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# Global solutions and asymptotic behavior for two dimensional gravity water waves 

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#### Abstract

This paper is devoted to the proof of a global existence result for the water waves equation with smooth, small, and decaying at infinity Cauchy data. We obtain moreover an asymptotic description in physical coordinates of the solution, which shows that modified scattering holds.

The proof is based on a bootstrap argument involving $L^{2}$ and $L^{\infty}$ estimates. The $L^{2}$ bounds are proved in the companion paper [5] of this article. They rely on a normal forms paradifferential method allowing one to obtain energy estimates on the Eulerian formulation of the water waves equation. We give here the proof of the uniform bounds, interpreting the equation in a semi-classical way, and combining Klainerman vector fields with the description of the solution in terms of semi-classical lagrangian distributions. This, together with the $L^{2}$ estimates of [5], allows us to deduce our main global existence result.


## Introduction

## 1 Main result

Consider an homogeneous and incompressible fluid in a gravity field, occupying a timedependent domain with a free surface. We assume that the motion is the same in every vertical section and consider the two-dimensional motion in one such section. At time $t$, the
fluid domain, denoted by $\Omega(t)$, is therefore a two-dimensional domain. We assume that its boundary is a free surface described by the equation $y=\eta(t, x)$, so that

$$
\Omega(t)=\{(x, y) \in \mathbb{R} \times \mathbb{R} ; y<\eta(t, x)\} .
$$

The velocity field is assumed to satisfy the incompressible Euler equations. Moreover, the fluid motion is assumed to have been generated from rest by conservative forces and is therefore irrotational in character. It follows that the velocity field $v: \Omega \rightarrow \mathbb{R}^{2}$ is given by $v=\nabla_{x, y} \phi$ for some velocity potential $\phi: \Omega \rightarrow \mathbb{R}$ satisfying

$$
\begin{equation*}
\Delta_{x, y} \phi=0, \quad \partial_{t} \phi+\frac{1}{2}\left|\nabla_{x, y} \phi\right|^{2}+P+g y=0, \tag{1}
\end{equation*}
$$

where $g$ is the modulus of the acceleration of gravity $(g>0)$ and where $P$ is the pressure term. Hereafter, the units of length and time are chosen so that $g=1$.

The problem is then given by two boundary conditions on the free surface:

$$
\begin{cases}\partial_{t} \eta=\sqrt{1+\left(\partial_{x} \eta\right)^{2}} \partial_{n} \phi & \text { on } \partial \Omega,  \tag{2}\\ P=0 & \text { on } \partial \Omega,\end{cases}
$$

where $\partial_{n}$ is the outward normal derivative of $\Omega$, so that $\sqrt{1+\left(\partial_{x} \eta\right)^{2}} \partial_{n} \phi=\partial_{y} \phi-\left(\partial_{x} \eta\right) \partial_{x} \phi$. The former condition expresses that the velocity of the free surface coincides with the one of the fluid particles. The latter condition is a balance of forces across the free surface.

Following Zakharov [68] and Craig and Sulem [27], we work with the trace of $\phi$ at the free boundary

$$
\psi(t, x):=\phi(t, x, \eta(t, x)) .
$$

To form a system of two evolution equations for $\eta$ and $\psi$, one introduces the DirichletNeumann operator $G(\eta)$ that relates $\psi$ to the normal derivative $\partial_{n} \phi$ of the potential by

$$
(G(\eta) \psi)(t, x)=\left.\sqrt{1+\left(\partial_{x} \eta\right)^{2}} \partial_{n} \phi\right|_{y=\eta(t, x)} .
$$

(This definition is made precise in the first section of the companion paper [5]. See proposition 1.2 below). Then $(\eta, \psi)$ solves (see [27]) the so-called Craig-Sulem-Zakharov system

$$
\left\{\begin{array}{l}
\partial_{t} \eta=G(\eta) \psi  \tag{3}\\
\partial_{t} \psi+\eta+\frac{1}{2}\left(\partial_{x} \psi\right)^{2}-\frac{1}{2\left(1+\left(\partial_{x} \eta\right)^{2}\right)}\left(G(\eta) \psi+\left(\partial_{x} \eta\right)\left(\partial_{x} \psi\right)\right)^{2}=0
\end{array}\right.
$$

In [4], it is proved that if $(\eta, \psi)$ is a classical solution of (3), such that $(\eta, \psi)$ belongs to $C^{0}\left([0, T] ; H^{s}(\mathbb{R})\right)$ for some $T>0$ and $s>3 / 2$, then one can define a velocity potential $\phi$ and a pressure $P$ satisfying (1) and (2). Thus it is sufficient to solve the Craig-Sulem-Zakharov formulation of the water waves equations.

Our main result is stated in full generality in the first section of this paper. A weaker statement is the following:

Main result. For small enough initial data of size $\varepsilon \ll 1$, sufficiently decaying at infinity, the Cauchy problem for (3) is globally in time well-posed. Moreover, $u=\left|D_{x}\right|^{\frac{1}{2}} \psi+i \eta$ admits the following asymptotic expansion as $t$ goes to $+\infty$ : There is a continuous function $\underline{\alpha}: \mathbb{R} \rightarrow \mathbb{C}$, depending of $\varepsilon$ but bounded uniformly in $\varepsilon$, such that

$$
u(t, x)=\frac{\varepsilon}{\sqrt{t}} \underline{\alpha}\left(\frac{x}{t}\right) \exp \left(\frac{i t}{4|x / t|}+\frac{i \varepsilon^{2}}{64} \frac{|\underline{\alpha}(x / t)|^{2}}{|x / t|^{5}} \log (t)\right)+\varepsilon t^{-\frac{1}{2}-\kappa} \rho(t, x)
$$

where $\kappa$ is some positive number and $\rho$ is a function uniformly bounded for $\left.t \geq 1, \varepsilon \in] 0, \varepsilon_{0}\right]$.
As an example of small enough initial data sufficiently decaying at infinity, consider

$$
\begin{equation*}
\left.\eta\right|_{t=1}=\varepsilon \eta_{0},\left.\quad \psi\right|_{t=1}=\varepsilon \psi_{0}, \tag{4}
\end{equation*}
$$

with $\eta_{0}, \psi_{0}$ in $C_{0}^{\infty}(\mathbb{R})$. Then there exists a unique solution $(\eta, \psi)$ in $C^{\infty}\left(\left[1,+\infty\left[; H^{\infty}(\mathbb{R})\right)\right.\right.$ of (3). In fact, in Theorem 1.4 we allow $\psi$ to be merely in some homogeneous Sobolev space.

The strategy of the proof will be explained in the following sections of this introduction. We discuss at the end of this paragraph some related previous works.

For the equations obtained by neglecting the nonlinear terms, the computation of the asymptotic behavior of the solutions was performed by Cauchy [17] who computed the phase of oscillations. The reader is referred to [31] and [30] for many historical comments on Cauchy's memoir.

Many results have been obtained in the study of the Cauchy problem for the water waves equations, starting from the pioneering work of Nalimov [54] who proved that the Cauchy problem is well-posed locally in time, in the framework of Sobolev spaces, under an additional smallness assumption on the data. We also refer the reader to Shinbrot [59], Yoshihara [67] and Craig [23]. Without smallness assumptions on the data, the well-posedness of the Cauchy problem was first proved by Wu for the case without surface tension (see [63, 64]) and by Beyer-Günther in [11] in the case with surface tension. Several extensions of their results have been obtained and we refer the reader to Córdoba, Córdoba and Gancedo [20], CoutandShkoller [22], Lannes [48, 49, 47], Linblad [50], Masmoudi-Rousset [52] and Shatah-Zeng [57, 58] for recent results on the Cauchy problem for the gravity water waves equation.

Our proof of global existence is based on the analysis of the Eulerian formulation of the water waves equations by means of microlocal analysis. In particular, the energy estimates discussed in [5] are influenced by the papers by Lannes [48] and Iooss-Plotnikov [43] and follow the paradifferential analysis introduced in [6] and further developed in $[2,1]$.

It is worth recalling that the only known coercive quantity for (3) is the hamiltonian, which reads (see $[68,27]$ )

$$
\begin{equation*}
\mathcal{H}=\frac{1}{2} \int \eta^{2} d x+\frac{1}{2} \int \psi G(\eta) \psi d x . \tag{5}
\end{equation*}
$$

We refer to the paper by Benjamin and Olver [10] for considerations on the conservation laws of the water waves equations. One can compare the hamiltonian with the critical threshold
given by the scaling invariance of the equations. Recall (see [10, 18]) that if $(\eta, \psi)$ solves (3), then the functions $\left(\eta_{\lambda}, \psi_{\lambda}\right)$ defined by

$$
\begin{equation*}
\eta_{\lambda}(t, x)=\lambda^{-2} \eta\left(\lambda t, \lambda^{2} x\right), \quad \psi_{\lambda}(t, x)=\lambda^{-3} \psi\left(\lambda t, \lambda^{2} x\right) \quad(\lambda>0) \tag{6}
\end{equation*}
$$

are also solutions of (3). In particular, one notices that the critical space for the scaling corresponds to $\eta_{0}$ in $\dot{H}^{3 / 2}(\mathbb{R})$. Since the hamiltonian (5) only controls the $L^{2}(\mathbb{R})$-norm of $\eta$, one sees that the hamiltonian is highly supercritical for the water waves equation and hence one cannot use it directly to prove global well-posedness of the Cauchy problem.

Given $\varepsilon \geq 0$, consider the solutions to the water waves system (3) with initial data satisfying (4). In her breakthrough result [65], Wu proved that the maximal time of existence $T_{\varepsilon}$ is larger or equal to $e^{c / \varepsilon}$ for $d=1$. Then Germain-Masmoudi-Shatah [35] and Wu [66] have shown that the Cauchy problem for three-dimensional waves is globally in time well-posed for $\varepsilon$ small enough (with linear scattering in Germain-Masmoudi-Shatah and no assumption about the decay to 0 at spatial infinity of $\left|D_{x}\right|^{\frac{1}{2}} \psi$ in Wu ). Germain-Masmoudi-Shatah recently proved global existence for pure capillary waves in dimension $d=2$ in [34].

There is at least one other case where the global existence of solutions is now understood, namely for the equations with viscosity (see [9], [36] and the references therein). Then global well-posedness is obtained by using the dissipation of energy. Without viscosity, the analysis of global well-posedness is based on dispersive estimates. Our approach follows a variant of the vector fields method introduced by Klainerman in [45, 44] to study the wave and Klein-Gordon equations (see the book by Hörmander in [38] or the Bourbaki seminar by Lannes [46] for an introduction to this method). More precisely, as it is discussed later in this introduction, we shall follow the approach introduced in [32] for the analysis of the KleinGordon equation in space dimension one, to cope with the fact that solutions of the equation do not scatter. Results for one dimensional Schrödinger equations, that display the same non scattering behavior, have been proved by Hayashi and Naumkin [37], and global existence for a simplified model of the water waves equation studied by Ionescu and Pusateri in [41].

Let us discuss two other questions related to our analysis : the possible emergence of singularities in finite time and the existence of solitary waves.

An important question is to determine whether the lifespan could be finite. Castro, Córdoba, Fefferman, Gancedo and Gómez-Serrano conjecture (see [15]) that blow-up in finite time is possible for some initial data. It is conjectured in [15] that there exists at least one water-wave solution such that, at time 0 , the fluid interface is a graph, at a later time $t_{1}>0$ the fluid interface is not a graph, and, at a later time $t_{2}>t_{1}$, the fluid self-intersects. Notice that, according to this conjecture, one does not expect global well-posedness for arbitrarily large initial data. One can quote several results supporting this conjecture (see [14, 16, 21]). In [14] (see also [21]), the authors prove the following result: there exists an initial data such the free surface is a self-intersecting curve, and such that solving backward in time the Cauchy problem, one obtains for small enough negative times a non self-intersecting curve of $\mathbb{R}^{2}$. On the other hand, it was conjectured that there is no blow-up in finite time for small enough, sufficiently decaying initial data (see the survey paper by Craig and Wayne [29]).

Our main result precludes the existence of solitary waves sufficiently small and sufficiently decaying at infinity. In this direction, notice that Sun [61] has shown that in infinitely deep
water, no two-dimensional solitary water waves exist. For further comments and references on solitary waves, we refer the reader to [25] as well as to $[13,39,53]$ for recent results.

We refer the reader to $[2,3,19]$ for the study of other dispersive properties of the water waves equations (Strichartz estimates and smoothing effect of surface tension).

Finally, let us mention that Ionescu and Pusateri [40] independently obtained a global existence result very similar to the one we get here. The main difference is that they assume less decay on the initial data, and get asymptotics not for the solution in physical space, with control of the remainders in $L^{\infty}$, but for its space Fourier transform, with remainders in $L^{2}$. These asymptotics, as well as ours, show that solutions do not scatter. To get asymptotics with remainders estimated in $L^{\infty}$, we shall commute iterated vector field $Z=t \partial_{t}+2 x \partial_{x}$ to the water waves equations. This introduces several new difficulties and requires that the initial data be sufficiently decaying at infinity.

## 2 General strategy of proof

Let us describe our general strategy, the difficulties one has to cope with, and the ideas used to overcome them. The general framework we use is the one of Klainerman vector fields. Consider as a (much) simplified model an equation of the form

$$
\begin{align*}
\left(D_{t}-P\left(D_{x}\right)\right) u & =N(u) \\
\left.u\right|_{t=1} & =\varepsilon u_{0}, \tag{7}
\end{align*}
$$

where $D_{t}=\frac{1}{i} \frac{\partial}{\partial t}, P(\xi)$ is a real valued symbol (for the linearized water waves equation, $P(\xi)$ would be $|\xi|^{1 / 2}$ ), and $N(u)$ is a nonlinearity vanishing at least at order two at zero. Recall that a Klainerman vector field for $D_{t}-P\left(D_{x}\right)$ is a space-time vector field $Z$ such that $\left[Z, D_{t}-P\left(D_{x}\right)\right]$ is zero (or a multiple of $D_{t}-P\left(D_{x}\right)$ ). For the water waves system, $Z$ will be $t \partial_{t}+2 x \partial_{x}$ or $D_{x}$. In that way, $\left(D_{t}-P\left(D_{x}\right)\right) Z^{k} u=Z^{k} N(u)$ for any $k$, and since $P(\xi)$ is real valued, an easy energy inequality shows that

$$
\begin{equation*}
\left\|Z^{k} u(t, \cdot)\right\|_{L^{2}} \leq\left\|Z^{k} u(1, \cdot)\right\|_{L^{2}}+\int_{1}^{t}\left\|Z^{k} N(u)(\tau, \cdot)\right\|_{L^{2}} d \tau \tag{8}
\end{equation*}
$$

for any $t \geq 1$. Assume first that $N(u)$ is cubic, so that

$$
\begin{equation*}
\left\|Z^{k} N(u)\right\|_{L^{2}} \leq C\|u\|_{L^{\infty}}^{2}\left\|Z^{k} u\right\|_{L^{2}}+C \sum_{\substack{k_{1}+k_{2}+k_{3} \leq k \\ k_{1}, k_{2} \leq k_{3} \leq k-1}}\left\|Z^{k_{1}} u\right\|_{L^{\infty}}\left\|Z^{k_{2}} u\right\|_{L^{\infty}}\left\|Z^{k_{3}} u\right\|_{L^{2}} \tag{9}
\end{equation*}
$$

Assuming an a priori $L^{\infty}$ bound, one can deduce from (8) an $L^{2}$ estimate. More precisely, introduce the following property, where $s$ is a large even integer:

$$
\text { For } t \text { in some interval }\left[1, T\left[,\|u(t, \cdot)\|_{L^{\infty}}=O(\varepsilon / \sqrt{t})\right.\right.
$$

$$
\begin{equation*}
\text { and for } k=0, \ldots, s / 2,\left\|Z^{k} u(t, \cdot)\right\|_{L^{\infty}}=O\left(\varepsilon t^{-\frac{1}{2}+\tilde{\delta}_{k}^{\prime}}\right) \tag{A}
\end{equation*}
$$

where $\tilde{\delta}_{k}^{\prime}$ are small positive numbers. Plugging these a priori bounds in (8), (9), we get

$$
\begin{align*}
\left\|Z^{k} u(t, \cdot)\right\|_{L^{2}} \leq\left\|Z^{k} u(1, \cdot)\right\|_{L^{2}} & +C \varepsilon^{2} \int_{1}^{t}\left\|Z^{k} u(\tau, \cdot)\right\|_{L^{2}} \frac{d \tau}{\tau} \\
& +C \varepsilon^{2} \int_{1}^{t}\left\|Z^{k-1} u(\tau, \cdot)\right\|_{L^{2}} \tau^{2 \tilde{\delta}_{k / 2}^{\prime}-1} d \tau \tag{10}
\end{align*}
$$

Gronwall inequality implies then that

$$
\begin{equation*}
\left\|Z^{k} u(t, \cdot)\right\|_{L^{2}}=O\left(\varepsilon t^{\delta_{k}}\right), k \leq s, \tag{B}
\end{equation*}
$$

for some small $\delta_{k}>0\left(\delta_{k}>C \varepsilon^{2}\right.$ and $\left.\delta_{k}>2 \tilde{\delta}_{k / 2}^{\prime}\right)$.
The proof of global existence is done classically using a bootstrap argument allowing one to to show that if $(A)$ and $(B)$ are assumed to hold on some interval, they actually hold on the same interval with smaller constants in the estimates.

We have outlined above the way of obtaining $(B)$, assuming $(A)$ for a solution of the model equation (7). In this subsection of the introduction, we shall explain, in a non technical way, the new difficulties that have to be solved to prove $(B)$ for the water waves equation. Actually, the proof of a long time energy inequality for system (3) faces two serious obstacles, that we describe now.

