, a delay is associated with each place. This delay is the time a token must stay in the place before contributing to the enabling of the downstream transition.

For a transition labelled x, we define the nondecreasing mapping called *dater* $k \mapsto x(k)$ where $x(k)$ is the date at which the transition x has been fired for the k^{th} time.

We introduce now the γ-transform of a dater (analogous to the z-transform in conventional linear system theory), which can be interpreted as a backward shift operator, formally, $\gamma\, v(k) = v(k-1)$. We note $V(\gamma) = \bigoplus_{k\in\mathbb{Z}} \gamma^k v(k)$ the γ transform of dater v.

TEG are described in $\overline{\mathbb{R}}_{max}[[\gamma]]$ (dioid of formal series of γ with power in $\mathbb{Z}$ and coefficients in $\overline{\mathbb{R}}_{max}$) by the following standard state equations:

$$\begin{cases} X(\gamma) = A\gamma X(\gamma) \oplus B\, U(\gamma) \\ Y(\gamma) = C X(\gamma) \end{cases}$$

where A, B, C are matrices.

From theorem 1, we deduce the transfer relation:

$$Y(\gamma) = C(A\gamma)^* B\, U(\gamma) = H(\gamma)\, U(\gamma)$$

where $H(\gamma)$ is called the *impulse response* of the system obtained with $U(\gamma) = e$ (transition u is infinitely fired at time 0).

Moreover, the impulse response $H(\gamma)$ can always be expressed as a *finite* sum of simple elements (see (Gallot *et al.*, 1997*a*) and (Gallot *et al.*, 1997*b*)):

$$H(\gamma) = \bigoplus_{i=1 \text{ to } l} t_i \gamma^{n_i} (\tau_i \gamma^{\nu_i})^*. \qquad (5)$$

A more detailed description on series can be found in (Baccelli *et al.*, 1992), (Gaubert, 1992), (Hardouin *et al.*, 1997).

4. IDENTIFICATION

4.1 *Motivation*

Systems (production lines for instance) are supposed represented by linear equations over $\overline{\mathbb{R}}_{max}$. We propose to transpose a classical method used in conventional linear systems theory into the max-plus algebra. We suppose that the structure of the model is known (n_i, ν_i of simple elements are fixed) and we estimate its parameters (t_i, τ_i) in order to minimize the error criterion. This method is well known as the *model method* (Fig. 1).

A quadratic criterion is used to compute the model parameters in the least square method (Ljung, 1987). We define an analogous criterion in order to identify the searched parameters when the model is given under an ARMA or a MA equation.

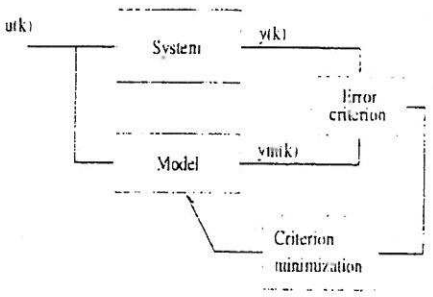

Fig. 1. Identification method.

In the following, to clearly describe the proposed identification methods, we suppose that the first non nul value ($\neq -\infty$) of u is $u(0)$.

4.2 *Input influence*

From identification point of view, it is obvious that the estimation quality of the model depends on the input behavior. For example, the drastic case $u(k) = T$, $\forall k$ does not allow computing the right parameters since $y(k) = T$, $\forall k$. In the opposite, the input $u(k) = e$, $\forall k \geqslant 0$ is such that output behaves as an impulse response. In other words, it makes appear all the impulse response parameters on the output, which allows an exact identification. An interesting limit to define is the greatest input $U(\gamma)$ such that the output of the system behaves as an impulse response, or formally $H(\gamma) \otimes U(\gamma) = H(\gamma)$. The greatest solution $U(\gamma) = H(\gamma) \backslash H(\gamma)$ of this relation is given by theorem 2.