## - Apparent loss of derivatives in energy inequalities

This difficulty already arises for local existence results, and was solved initially by Nalimov [54] and $\mathrm{Wu}[63,64]$. For long time existence problems, Wu [66] uses arguments combining the Eulerian and Lagrangian formulations of the system. The approach followed in our companion paper [5] is purely Eulerian. We explain the idea on the model obtained from (3) paralinearizing the equations and keeping only the quadratic terms. If we denote $U=$ $\left[\begin{array}{c}\eta \\ \left|D_{x}\right|^{1 / 2} \psi\end{array}\right]$, such a model may be written as

$$
\partial_{t} U=T_{A} U
$$

where $T_{A}$ is the paradifferential operator with symbol $A$, and where $A(U, x, \xi)$ is a matrix of symbols $A(U, x, \xi)=A_{0}(U, x, \xi)+A_{1}(U, x, \xi)$, with

$$
\left.A_{0}(U, x, \xi)=\left[\begin{array}{cc}
-i\left(\partial_{x} \psi\right) \xi & |\xi|^{1 / 2} \\
-|\xi|^{1 / 2} & -i\left(\partial_{x} \psi\right) \xi
\end{array}\right], A_{1}(U, x, \xi)=\left(\left|D_{x}\right| \psi\right)|\xi| \begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array}\right] .
$$

Because of the $A_{1}$ contribution, which is self-adjoint, the eigenvalues of $A(U, x, \xi)$ are not purely imaginary. For large $|\xi|$, there is one eigenvalue with positive real part, which shows that one cannot expect for the solution of $\partial_{t} U=T_{A} U$ energy inequalities without derivative losses. A way to circumvent this difficulty is well known, and consists in using the "good unknown" of Alinhac [7]. For our quadratic model, this just means introducing as a new unknown $\tilde{U}=\left[\begin{array}{c}\eta \\ \left|D_{x}\right|^{1 / 2} \omega\end{array}\right]$, where $\omega=\psi-T_{\left|D_{x}\right| \psi} \eta$ is the (quadratic approximation of the) good unknown. In that way, ignoring again remainders and terms which are at least cubic,
one gets for $\tilde{U}$ an evolution equation $\partial_{t} \tilde{U}=T_{A_{0}} \tilde{U}$. Since $A_{0}$ is anti-self-adjoint, one gets $L^{2}$ or Sobolev energy inequalities for $\tilde{U}$. In particular, if for some $s,\left\|\left|D_{x}\right|^{1 / 2} \omega\right\|_{H^{s}}+\|\eta\|_{H^{s}}$ is under control, and if one has also an auxiliary bound for $\left\|\left|D_{x}\right| \psi\right\|_{L^{\infty}}$, one gets an estimate for $\left\|\left|D_{x}\right|^{1 / 2} \psi\right\|_{H^{s-1 / 2}}+\|\eta\|_{H^{s}}$.

## - Quadratic terms in the nonlinearity

In the model equation (7) discussed above, we considered a cubic nonlinearity: this played an essential role to make appear in the first integral in the right hand side of (10) the almost integrable factor $1 / \tau$. For a quadratic nonlinearity, we would have had instead a $1 / \sqrt{\tau}$-factor, which would have given in $(B)$, through Gronwall, a $O\left(e^{\varepsilon \sqrt{t}}\right)$-bound, instead of $O\left(\varepsilon t^{\delta_{k}}\right)$. The way to overcome such a difficulty is well known since the work of Shatah [56] devoted to the non-linear Klein-Gordon equation: it is to use a normal forms method to eliminate the quadratic part of the nonlinearity, up to terms that do not contribute to the Sobolev energy inequality.

In practice, one looks for a local diffeomorphism at 0 in $H^{s}$, for $s$ large enough, so that the Sobolev energy inequality written for the equation obtained by conjugation by this diffeomorphim be of the form (10). Nonlinear changes of unknowns, reducing the water waves system to a cubic equation, have been known for quite a time (see Craig [24] or Iooss and Plotnikov [42, Lemma 1]). However, these transformations were losing derivatives, as a consequence of the quasi-linear character of the problem. Nevertheless, one can construct a bona fide change of unknown, without derivatives losses, if one notices that it is not necessary to eliminate the whole quadratic part of the nonlinearity, but only the part of it that would bring non zero contributions in a Sobolev energy inequality. This is what we do in our companion paper [5]. Let us also mention that the analysis of normal forms for the water waves system is motivated by physical considerations, such as the derivations of various equations in asymptotic regimes (see [28, 26, 55, 62]).

Our proof of $L^{2}$-estimates of type $(B)$, assuming that a priori inequalities of type $(A)$ hold, is performed in [5] using the ideas that we just outlined. Of course, the models we have discussed so far do not make justice to the full complexity of the water waves system. In particular, the good unknown $\omega$ is given by a more involved formula than the one indicated above, and one also needs to define precisely the Dirichlet-Neumann operator. The latter is done in [5]. We recall in section 1 below the main properties of the operator $G(\eta)$ when $\eta$ belongs to a space of the form $C^{\gamma}(\mathbb{R}) \cap L^{2}(\mathbb{R})$ with $\gamma>2$, and is small enough. Once $G(\eta) \psi$ has been defined, one can introduce functions of $(\eta, \psi), B=\left.\left(\partial_{y} \phi\right)\right|_{y=\eta}, V=\left.\left(\partial_{x} \phi\right)\right|_{y=\eta}$, where $\phi$ is the harmonic potential solving (1). Explicit expressions of these quantities are given by

$$
B=\frac{G(\eta) \psi+\left(\partial_{x} \eta\right)\left(\partial_{x} \psi\right)}{1+\left(\partial_{x} \eta\right)^{2}}, V=\partial_{x} \psi-B \partial_{x} \eta \text {. }
$$

The good unknown for the water waves equation is given by $\omega=\psi-T_{B} \eta$. Following the analysis in $[1,2,6]$, we prove in [5] an expression for $G(\eta) \psi$ in terms of $\omega$ :

$$
G(\eta) \psi=\left|D_{x}\right| \omega-\partial_{x}\left(T_{V} \eta\right)+F(\eta) \psi
$$

where $F(\eta) \psi$ is a quadratic smoothing term, that belongs to $H^{s+\gamma-4}$ if $\eta$ is in $C^{\gamma} \cap H^{s}$ and $\left|D_{x}\right|^{1 / 2} \psi$ belongs to $C^{\gamma-1 / 2} \cap H^{s-1 / 2}$. This gives a quite explicit expression for the main contributions to $G(\eta) \psi$. Moreover, we prove as well tame estimates, that complement similar results due to Craig, Schanz and Sulem (see [26], [60, Chapter 11] and [8, 43]), and establish bounds for the approximation of $G(\eta) \psi$ (resp. $F(\eta) \psi$ ) by its Taylor expansion at order two $G_{\leq 2}(\eta) \psi\left(\right.$ resp. $\left.F_{\leq 2}(\eta) \psi\right)$.

## 3 Klainerman-Sobolev inequalities

As previously mentioned, the proof of global existence relies on a bootstrap argument on properties $(A)$ and $(B)$. We have indicated in the preceding section how $(B)$ may be deduced from $(A)$. On the other hand, one has to prove that conversely, $(A)$ and $(B)$ imply that $(A)$ holds with smaller constants in the inequalities. The first step is to show that if the $L^{2}$ estimate $(B)$ holds for $k \leq s$, then bounds of the form

$$
\left\|Z^{k} u(t, \cdot)\right\|_{L^{\infty}}=O\left(\varepsilon t^{-\frac{1}{2}+\delta_{k}^{\prime}}\right), k \leq s-100
$$

are true, for small positive $\delta_{k}^{\prime}$. This is not $(A)$, since the $\delta_{k}^{\prime}$ may be larger that the $\tilde{\delta}_{k}^{\prime}$ of $(A)$, and since this does not give a uniform bound for $\|u(t, \cdot)\|_{L^{\infty}}$. But this first information will allow us to deduce, in the last step of the proof, estimates of the form $(A)$ from $\left(A^{\prime}\right)$ and the equation.

Let us make a change of variables $x \rightarrow x / t$ in the water waves system. If $u(t, x)$ is given by $u(t, x)=\left(\left|D_{x}\right|^{1 / 2} \psi+i \eta\right)(t, x)$, we define $v$ by $u(t, x)=\frac{1}{\sqrt{t}} v(t, x / t)$. We set $h=1 / t$ and eventually consider $v$ as a family of functions of $x$ depending on the semi-classical parameter $h$. Moreover, for $a(x, \xi)$ a function satisfying convenient symbol estimates, and $\left(v_{h}\right)_{h}$ a family of functions on $\mathbb{R}$, we define

$$
\mathrm{Op}_{h}(a) v_{h}=a(x, h D) v_{h}=\frac{1}{2 \pi} \int e^{i x \xi} a(x, h \xi) \hat{v}_{h}(\xi) d \xi
$$

Then the water waves system is equivalent to the equation

$$
\begin{equation*}
\left(D_{t}-\mathrm{Op}_{h}\left(x \xi+|\xi|^{1 / 2}\right)\right) v=\sqrt{h} Q_{0}(V)+h\left[C_{0}(V)-\frac{i}{2} v\right]+h^{1+\kappa} R(V) \tag{11}
\end{equation*}
$$

where we used the following notations

- $Q_{0}$ (resp. $C_{0}$ ) is a nonlocal quadratic (resp. cubic) form of $V=(v, \bar{v})$ that may be written as a linear combination of expressions $\mathrm{Op}_{h}\left(b_{0}\right)\left[\prod_{j=1}^{\ell} \mathrm{Op}_{h}\left(b_{j}\right) v_{ \pm}\right], \ell=2$ (resp. $\ell=3$ ), where $b_{\ell}(\xi)$ are homogeneous functions of degree $d_{\ell} \geq 0$ with $\sum_{0}^{2} d_{\ell}=3 / 2$ (resp. $\sum_{0}^{3} d_{\ell}=5 / 2$ ) and $v_{+}=v, v_{-}=\bar{v}$.
- $R(V)$ is a remainder, made of the contributions vanishing at least at order four at $V=0$.

To simplify the exposition in this introduction, we shall assume that $v$ satisfies $\varphi(h D) v=v$ for some $C_{0}^{\infty}(\mathbb{R}-\{0\})$-function $\varphi$, equal to one on a large enough compact subset of of $\mathbb{R}-\{0\}$.

Such a property is not satisfied by solutions of (11), but one can essentially reduce to such a situation performing a dyadic decomposition $v=\sum_{j \in \mathbb{Z}} \varphi\left(2^{-j} h D\right) v$.

The Klainerman vector field associated to the linearization of the water waves equation may be written, in the new coordinates that we are using, as $Z=t \partial_{t}+x \partial_{x}$. Remembering $h=1 / t$ and expressing $\partial_{t}$ from $Z$ in equation (11), we get

$$
\begin{equation*}
\mathrm{Op}_{h}\left(2 x \xi+|\xi|^{1 / 2}\right) v=-\sqrt{h} Q_{0}(V)+h\left[\frac{i}{2} v-i Z v-C_{0}(V)\right]-h^{1+\kappa} R(V) \tag{12}
\end{equation*}
$$

Since we factored out the expected decay in $1 / \sqrt{t}$, our goal is to deduce from assumptions $(A)$ and $(B)$ estimates of the form $\left\|Z^{k} v\right\|_{L^{\infty}}=O\left(\varepsilon h^{-\delta_{k}^{\prime}}\right)$ for $k \leq s-100$.

Proposition. Assume that for $t$ in some interval $\left[T_{0}, T[\right.$ (i.e. for $h$ in some interval $\left.] h^{\prime}, h_{0}\right]$ ), one has estimates $(A)$ and $(B)$ :

$$
\begin{equation*}
\left\|Z^{k} v\right\|_{L^{\infty}}=O\left(\varepsilon h^{-\tilde{\delta}_{k}^{\prime}}\right), k \leq s / 2,\left\|Z^{k} v\right\|_{L^{2}}=O\left(\varepsilon h^{-\delta_{k}}\right), k \leq s \tag{13}
\end{equation*}
$$

Denote $\Lambda=\left\{(x, d \omega(x)) ; x \in \mathbb{R}^{*}\right\}$ where $\omega(x)=1 /(4|x|)$. Then, if $\gamma_{\Lambda}$ is smooth, supported close to $\Lambda$ and equal to one on a neighborhood of $\Lambda$, and if $\gamma_{\Lambda}^{c}=1-\gamma_{\Lambda}$, we have for $k \leq s-100$

$$
\begin{gather*}
\left\|Z^{k} \mathrm{Op}_{h}\left(\gamma_{\Lambda}^{c}\right) v\right\|_{L^{2}}=O\left(\varepsilon h^{\frac{1}{2}-\delta_{k}^{\prime}}\right)  \tag{14}\\
\left\|\left(h D_{x}-d \omega\right) Z^{k} \mathrm{Op}_{h}\left(\gamma_{\Lambda}\right) v\right\|_{L^{2}}=O\left(\varepsilon h^{1-\delta_{k}^{\prime}}\right)  \tag{15}\\
\left\|Z^{k} v\right\|_{L^{\infty}}=O\left(\varepsilon h^{-\delta_{k}^{\prime}}\right) \tag{16}
\end{gather*}
$$

Idea of proof. One applies $k$ vector fields $Z$ to (12) and uses their commutation properties to the linearized equation. In that way, taking into account the assumptions, one gets

$$
\begin{equation*}
\mathrm{Op}_{h}\left(2 x \xi+|\xi|^{1 / 2}\right) Z^{k} v=O_{L^{2}}\left(\varepsilon h^{\frac{1}{2}-\delta_{k}^{\prime}}\right) \tag{17}
\end{equation*}
$$

for some small $\delta_{k}^{\prime}>0$. One remarks that $2 x \xi+|\xi|^{1 / 2}$ vanishes exactly on $\Lambda$. Consequently, this symbol is elliptic on the support of $\gamma_{\Lambda}^{c}$, and this allows one to get (14) by ellipticity.

To prove the second inequality, one uses the fact that,

$$
\begin{equation*}
\mathrm{Op}_{h}\left(2 x \xi+|\xi|^{1 / 2}\right) Z^{k} \mathrm{Op}_{h}\left(\gamma_{\Lambda}\right) v=-\sqrt{h} \mathrm{Op}_{h}\left(\gamma_{\Lambda}\right) Z^{k} Q_{0}(V)+O\left(\varepsilon h^{1-\delta_{k}^{\prime}}\right) \tag{18}
\end{equation*}
$$

We may decompose $v=v_{\Lambda}+v_{\Lambda^{c}}$ where $v_{\Lambda}=\mathrm{Op}_{h}\left(\gamma_{\Lambda}\right) v$ and $v_{\Lambda^{c}}=\mathrm{Op}_{h}\left(\gamma_{\Lambda}^{c}\right) v$. We may write $Z^{k} Q_{0}(V)-Z^{k} Q_{0}\left(v_{\Lambda}, \bar{v}_{\Lambda}\right)=B\left(v_{\Lambda}, Z^{k} v_{\Lambda^{c}}\right)+\cdots$ where $B$ is the polar form of $Q_{0}$. By (14), $\left\|Z^{k} v_{\Lambda^{c}}\right\|_{L^{2}}=O\left(\varepsilon h^{\frac{1}{2}-\delta_{k}^{\prime}}\right)$, and by assumption $\left\|v_{\Lambda}\right\|_{L^{\infty}}=O(\varepsilon)$. It follows that $\left\|B\left(v_{\Lambda}, Z^{k} v_{\Lambda} c\right)\right\|_{L^{2}}=O\left(\varepsilon h^{\frac{1}{2}-\delta_{k}^{\prime}}\right)$. The other contributions to $Z^{k} Q_{0}(V)-Z^{k} Q_{0}\left(v_{\Lambda}, \bar{v}_{\Lambda}\right)$ may be estimated in a similar way, up to extra contributions, that we do not write explicitly in this
outline, and that may be absorbed in the left hand side of (16) at the end of the reasoning. The right hand side of (18) may thus be written

$$
\begin{equation*}
-\sqrt{h} \mathrm{Op}_{h}\left(\gamma_{\Lambda}\right) Z^{k} Q_{0}\left(V_{\Lambda}\right)+O_{L^{2}}\left(\varepsilon h^{1-\delta_{k}^{\prime}}\right), \tag{19}
\end{equation*}
$$

where $V_{\Lambda}=\left(v_{\Lambda}, \bar{v}_{\Lambda}\right)$. One notices then that since $v_{\Lambda}$ (resp. $\bar{v}_{\Lambda}$ ) is microlocally supported close to $\Lambda$ (resp. $-\Lambda$ ), $Q_{0}\left(V_{\Lambda}\right)$ is microlocally supported close to the union of $2 \Lambda, 0 \Lambda$ and $-2 \Lambda$, so far away from the support of the cut-off $\gamma_{\Lambda}$ (where $\left.\ell \Lambda=\left\{(x, \ell d \omega(x)) ; x \in \mathbb{R}^{*}\right\}\right)$.

Consequently, the first term in (19) vanishes, and we get

$$
\mathrm{Op}_{h}\left(2 x \xi+|\xi|^{1 / 2}\right) Z^{k} \mathrm{Op}_{h}\left(\gamma_{\Lambda}\right) v=O_{L^{2}}\left(\varepsilon h^{1-\delta_{k}^{\prime}}\right) .
$$

Since $2 x \xi+|\xi|^{1 / 2}$ and $\xi-d \omega(x)$ have the same zero set, namely $\Lambda$, one deduces (15) from this estimate using symbolic calculus.

Finally, to obtain (16), we write

$$
\left\|Z^{k} v_{\Lambda}\right\|_{L^{\infty}}=\left\|e^{-i \omega / h} Z^{k} v_{\Lambda}\right\|_{L^{\infty}} \leq C\left\|e^{-i \omega / h} Z^{k} v_{\Lambda}\right\|_{L^{2}}^{1 / 2}\left\|D_{x}\left(e^{-i \omega / h} Z^{k} v_{\Lambda}\right)\right\|_{L^{2}}^{1 / 2}
$$

The last factor is $h^{-1 / 2}\left\|\left(h D_{x}-d \omega\right) Z^{k} v_{\Lambda}\right\|_{L^{2}}^{1 / 2}$, which is $O\left(\sqrt{\varepsilon} h^{-\delta_{k}^{\prime} / 2}\right)$ by (15). Moreover, (14) and Sobolev inequality imply that $\left\|Z^{k} v_{\Lambda^{c}}\right\|_{L^{\infty}}=O\left(\varepsilon h^{-\delta_{k}^{\prime}}\right)$, since we have assumed that $v$ is spectrally localized for $|\xi| \sim 1 / h$. This gives (16).