The computation of $H(\gamma) \backslash H(\gamma)$ is not possible since the system impulse response $H(\gamma)$ is unknown. However, we can compute an approach limit by replacing $H(\gamma)$ by a supposed initially known model.

4.3 *Identification of one simple element*

A system represented by a TEG can always be written as a finite sum of simple elements (see eq. (5)). Before dealing with the simultaneous identification of several simple elements, we propose to compute the parameters t and τ (ν and n given) of one simple element through two identification methods.

Let us consider the output $Y_m(\gamma)$ of a simple element (the model) to an input $U(\gamma)$:

$$Y_m(\gamma) = t\gamma^n (\tau\gamma^\nu)^*\, U(\gamma). \qquad (6)$$

Developing the $(\tau\gamma^\nu)^*$ term, we obtain:

$$Y_m(\gamma) = \left[t\gamma^n \oplus t\tau\gamma^{(n+\nu)} \oplus t\tau^2\gamma^{(n+2\nu)} \oplus \ldots \right] U(\gamma)$$

which corresponds in $\overline{\mathbb{R}}_{\max}$ to the following MA equation:

$$y_m(k) = tu(k-n) \oplus t\tau u(k-n-\nu) \oplus t\tau^2 u(k-n-2\nu) \oplus \ldots \quad (7)$$

Applying theorem 1 to eq. (6), we obtain the least solution:

$$Y_m(\gamma) = t\gamma^n U(\gamma) \oplus \tau\gamma^\nu Y_m(\gamma)$$

which leads in $\overline{\mathbb{R}}_{\max}$ to the particular ARMA equation:

$$y_m(k) = t\,u(k-n) \oplus \tau\, y_m(k-\nu). \quad (8)$$

By analogy with the identification method based on the prediction error of the conventional linear system theory, we consider the gap between the system output y and the model output y_m at event k, i.e.,

$$Er(k) = y(k) - y_m(k). \quad (9)$$

4.3.1. *Identification using the ARMA equation* We are interested in the error $Er(k)$ (eq. (9)) when $y_m(k)$ is computed from the simple element ARMA model (eq. (8)). Let us replace t, τ, y_m by t_{ar}, τ_{ar}, y_{ar} respectively and consider the model $y_{ar}(k) = \Phi_k^T \hat{\Theta}_{ar}$ with $\Phi_k^T = [u(k-n) \quad y_{ar}(k-\nu)]$ the regressive vector at event k, where y_{ar} is the model output, u the input and $\hat{\Theta}_{ar} = [\hat{t}_{ar} \quad \hat{\tau}_{ar}]^T$ the estimated parameters vector.

For the set of measured data up to event N ($N \geqslant n$), we obtain the matrix expression:

$$Er = Y - (M \otimes \hat{\Theta}_{ar}),$$

with $Er = [Er(n) \ldots Er(N)]^T$ the prediction errors vector, $Y = [y(n) \ldots y(N)]^T$ the measured system outputs and $M = [\Phi_n \ldots \Phi_N]^T$ the model regressive vectors matrix.

We consider the criterion J defined as:

$$J(\hat{\Theta}_{ar}) = \bigoplus_{l=n}^{N} Er(l) \text{ with } Er(l) \succeq 0.$$

A basic result of the Residuation theory (Cuninghame-Green, 1979) shows that $M \backslash Y$, the greatest solution of inequation $M \otimes \hat{\Theta}_{ar} \preceq Y$, minimizes criterion J.