## 4 Optimal $L^{\infty}$ bounds

As seen in the preceding section, one can deduce from the $L^{2}$-estimates $(B)$ some $L^{\infty}$-estimates (16), which are not the optimal estimates of the form $(A)$ that we need (because the exponents $\delta_{k}^{\prime}$ are larger than $\tilde{\delta}_{k}^{\prime}$, and because $\delta_{0}^{\prime}$ is positive, while we need a uniform estimate when no $Z$ field acts on $v$ ). In order to get $(A)$, we deduce from the PDE (11) an ODE satisfied by $v$.

Proposition. Under the conclusions of the preceding proposition, we may write

$$
\begin{equation*}
v=v_{\Lambda}+\sqrt{h}\left(v_{2 \Lambda}+v_{-2 \Lambda}\right)+h\left(v_{3 \Lambda}+v_{-\Lambda}+v_{-3 \Lambda}\right)+h^{1+\kappa} g, \tag{20}
\end{equation*}
$$

where $\kappa>0$, $g$ satisfies bounds of the form $\left\|Z^{k} g\right\|_{L^{\infty}}=O\left(\varepsilon h^{-\delta_{k}^{\prime}}\right)$, and $v_{\ell \Lambda}$ is microlocally supported close to $\ell \Lambda$ and is a semi-classical lagrangian distribution along $\ell \Lambda$, as well as $Z^{k} w_{\ell \Lambda}$ for $k \leq s / 2$, in the following sense

$$
\begin{gather*}
\left\|\mathrm{Op}_{h}\left(e_{\ell}(x, \xi)\right) Z^{k} v_{\ell \Lambda}\right\|_{L^{\infty}}=O\left(\varepsilon h^{1-\delta_{k}^{\prime}}\right), \ell \in\{1,-2,2\},  \tag{22}\\
\left\|\mathrm{Op}_{h}\left(e_{\ell}(x, \xi)\right) Z^{k} v_{\ell \Lambda}\right\|_{L^{\infty}}=O\left(\varepsilon h^{\frac{1}{2}-\delta_{k}^{\prime}}\right), \ell \in\{-3,-1,3\}, \tag{23}
\end{gather*}
$$

if $e_{\ell}$ vanishes on $\ell \Lambda$.

Remark. Consider a function $w=\alpha(x) \exp (i \omega(x) / h)$. If $\alpha$ is smooth and bounded as well as its derivatives, we see that $\left(h D_{x}-d \omega(x)\right) w=O_{L^{\infty}}(h)$ i.e. $w$ satisfies the second of the above conditions with $\ell=1$, where $e_{1}(x, \xi)=\xi-d \omega(x)$ is an equation of $\Lambda$. The conclusion of the proposition thus means that $v_{\ell \Lambda}$ enjoys a weak form of such an oscillatory behavior.

The proposition is proved using equation (12). For instance, the bound (22) for $v_{\Lambda}=$ $\mathrm{Op}_{h}\left(\gamma_{\Lambda}\right) v$ is proved in the same way as (15), with $L^{2}$-norms replaced by $L^{\infty}$ ones, using (16) to estimate the right hand side. In the same way, one defines $v_{ \pm 2 \Lambda}$ as the cut-off of $v$ close to $\pm 2 \Lambda$. As in the proof of (14), one shows an $O_{L^{\infty}}\left(h^{\frac{1}{2}-\delta_{k}^{\prime}}\right)$ bound for $Z^{k} v_{\Lambda^{c}}$, which implies that the main contribution to $Q_{0}(v, \bar{v})$ is $Q_{0}\left(v_{\Lambda}, \bar{v}_{\Lambda}\right)$. Localizing (12) close to $\pm 2 \Lambda$, one gets an elliptic equation that allows to determine $v_{ \pm 2 \Lambda}$ as a quadratic function of $v_{\Lambda}, \bar{v}_{\Lambda}$. Iterating the argument, one gets the expansion of the proposition. One does not get in the $\sqrt{h}$-terms of the expansion a contribution associated to $0 \Lambda$ because $Q_{0}(V)$ may be factored out by a Fourier multiplier vanishing on the zero section. Consequently, non oscillating terms form part of the $O\left(h^{1+\kappa}\right)$ remainder.

Let us use the result of the preceding proposition to obtain an ODE satisfied by $v$ :
Proposition. The function $v$ satisfies an ODE of the form

$$
\begin{align*}
D_{t} v= & \frac{1}{2}\left(1-\chi\left(h^{-\beta} x\right)\right)|d \omega|^{1 / 2} v-i \sqrt{h}\left(1-\chi\left(h^{-\beta} x\right)\right)\left[\Phi_{2}(x) v^{2}+\Phi_{-2}(x) \bar{v}^{2}\right] \\
& +h\left(1-\chi\left(h^{-\beta} x\right)\right)\left[\Phi_{3}(x) v^{3}+\Phi_{1}(x)|v|^{2} v+\Phi_{-1}(x)|v|^{2} \bar{v}+\Phi_{-3}(x) \bar{v}^{3}\right]  \tag{24}\\
& +O\left(\varepsilon h^{1+\kappa}\right),
\end{align*}
$$

where $\kappa>0, \beta>0$ are small, $\Phi_{\ell}$ are real valued functions of $x$ defined on $\mathbb{R}^{*}$ and $\chi$ is in $C_{0}^{\infty}(\mathbb{R})$, equal to one close to zero.

To prove the proposition, one plugs expansion (20) in equation (11). The key point is to use (22), (23) to express all (pseudo-)differential terms from multiplication operators and remainders. For instance, if $b(\xi)$ is some symbol, one may write $b(\xi)=\left.b\right|_{\ell \Lambda}+e_{\ell}$ where $e_{\ell}$ vanishes on $\ell \Lambda=\{\xi=\ell d \omega\}$. Consequently

$$
\mathrm{Op}_{h}(b) v_{\ell \Lambda}=b(\ell d \omega) v_{\ell \Lambda}+\operatorname{Op}_{h}\left(e_{\ell}\right) v_{\ell \Lambda},
$$

and by (22), when $\ell=-2,1,2$, one gets $\left\|\mathrm{Op}_{h}\left(e_{\ell}\right) v_{\ell \Lambda}\right\|_{L^{\infty}}=O\left(\varepsilon h^{1-\delta_{0}^{\prime}}\right)$. Since $Q_{0}\left(v_{\Lambda}, \bar{v}_{\Lambda}\right)$ is made of expressions of type

$$
S=\mathrm{Op}_{h}\left(b_{0}\right)\left[\left(\mathrm{Op}_{h}\left(b_{1}\right) v_{\Lambda}\right)\left(\mathrm{Op}_{h}\left(b_{2}\right) v_{\Lambda}\right)\right]
$$

(and similar ones replacing $v_{\Lambda}$ by $\bar{v}_{\Lambda}$ ), one gets, using that $v_{\Lambda}^{2}$ is lagrangian along $2 \Lambda$,

$$
S=b_{0}(2 d \omega) b_{1}(d \omega) b_{2}(d \omega) v_{\Lambda}^{2}+O_{L^{\infty}}\left(h^{1-\delta_{0}^{\prime}}\right) .
$$

One applies a similar procedure to the other pseudo-differential terms of equation (11), namely $\mathrm{Op}_{h}\left(x \xi+|\xi|^{1 / 2}\right) v$ and $C_{0}(V)$, where $v$ is expressed using (20) in which the $v_{\ell \Lambda}$ are written
as explicit quadratic or cubic forms in $\left(v_{\Lambda}, \bar{v}_{\Lambda}\right)$. This permits to write all those terms as polynomial expressions in $\left(v_{\Lambda}, \bar{v}_{\Lambda}\right)$ with $x$-depending coefficients, up to a remainder vanishing like $h^{1+\kappa}$ when $h$ goes to zero. Expressing back $v_{\Lambda}$ from $v$, one gets the ODE (24).

As soon as the preceding proposition has been established, the proof of optimal $L^{\infty}$-estimates for $v$ is straightforward. Applying a Poincaré normal forms method to (24), one is reduced to an equivalent ODE of the form

$$
D_{t} f=\frac{1}{2}\left(1-\chi\left(h^{-\beta} x\right)\right)|d \omega|^{1 / 2}\left[1+\frac{|d \omega|^{2}}{t}|f|^{2}\right] f+O\left(\varepsilon t^{-1-\kappa}\right) .
$$

This implies that $\partial_{t}|f|^{2}$ is integrable in time, whence a uniform bound for $f$ and explicit asymptotics when $t$ goes to infinity. Expressing $v$ in terms of $f$, and writing $u(t, x)=$ $\frac{1}{\sqrt{t}} v(t, x / t)$, one obtains the uniform $O\left(t^{-1 / 2}\right)$ bound for $u$ given in $(A)$ as well as the asymptotics of the statement of the main theorem. Estimates for $Z^{k} u$ are proved in the same way.

## 1 Statement of the main result

We have already written in the introduction the water waves equations under the form of the Craig-Sulem-Zakharov system (3). We shall give here the precise definition of the DirichletNeuman operator that is used in that system, and state some of its properties that are used in the rest of this paper, as well as in the companion paper [5]. Theses properties, that are essentially well known, are proved in that reference. Once the Dirichlet-Neuman operator has been properly defined, we give the precise statement of our global existence result. Next, we explain the strategy of proof, which relies on a bootstrap argument on some a priori $L^{2}$ and $L^{\infty}$ estimates. The $L^{2}$ bounds are proved in the companion paper [5]. The $L^{\infty}$ ones, that represent the main novelty of our method, are established in sections 2 to 6 of the present paper.

### 1.1 Dirichlet-Neumann operator

Let $\eta: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth enough function and consider the open set

$$
\Omega:=\{(x, y) \in \mathbb{R} \times \mathbb{R} ; y<\eta(x)\} .
$$

It $\psi: \mathbb{R} \rightarrow \mathbb{R}$ is another function, and if we call $\phi: \Omega \rightarrow \mathbb{R}$ the unique solution of $\Delta \phi=0$ in $\Omega$ satisfying $\left.\phi\right|_{y=\eta(x)}=\psi$ and a convenient vanishing condition at $y \rightarrow-\infty$, one defines the Dirichlet-Neumann operator $G(\eta)$ by

$$
G(\eta) \psi=\left.\sqrt{1+\left(\partial_{x} \eta\right)^{2}} \partial_{n} \phi\right|_{y=\eta},
$$

where $\partial_{n}$ is the outward normal derivative on $\partial \Omega$, so that

$$
G(\eta) \psi=\left(\partial_{y} \phi\right)(x, \eta(x))-\left(\partial_{x} \eta\right)\left(\partial_{x} \phi\right)(x, \eta(x)) .
$$

In this subsection, we recall the estimates obtained in [5] for $G(\eta)$.
One may reduce the problem to the negative half-space through the change of coordinates $(x, y) \mapsto(x, z=y-\eta(x))$, which sends $\Omega$ on $\left\{(x, z) \in \mathbb{R}^{2} ; z<0\right\}$. Then $\phi(x, y)$ solves $\Delta \phi=0$ if and only if $\varphi(x, z)=\phi(x, z+\eta(x))$ is a solution of $P \varphi=0$ in $z<0$, where

$$
\begin{equation*}
P=\left(1+\eta^{\prime 2}\right) \partial_{z}^{2}+\partial_{x}^{2}-2 \eta^{\prime} \partial_{x} \partial_{z}-\eta^{\prime \prime} \partial_{z} \tag{1.1}
\end{equation*}
$$

(we denote by $\eta^{\prime}$ the derivative $\partial_{x} \eta$ ). The boundary condition becomes $\varphi(x, 0)=\psi(x)$ and $G(\eta)$ is given by

$$
G(\eta) \psi=\left.\left[\left(1+\eta^{\prime 2}\right) \partial_{z} \varphi-\eta^{\prime} \partial_{x} \varphi\right]\right|_{z=0} .
$$

It is convenient and natural to try to solve the boundary value problem

$$
P \varphi=0,\left.\quad \varphi\right|_{z=0}=\psi
$$

when $\psi$ lies in homogeneous Sobolev spaces. Let us introduce them and fix some notation.
We denote by $\mathcal{S}_{\infty}^{\prime}(\mathbb{R})\left(\right.$ resp. $\left.\mathcal{S}_{1}^{\prime}(\mathbb{R})\right)$ the quotient space $\mathcal{S}^{\prime}(\mathbb{R}) / \mathbb{C}[X]\left(\right.$ resp. $\left.\mathcal{S}^{\prime}(\mathbb{R}) / \mathbb{C}\right)$. If $\mathcal{S}_{\infty}(\mathbb{R})$ (resp. $\mathcal{S}_{1}(\mathbb{R})$ ) is the subspace of $\mathcal{S}(\mathbb{R})$ made of the functions orthogonal to any polynomial
(resp. to the constants), $\mathcal{S}_{\infty}^{\prime}(\mathbb{R})$ (resp. $\mathcal{S}_{1}^{\prime}(\mathbb{R})$ ) is the dual of $\mathcal{S}_{\infty}(\mathbb{R})$ (resp. $\mathcal{S}_{1}(\mathbb{R})$ ). Since the Fourier transform realizes an isomorphism from $\mathcal{S}_{\infty}(\mathbb{R})$ (resp. $\mathcal{S}_{1}(\mathbb{R})$ ) to

$$
\widehat{\mathcal{S}}_{\infty}(\mathbb{R})=\left\{u \in \mathcal{S}(\mathbb{R}) ; u^{(k)}(0)=0 \text { for any } k \text { in } \mathbb{N}\right\}
$$

(resp. $\widehat{\mathcal{S}}_{1}(\mathbb{R})=\{u \in \mathcal{S}(\mathbb{R}) ; u(0)=0\}$ ), we get by duality that the Fourier transform defines an isomorphism from $\mathcal{S}_{\infty}^{\prime}(\mathbb{R})$ to $\left(\widehat{\mathcal{S}}_{\infty}(\mathbb{R})\right)^{\prime}$, which is the quotient of $\mathcal{S}^{\prime}(\mathbb{R})$ by the subspace of distributions supported in $\{0\}$ (resp. from $\mathcal{S}_{1}^{\prime}(\mathbb{R})$ to $\left(\widehat{\mathcal{S}}_{1}(\mathbb{R})\right)^{\prime}=\mathcal{S}^{\prime}(\mathbb{R}) / \operatorname{Vect}\left(\delta_{0}\right)$ ).

Let $\phi: \mathbb{R} \rightarrow \mathbb{R}$ be a function defining a Littlewood-Paley decomposition and set for $j \in \mathbb{Z}$, $\Delta_{j}=\phi\left(2^{-j} D\right)$. Then for any $u$ in $\mathcal{S}_{\infty}^{\prime}(\mathbb{R})$, the series $\sum_{j \in \mathbb{Z}} \Delta_{j} u$ converges to $u$ in $\mathcal{S}_{\infty}^{\prime}(\mathbb{R})$ (for the weak-* topology associated to the natural topology on $\mathcal{S}_{\infty}(\mathbb{R})$ ). Let us recall (an extension of) the usual definition of homogeneous Sobolev or Hölder spaces.

Definition 1.1. Let $s^{\prime}, s$ be real numbers. One denotes by $\dot{H}^{s^{\prime}, s}(\mathbb{R})\left(\right.$ resp. $\left.\dot{C}^{s^{\prime}, s}(\mathbb{R})\right)$ the space of elements $u$ in $\mathcal{S}_{\infty}^{\prime}(\mathbb{R})$ such that there is a sequence $\left(c_{j}\right)_{j \in \mathbb{Z}}$ in $\ell^{2}(\mathbb{Z})$ (resp. a constant $C>0$ ) with for any $j$ in $\mathbb{Z}$,

$$
\left\|\Delta_{j} u\right\|_{L^{2}} \leq c_{j} 2^{-j s^{\prime}-j_{+} s}
$$

(resp.

$$
\left.\left\|\Delta_{j} u\right\|_{L^{\infty}} \leq C 2^{-j s^{\prime}-j_{+} s}\right)
$$

where $j_{+}=\max (j, 0)$. We set $\dot{H}^{s^{\prime}}$ (resp. $\left.\dot{C}^{s^{\prime}}\right)$ when $s=0$.

The series $\sum_{j=0}^{+\infty} \Delta_{j} u$ always converges in $\mathcal{S}^{\prime}(\mathbb{R})$ under the preceding assumptions, but the same is not true for $\sum_{j=-\infty}^{-1} \Delta_{j} u$. If $u$ is in $\dot{H}^{s^{\prime}, s}(\mathbb{R})$ with $s^{\prime}<1 / 2$ (resp. in $\dot{C}^{s^{\prime}, s}(\mathbb{R})$ with $s^{\prime}<0$ ), then $\sum_{j=-\infty}^{-1} \Delta_{j} u$ converges normally in $L^{\infty}$, so in $\mathcal{S}^{\prime}(\mathbb{R})$, and $u \rightarrow \sum_{-\infty}^{+\infty} \Delta_{j} u$ gives the unique dilation and translation invariant realization of $\dot{H}^{s^{\prime}, s}\left(\right.$ resp. $\left.\dot{C}^{s^{\prime}, s}(\mathbb{R})\right)$ as a subspace of $\mathcal{S}^{\prime}(\mathbb{R})$. One the other hand, if $s^{\prime} \in\left[1 / 2,3 / 2\left[\right.\right.$ (resp. $s^{\prime} \in\left[0,1[)\right.$, the space $\dot{H}^{s^{\prime}}(\mathbb{R})$ (resp. $\left.\dot{C}^{s^{\prime}}(\mathbb{R})\right)$ admits no translation commuting realization as a subspace of $\mathcal{S}^{\prime}(\mathbb{R})$, but the map $u \rightarrow \sum_{-\infty}^{+\infty} \Delta_{j} u$ defines a dilation and translation commuting realization of these spaces as subspaces of $\mathcal{S}_{1}^{\prime}(\mathbb{R})$. We refer to Bourdaud [12] for these properties.