We compute $\hat{\Theta}_{ar} = M \backslash Y$ using theorem 3:

$$\hat{\Theta}_{ar_j} = \bigwedge_{l=1}^{N+1-n} \frac{Y_l}{M_{lj}} = \bigwedge_{l=n}^{N} \frac{y(l)}{\Phi_{lj}^T} \quad j = 1, 2,$$

or equivalently:

$$\hat{\Theta}_{ar} = \begin{bmatrix} \hat{t}_{ar} \\ \hat{\tau}_{ar} \end{bmatrix} = \begin{bmatrix} \bigwedge_{l=n}^{N} \dfrac{y(l)}{u(l-n)} \\ \bigwedge_{l=n}^{N} \dfrac{y(l)}{y_{ar}(l-\nu)} \end{bmatrix}. \quad (10)$$

Practical computation of $\hat{\Theta}_{ar}$ is given by the following recursive algorithm:

Algorithm 1

$\underset{(n-1)}{\hat{\Theta}_{ar}} = [-\infty \quad +\infty]^T$,
FOR $l = n$ to N
  FOR $i = n$ to $l-\nu$, $(l-\nu \geqslant n)$
    $\underset{(l)}{y_{ar}}(i) = [u(i-n) \quad \underset{(l)}{y_{ar}}(i-\nu)] \; \underset{(l-1)}{\hat{\Theta}_{ar}}$
  END
  $\underset{(l)}{\hat{\Theta}_{ar}} = \underset{(l-1)}{\hat{\Theta}_{ar}} \wedge [u(l-n) \backslash y(l) \quad \underset{(l)}{y_{ar}}(l-\nu) \backslash y(l)]^T$
END
$\underset{(N)}{\hat{\Theta}_{ar}} = \underset{(N)}{\hat{\Theta}_{ar}}$

where the estimation $\underset{(l)}{\hat{\Theta}_{ar}} = [\underset{(l)}{\hat{t}_{ar}} \quad \underset{(l)}{\hat{\tau}_{ar}}]^T$ is obtained using the system outputs $y(n), \ldots, y(l)$. The input u being such that $u(j) = -\infty$ for $j < 0$, we have $\underset{(l)}{y_{ar}}(j) = -\infty$ for $j < n$.

Remark 4. By construction the series $\underset{(l)}{\hat{\Theta}_{ar}}$, $l = n$ to N, is decreasing.

Because at each step l ($l \geqslant n + 2\nu$), the computation of $\underset{(l)}{y_{ar}}(l-\nu)$ needs to calculate, using eq. (8), $\underset{(l)}{y_{ar}}(l-2\nu), \ldots, \underset{(l)}{y_{ar}}(l-v\nu)$ with $v = \lfloor \frac{l-n}{\nu} \rfloor$ ($\lfloor x \rfloor$ is the truncated part of x), the complexity of this algorithm is $O(N^2)$. In order to reduce the complexity to $O(N)$, the following algorithm proposes to only evaluate the previous model output as it is usually done in conventionnal linear systems theory:

Algorithm 2

$\underset{(n-1)}{\hat{\Theta}_{ar}} = [-\infty \quad +\infty]^T$,
FOR $l = n$ to N
  $\underset{(l)}{y_{ar}}(l-\nu) = \underset{(l-1)}{\hat{t}_{ar}}\, u(l-n-\nu) \oplus \underset{(l-1)}{\hat{\tau}_{ar}}\, \underset{(l-1)}{y_{ar}}\,(l-2\nu)$
  $\underset{(l)}{\hat{\Theta}_{ar}} = \underset{(l-1)}{\hat{\Theta}_{ar}} \wedge [u(l-n) \backslash y(l) \quad \underset{(l)}{y_{ar}}(l-\nu) \backslash y(l)]^T$
END
$\underset{(N)}{\hat{\Theta}_{ar}} = \underset{(N)}{\hat{\Theta}_{ar}}$

with $\underset{(l)}{y_{ar}}(j) = -\infty$ for $j < n$.

Remark 5. If the regressive vector at event k is such that $\Phi_k^T = [u(k-n) \quad y(k-\nu)]$, where y is the system output, we obtain a worse estimated parameter vector (see (Gallot *et al.*, 1997*b*), (Gallot *et al.*, 1997*a*)).