Recall also that if $s$ is in $\mathbb{R}$ (resp. $\gamma$ is in $\mathbb{R}-\mathbb{N}$ ), the usual Sobolev space $H^{s}(\mathbb{R})$ (resp. the space $\left.C^{\gamma}(\mathbb{R})\right)$ is defined as the space of elements $u$ of $\mathcal{S}^{\prime}(\mathbb{R})$ satisfying, for any $j$ in $\mathbb{N}$, $\left\|\Delta_{j} u\right\|_{L^{2}} \leq c_{j} 2^{-j s}$ (resp. $\left\|\Delta_{j} u\right\|_{L^{\infty}} \leq C 2^{-j s}$ ) for some $\ell^{2}(\mathbb{N})$-sequence $\left(c_{j}\right)_{j}$ (resp. some constant $C$ ), and $\chi(D) u \in L^{2}$ (resp. $\chi(D) u \in L^{\infty}$ ) for some $C_{0}^{\infty}(\mathbb{R})$-function $\chi$ equal to one on a large enough neighborhood of zero. Moreover, if $\gamma$ is in $\mathbb{N}$, we denote by $C^{\gamma}(\mathbb{R})$ the space of $\gamma$ times continuously differentiable functions, which are bounded as well as their derivatives (endowed with the natural norms).

The main result about the Dirichlet-Neumann operator that we shell use in that paper is the following proposition, which is proved in the companion paper [5] (see Corollary 1.1.8.):

Proposition 1.2. Let $\gamma$ be a real number, $\gamma>2, \gamma \notin \frac{1}{2} \mathbb{N}$. There is some $\delta>0$ such that, for any $\eta$ in $L^{2} \cap C^{\gamma}(\mathbb{R})$ satisfying $\left\|\eta^{\prime}\right\|_{C^{\gamma-1}}+\left\|\eta^{\prime}\right\|_{C^{-1}}^{1 / 2}\left\|\eta^{\prime}\right\|_{H^{-1}}^{1 / 2}<\delta$, one may define for $\psi$
in $\dot{H}^{1 / 2}(\mathbb{R})$ the Dirichlet-Neumann operator $G(\eta)$ as a bounded operator from $\dot{H}^{1 / 2}(\mathbb{R})$ to $\dot{H}^{-1 / 2}(\mathbb{R})$ that satisfies an estimate

$$
\begin{equation*}
\|G(\eta) \psi\|_{\dot{H}^{-1 / 2}} \leq C\left(\left\|\eta^{\prime}\right\|_{C^{\gamma-1}}\right)\left\|\left|D_{x}\right|^{\frac{1}{2}} \psi\right\|_{L^{2}} . \tag{1.2}
\end{equation*}
$$

In particular, if we define $G_{1 / 2}(\eta)=\left|D_{x}\right|^{-\frac{1}{2}} G(\eta)$, we obtain a bounded operator from $\dot{H}^{1 / 2}(\mathbb{R})$ to $L^{2}(\mathbb{R})$ satisfying

$$
\begin{equation*}
\left\|G_{1 / 2}(\eta) \psi\right\|_{L^{2}} \leq C\left(\left\|\eta^{\prime}\right\|_{C^{\gamma-1}}\right)\left\|\left|D_{x}\right|^{\frac{1}{2}} \psi\right\|_{L^{2}} \tag{1.3}
\end{equation*}
$$

Moreover, $G(\eta)$ satisfies when $\psi$ is in $\dot{C}^{\frac{1}{2}, \gamma-\frac{1}{2}}(\mathbb{R})$

$$
\begin{equation*}
\|G(\eta) \psi\|_{C^{\gamma-1}} \leq C\left(\left\|\eta^{\prime}\right\|_{C^{\gamma-1}}\right)\left\|\left|D_{x}\right|^{\frac{1}{2}} \psi\right\|_{C^{\gamma-\frac{1}{2}}} . \tag{1.4}
\end{equation*}
$$

where $C(\cdot)$ is a non decreasing continuous function of its argument.
If we assume moreover that for some $0<\theta^{\prime}<\theta<\frac{1}{2},\left\|\eta^{\prime}\right\|_{H^{-1}}^{1-2 \theta^{\prime}}\left\|\eta^{\prime}\right\|_{C^{-1}}^{2 \theta^{\prime}}$ is bounded, then $\left|D_{x}\right|^{-\frac{1}{2}+\theta} G(\eta)$ satisfies

$$
\begin{equation*}
\left\|\left|D_{x}\right|^{-\frac{1}{2}+\theta} G(\eta) \psi\right\|_{C^{\gamma-\frac{1}{2}-\theta}} \leq C\left(\left\|\eta^{\prime}\right\|_{C^{\gamma-1}}\right)\left\|\left|D_{x}\right|^{\frac{1}{2}} \psi\right\|_{C^{\gamma-\frac{1}{2}}} \tag{1.5}
\end{equation*}
$$

### 1.2 Global existence result

The goal of this paper is to prove global existence of small solutions with decaying Cauchy data of the Craig-Sulem-Zakharov system. We thus look for a couple of real valued functions $(\eta, \psi)$ defined on $\mathbb{R} \times \mathbb{R}$ satisfying for $t \geq 1$ the system

$$
\left\{\begin{array}{l}
\partial_{t} \eta=G(\eta) \psi  \tag{1.6}\\
\partial_{t} \psi+\eta+\frac{1}{2}\left(\partial_{x} \psi\right)^{2}-\frac{1}{2\left(1+\left(\partial_{x} \eta\right)^{2}\right)}\left(G(\eta) \psi+\partial_{x} \eta \partial_{x} \psi\right)^{2}=0,
\end{array}\right.
$$

with Cauchy data small enough in a convenient space.
The operator $G(\eta)$ in (1.6) being defined as in the preceding subsection, we set, for $\eta, \psi$ smooth enough and small enough functions

$$
\begin{equation*}
B(\eta) \psi=\frac{G(\eta) \psi+\partial_{x} \eta \partial_{x} \psi}{1+\left(\partial_{x} \eta\right)^{2}} \tag{1.7}
\end{equation*}
$$

Before stating our global existence result, let us recall a known local existence theorem (see [63, 47, 1]).

Proposition 1.3. Let $\gamma$ be in $] 7 / 2,+\infty\left[\backslash \frac{1}{2} \mathbb{N}, s \in \mathbb{N}\right.$ with $s>2 \gamma-1 / 2$. There are $\delta_{0}>0$, $T>1$ such that for any couple $\left(\eta_{0}, \psi_{0}\right)$ in $H^{s}(\mathbb{R}) \times \dot{H}^{\frac{1}{2}, \gamma}(\mathbb{R})$ satisfying

$$
\begin{equation*}
\psi_{0}-T_{B\left(\eta_{0}\right) \psi_{0}} \eta_{0} \in \dot{H}^{\frac{1}{2}, s}(\mathbb{R}), \quad\left\|\eta_{0}\right\|_{C^{\gamma}}+\left\|\left|D_{x}\right|^{\frac{1}{2}} \psi_{0}\right\|_{C^{\gamma-\frac{1}{2}}}<\delta_{0} \tag{1.8}
\end{equation*}
$$

equation (1.6) with Cauchy data $\left.\eta\right|_{t=1}=\eta_{0},\left.\psi\right|_{t=1}=\psi_{0}$ has a unique solution $(\eta, \psi)$ which is continuous on $[1, T]$ with values in

$$
\begin{equation*}
\left\{(\eta, \psi) \in H^{S}(\mathbb{R}) \times \dot{H}^{\frac{1}{2}, \gamma}(\mathbb{R}) ; \psi-T_{B(\eta) \psi} \eta \in \dot{H}^{\frac{1}{2}, s}(\mathbb{R})\right\} . \tag{1.9}
\end{equation*}
$$

Moreover, if the data are $O(\varepsilon)$ on the indicated spaces, then $T \geq c / \varepsilon$.
Remarks. - The assumption $\psi_{0} \in \dot{H}^{\frac{1}{2}, \gamma}$ implies that $\psi_{0}$ is in $\dot{C}^{\frac{1}{2}, \gamma-\frac{1}{2}}$ so that Proposition 1.2 shows that $G\left(\eta_{0}\right) \psi_{0}$ whence $B\left(\eta_{0}\right) \psi_{0}$ is in $C^{\gamma-1} \subset L^{\infty}$. Consequently, by the first half of (1.8), $\left|D_{x}\right|^{\frac{1}{2}} \psi$ is in $H^{s-\frac{1}{2}} \subset C^{\gamma-\frac{1}{2}}$ as our assumption on $s$ implies that $s>\gamma+1 / 2$. This gives sense to the second assumption (1.8).

- As already mentioned in the introduction, the difficulty in the analysis of equation (1.6) is that writing energy inequalities on the function $\left(\eta,\left|D_{x}\right|^{\frac{1}{2}} \psi\right)$ makes appear an apparent loss of half a derivative. A way to circumvent that difficulty is to bound the energy not of $\left(\eta,\left|D_{x}\right|^{\frac{1}{2}} \psi\right)$, but of $\left(\eta,\left|D_{x}\right|^{\frac{1}{2}} \omega\right)$, where $\omega$ is the "good unknown" of Alinhac, defined by $\omega=\psi-T_{B(\eta) \psi} \eta$ (see subsection 2 of the introduction). This explains why the regularity assumption (1.8) on the Cauchy data concerns $\psi_{0}-T_{B\left(\eta_{0}\right) \psi_{0}} \eta_{0}$ and not $\psi_{0}$ itself. Notice that this function is in $\dot{H}^{\frac{1}{2}, s}$ while $\psi_{0}$ itself, written from $\psi_{0}=\omega_{0}+T_{B\left(\eta_{0}\right) \psi_{0}} \eta_{0}$ is only in $\dot{H}^{\frac{1}{2}, s-\frac{1}{2}}$, because of the $H^{s}$-regularity of $\eta_{0}$.
- By (1.4) if $\psi$ is in $\dot{C}^{\frac{1}{2}, \gamma-\frac{1}{2}}$ and $\eta$ is in $C^{\gamma}, G(\eta) \psi$ is in $C^{\gamma-1}$, so $B(\eta) \psi$ is also in $C^{\gamma-1}$ with $\|B(\eta) \psi\|_{C^{\gamma-1}} \leq C\left(\left\|\eta^{\prime}\right\|_{C^{\gamma-1}}\right)\left\|\left|D_{x}\right|^{\frac{1}{2}} \psi\right\|_{C^{\gamma-\frac{1}{2}}}$. In particular, as a paraproduct with an $L^{\infty}$-function acts on any Hölder space,

$$
\left\|\left|D_{x}\right|^{\frac{1}{2}} T_{B(\eta) \psi} \eta\right\|_{C^{\gamma-\frac{1}{2}}} \leq C\left(\left\|\eta^{\prime}\right\|_{C^{\gamma-1}}\right)\|\eta\|_{C^{\gamma}}\left\|\left|D_{x}\right|^{\frac{1}{2}} \psi\right\|_{C^{\gamma-\frac{1}{2}}} .
$$

This shows that for $\|\eta\|_{C^{\gamma}}$ small enough, $\psi \rightarrow \psi-T_{B(\eta) \psi} \eta$ is an isomorphism from $\dot{C}^{\frac{1}{2}, \gamma-\frac{1}{2}}$ to itself. In particular, if we are given a small enough $\omega$ in $\dot{H}^{\frac{1}{2}, s} \subset \dot{C}^{\frac{1}{2}, \gamma-\frac{1}{2}}$, we may find a unique $\psi$ in $\dot{C}^{\frac{1}{2}, \gamma-\frac{1}{2}}$ such that $\omega=\psi-T_{B(\eta) \psi} \eta$. In other words, when interested only in $C^{\gamma-\frac{1}{2}}$-estimates for $\left|D_{x}\right|^{\frac{1}{2}} \omega$, we may as well establish them on $\left|D_{x}\right|^{\frac{1}{2}} \psi$ instead, as soon as $\|\eta\|_{C^{\gamma}}$ stays small enough.

Let us state now our main result.
We fix real numbers $s, s_{1}, s_{0}$ satisfying, for some large enough numbers $a$ and $\gamma$ with $\gamma \notin \frac{1}{2} \mathbb{N}$ and $a \gg \gamma$, the following conditions

$$
\begin{equation*}
s, s_{0}, s_{1} \in \mathbb{N}, \quad s-a \geq s_{1} \geq s_{0} \geq \frac{s}{2}+\gamma \tag{1.10}
\end{equation*}
$$

Theorem 1.4. There is $\varepsilon_{0}>0$ such that for any $\left.\left.\varepsilon \in\right] 0, \varepsilon_{0}\right]$, any couple of functions ( $\eta_{0}, \psi_{0}$ ) satisfying for any integer $p \leq s_{1}$

$$
\begin{align*}
& \left(x \partial_{x}\right)^{p} \eta_{0} \in H^{s-p}(\mathbb{R}), \quad\left(x \partial_{x}\right)^{p} \psi_{0} \in \dot{H}^{\frac{1}{2}, s-p-\frac{1}{2}}(\mathbb{R}), \\
& \left(x \partial_{x}\right)^{p}\left(\psi_{0}-T_{\left.B\left(\eta_{0}\right) \psi_{0} \eta_{0}\right) \in \dot{H}^{\frac{1}{2}, s-p}(\mathbb{R}),}\right. \tag{1.11}
\end{align*}
$$

and such that the norm of the above functions in the indicated spaces is smaller than 1 , equation (1.6) with the Cauchy data $\left.\eta\right|_{t=1}=\varepsilon \eta_{0},\left.\psi\right|_{t=1}=\varepsilon \psi_{0}$ has a unique solution $(\eta, \psi)$ which is defined and continuous on $[1,+\infty[$ with values in the set (1.9).

Moreover, $u=\left|D_{x}\right|^{\frac{1}{2}} \psi+i \eta$ admits the following asymptotic expansion as $t$ goes to $+\infty$ :
There is a continuous function $\underline{\alpha}: \mathbb{R} \rightarrow \mathbb{C}$, depending of $\varepsilon$ but bounded uniformly in $\varepsilon$, such that

$$
\begin{equation*}
u(t, x)=\frac{\varepsilon}{\sqrt{t}} \underline{\alpha}\left(\frac{x}{t}\right) \exp \left(\frac{i t}{4|x / t|}+\frac{i \varepsilon^{2}}{64} \frac{|\underline{\alpha}(x / t)|^{2}}{|x / t|^{5}} \log (t)\right)+\varepsilon t^{-\frac{1}{2}-\kappa} \rho(t, x) \tag{1.12}
\end{equation*}
$$

where $\kappa$ is some positive number and $\rho$ is a function uniformly bounded for $\left.t \geq 1, \varepsilon \in] 0, \varepsilon_{0}\right]$.
Remark. If the integers $s, s_{1}, s_{0}$ are large enough, we shall see in section 6 that $\underline{\alpha}(x / t)$ vanishes when $x / t$ goes to zero at an order that increases with these integers. Because of that, we see that the singularity of the phase at $x / t=0$ is quite irrelevant: for $|x / t|$ small enough, the first term in the expansion is not larger than the remainder.

### 1.3 Strategy of the proof

The proof of the main theorem relies on the simultaneous propagation through a bootstrap of $L^{\infty}$ and $L^{2}$-estimates. We state here these two results. The first one is proved in the companion paper [5]. The proof of the second one is the bulk of the present paper. We show below how these two results together imply Theorem 1.4.

The main point will be to prove $L^{2}$ and $L^{\infty}$-estimates for the action of the vector field

$$
\begin{equation*}
Z=t \partial_{t}+2 x \partial_{x} \tag{1.13}
\end{equation*}
$$

on the unknown in equation (1.6). We introduce the following notation:
We assume given $\gamma, s, a, s_{0}, s_{1}$ satisfying (1.10). For $(\eta, \psi)$ a local smooth enough solution of (1.6), we set $\omega=\psi-T_{B(\eta) \psi} \eta$ and for any integer $k \leq s_{1}$,

$$
\begin{equation*}
M_{s}^{(k)}(t)=\sum_{p=0}^{k}\left(\left\|Z^{p} \eta(t, \cdot)\right\|_{H^{s-p}}+\left\|\left|D_{x}\right|^{\frac{1}{2}} Z^{p} \omega(t, \cdot)\right\|_{H^{s-p}}\right) . \tag{1.14}
\end{equation*}
$$

In the same way, for $\rho$ a positive number (that will be larger than $s_{0}$ ), we set for $k \leq s_{0}$,

$$
\begin{equation*}
N_{\rho}^{(k)}(t)=\sum_{p=0}^{k}\left(\left\|Z^{p} \eta(t, \cdot)\right\|_{C^{\rho-p}}+\left\|\left|D_{x}\right|^{\frac{1}{2}} Z^{p} \psi(t, \cdot)\right\|_{C^{\rho-p}}\right) . \tag{1.15}
\end{equation*}
$$

By local existence theory, for any given $T_{0}>1$, there is $\varepsilon_{0}^{\prime}>0$ such that if $\varepsilon<\varepsilon_{0}^{\prime}$, equation (1.6) has a solution for $t \in\left[1, T_{0}\right]$. Moreover, assumptions (1.11) remain valid at $t=T_{0}$ (see Proposition A.4.2. in the companion paper). Consequently, it is enough to prove Theorem 1.4 with Cauchy data at $t=T_{0}$.

The $L^{2}$ estimates that we need are given by the following theorem, that is proved in the companion paper [5] (see Theorem 1.2.2. of that paper).