4.3.2. *Identification using the MA equation* The error $Er(k)$ (eq. (9)) is now calculated using the MA form of the model (eq. (7)). We replace t, τ, y_m by t_{ma}, τ_{ma} and y_{ma} respectively and consider $y_{ma}(k) = \Psi_k^T \hat{\Delta}_{ma}$

with $\hat{\Delta}_{ma} = [\hat{t}_{ma} \quad \hat{t}_{ma}\hat{\tau}_{ma} \quad \hat{t}_{ma}\hat{\tau}_{ma}^2 \quad \ldots]^T$ the estimated parameters vector and $\Psi_k^T = [u(k-n) \quad u(k-n-\nu) \quad u(k-n-2\nu) \quad \ldots]$ the regressive vector at event k.

For the set of measured data up to event N ($N \geqslant n$), we obtain the matrix expression:

$$Er = Y - (M' \otimes \hat{\Delta}_{ma}),$$

with $M' = [\Psi_n \ldots \Psi_N]^T$ the measured regressive vector matrix.

The minimization of $\sum_{k=n}^{N} Er(k)$ is a linear programming problem when the input is an impulse ((Gallot *et al.*, 1997*a*)).

Fixing $\hat{t}_{ma} = \hat{t}_{ar} = \bigwedge_{l=n}^{N} \frac{y(l)}{u(l-n)}$, the residuation theory allows us to minimize $J_1(\hat{\tau}_{ma}) = \bigoplus_{k=n}^{N} Er(k)$ for any input. $\hat{t}_{ma}$ is the greatest parameter such that $y(l) \succeq \hat{t}_{ma}u(l-n)$, $\forall l \in [n, N]$. Under this assumption, the resulting model is always causal.

Proposition 6. The estimated parameter $\hat{\tau}_{ma}$ which minimizes criterion $J_1(\hat{\tau}_{ma}) = \bigoplus_{k=n}^{N} Er(k)$ ($Er(k) \succeq 0$, $\forall k$) with $\hat{t}_{ma} = \hat{t}_{ar}$ is given by:

$$\hat{\tau}_{ma} = \bigwedge_{i=1}^{p} \left(\bigwedge_{l=n}^{N} \frac{y(l)}{\hat{t}_{ma}\, u(l-n-i\nu)} \right)^{\otimes \frac{1}{i}} \tag{11}$$

where $p = \left\lfloor \frac{N-n}{\nu} \right\rfloor$.

Proof The problem is to obtain $\hat{\tau}_{ma}$ such that:
$y(l) \succeq \hat{t}_{ma}u(l-n) \oplus \hat{t}_{ma}\hat{\tau}_{ma}u(l-n-\nu) \oplus \ldots$, $\forall l \in [n, N]$ with $\hat{t}_{ma}$ fixed. Then, the expression of $y(l)$ becomes:

$$\begin{aligned} y(l) \succeq\ & \hat{t}_{ma}\hat{\tau}_{ma}u(l-n-\nu) \oplus \\ & \hat{t}_{ma}\hat{\tau}_{ma}^2 u(l-n-2\nu) \oplus \ldots, \ \forall l \in [n, N] \end{aligned} \tag{12}$$

The residuation theory leads to the greatest vector which satisfies all the following equalities:

$$\begin{bmatrix} \hat{\tau}_{ma_1} \\ \hat{\tau}_{ma_2}^2 \\ \vdots \\ \hat{\tau}_{ma_p}^p \end{bmatrix} = \begin{bmatrix} \bigwedge_{l=n}^{N} \hat{t}_{ma}u(l-n-\nu) \backslash y(l) \\ \bigwedge_{l=n}^{N} \hat{t}_{ma}u(l-n-2\nu) \backslash y(l) \\ \vdots \\ \bigwedge_{l=n}^{N} \hat{t}_{ma}u(l-n-p\nu) \backslash y(l) \end{bmatrix}$$

or:

$$\hat{\tau}_{ma_i} = \left(\bigwedge_{l=n}^{N} \frac{y(l)}{\hat{t}_{ma}\, u(l-n-i\nu)} \right)^{\otimes \frac{1}{i}}.$$