Theorem 1.5. There is a constant $B_{2}>0$ such that $M_{s}^{\left(s_{1}\right)}\left(T_{0}\right)<\frac{1}{4} B_{2} \varepsilon$, and for any constants $B_{\infty}>0, B_{\infty}^{\prime}>0$ there is $\varepsilon_{0}$ such that the following holds: Let $T>T_{0}$ be a number such that equation (1.6) with Cauchy data satisfying (1.11) has a solution satisfying the regularity properties of Proposition 1.3 on $\left[T_{0}, T[\times \mathbb{R}\right.$ and such that
i) For any $t \in\left[T_{0}, T[\right.$, and any $\left.\varepsilon \in] 0, \varepsilon_{0}\right]$,

$$
\begin{equation*}
\left\|\left|D_{x}\right|^{\frac{1}{2}} \psi(t, \cdot)\right\|_{C^{\gamma-\frac{1}{2}}}+\|\eta(t, \cdot)\|_{C^{\gamma}} \leq B_{\infty} \varepsilon t^{-\frac{1}{2}} \tag{1.16}
\end{equation*}
$$

ii) For any $t \in\left[T_{0}, T[\right.$, any $\left.\varepsilon \in] 0, \varepsilon_{0}\right]$

$$
\begin{equation*}
N_{\rho}^{\left(s_{0}\right)}(t) \leq B_{\infty} \varepsilon t^{-\frac{1}{2}+B_{\infty}^{\prime} \varepsilon^{2}} . \tag{1.17}
\end{equation*}
$$

Then, there is an increasing sequence $\left(\delta_{k}\right)_{0 \leq k \leq s_{1}}$, depending only on $B_{\infty}^{\prime}$ and $\varepsilon$ with $\delta_{s_{1}}<1 / 32$ such that for any $t$ in $\left[T_{0}, T[\right.$, any $\varepsilon$ in $\left.] 0, \varepsilon_{0}\right]$, any $k \leq s_{1}$,

$$
\begin{equation*}
M_{s}^{(k)}(t) \leq \frac{1}{2} B_{2} \varepsilon t^{\delta_{k}} \tag{1.18}
\end{equation*}
$$

Remark. We do not get for the $L^{2}$-quantities $M_{s}^{(k)}(t)$ a uniform estimate when $t \rightarrow+\infty$. Actually, the form of the principal term in the expansion (1.12) shows that the action of a $Z$-vector field on it generates a $\log (t)$-loss, so that one cannot expect (1.18) to hold true with $\delta_{k}=0$. For similar reasons, one could not expect that $N_{\rho}^{\left(s_{0}\right)}(t)$ in (1.17) be $O\left(t^{-1 / 2}\right)$ when $t \rightarrow+\infty$. Such an estimate can be true only if no $Z$-derivative acts on the solution, as in (1.16).

Let us write down next the $L^{\infty}$-estimates.
Theorem 1.6. Let $T>T_{0}$ be a number such that the equation (1.6) with Cauchy data satisfying (1.11) has a solution on $\left[T_{0}, T[\times \mathbb{R}\right.$ satisfying the regularity properties of Proposition 1.3. Assume that, for some constant $B_{2}>0$, for any $t \in\left[T_{0}, T[\right.$, any $\varepsilon$ in $\left.] 0,1\right]$, any $k \leq s_{1}$,

$$
\begin{align*}
& M_{s}^{(k)}(t) \leq B_{2} \varepsilon t^{\delta_{k}} \\
& N_{\rho}^{\left(s_{0}\right)}(t) \leq \sqrt{\varepsilon}<1 \tag{1.19}
\end{align*}
$$

Then there are constants $B_{\infty}, B_{\infty}^{\prime}>0$ depending only on $B_{2}$ and some $\left.\left.\varepsilon_{0}^{\prime} \in\right] 0,1\right]$, independent of $B_{2}$, such that, for any $t$ in $\left[T_{0}, T[\right.$, any $\varepsilon$ in $\left.] 0, \varepsilon_{0}^{\prime}\right]$,

$$
\begin{align*}
& N_{\rho}^{\left(s_{0}\right)}(t) \leq \frac{1}{2} B_{\infty} \varepsilon t^{-\frac{1}{2}+\varepsilon^{2} B_{\infty}^{\prime}}, \\
& \left\|\left|D_{x}\right|^{\frac{1}{2}} \psi(t, \cdot)\right\|_{C^{\gamma-\frac{1}{2}}}+\|\eta(t, \cdot)\|_{C^{\gamma}} \leq \frac{1}{2} B_{\infty} \varepsilon t^{-\frac{1}{2}} . \tag{1.20}
\end{align*}
$$

We deduce form the above results the global existence statements in Theorem 1.7.

Proof of Theorem 1.4. We take for $B_{2}$ the constant given by Theorem 1.5. Then Theorem 1.6 provides constants $B_{\infty}>0, B_{\infty}^{\prime}>0$, and given these $B_{\infty}, B_{\infty}^{\prime}$, Theorem 1.5 brings a small positive number $\varepsilon_{0}$. We denote by $T_{*}$ the supremum of those $T>T_{0}$ such that a solution exists over the interval $\left[T_{0}, T[\right.$, satisfies over this interval the regularity conditions of Proposition 1.3 and the estimates

$$
\begin{align*}
& M_{s}^{(k)}(t) \leq B_{2} \varepsilon t^{\delta_{k}} \quad \text { for } k \leq s_{1}, \\
& N_{\rho}^{\left(s_{0}\right)}(t) \leq B_{\infty} \varepsilon t^{-\frac{1}{2}+\varepsilon^{2} B_{\infty}^{\prime}},  \tag{1.21}\\
& \left\|\left|D_{x}\right|^{\frac{1}{2}} \psi(t, \cdot)\right\|_{C^{\gamma-\frac{1}{2}}}+\|\eta(t, \cdot)\|_{C^{\gamma}} \leq B_{\infty} \varepsilon t^{-\frac{1}{2}}
\end{align*}
$$

We have $T_{*}>T_{0}$ : By the choice of $B_{2}$ in the statement of Theorem 1.5, the first estimate (1.21) holds at $t=T_{0}$ with $B_{2}$ replaced by $B_{2} / 2$. If $B_{\infty}$ is chosen from start large enough, we may as well assume that at $t=T_{0}$, the second and third inequalities in (1.21) hold with $B_{\infty}$ replaced by $B_{\infty} / 2$. Consequently, the local existence results of Appendix A. 4 in the companion paper [5] show that a solution exists on some interval [ $T_{0}, T_{0}+\delta[$, and will satisfy (1.21) on that interval if $\delta$ is small enough.

If $T_{*}<+\infty$, and if we take $\varepsilon_{0}$ small enough so that $B_{\infty} \sqrt{\varepsilon_{0}}<1$, we see that (1.21) implies that assumptions (1.16), (1.17), and (1.19) of Theorem 1.5 and 1.6 are satisfied. Consequently, (1.18) and (1.20) hold on the interval [ $T_{0}, T_{*}$ [ i.e. (1.21) is true on this interval with $B_{2}$ (resp. $B_{\infty}$ ) replaced by $B_{2} / 2$ (resp. $B_{\infty} / 2$ ). This contradicts the maximality of $T_{*}$. So $T_{*}=+\infty$ and the solution is global. We postpone the proof of (1.12) to the end of Section 6.

The rest of this paper will be devoted to the proof of Theorem 1.6. Theorem 1.5 is proved in the companion paper [5].

## 2 Classes of Lagrangian distributions

We denote by $h$ a semi-classical parameter belonging to $] 0,1]$. If $(x, \xi) \mapsto m(x, \xi)$ is an order function from $T^{*} \mathbb{R}$ to $\mathbb{C}$, as defined in Appendix A, and if $a$ is a symbol in the class $S(m)$ of Definition A.1, we set, for $\left(u_{h}\right)_{h}$ any family of elements of $\mathcal{S}^{\prime}(\mathbb{R})$

$$
\begin{equation*}
\mathrm{Op}_{h}(a) u=\frac{1}{2 \pi} \int e^{i x \xi} a(x, h \xi, h) \widehat{u}(\xi) d \xi \tag{2.1}
\end{equation*}
$$

It turns out that we shall need extensions of this definition to more general classes of symbols. On the one hand, we notice that if $a$ is a continuous function such that $|a(x, \xi, h)| \leq m(x, \xi)$ and if $u$ is in $L^{2}(\mathbb{R}),(2.1)$ is still meaningful.

We shall also use a formula of type (2.1) when the symbol $a$ is defined only on a subset of $T^{*} \mathbb{R}$. Denote by $\pi_{1}:(x, \xi) \mapsto x$ and $\pi_{2}:(x, \xi) \mapsto \xi$ the two projections. For $F$ a closed subset
of $T^{*} \mathbb{R}$, and $r>0$, we set

$$
F_{r}=\left\{(x, \xi) \in T^{*} \mathbb{R} ; d((x, \xi), F)<r\right\}
$$

where $d$ is the euclidian distance.
Definition 2.1. Let $m$ be an order function on $T^{*} \mathbb{R}, F$ a closed non empty subset of $T^{*} \mathbb{R}$ such that $\pi_{2}(F)$ is compact. We denote by $S(m, F)$ the space of functions $(x, \xi, h) \mapsto a(x, \xi, h)$, defined on $\left.\left.F_{r_{0}} \times\right] 0,1\right]$ for some $r_{0}>0$, and satisfying for any $\alpha, \beta$ in $\mathbb{N}$, any $(x, \xi)$ in $F_{r_{0}}$, any $h$ in 10,1 ],

$$
\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} a(x, \xi, h)\right| \leq C_{\alpha, \beta} m(x, \xi) .
$$

We define next the notion of a family of functions microlocally supported close to a subset $F$ as above.

Definition 2.2. i) Let $p$ be in $[1,+\infty]$. We denote by $E_{\emptyset}^{p}$ the space of families of $L^{p}$ functions $\left(v_{h}\right)_{h}$ indexed by $\left.\left.h \in\right] 0,1\right]$, defined on $\mathbb{R}$ with values in $\mathbb{C}$ such that for any $N$ in $\mathbb{N}$, there is $C_{N}>0$ with $\left\|v_{h}\right\|_{L^{p}} \leq C_{N} h^{N}$ for any $h$ in $\left.] 0,1\right]$.
ii) Let $F$ be a closed non empty subset of $T^{*} \mathbb{R}$ such that $\pi_{2}(F)$ is compact. We denote by $E_{F}^{p}$ the space of families of functions $\left(v_{h}\right)_{h}$ of $L^{p}(\mathbb{R})$ satisfying

- There are $N_{0}$ in $\mathbb{N}, C_{0}>0$ and for any $h$ in $\left.] 0,1\right],\left\|v_{h}\right\|_{L^{p}} \leq C_{0} h^{-N_{0}}$.
- For any $r>r^{\prime}>0$, there is an element $\phi$ of $S(1)$ supported in $F_{r}$, equal to one on $\overline{F_{r^{\prime}}}$ such that $\left(\mathrm{Op}_{h}(\phi) v_{h}-v_{h}\right)_{h}$ belongs to $E_{\emptyset}^{p}$. We say that $\left(v_{h}\right)_{h}$ is microlocally suported close to $F$.

Remarks. - Notice that definition 2.2 is non empty only if there exists at least one function $\phi$ in $S(1)$ supported in $F_{r}$, equal to one on $\overline{F_{r^{\prime}}}$. This holds if $F$ is not "too wild" when $|x|$ goes to infinity, for instance if $F$ is compact, or if $F=\pi_{2}^{-1}(K)$ for some compact subset $K$ of $\mathbb{R}$. In the sequel, we shall always implicitly assume that such a property holds for the closed subsets in which are microlocally supported the different classes of distributions we shall define.

- It follows from Theorem A. 2 of Appendix A that the last condition in Definition 2.2 will hold for any element $\phi$ of $S(1)$, supported in $F_{r}$, equal to one on $\overline{F_{r^{\prime}}}$.

We may define the action of operators associated to symbols belonging to the class $S(1, F)$ on functions microlocally supported close to $F$, modulo elements of $E_{\emptyset}^{p}$. Let us notice first that if $a$ is in $S(1)$ and is supported in a domain $\{(x, \xi, h) ;|\xi| \leq C\}$ for some $C>0$, then (2.1) defines an operator bounded on $L^{p}(\mathbb{R})$ for any $p$, uniformly in $h$. It follows then from the theorem of symbolic calculus A. 2 of Appendix A that, if $a$ is in $S(1, F)$, if $\tilde{\phi}$ is in $S(1)$ supported in $F_{r} \cap\left(\overline{F_{r^{\prime}}}\right)^{c}$, for some $0<r^{\prime}<r \ll 1$, then $\left(\mathrm{Op}_{h}(a \tilde{\phi}) v_{h}\right)_{h}$ is in $E_{\emptyset}^{p}$ for any $\left(v_{h}\right)_{h}$ in $E_{F}^{p}$. We may thus state:

Definition 2.3. Let $F$ be a closed set as in Definition 2.2, a be an element of $S(1, F)$. For $\left(v_{h}\right)_{h}$ in $E_{F}^{p}$, we define

$$
\begin{equation*}
\mathrm{Op}_{h}(a) v_{h}=\mathrm{Op}_{h}(a \phi) v_{h} . \tag{2.2}
\end{equation*}
$$

where the right-hand side is defined by (2.1) and where $\phi$ is in $S(1)$, supported in $\left.\left.F_{r} \times\right] 0,1\right]$ for small enough $r>0$ and equal to one on $\left.\left.\overline{F_{r^{\prime}}} \times\right] 0,1\right]$, for some $\left.r^{\prime} \in\right] 0, r[$. The definition is independent of the choice of $\phi$ modulo $E_{\emptyset}^{p}$, so that $\operatorname{Op}_{h}(a)$ is well defined from $E_{F}^{p} / E_{\emptyset}^{p}$ to itself.

Let $K$ be a compact subset of $T^{*} \mathbb{R}, K_{1}=\pi_{1}(K)$ and let $\omega$ be a real valued function defined on an open neighborhood $U$ of $K_{1}$. Denote by $\chi_{\omega}$ the canonical transformation

$$
\begin{aligned}
\chi_{\omega}: & T^{*} U
\end{aligned} \rightarrow T^{*} U .
$$

Set $K^{\prime}=\chi_{\omega}(K)$.
Lemma 2.4. $i$ ) Let a be in $S\left(1, K^{\prime}\right)$. There is a symbol $b$ in $S(1, K)$ such that for any $\left(v_{h}\right)_{h}$ in $E_{K}^{p}$, we may write

$$
\begin{equation*}
e^{i \omega(x) / h} \operatorname{Op}_{h}(a)\left(e^{-i \omega(x) / h} v_{h}\right)=\operatorname{Op}_{h}\left(a \circ \chi_{\omega}(x, \xi)\right) v_{h}+h \operatorname{Op}_{h}(b) v_{h} \tag{2.3}
\end{equation*}
$$

modulo $E_{\emptyset}^{p}$.
ii) If $\left(v_{h}\right)_{h}$ is in $E_{K}^{p}$, then $\left(e^{-i \omega(x) / h} v_{h}\right)_{h}$ is in $E_{K^{\prime}}^{p}$.

Proof. We shall prove both assertions at the same time. Remark first that since $\left(\theta v_{h}-v_{h}\right)_{h}$ is in $E_{\emptyset}^{p}$ if $\theta$ is in $C_{0}^{\infty}(U)$ equal to one on a neighborhood of $K_{1}$, we may always assume that $v_{h}$ is compactly supported in $U$. By symbolic calculus, and the assumption $a \in S\left(1, K^{\prime}\right)$, we may also assume that $a$ is compactly supported and that the first projection of the support is contained in $U$. Consequently, we may replace in (2.3) $\omega$ by a $C_{0}^{\infty}(\mathbb{R})$ function, equal to the given phase in a neighborhood of $K_{1}$.

We compute

$$
\begin{equation*}
e^{i \omega(x) / h} \operatorname{Op}_{h}(a)\left(e^{-i \omega(x) / h} v_{h}\right)=\frac{1}{2 \pi} \int e^{i x \xi} c(x, h \xi, h) \widehat{v_{h}}(\xi) d \xi \tag{2.4}
\end{equation*}
$$

with

$$
\begin{align*}
c(x, \xi, h) & =\frac{1}{2 \pi h} \int e^{-i[y \eta-(\omega(x)-\omega(x-y))] / h} a(x, \xi-\eta, h) d y d \eta  \tag{2.5}\\
& =\frac{1}{2 \pi h} \int e^{-i y \eta / h} a(x, \xi-\eta-\theta(x, y), h) d y d \eta
\end{align*}
$$

where $\theta(x, y)=\frac{\omega(x)-\omega(x-y)}{y}$. Let $\kappa$ be a smooth function supported in a small neighborhood of zero in $\mathbb{R}$ and equal to one close to zero. We insert under the last oscillatory integral in (2.5) a factor $\kappa(y) \kappa(\eta)$. The error introduced in that way is a symbol in $h^{\infty} S\left(\langle\xi\rangle^{-\infty}\right)$. The action of the associated operator on $v_{h}$ gives an element of $E_{\emptyset}^{p}$. We have reduced ourselves to

$$
\begin{equation*}
\frac{1}{2 \pi h} \int e^{-i y \eta / h} a(x, \xi-\eta-\theta(x, y), h) \kappa(y) \kappa(\eta) d y d \eta \tag{2.6}
\end{equation*}
$$

The argument of $a$ belongs to $K_{r}^{\prime}$ if $(x, \xi)$ is in $K_{r^{\prime}}$ with $r^{\prime} \ll r$ and the support of $\kappa$ has been taken small enough. Moreover, it is given by $(x, \xi-\mathrm{d} \omega(x)+O(y)+O(\eta))$, so that an integration by parts shows that (2.6) may be written $a \circ \chi_{\omega}+h b$ for some symbol $b$ in $S(1, K)$. This gives $i$ ).

To check $i i$ ) we apply (2.4) with $a=\phi^{\prime}$ an element of $S(1)$ supported in $\left.K_{r_{0}^{\prime}}^{\prime} \times\right] 0,1$, equal to one on $\left.\left.K_{r_{1}^{\prime}}^{\prime} \times\right] 0,1\right]$ for some $0<r_{1}^{\prime}<r_{0}^{\prime}$. We assume that $\left\|\mathrm{Op}_{h}(\phi) v_{h}-v_{h}\right\|_{L^{p}}=O\left(h^{\infty}\right)$ for some $\phi$ in $S(1)$ supported in $\left.\left.K_{r_{0}} \times\right] 0,1\right]$, $\phi \equiv 1$ on $\left.\left.K_{r_{1}} \times\right] 0,1\right]$ with $r_{1}<r_{0} \ll r_{1}^{\prime}$. Then if $(x, \xi)$ is in $K_{r_{0}}$ and $\operatorname{Supp} \kappa$ has been taken small enough in (2.6), we see that this integral is equal to

$$
\frac{1}{2 \pi h} \int e^{-i y \eta / h} \kappa(y) \kappa(\eta) d y d \eta
$$

which is equal to one modulo $O\left(h^{\infty}\right)$. We conclude from (2.4) where we replaced $v_{h}$ by $\mathrm{Op}_{h}(\phi) v_{h}$ modulo $O\left(h^{\infty}\right)$ and from symbolic calculus that

$$
\left\|e^{i \omega / h} \operatorname{Op}_{h}\left(\phi^{\prime}\right)\left(e^{-i \omega / h} v_{h}\right)-v_{h}\right\|_{L^{p}}=O\left(h^{\infty}\right)
$$

which is the wanted conclusion.