The greatest $\hat{\tau}_{ma}$ satisfying equation (12) is $\hat{\tau}_{ma} = \bigwedge_{i=1}^{p} \hat{\tau}_{ma_i}$ (due to the isotony of function $x \mapsto x^b$). ■

Notations
In the following, the values $\hat{t}_{ma}$, $\hat{\tau}_{ma}$ computed at event k ($k \geqslant n$) will be noted:

$$\hat{t}_{ma} = \bigwedge_{l=n}^{k} u(l-n) \backslash y(l)$$

$$\hat{\tau}_{ma} = \bigwedge_{i=1}^{\left\lfloor \frac{k-n}{\nu} \right\rfloor} \frac{1}{i} \times \hat{\tau}_{ma_i}^i$$

with:

$$\hat{\tau}_{ma_i}^i = \bigwedge_{l=n}^{k} \frac{y(l)}{\hat{t}_{ma}\, u(l-n-i\nu)}. \tag{13}$$

Let us recall that $\hat{t}_{ar} = \hat{t}_{ma}$ and $\hat{\tau}_{ar} = \bigwedge_{l=n}^{k} y_{ar}(l-\nu) \backslash y(l)$.

Proposition 7. The model parameter $\hat{\tau}$ obtained using the MA model of the simple element is greater or equal to the one given by the ARMA model, i.e., for $k \geqslant n$

$$\hat{\tau}_{ma} \succeq \hat{\tau}_{ar}$$

Proof
Due to remark 4 and eq. (8), we deduce $y_{ar}(l-i\nu) \succeq \hat{t}\, u(l-n-i\nu) \succeq \hat{t}\, u(l-n-i\nu)$ since $k \geqslant l$, $l \in [n, N]$.

Because the mapping $x \mapsto \frac{a}{x}$ is antitone (see (Max-Plus, 1991)), eq. (13) becomes:

$$\hat{\tau}_{ma_i}^i \succeq \bigwedge_{l=n}^{k} \frac{y(l)}{y_{ar}(l-i\nu)}. \tag{14}$$

We introduce $y_{ar}(l-\nu), \ldots, y_{ar}(l-(i-1)\nu)$ into eq. (14):

$$\begin{aligned} \hat{\tau}_{ma_i}^i \succeq \bigwedge_{l=n}^{k} \Bigg(& \frac{y(l)}{y_{ar}(l-\nu)} \otimes \frac{y_{ar}(l-\nu)}{y_{ar}(l-2\nu)} \otimes \ldots \\ & \ldots \otimes \frac{y_{ar}(l-(i-1)\nu)}{y_{ar}(l-i\nu)} \Bigg) \end{aligned}$$

which can be expressed as (cf. eq. 4):

$$\hat{\tau}_{ma_i}^i \succeq \underbrace{\bigwedge_{l=n}^{k} \frac{y(l)}{y_{ar}(l-\nu)}}_{\hat{\tau}_{ar}} \oslash \underbrace{\bigwedge_{l=n}^{k} \frac{y_{ar}(l-\nu)}{y_{ar}(l-2\nu)} \otimes \ldots \otimes \bigwedge_{l=n}^{k} \frac{y_{ar}(l-(i-1)\nu)}{y_{ar}(l-i\nu)}}_{i-1 \text{ elements}}. \tag{15}$$

Eq. (8) implies also that $\underset{(l)}{y_{ar}}(j) \succeq \underset{(l-1)}{\hat{\tau}_{ar}} \underset{(l)}{y_{ar}}(j-\nu)$, $\forall j$ which can be written as $\underset{(l)}{y_{ar}}(j-\nu) \backslash \underset{(l)}{y_{ar}}(j) \succeq \underset{(l-1)}{\hat{\tau}_{ar}}$, $\forall l = \{n, \ldots, k\}$, $\forall j$. Since $\underset{(l-1)}{\hat{\tau}_{ar}}$ is decreasing (remark (4)), we have $\underset{(l-1)}{\hat{\tau}_{ar}} \succeq \underset{(k)}{\hat{\tau}_{ar}}$, $\forall\, l = \{n, \ldots, k\}$. Consequently, the $i-1$ elements in eq. (15) verify the set of inequations $\bigwedge_{l=n}^{k} \frac{\underset{(l)}{y_{ar}}(l-(j-1)\nu)}{\underset{(l)}{y_{ar}}(l-j\nu)} \succeq \underset{(k)}{\hat{\tau}_{ar}}$ for $j = 2$ to i. Then eq. (15) leads to:

$$\underset{(k)}{\hat{\tau}^{i}_{ma_i}} \succeq \underbrace{\underset{(k)}{\hat{\tau}_{ar}} \otimes \ldots \otimes \underset{(k)}{\hat{\tau}_{ar}}}_{i \text{ times}} = \underset{(k)}{\hat{\tau}^{i}_{ar}}.$$

As $\underset{(k)}{\hat{\tau}_{ma}} = \bigwedge_{i=1}^{p} \frac{1}{i} \times \underset{(k)}{\hat{\tau}^{i}_{ma_i}}$ then $\underset{(k)}{\hat{\tau}_{ma}} \succeq \bigwedge_{i=1}^{p} \frac{1}{i} \times \underset{(k)}{\hat{\tau}^{i}_{ar}} = \bigwedge_{i=1}^{p} \underset{(k)}{\hat{\tau}_{ar}}$.

Since $\underset{(k)}{\hat{\tau}_{ar}}$ does not depend on i, we obtain $\underset{(k_i)}{\hat{\tau}_{ma}} \succeq \underset{(k)}{\hat{\tau}_{ar}}$, $\forall k$ ■

As for the identification method using ARMA model, we propose a recursive algorithm to estimate $\hat{\theta}_{ma} = [\hat{t}_{ma}, \hat{\tau}_{ma}]^T$ at each event l ($l \geqslant n$).

The practical computation of $\hat{\Theta}_{ma}$ is given by the following recursive algorithm:

Algorithm 3

$\underset{(n-1)}{\hat{t}_{ma}} = +\infty$

$\underset{(n)}{\hat{\tau}_{ma}} = \ldots = \underset{(N)}{\hat{\tau}_{ma}} = -\infty$

FOR $l = n$ to N

  $\underset{(l)}{\hat{t}_{ma}} = \underset{(l-1)}{\hat{t}_{ma}} \wedge (u(l-n) \backslash y(l))$

  FOR $i = 1$ to $\lfloor \frac{l-n}{\nu} \rfloor$, $(l \geqslant n + \nu)$

    $\tau(i) = +\infty$

    FOR $j = n$ to $l - i\nu$, $(l \geqslant n + i\nu)$

      $\tau(i) = \tau(i) \wedge ((\underset{(l)}{\hat{t}_{ma}} u(j-n-i\nu)) \backslash y(j))$

    END

    $\underset{(l)}{\hat{\tau}_{ma}} = \underset{(l)}{\hat{\tau}_{ma}} \wedge (\frac{1}{i} \times \tau(i))$

  END

END

$\hat{\Theta}_{ma} = [\underset{(N)}{\hat{t}_{ma}} \quad \underset{(N)}{\hat{\tau}_{ma}}]^T$

The complexity of this algorithm is $O(N^3)$.

4.4 *Identification of several simple elements*

We consider the simultaneous identification of the parameters of $\mathcal{N}$ simple elements $(t_i, \tau_i)_{i=1 \text{ to } \mathcal{N}}$ with $(n_i, \nu_i)_{i=1 \text{ to } \mathcal{N}}$ given. The model output is defined by:

$$Y_m(\gamma) = [\bigoplus_{i=1}^{\mathcal{N}} t_i \gamma^{n_i} (\tau_i \gamma^{\nu_i})^*] U(\gamma)$$

which corresponds in $\overline{\mathbb{R}}_{max}$ to:

$$y_m(k) = \bigoplus_{i=1}^{\mathcal{N}} y_{m_i}(k)$$

where $y_{m_i}(k)$ is expressed as the MA equation:

$$y_{ma_i}(k) = t_{ma_i} u(k-n_i) \oplus t_{ma_i} \tau_{ma_i} u(k-n_i\nu_i) \oplus \ldots$$

or by the ARMA equation:

$$y_{ar_i}(k) = t_{ar_i} u(k-n_i) \oplus \tau_{ar_i} y_{ar_i}(k-\nu_i)$$

As in sec. 4.3, the idenfication methods are still based on the predictive error $Er(k) = y(k) - y_m(k)$.