Lagrangian distributions We consider $\Lambda$ a Lagrangian submanifold of $T^{*}(\mathbb{R} \backslash\{0\})$ that is, since we are in a one-dimensional setting, a smooth curve of $T^{*}(\mathbb{R} \backslash\{0\})$. We shall assume that

$$
\Lambda=\left\{(x, \mathrm{~d} \omega(x)) ; w \in \mathbb{R}^{*}\right\}
$$

for $\omega$ a smooth function from $\mathbb{R}^{*}$ to $\mathbb{R}$. We want to define semi-classical lagrangian distributions on $\Lambda$ i.e. distributions generalizing families of oscillating functions $\left(\theta(x) e^{i \omega(x) / h}\right)_{h}$. Since in our applications $\omega$ will be homogeneous of degree -1 , so will have a singularity at zero, we shall define in a first step these distributions above a compact subset of $\mathbb{R} \backslash\{0\}$. In a second step, the lagrangian distributions along $\Lambda$ will be defined as sums of conveniently rescaled Lagrangian distributions on a compact set.

We fix $\sigma, \beta$ two small positive numbers and consider two Planck constants $h$ and $\hbar$ satisfying the inequalities

$$
\begin{equation*}
0<C_{0}^{-1} h^{1+\beta} \leq \hbar \leq C_{0} h^{\sigma} \leq 1 \tag{2.7}
\end{equation*}
$$

for some constant $C_{0}>0$. Notice that these inequalities imply that $O\left(\hbar^{\infty}\right)$ remainders will be also $O\left(h^{\infty}\right)$ remainders.

Definition 2.5. Let $F$ be a closed nonempty subset of $T^{*} \mathbb{R}$ such that $\pi_{2}(F)$ is compact. Let $\nu, \mu$ be in $\mathbb{R}, \gamma \in \mathbb{R}_{+}, p \in[1,+\infty]$.
i) One denotes by $h^{\nu} \mathcal{B}_{p}^{\mu, \gamma}[F]$ the space of elements $\left(v_{\hbar}\right)_{\hbar}$ of $E_{F}^{p} / E_{\emptyset}^{p}$, indexed by $\hbar$ and depending on $h$, such that there is $C>0$ and for any $h, \hbar$ in $] 0,1]$ satisfying (2.7)

$$
\begin{equation*}
\left\|v_{\hbar}\right\|_{L^{p}} \leq C h^{\nu}\left(\frac{h}{\hbar}\right)^{\mu+\frac{1}{p}}\left(1+\frac{h}{\hbar}\right)^{-2 \gamma} \tag{2.8}
\end{equation*}
$$

We denote

$$
h^{\nu} \mathcal{B}_{p}^{\mu, \gamma}=\bigcup_{F} h^{\nu} \mathcal{B}_{p}^{\mu, \gamma}[F]
$$

where the union is taken over all closed non empty subsets $F$ of $T^{*} \mathbb{R}$ such that $\pi_{2}(F)$ is compact.
ii) Let $K$ be a compact subset of $T^{*}(\mathbb{R} \backslash\{0\})$ such that $K \cap \Lambda \neq \emptyset$. Denote by e an equation of $\Lambda$ defined on a neighborhood of $K$. One denotes by $h^{\nu} L^{p} I_{\Lambda}^{\mu, \gamma}[K]\left(r e s p . h^{\nu} L^{p} J_{\Lambda}^{\mu, \gamma}[K]\right)$ the subspace of $h^{\nu} \mathcal{B}_{p}^{\mu, \gamma}[K]$ made of those families of functions $\left(v_{\hbar}\right)_{\hbar}$ such that there is $C>0$ and for any $h, \hbar$ in ]0,1] satisfying (2.7), one has the inequality

$$
\begin{equation*}
\left\|\mathrm{Op}_{\hbar}(e) v_{\hbar}\right\|_{L^{p}} \leq C h^{\nu}\left(\frac{h}{\hbar}\right)^{\mu+\frac{1}{p}}\left(1+\frac{h}{\hbar}\right)^{-2 \gamma}\left[h^{\frac{1}{2}}+\hbar\right] \tag{2.9}
\end{equation*}
$$

respectively, the inequality

$$
\begin{equation*}
\left\|\mathrm{Op}_{\hbar}(e) v_{\hbar}\right\|_{L^{p}} \leq C h^{\nu}\left(\frac{h}{\hbar}\right)^{\mu+\frac{1}{p}}\left(1+\frac{h}{\hbar}\right)^{-2 \gamma} \hbar \tag{2.10}
\end{equation*}
$$

Notice that by definition $h^{\nu} L^{p} J_{\Lambda}^{\mu, \gamma}[K]$ is included in $h^{\nu} L^{p} I_{\Lambda}^{\mu, \nu}[K]$. If $\tilde{e}$ is another equation of $\Lambda$ close to $K$, we may write $\tilde{e}=a e$ for some symbol $a \in S(1, K)$ on a neighborhood of $K$. By Theorem A. $2, \mathrm{Op}_{\hbar}(\tilde{e})=\mathrm{Op}_{\hbar}(a) \mathrm{Op}_{\hbar}(e)+\hbar \mathrm{Op}_{\hbar}(b)$ for another symbol $b$ in $S(1, K)$. Consequently (2.8) and (2.9) imply that the same estimate holds with $e$ replaced by $\tilde{e}$, so that the space $h^{\nu} L^{p} I_{\Lambda}^{\mu, \gamma}[K]$ depends only on $\Lambda$. The same holds for $h^{\nu} L^{p} J_{\Lambda}^{\mu, \gamma}[K]$. In particular, because of our definition of $\Lambda$, we may take $e(x, \xi)=\xi-\mathrm{d} \omega(x)$.
Example 2.6. Let $\theta$ be in $C_{0}^{\infty}(\mathbb{R} \backslash\{0\})$ and set $v_{\hbar}(x)=\theta(x) e^{i \omega(x) / \hbar}$. Then $\left(v_{\hbar}\right)_{\hbar}$ is in $L^{\infty} J_{\Lambda}^{0,0}[K]$ for any compact subset of $T^{*}(\mathbb{R} \backslash\{0\})$ meeting $\Lambda$ such that $\operatorname{Supp} \theta \subset \pi_{1}(K)$. Actually $\left(v_{\hbar}\right)_{\hbar}$ is microlocally supported close to $K$ and Lemma 2.4 shows that $\mathrm{Op}_{\hbar}(\xi-$ $\mathrm{d} \omega(x)) v_{\hbar}$ satisfies estimate $(2.10)$ with $p=\infty, \nu=\mu=\gamma=0$. Notice that in this example, one could apply $\mathrm{Op}_{\hbar}(\xi-\mathrm{d} \omega(x))$ several times to $v_{\hbar}$, and gain at each step one factor $\hbar$ in the $L^{\infty}$ estimates. It turns out that the lagrangian distributions we shall have to cope with will not satisfy such a strong statement, but only estimates of type (2.9) or (2.10).
Proposition 2.7. Let $p \in[1,+\infty]$, $\mu, \gamma$ in $\mathbb{R}, K$ a compact subset of $T^{*}(\mathbb{R} \backslash\{0\})$ with $\Lambda \cap K \neq \emptyset$.
i) Let a be in $S(1, K)$ and $\left(v_{\hbar}\right)_{\hbar}$ an element of $L^{p} I_{\Lambda}^{\mu, \gamma}[K]$. Then $\left(\left(\mathrm{Op}_{\hbar}(a)-a(x, \mathrm{~d} \omega(x)) v_{\hbar}\right)_{\hbar}\right.$ is in $\left(h^{1 / 2}+\hbar\right) \mathcal{B}_{p}^{\mu, \gamma}[K]$.

Assume we are given a vector field $Z=\alpha(\hbar, x) D_{\hbar}+\beta(\hbar, x) D_{x}$ satisfying the following conditions: $\|Z \hbar\|_{L^{\infty}}=O(\hbar)$ and if $e$ is a symbol in $S(1, K)$ (resp. that vanishes on $\Lambda$ ), then $\left[Z, \mathrm{Op}_{\hbar}(e)\right]=\mathrm{Op}_{\hbar}(\tilde{e})$ for some other symbol $\tilde{e}$ in $S(1, K)$ (resp. that vanishes on $\Lambda$ ). Assume also that for some integer $k, Z^{k^{\prime}} v_{\hbar}$ is in $L^{p} I_{\Lambda}^{\mu, \gamma}[K]$ for $0 \leq k^{\prime} \leq k$. Then $Z^{k} \mathrm{Op}_{\hbar}(a) v_{\hbar}$ is in $L^{p} I_{\Lambda}^{\mu, \gamma}[K]$ and $Z^{k}\left[\left(\mathrm{Op}_{\hbar}(a)-a(x, \mathrm{~d} \omega)\right) v_{\hbar}\right]$ is in $\left(h^{1 / 2}+\hbar\right) \mathcal{B}_{p}^{\mu, \gamma}[K]$.
ii) Denote by $\Lambda_{0}$ the zero section of $T^{*} \mathbb{R}$. Let $\left(v_{\hbar}\right)_{\hbar}$ be in $L^{p} I_{\Lambda}^{\mu, \gamma}[K]$. Then $\left(e^{-i \omega / \hbar} v_{\hbar}\right)_{\hbar}$ is $L^{p} I_{\Lambda_{0}}^{\mu, \gamma}\left[K_{0}\right]$ where $K_{0}=\chi_{\omega}(K)$. Conversely, if $\left(v_{\hbar}\right)_{\hbar}$ is in $L^{p} I_{\Lambda_{0}}^{\mu, \gamma}\left[K_{0}\right]$, $\left(e^{i \omega / \hbar} v_{\hbar}\right)_{\hbar}$ is in $L^{p} I_{\Lambda}^{\mu, \gamma}[K]$.
iii) Let $\Lambda_{1}, \Lambda_{2}$ be two Lagrangian submanifolds of $T^{*}(\mathbb{R} \backslash\{0\})$ satisfying the same assumptions as $\Lambda$, let $K_{1}, K_{2}$ be compact subsets of $T^{*}(\mathbb{R} \backslash\{0\})$ with $\Lambda_{1} \cap K_{1} \neq \emptyset, \Lambda_{2} \cap K_{2} \neq \emptyset$. Set

$$
\Lambda_{1}+\Lambda_{2}=\left\{\left(x, \xi_{1}+\xi_{2}\right) ;\left(x, \xi_{1}\right) \in \Lambda_{1},\left(x, \xi_{2}\right) \in \Lambda_{2}\right\}
$$

and define in the same way $K_{1}+K_{2}$. Let $p_{1}, p_{2}$ be in $[1, \infty]$ with $\frac{1}{p_{1}}+\frac{1}{p_{2}}=\frac{1}{p}, \mu_{1}, \mu_{2}, \gamma_{1}, \gamma_{2}$ in $\mathbb{R}$ with $\mu_{1}+\mu_{2}=\mu, \gamma_{1}+\gamma_{2}=\gamma$. Let $\left(v_{\hbar}^{\ell}\right)_{\hbar}$ be in $L^{p_{\ell}} I_{\Lambda_{\ell}}^{\mu_{\ell}, \gamma_{\ell}}\left[K_{\ell}\right]$ for $\ell=1,2$. Then $\left(v_{\hbar}^{1} \cdot v_{\hbar}^{2}\right)_{\hbar}$ is in $L^{p} I_{\Lambda_{1}+\Lambda_{2}}^{\mu, \gamma}\left[K_{1}+K_{2}\right]$.

A similar statement holds for the classes $L^{p} J_{\Lambda}^{\mu, \gamma}[K]$ and $\mathcal{B}_{p}^{\mu, \gamma}[K]$.

Proof. i) We may always modify $\omega$ outside a neighborhood of $\pi_{1}(K)$ so that it is compactly supported, and this will modify the quantities at hand only by an element of $E_{\emptyset}^{p}$. We may find a symbol $b$ in $S(1, K)$ so that

$$
a(x, \xi)-a(x, \mathrm{~d} \omega(x))=b(x, \xi)(\xi-\mathrm{d} \omega(x))
$$

in $K_{r}$ for some small $r$. By the symbolic calculus of appendix A,

$$
\mathrm{Op}_{\hbar}(a) v_{\hbar}-a(x, \mathrm{~d} \omega) v_{\hbar}=\mathrm{Op}_{\hbar}(b) \mathrm{Op}_{\hbar}(\xi-\mathrm{d} \omega(x)) v_{\hbar}+\hbar \mathrm{Op}_{\hbar}(c) v_{\hbar}
$$

for a new symbol $e$ in $S(1, K)$. The conclusion follows from estimate (2.9).
If we make act a vector field $Z$ as in the statement on the last equality and use the commutation assumptions, we obtain the last statement of $i$ ).
ii) We have seen in Lemma 2.4 that $\left(e^{-i \omega / \hbar} v_{\hbar}\right)_{\hbar}$ is in $E_{K_{0}}^{p} / E_{\emptyset}^{p}$. Since $\mathrm{Op}_{\hbar}(\xi)\left(e^{-i \omega / \hbar} v_{\hbar}\right)=$ $e^{-i \omega / \hbar} \mathrm{Op}_{\hbar}(\xi-\mathrm{d} \omega(x)) v_{\hbar}$, we deduce from (2.9) the statement.
iii) Denote by $\omega_{1}, \omega_{2}$ two smooth functions, that may be assumed to be compactly supported close to $\pi_{1}\left(K_{1}\right), \pi_{1}\left(K_{2}\right)$ respectively, such that $\Lambda_{\ell}=\left\{\left(x, \mathrm{~d} \omega_{\ell}(x)\right)\right\}$ close to $K_{\ell}, \ell=1,2$. Then $\omega=\omega_{1}+\omega_{2}$ parametrizes $\Lambda=\Lambda_{1}+\Lambda_{2}$ close to $K_{1}+K_{2}$. We define $w_{\hbar}^{\ell}=e^{-i \omega^{\ell} / \hbar} v_{\hbar}^{\ell}$. By $\left.i i\right)$, $\left(w_{\hbar}^{\ell}\right)_{\hbar}$ is in $L^{p_{\ell}} I_{\Lambda_{0}}^{\mu_{\ell} \gamma_{\ell}}\left[K_{\ell, 0}\right]$, where $K_{\ell, 0}=\chi_{\omega_{\ell}}\left(K_{\ell}\right)$. Writing a product from the convolution of the Fourier transforms of the factors, we see that $\left(w_{\hbar}^{1} w_{\hbar}^{2}\right)_{\hbar}$ is in $E_{K_{1,0}+K_{2,0}}^{p} / E_{\emptyset}^{p}$. Let us check that $w_{\hbar}^{1} w_{\hbar}^{2}$ satisfies estimate (2.9) when $e$ is an equation of $\Lambda_{0}$ i.e. $e(x, \xi)=\xi$ so that $\mathrm{Op}_{\hbar}(e)=\hbar D_{x}$. We write

$$
\left\|\hbar D_{x}\left(w_{\hbar}^{1} w_{\hbar}^{2}\right)\right\|_{L^{p}} \leq\left\|\hbar D_{x} w_{\hbar}^{1}\right\|_{L^{p_{1}}}\left\|w_{\hbar}^{2}\right\|_{L^{p_{2}}}+\left\|w_{\hbar}^{1}\right\|_{L^{p_{1}}}\left\|\hbar D_{x} w_{\hbar}^{2}\right\|_{L^{p_{2}}}
$$

and use (2.8), (2.9) for each factor to get that $\left(w_{\hbar}^{1} w_{\hbar}^{2}\right)_{\hbar}$ is in $L^{p} I_{\Lambda_{0}}^{\mu, \gamma}\left[K_{1,0}+K_{2,0}\right]$. We just have to apply again $i i)$ to $v_{\hbar}=e^{i \omega / \hbar}\left(w_{\hbar}^{1} w_{\hbar}^{2}\right)$ to get the conclusion. The proof is similar for classes $L^{p} J_{\Lambda}^{\mu, \gamma}[K]$ and $\mathcal{B}_{p}^{\mu, \gamma}[K]$.

We have defined, up to now, classes of Lagrangian distributions microlocally supported close to a compact set of the phase space. We introduce next classes of Lagrangian distributions that do not obey such a localization property.