4.4.1. *Identification using the ARMA equation* We consider the ARMA model ouput $y_{ar}(k) = \Phi_k^T \hat{\Theta}_{ar}$ with $\Phi_k^T = [u(k-n_1) \ldots u(k-n_{\mathcal{N}}) \quad y_{ar_1}(k-\nu_1) \ldots y_{ar_{\mathcal{N}}}(k-\nu_{\mathcal{N}})]$ the regressive vector at event k where y_{ar_i} is the i-th simple element output, u the input and $\hat{\Theta}_{ar} = [\hat{t}_{ar_1} \ldots \hat{t}_{ar_{\mathcal{N}}} \quad \hat{\tau}_{ar_1} \ldots \hat{\tau}_{ar_{\mathcal{N}}}]^T$ the estimated parameters vector.

Proposition 8. The components of vector $\hat{\Theta}_{ar}$ minimizing criterion J defined in sec. 4.3.1, are given by:

$$\begin{bmatrix} \hat{t}_{ar_i} \\ \hat{\tau}_{ar_i} \end{bmatrix} = \begin{bmatrix} \bigwedge_{l=n_i}^{\mathcal{N}} \frac{y(l)}{u(l-n_i)} \\ \bigwedge_{l=n_i}^{\mathcal{N}} \frac{y(l)}{y_{ar_i}(l-\nu_i)} \end{bmatrix}, \; i = 1 \text{ to } \mathcal{N}. \qquad (16)$$

Proof The proof is similar to the one given in sec. 4.3.1■

4.4.2. *Identification using the MA equation* We consider the model $y_{ma}(k) = \Psi_k^T \hat{\Delta}_{ma}$ where

$$\hat{\Delta}_{ma} = [\hat{t}_{ma_1} \quad \hat{t}_{ma_1}\hat{\tau}_{ma_1} \quad \hat{t}_{ma_1}\hat{\tau}^2_{ma_1} \quad \ldots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \hat{t}_{ma_{\mathcal{N}}} \quad \hat{t}_{ma_{\mathcal{N}}}\hat{\tau}_{ma_{\mathcal{N}}} \quad \hat{t}_{ma_{\mathcal{N}}}\hat{\tau}^2_{ma_{\mathcal{N}}} \quad \ldots]^T$$

the estimated parameters vector and

$$\Psi_k^T = [u(k-n_1) \quad u(k-n_1\nu_1) \quad u(k-n_1\nu_1^2) \quad \ldots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ u(k-n_{\mathcal{N}}) \quad u(k-n_{\mathcal{N}}\nu_{\mathcal{N}}) \quad u(k-n_{\mathcal{N}}\nu_{\mathcal{N}}^2) \quad \ldots]$$

the regressive vector at event k with the input u.

Proposition 9. The estimated parameters $\hat{\tau}_{ma_j}$, $j = 1$ to $\mathcal{N}$ which minimize the criterion $\bigoplus_{k=n}^{\mathcal{N}} Er(k)$ ($Er(k) \succeq 0$, $\forall k$) with $\hat{t}_{ma_j} = \hat{t}_{ar_j}$, are given by:

$$\hat{\tau}_{ma_j} = \bigwedge_{i=1}^{p_j} \left(\bigwedge_{l=n_j}^{\mathcal{N}} \frac{y(l)}{\hat{t}_{ma_j}\, u(l-n_j-i\nu_j)} \right)^{\otimes \frac{1}{i}}, \; j = 1 \ldots \mathcal{N}$$

where $p_j = \lfloor \frac{\mathcal{N}-n_j}{\nu_j} \rfloor$.

Proof The proof is similar to the one of proposition 6■

5. EXAMPLE

We illustrate the two proposed identification methods to modelise a GET of impulse response $H(\gamma) = (1 \ominus 3\gamma \oplus 5\gamma^2 \ominus 7\gamma^3 \oplus 9\gamma^4) \oplus 11\gamma^5(4\gamma^2)^*$.

Given the input sequence $u(k) = -\infty$ for $k < 0$, $u(k) = 0$ for $0 \leqslant k \leqslant 9$ and $u(k) = +\infty$ for $k > 9$ (truncated impulsion), we consider a model constitued by 2 simple elements ($n_1 = 0$, $n_2 = 3$, $\nu_1 = \nu_2 = 2$).

The result obtained respectively by the two methods are:

$$\hat{\Theta}_{ar} = [\hat{t}_{ar_1} \quad \hat{t}_{ar_2} \quad \hat{\tau}_{ar_1} \quad \hat{\tau}_{ar_2}]^T = [1 \quad 7 \quad 2 \quad 4]^T$$
$$\hat{\Theta}_{ma} = [\hat{t}_{ma_1} \quad \hat{t}_{ma_2} \quad \hat{\tau}_{ma_1} \quad \hat{\tau}_{ma_2}]^T = [1 \quad 7 \quad 3.33 \quad 4]^T$$

which correspond to the models (Fig 2 and 3):

$$Y_{ar}(\gamma) = 1\gamma^0(2\gamma^2)^* \oplus 7\gamma^3(4\gamma^2)^*$$
$$Y_{ma}(\gamma) = 1\gamma^0(3.33\gamma^2)^* \oplus 7\gamma^3(4\gamma^2)^*$$

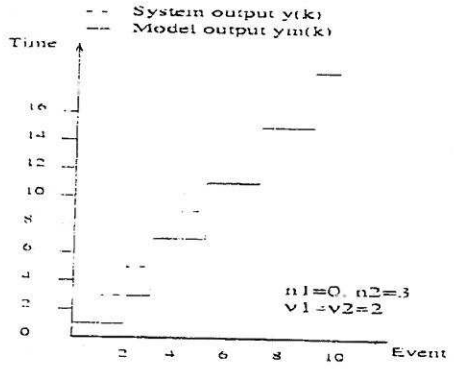

Fig. 2. Identification method using the ARMA model.

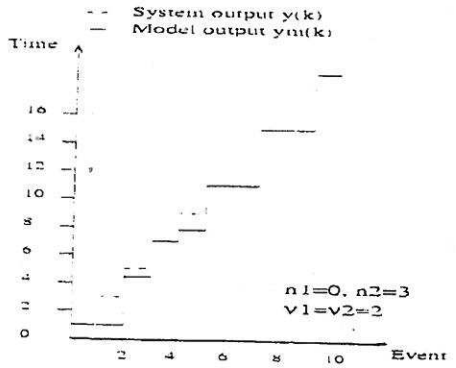

Fig. 3. Identification method using the MA model.

Here, both methods lead to the same value of the criterion $J = J1 = 2$.

6. CONCLUSION

We have proposed two identification methods of one simple element from the response of a max-plus linear system. The first one uses the ARMA model whereas the second one uses the MA model. Moreover, these methods have been extended to the simultaneous identification of several simple elements. Both approaches are based on a basic property of the Residuation theory.

We have shown that the approach based on the MA model yields a better model output but the complexity of the algorithm used to compute the model parameters is $O(N^3)$. The algorithm used to compute the parameters of the ARMA model, is of order $O(N^2)$. It can be reduced into an algorithm of order $O(N)$ but the obtained model output is farther from the system output.

We are now able to identify several simple elements of a given structure. We plan to use these methods in order to modelise a max-plus linear system which will probably involve the residuation of the law $\oplus$.

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