From now on, we consider phase functions $\omega: \mathbb{R} \backslash\{0\} \rightarrow \mathbb{R}$ which are smooth, non zero, and positively homogeneous of degree -1 . We set

$$
\begin{equation*}
\Lambda=\{(x, \mathrm{~d} \omega(x)) ; x \in \mathbb{R} \backslash\{0\}\} \subset T^{*}(\mathbb{R} \backslash\{0\}) \tag{2.11}
\end{equation*}
$$

so that $\Lambda$ is invariant under the action of $\mathbb{R}_{+}^{*}$ on $T^{*}(\mathbb{R} \backslash\{0\})$ given by $\lambda \cdot(x, \xi)=\left(\lambda x, \lambda^{-2} \xi\right)$. For $h \in] 0,1], C$ a positive constant, we introduce the notations

$$
\begin{align*}
& J(h, C)=\left\{j \in \mathbb{Z} ; C^{-1} h^{2(1-\sigma)} \leq 2^{j} \leq C h^{-2 \beta}\right\} \\
& h_{j}=h 2^{-j / 2} \text { if } j \in J(h, C)  \tag{2.12}\\
& j_{0}(h, C)=\min (J(h, C))-1, \quad j_{1}(h, C)=\max (J(h, C))+1 .
\end{align*}
$$

We note that (2.7) is satisfied by $\hbar=h_{j}$ if $j \in J(h, C)$ (for a constant $C=C_{0}^{2}$ ). For $j \in \mathbb{Z}, v$ a distribution on $\mathbb{R}$, we set

$$
\Theta_{j}^{*} v=v\left(2^{j / 2} \cdot\right)
$$

so that in particular, if $p \in[1, \infty],\left\|\Theta_{j}^{*}\right\|_{\mathcal{L}\left(L^{p}, L^{p}\right)}=2^{-j /(2 p)}$. If $a$ belongs to the class of symbols $S(m)$ and if $a_{j}(x, \xi)=a\left(2^{-j / 2} x, 2^{j} \xi\right)$ we notice that for $j \in J(h, C)$

$$
\begin{equation*}
\Theta_{-j}^{*} \mathrm{Op}_{h}(a) \Theta_{j}^{*}=\mathrm{Op}_{h_{j}}\left(a_{j}\right) \tag{2.13}
\end{equation*}
$$

We fix a function $\varphi$ in $C_{0}^{\infty}\left(\mathbb{R}^{d}\right)$ such that $\sum_{j \in \mathbb{Z}} \varphi\left(2^{-j} \xi\right) \equiv 1$. We define

$$
\varphi_{0}(\xi)=\sum_{j=-\infty}^{-1} \varphi\left(2^{-j} \xi\right), \quad \Delta_{j}^{h}=\operatorname{Op}_{h}\left(\varphi\left(2^{-j} \xi\right)\right)=\varphi\left(2^{-j} h D\right)
$$

Definition 2.8. Let $\nu \in \mathbb{R}, p \in[1, \infty], b \in \mathbb{R}$. One denotes by $h^{\nu} \mathcal{R}_{p}^{b}$ the space of families of $L^{p}$-functions $\left(v_{\hbar}\right)_{\hbar}$ such that there is $C>0$ and

$$
\begin{align*}
& \left\|\Delta_{j}^{h} v_{h}\right\|_{L^{p}} \leq C_{0} h^{\nu} 2^{-j+b} \quad \text { for } j \geq j_{0}(h, C)  \tag{2.14}\\
& \left\|\operatorname{Op}_{h}\left(\varphi_{0}\left(2^{-j_{0}(h, C)} \xi\right)\right) v_{h}\right\|_{L^{p}} \leq C h^{\nu}
\end{align*}
$$

where $j_{+}=\max (j, 0)$.
Clearly the definition is independent of the choice of $\varphi_{0}$.
Definition 2.9. Let $\Lambda$ be a lagrangian submanifold of form (2.11), $K$ a compact subset of $T^{*}(\mathbb{R} \backslash\{0\})$ meeting $\Lambda$. Let $\nu, \mu$ be in $\mathbb{R}, \gamma \in \mathbb{R}_{+}$, $F$ a closed non empty subset of $T^{*} \mathbb{R}$ such that $\pi_{2}(F)$ is compact in $\mathbb{R}$. One denotes by $h^{\nu} L^{p} \tilde{I}_{\Lambda}^{\mu, \gamma}[K]$ (resp. $h^{\nu} L^{p} \tilde{J}_{\Lambda}^{\mu, \gamma}[K]$, resp. $h^{\nu} \tilde{\mathcal{B}}_{p}^{\mu, \gamma}[F]$ ) the space of families of functions $\left(v_{h}\right)_{h \in] 0,1]}$ such that

- For any $j \in J(h, C)$, there is a family $\left(v_{h_{j}}^{j}\right)_{h_{j}}$, indexed by

$$
\left.\left.h_{j} \in\right] 0, \min \left(C_{0}^{\frac{1}{1-\sigma}} 2^{\frac{j \sigma}{2(1-\sigma)}}, C_{0}^{\frac{1}{\beta}} 2^{-j \frac{1+\beta}{2 \beta}}\right)\right]
$$

which is an element of $h^{\nu} L^{p} I_{\Lambda}^{\mu, \gamma}[K]$ (resp. $h^{\nu} L^{p} J_{\Lambda}^{\mu, \gamma}[K]$, resp. $h^{\nu} \mathcal{B}_{p}^{\mu, \gamma}[F]$ ) with the constants in (2.8), (2.9), (2.10) uniform in $j \in J(h, C)$.

- For any $h \in] 0,1], v_{h}=\sum_{j \in J(h, C)} \Theta_{j}^{*} v_{h_{j}}^{j}$.

One defines $h^{\nu} \tilde{\mathcal{B}}_{p}^{\mu, \gamma}=\bigcup h^{\nu} \tilde{\mathcal{B}}_{p}^{\mu, \gamma}[F]$ where the union is taken over the sets $F$ which are closed with $\pi_{2}(F)$ compact in $\mathbb{R}$.

Remark 2.10. - The interval of variation imposed to $h_{j}$ in the preceding definition is the one deduced from (2.7) with $\hbar=h_{j}$.

- The building blocks $\left(v_{h_{j}}^{j}\right)_{h_{j}}$ in the above definition are defined modulo $O\left(h_{j}^{\infty}\right)$ so modulo $O\left(h^{\infty}\right)$ since $h_{j} \leq h^{\sigma}$. Since the cardinal of $J(h, C)$ is $O(|\log h|)$, we see that the classes introduced in the above definition are well defined modulo $O\left(h^{\infty}\right)$.
- It follows from the above two definitions that $h^{\nu} \tilde{\mathcal{B}}_{p}^{0, b} \subset h^{\nu} \mathcal{R}_{p}^{b}$. Moreover, by (2.14) and the fact that the cardinal of $\mathbb{Z}_{-} \cap\left\{j \geq j_{0}(h, C)\right\}$ is $O(|\log h|)$, we see that if $u, v$ are in $\mathcal{R}_{\infty}^{b}$ with $b>0$, then $u v$ is in $h^{-0} \mathcal{R}_{\infty}^{b}:=\cap_{\theta>0} h^{-\theta} \mathcal{R}_{\infty}^{b}$.

Let us prove a statement similar to $i$ ) of Proposition 2.7 for elements of the classes of distributions we just defined.

Proposition 2.11. We assume that the function $\omega$ defining $\Lambda$ satisfies either $\omega(x) \neq 0$ for all $x \in \mathbb{R}^{*}$ or $\omega \equiv 0$. In the first (resp. second) case we denote by $K$ a compact subset of $T^{*}(\mathbb{R} \backslash\{0\}) \backslash 0$ (resp. of $T^{*}(\mathbb{R} \backslash\{0\})$ ) such that $K \cap \Lambda \neq \emptyset$. Let $\mu \in \mathbb{R}, \gamma \in \mathbb{R}_{+}, p \in[1, \infty]$, $k \in \mathbb{N}$ be given. Consider a function $(x, \xi) \mapsto a(x, \xi)$ smooth on $\mathbb{R}^{*} \times \mathbb{R}^{*}\left(\right.$ resp. $\left.\mathbb{R}^{*} \times \mathbb{R}\right)$ satisfying for some real numbers $\ell, \ell^{\prime}, d, d^{\prime}$ (resp. $\ell, \ell^{\prime}, d \geq 0, d^{\prime} \geq 0$ ) and all $\alpha, \beta$ in $\mathbb{N}$

$$
\begin{equation*}
\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} a(x, \xi)\right| \leq C_{\alpha \beta}|x|^{\ell-\alpha}\langle x\rangle^{\ell^{\prime}}|\xi|^{d-\beta}\langle\xi\rangle^{d^{\prime}} \tag{2.15}
\end{equation*}
$$

when $(x, \xi) \in \mathbb{R}^{*} \times \mathbb{R}^{*}$ (resp. (2.15) when $\beta \leq d$ and

$$
\begin{equation*}
\partial_{x}^{\alpha} \partial_{\xi}^{\beta} a(x, \xi) \equiv 0 \quad \text { for } \beta>d, \tag{2.16}
\end{equation*}
$$

when $\left.(x, \xi) \in \mathbb{R}^{*} \times \mathbb{R}\right)$.
Denote $Z=-h D_{h}+x D_{x}$ and let $\left(v_{h}\right)_{h}$ satisfy for any $k^{\prime} \leq k,\left(Z^{k^{\prime}} v_{h}\right)_{h} \in L^{p} \tilde{I}_{\Lambda}^{\mu, \nu}[K]$ (resp. $\left.\tilde{\mathcal{B}}_{p}^{\mu, \nu}[K]\right)$. Then $\left(Z^{k}\left(\mathrm{Op}_{h}(a) v_{h}\right)\right)_{h}$ belongs to $L^{p} \tilde{I}_{\Lambda}^{\tilde{\mu}, \tilde{\gamma}}[K]$ (resp. $\left.\tilde{\mathcal{B}}_{p}^{\tilde{,}, \tilde{\nu}}[K]\right)$ where $\tilde{\mu}=\mu+2 d-\ell-\ell^{\prime}$, $\tilde{\gamma}=\gamma-\frac{\ell^{\prime}}{2}-d^{\prime}$.

Moreover, under the assumption $\left(Z^{k^{\prime}} v_{h}\right)_{h} \in L^{p} \tilde{I}_{\Lambda}^{\mu, \gamma}[K]$, if $\chi$ is in $C_{0}^{\infty}(\mathbb{R})$ is equal to one close to zero, and has small enough support, $Z^{k}\left((1-\chi)\left(x h^{-\beta}\right) a(x, \mathrm{~d} \omega) v_{h}\right)$ is also in $L^{p} \tilde{I}_{\Lambda}^{\tilde{\mu}} \tilde{\gamma}[K]$ and

$$
\begin{equation*}
Z^{k}\left[\mathrm{Op}_{h}(a) v_{h}-(1-\chi)\left(x h^{-\beta}\right) a(x, \mathrm{~d} \omega) v_{h}\right] \tag{2.17}
\end{equation*}
$$

belongs to $h^{1 / 2} \tilde{\mathcal{B}}_{p}^{\tilde{\mu}, \tilde{\gamma}}[K]+h \tilde{\mathcal{B}}_{p}^{\tilde{\mu}-1, \tilde{\gamma}}[K]$.

If we assume that $\left(Z^{k^{\prime}} v_{h}\right)_{h}$ is in $L^{p} \tilde{J}_{\Lambda}^{\mu, \gamma}[K]$ for $k^{\prime} \leq k$, we obtain instead that $\left(Z^{k} \mathrm{Op}_{h}(a) v_{h}\right)_{h}$ and $\left(Z^{k}(1-\chi)\left(x h^{-\beta}\right) a(x, \mathrm{~d} \omega) v_{h}\right)_{h}$ belong to $L^{p} \tilde{J}_{\Lambda}^{\tilde{,}, \tilde{\gamma}}[K]$ and that (2.17) is in $h \tilde{\mathcal{B}}_{p}^{\tilde{\mu}-1, \tilde{\gamma}}[K]$.

When $\omega \equiv 0$, if $Z^{k^{\prime}} v_{h}$ is in $\tilde{\mathcal{B}}_{p}^{\mu, \gamma}[K]$, we obtain that $\left(Z^{k} \operatorname{Op}_{h}(a) v_{h}\right)_{h}$ is in $\tilde{\mathcal{B}}_{p}^{\tilde{,}, \tilde{\gamma}}[K]$.
The same results hold if we quantize a by $\mathrm{Op}_{h}(\bar{a})^{*}$ instead of $\mathrm{Op}_{h}(a)$ i.e. under the same assumptions as above $\left(Z^{k} \mathrm{Op}_{h}(\bar{a})^{*}\left(v_{h}\right)\right)$ belongs to $L^{p} \tilde{I}_{\Lambda}^{\tilde{\mu}} \tilde{\gamma}[K]$ and

$$
\begin{equation*}
Z^{k}\left[\mathrm{Op}_{h}(\bar{a})^{*} v_{h}-(1-\chi)\left(x h^{-\beta}\right) a(x, \mathrm{~d} \omega) v_{h}\right] \tag{2.18}
\end{equation*}
$$

belongs to the same spaces as indicated above after (2.17). In the same way, when $\omega \equiv 0$, and when $\left(Z^{k^{\prime}} v_{h}\right)_{h}$ is in $\tilde{\mathcal{B}}_{p}^{\mu, \gamma}[K]$, for $k^{\prime} \leq k,\left(Z^{k}\left(\operatorname{Op}_{h}(\bar{a})^{*} v_{h}\right)\right)_{h}$ is in $\tilde{\mathcal{B}}_{p}^{\mu, \gamma}[K]$.

Proof. According to Definition 2.9, we represent $v_{h}=\sum_{j \in J(h, C)} \Theta_{j}^{*} v_{h_{j}}^{j}$ where $\left(v_{h_{j}}^{j}\right)_{h_{j}}$ is a bounded sequence of elements of $L^{p} I_{\Lambda}^{\mu, \gamma}[K]$, as well as $\left(Z^{k^{\prime}} v_{h_{j}}\right)_{h_{j}}$ for $k^{\prime} \leq k$. By (2.13) and (2.15),

$$
\begin{equation*}
\mathrm{Op}_{h}(a) v_{h}=\sum_{j \in J(h, C)} \Theta_{j}^{*} w_{h_{j}}^{j} \tag{2.19}
\end{equation*}
$$

with $w_{h_{j}}^{j}=\operatorname{Op}_{h_{j}}\left(a_{j}\right) v_{h_{j}}^{j}$ and

$$
a_{j}(x, \xi)=a\left(2^{-j / 2} x, 2^{j} \xi\right)=2^{j\left(d-\frac{\ell+\ell^{\prime}}{2}\right)+j_{+}\left(d^{\prime}+\frac{\ell^{\prime}}{2}\right)} b_{j}(x, \xi)
$$

When $(x, \xi)$ stays in a compact subset of $T^{*}(\mathbb{R} \backslash\{0\}) \backslash 0,(2.15)$ shows that $\partial_{x}^{\alpha} \partial_{\xi}^{\beta} b_{j}=O(1)$ uniformly in $j$. In the same way when $(x, \xi)$ stays in a compact subset of $T^{*}(\mathbb{R} \backslash\{0\})$, (2.15) for $\beta \leq d$ and (2.16) show also that $\partial_{x}^{\alpha} \partial_{\xi}^{\beta} b_{j}=O(1)$ uniformly in $j$. Since $2^{j}=\left(h / h_{j}\right)^{2}$, it follows from Theorem A. 2 of the appendix that $\left(w_{h_{j}}^{j}\right)_{h_{j}}$ is a bounded sequence indexed by $j \in J(h, C)$ of elements of $L^{p} I_{\Lambda}^{\tilde{\mu}, \tilde{\gamma}}[K]$. Moreover, the vector field $Z$ satisfies for any symbol $e$

$$
\left[Z, \mathrm{Op}_{h_{j}}(e)\right]=\mathrm{Op}_{h_{j}}\left(\left(x D_{x}-2 \xi D_{\xi}\right) e\right)
$$

Since either $\Lambda=\{(x, \mathrm{~d} \omega(x))\}$ with $\omega$ homogeneous of degree -1 or $\Lambda=\{(x, 0)\}$, we see that $\left(x D_{x}-2 \xi D_{\xi}\right) e$ vanishes on $\Lambda$ if $e$ does. Consequently, the assumption of the last statement in $i$ ) of Proposition 2.7 is satisfied and we conclude that $\left(Z w_{h_{j}}^{j}\right)_{h_{j}}$ is a bounded sequence of elements of $L^{p} I_{\Lambda}^{\tilde{\mu}, \tilde{d}}[K]$.

To prove (2.17), we use that again by $i$ ) of Proposition 2.7,

$$
w_{h_{j}}^{j}=a_{j}(x, \mathrm{~d} \omega) v_{h_{j}}^{j}+\left(h^{1 / 2}+h_{j}\right) r_{h_{j}}^{j}
$$

where $\left(r_{h_{j}}^{j}\right)_{h_{j}}$ is a bounded sequence indexed by $j \in J(h, C)$ of elements of $\mathcal{B}_{p}^{\tilde{\mu}, \tilde{\gamma}}[K]$, that stay in that space if one applies $Z^{k^{\prime}}\left(k^{\prime} \leq k\right)$ on them. Let $\chi$ be as in the statement of the
proposition, with small enough support. Then, if $x$ is close to $\pi_{1}(K)$ and $j$ is in $J(h, C)$, $(1-\chi)\left(2^{-j / 2} x h^{-\beta}\right) \equiv 1$. Consequently, since $\left(v_{h_{j}}^{j}\right)_{h_{j}}$, as well as $\left(Z^{k^{\prime}} v_{h_{j}}^{j}\right)_{h_{j}}$ is microlocalized close to $K$, we may write $v_{h_{j}}^{j}=v_{h_{j}}^{j}(1-\chi)\left(2^{-j / 2} x h^{-\beta}\right)$ modulo a remainder which is $O\left(h_{j}^{\infty}\right)=$ $O\left(h^{\infty}\right)$ in $L^{p}$, as well as its $Z^{k^{\prime}}$-derivatives, $0 \leq k^{\prime} \leq k$. Integrating such a remainder in the $r_{h_{j}}^{j}$ contributions, we may write, using that $\mathrm{d} \omega$ is homogeneous of degree -2 ,

$$
w_{h_{j}}^{j}=(1-\chi)\left(2^{-j / 2} x h^{-\beta}\right) a\left(2^{-j / 2} x, \mathrm{~d} \omega\left(2^{-j / 2} x\right)\right) v_{h_{j}}^{j}+h^{1 / 2} r_{h_{j}}^{j}+h \tilde{r}_{h_{j}}^{j}
$$

where $\left(r_{h_{j}}^{j}\right)_{h_{j}}$ is as above and $\tilde{r}_{h_{j}}^{j}=2^{-j / 2} r_{h_{j}}^{j}$ is such that $\left(Z^{k^{\prime}} \tilde{r}_{h_{j}}^{j}\right)_{h_{j}}$ is in $\mathcal{B}_{p}^{\tilde{\mu}-1, \tilde{\gamma}}[K]$ for $k^{\prime} \leq k$. This gives (2.17) if we plug this expansion in (2.19).

To check that $\left(Z^{k}\left((1-\chi)\left(x h^{-\beta}\right) a(x, \mathrm{~d} \omega) v_{h}\right)\right)_{h}$ is also in $L^{p} \tilde{I}_{\Lambda}^{\mu}, \tilde{\gamma}[K]$, we write the function on which acts $Z^{k}$ as

$$
\sum_{j \in J(h, C)} \Theta_{j}^{*}\left[a\left(2^{-j / 2} x, \mathrm{~d} \omega\left(2^{-j / 2} x\right)\right)(1-\chi)\left(x 2^{-j / 2} h^{-\beta}\right) v_{h_{j}}^{j}\right]
$$

and remark that, as above, the assumptions of microlocal localization of $\left(v_{h_{j}}^{j}\right)_{h_{j}}$ allow one to remove the cut-off $(1-\chi)$ up to $O\left(h^{\infty}\right)$ remainders. Since $\mathrm{d} \omega$ is homogeneous of degree -2 ,

$$
a\left(2^{-j / 2} x, \mathrm{~d} \omega\left(2^{-j / 2} x\right)\right)=O\left(2^{j\left(d-\frac{\ell+\ell^{\prime}}{2}\right)+j+\left(d^{\prime}+\frac{\ell^{\prime}}{2}\right)}\right)
$$

when $x$ stays in a compact subset of $\mathbb{R}^{*}$, so that the above sum defines an element of $L^{p} \tilde{I}_{\Lambda}^{\tilde{\mu}} \tilde{\gamma}[K]$.
The statement of the proposition concerning the case when $\left(Z^{k^{\prime}} v_{h}\right)_{h}$ is in $L^{p} \tilde{J}_{\Lambda}[K]$ is proved similarly, as well as the one about $\tilde{\mathcal{B}}_{p}^{\mu, \gamma}[K]$.

Finally, the statements concerning $\mathrm{Op}_{h}(\bar{a})^{*}$ instead of $\mathrm{Op}_{h}(a)$ are proved in the same way: one may write (2.19) with $w_{h_{j}}^{j}$ given by $\mathrm{Op}_{h}\left(\overline{a_{j}}\right)^{*} v_{h_{j}}^{j}$. By Theorem A. 2 in the appendix, we know that there is a symbol $b_{j}$ in $S(1, K)$ uniformly in $j$, such that $\mathrm{Op}_{h_{j}}\left(\overline{a_{j}}\right)^{*}=\mathrm{Op}_{h_{j}}\left(b_{j}\right)$. Moreover, $b_{j}(x, \xi)=a_{j}(x, \xi)+h_{j} c_{j}(x, \xi)$ for some other symbol $c_{j}$ in $S(1, K)$ uniformly in $j$. The statements concerning $\mathrm{Op}_{h}(\bar{a})^{*} v_{h}$ thus follows from those we just proved for $\mathrm{Op}_{h}(a) v_{h}$.

Let us study products.
Proposition 2.12. Let $p_{1}, p_{2}, p$ be in $[1,+\infty]$ with $\frac{1}{p_{1}}+\frac{1}{p_{2}}=\frac{1}{p}, \gamma_{1}, \gamma_{2}$ in $\mathbb{R}, \Lambda_{1}, \Lambda_{2}$ be two Lagrangian submanifolds of $T^{*}(\mathbb{R} \backslash\{0\})$ of the form (2.11), defined in terms of phase functions $\omega_{1}, \omega_{2}$ homogeneous of degree $-1, \omega_{1} \not \equiv 0, \omega_{2} \not \equiv 0$. Let $K_{1}, K_{2}$ be two compact subsets of $T^{*}(\mathbb{R} \backslash\{0\})$ with $K_{\ell} \cap \Lambda_{\ell} \neq \emptyset, \ell=1,2$. Let $\left(v_{h}^{\ell}\right)_{h}$ be an element of $\tilde{\mathcal{B}}_{p_{\ell}}^{\mu_{\ell}, \gamma_{\ell}}\left[K_{\ell}\right]$ (resp. $L^{p_{\ell}} \tilde{I}_{\Lambda_{\ell}}^{\mu_{\ell}, \gamma_{\ell}}\left[K_{\ell}\right]$, resp. $\left.L^{p_{\ell}} \tilde{J}_{\Lambda_{\ell}}^{\mu_{\ell}, \gamma_{\ell}}\left[K_{\ell}\right]\right) \ell=1,2$.

There is a compact subset $K$ of $T^{*}(\mathbb{R} \backslash\{0\})$ with $K \cap\left(\Lambda_{1}+\Lambda_{2}\right) \neq \emptyset$ such that $\left(v_{h}^{1} \cdot v_{h}^{2}\right)_{h}$ belongs to $\tilde{\mathcal{B}}_{p}^{\mu, \gamma}[K]$ (resp. $L^{p} \tilde{I}_{\Lambda_{1}}^{\mu, \gamma},[K]$, resp. $L^{p} \widetilde{J}_{\Lambda_{1}+\Lambda_{2}}^{\mu, \gamma}[K]$ ) with $\mu=\mu_{1}+\mu_{2}, \gamma=\gamma_{1}+\gamma_{2}$. Moreover, for any neighborhood $\Omega$ of $\Lambda_{1}+\Lambda_{2}$, any compact subset $L$ of $\mathbb{R} \backslash\{0\}$, there are neighborhoods $\Omega_{\ell}$ of $\Lambda_{\ell}, \ell=1,2$, such that if $K_{\ell} \subset \Omega_{\ell} \cap \pi_{1}^{-1}(L), \ell=1,2$, then $K \subset \Omega$.

Proof. By Definition 2.9, we may write for $\ell=1,2$,

$$
v_{h}^{\ell}=\sum_{j_{\ell} \in J(h, C)} \Theta_{j_{\ell}}^{*} v_{h_{j_{\ell}}}^{\ell, j_{\ell}}
$$

where $\left(v_{h_{j_{\ell}}}^{\ell, j_{\ell}}\right)_{h_{j_{\ell}}}$ is a bounded sequence of $L^{p_{\ell}} I_{\Lambda_{\ell}}^{\mu_{\ell}, \gamma_{\ell}}\left[K_{\ell}\right]$. We write

$$
\begin{equation*}
v_{h}^{1} \cdot v_{h}^{2}=\sum_{j_{1} \in J(h, C)} \Theta_{j_{1}}^{*} w_{h_{j_{1}}}^{j_{1}} \tag{2.20}
\end{equation*}
$$

with

$$
\begin{equation*}
w_{h_{j_{1}}}^{j_{1}}=v_{h_{j_{1}}}^{1, j_{1}} \sum_{j_{2} \in J(h, C)} \Theta_{j_{2}-j_{1}}^{*} v_{h_{j_{2}}}^{2, j_{2}} . \tag{2.21}
\end{equation*}
$$

Because of the microlocal localization properties of $\left(v_{h_{j_{\ell}}}^{\ell, j_{\ell}}\right)_{h_{j_{\ell}}}$ we may, up to an $O\left(h^{\infty}\right)$ remainder in $L^{p_{\ell}}$, replace $\left(v_{h_{j_{\ell}}}^{\ell, j_{\ell}}\right)_{h_{j_{\ell}}}$ by $\left(\theta v_{h_{j_{\ell}}}^{\ell, j_{\ell}}\right)_{h_{j_{\ell}}}$ where $\theta$ is in $C_{0}^{\infty}(\mathbb{R})$ and is equal to one on a large enough compact subset of $\mathbb{R}^{*}$. This shows that in $(2.21)$, we may limit the summation to those $j_{2}$ such that $\left|j_{2}-j_{1}\right| \leq C_{0}$ for some large enough $C_{0}$, up to remainders which are $O\left(h^{\infty}\right)$ in $L^{p}$. Define

$$
\tilde{v}_{h_{j_{1}}}^{1, j_{1}}=\sum_{\substack{j_{2} \in J(h, C) \\\left|j_{1}-j_{2}\right| \leq C_{0}}} \Theta_{j_{2}-j_{1}}^{*} v_{h_{j_{2}}}^{2, j_{2}}
$$

Then $\left(\tilde{v}_{h_{j_{1}}}^{1, j_{1}}\right)_{h_{j_{1}}}$ is a bounded sequence of $\mathcal{B}_{p_{2}}^{\mu_{2}, \gamma_{2}}\left[\tilde{K}_{2}\right]$ (resp. $L^{p_{2}} I_{\Lambda_{2}}^{\mu_{2}, \gamma_{2}}\left[\tilde{K}_{2}\right]$, resp. $\left.L^{p_{2}} J_{\Lambda_{2}}^{\mu_{2}, \gamma_{2}}\left[\tilde{K}_{2}\right]\right)$ for some large enough compact subset $\tilde{K}_{2}$ of $T^{*}(\mathbb{R} \backslash\{0\})$, as follows from (2.13) and the homogeneity properties of $\Lambda_{2}$. We just need to apply $i i i$ ) of Proposition 2.7 to conclude that $\left(w_{h_{j_{1}}}^{j_{1}}\right)_{h_{j_{1}}}$ is a bounded sequence of elements of $\mathcal{B}_{p}^{\mu, \gamma}\left[K_{1}+\tilde{K}_{2}\right]$ (resp. $L^{p} I_{\Lambda_{1}+\Lambda_{2}}^{\mu, \gamma}\left[K_{1}+\tilde{K}_{2}\right]$, resp. $\left.L^{p} J_{\Lambda_{1}+\Lambda_{2}}^{\mu, \gamma}\left[K_{1}+\tilde{K}_{2}\right]\right)$.

The last statement of the proposition follows from the fact that $K=K_{1}+\tilde{K}_{2}$, and that $\tilde{K}_{2}$ mat be taken in an arbitrary neighborhood of $\Lambda_{2}$ if $K_{2}$ is contained in an even smaller neighborhood of that submanifold.

Proposition 2.13. Let $F_{1}, F_{2}$ be closed subsets of $T^{*} \mathbb{R}$ such that $\pi_{2}\left(F_{\ell}\right)$ is compact, $\ell=1,2$. Let $\left(v_{h}^{\ell}\right)$ be in $\tilde{\mathcal{B}}_{\infty}^{\mu_{\ell}, \gamma}\left[F_{\ell}\right]$ with $\mu_{\ell} \geq 0, \gamma>\mu_{\ell}$. Then $v_{h}^{1} \cdot v_{h}^{2}$ is in $h^{-\theta} \mathcal{B}_{\infty}^{\mu, \gamma}[F]$ with $\mu=\mu_{1}+\mu_{2}$ for any $\theta>0$ and some closed subset $F$ of $T^{*} \mathbb{R}$ whose second projection is compact.

Proof. We write (2.20)

$$
v_{h}^{1} \cdot v_{h}^{2}=\sum_{j_{1} \in J(h, C)} \Theta_{j_{1}}^{*} w_{h_{j_{1}}}^{1}+\sum_{j_{2} \in J(h, C)} \Theta_{j_{2}}^{*} w_{h_{j_{2}}}^{2}
$$

with

$$
w_{h_{j_{1}}}^{1}=v_{h_{j_{1}}}^{1, j_{1}} \sum_{\substack{j_{2} \in J(h, C) \\ j_{2} \leq j_{1}}} \Theta_{j_{2}-j_{1}}^{*} v_{h_{j_{2}}}^{2, j_{2}}
$$

and a symmetric expression for $w_{h_{j_{2}}}^{2}$. Then $w_{h_{j_{1}}}^{1}$ is microlocally supported in some closed subset of $T^{*} \mathbb{R}$, whose $\xi$ projection is compact, as

$$
\mathrm{Op}_{h_{j_{1}}}(\tilde{\varphi})\left[\Theta_{j_{2}-j_{1}}^{*} v_{h_{j_{2}}}^{2, j_{2}}\right]=\Theta_{j_{2}-j_{1}}^{*} v_{h_{j_{2}}}^{2, j_{2}}
$$

if $\tilde{\varphi}$ is supported for $|\xi| \leq C$, equal to one on $|\xi| \leq C / 2$, for some $C>0$. The $L^{\infty}$-norm of $w_{h_{j_{1}}}^{1}$ is bounded from above by

$$
2^{j_{1} \mu_{1} / 2-j_{1+} \gamma} \sum_{\substack{j_{2} \in J(h, C) \\ j_{2} \leq j_{1}}} 2^{j_{2} \mu_{2} / 2} 2^{-j_{2+} \gamma} \leq C|\log h| 2^{j_{1} \mu / 2-j_{1+\gamma} \gamma}
$$

since $\mu_{\ell} \geq 0, \gamma>\mu_{\ell} / 2$.

## 3 The semi-classical water waves equation

Let us recall an equivalent form of the water waves equation that is obtained in the companion paper [5] (see Corollary 4.3.13. in that paper). If $(\eta, \psi)$ is a solution of the water waves equation, if $\mathcal{Z}$ denotes the collection of vector fields $\mathcal{Z}=\left(Z, \partial_{x}\right)$ and if we assume that $u=\left|D_{x}\right|^{\frac{1}{2}} \psi+i \eta$ satisfies for $k$ smaller than some integer $s_{0}$, for $\alpha>0$ large enough and $d \in \mathbb{N}$,

$$
\sup _{\left[T_{0}, T\right]}\left\|\mathcal{Z}^{k} u(t, \cdot)\right\|_{H^{d+\alpha}}<+\infty, \sup _{\left[T_{0}, T\right]}\left\|\mathcal{Z}^{k} u(t, \cdot)\right\|_{C^{d+\alpha}}<+\infty
$$

on an interval $\left[T_{0}, T\right]$, we may write, using the notation $\mathcal{U}=(u, \bar{u})$,

$$
\begin{equation*}
D_{t} u=\left|D_{x}\right|^{\frac{1}{2}} u+\widetilde{Q}_{0}(\mathcal{U})+\widetilde{C}_{0}(\mathcal{U})+\widetilde{\mathcal{R}}_{0}(\mathcal{U}) \tag{3.1}
\end{equation*}
$$

where $\widetilde{Q}_{0}(\mathcal{U})$ denotes the quadratic part of the nonlinearity

$$
\begin{align*}
\widetilde{Q}_{0}(\mathcal{U})= & -\frac{i}{8}\left|D_{x}\right|^{\frac{1}{2}}\left[\left(D_{x}\left|D_{x}\right|^{-\frac{1}{2}}(u+\bar{u})\right)^{2}+\left(\left|D_{x}\right|^{\frac{1}{2}}(u+\bar{u})\right)^{2}\right] \\
& +\frac{i}{4}\left|D_{x}\right|\left((u-\bar{u})\left|D_{x}\right|^{\frac{1}{2}}(u+\bar{u})\right)-\frac{i}{4} D_{x}\left((u-\bar{u}) D_{x}\left|D_{x}\right|^{-\frac{1}{2}}(u+\bar{u})\right), \tag{3.2}
\end{align*}
$$

$\widetilde{C}_{0}(\mathcal{U})$ stands for the cubic contribution

$$
\begin{align*}
\widetilde{C}_{0}(\mathcal{U})= & \frac{1}{8}\left|D_{x}\right|^{\frac{1}{2}}\left[\left(\left|D_{x}\right|^{\frac{1}{2}}(u+\bar{u})\right)\left|D_{x}\right|\left((u-\bar{u})\left|D_{x}\right|^{\frac{1}{2}}(u+\bar{u})\right)\right] \\
& -\frac{1}{8}\left|D_{x}\right|^{\frac{1}{2}}\left[\left(\left|D_{x}\right|^{\frac{1}{2}}(u+\bar{u})\left((u-\bar{u})\left|D_{x}\right|^{\frac{3}{2}}(u+\bar{u})\right)\right]\right. \\
& -\frac{1}{8}\left|D_{x}\right|\left[(u-\bar{u})\left|D_{x}\right|\left((u-\bar{u})\left|D_{x}\right|^{\frac{1}{2}}(u+\bar{u})\right)\right]  \tag{3.3}\\
& +\frac{1}{16}\left|D_{x}\right|\left[(u-\bar{u})^{2}\left|D_{x}\right|^{\frac{3}{2}}(u+\bar{u})\right]+\frac{1}{16}\left|D_{x}\right|^{2}\left[(u-\bar{u})^{2}\left|D_{x}\right|^{\frac{1}{2}}(u+\bar{u})\right]
\end{align*}
$$

and where $\widetilde{\mathcal{R}}_{0}(\mathcal{U})$ is a remainder, vanishing at least at order 4 at $\mathcal{U}=0$, which satisfies for $k \leq s_{0}$ the following estimates

$$
\begin{equation*}
\left\|\mathcal{Z}^{k}\left|D_{x}\right|^{-\frac{1}{2}} \widetilde{R}_{0}(\mathcal{U})\right\|_{H^{d}} \leq C_{k}[u]_{\substack{k_{1}+\ldots+k_{j} \leq k \\ k_{1}, k_{2}, k_{3} \leq k_{4}}} \prod_{j=1}^{3}\left\|\mathcal{Z}^{k_{j}} u\right\|_{C^{d+\alpha}}\left\|\mathcal{Z}^{k_{4}} u\right\|_{H^{d+\alpha}} \tag{3.4}
\end{equation*}
$$

with a constant $C_{k}[u]$ depending only on $\left\|\mathcal{Z}^{(k-1)+} u\right\|_{C^{d+\alpha}}$ and, if $\theta>0$ is small enough,

$$
\begin{equation*}
\left\|\mathcal{Z}^{k}\left|D_{x}\right|^{-\frac{1}{2}+\theta} \widetilde{R}_{0}(\mathcal{U})\right\|_{C^{d}} \leq C_{k}[u] \sum_{k_{1}+\cdots+k_{4} \leq k} \prod_{j=1}^{4}\left\|\mathcal{Z}^{k_{j}} u\right\|_{C^{d+\alpha}} \tag{3.5}
\end{equation*}
$$

where $C_{k}[u]$ depends only on $\left\|\mathcal{Z}^{(k-1)_{+}} u\right\|_{C^{d+\alpha}}$ and on a bound on $\|u\|_{L^{2}}^{1-2 \theta^{\prime}}\|u\|_{L^{\infty}}^{2 \theta^{\prime}}$ for some $\left.\theta^{\prime} \in\right] 0, \theta[$.

We make the change of variables $t=t^{\prime}, x=t^{\prime} x^{\prime}$ and set $h=t^{\prime-1}, u(t, x)=h^{1 / 2} v\left(t^{\prime}, x^{\prime}\right)$, so that

$$
D_{t} u=h^{\frac{1}{2}}\left[\left(D_{t^{\prime}}-x^{\prime} h D_{x^{\prime}}\right) v+\frac{i}{2} h v\right] .
$$

The vector field $Z=t \partial_{t}+2 x \partial_{x}$ becomes $Z=t^{\prime} \partial_{t^{\prime}}+x^{\prime} \partial_{x^{\prime}}$. We deduce from (3.1) the following equation for $v$, in which we write $(t, x)$ instead of $\left(t^{\prime}, x^{\prime}\right)$, since we shall not go back to the old coordinates

$$
\begin{equation*}
\left(D_{t}-\mathrm{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right)\right) v=\sqrt{h} Q_{0}(V)+h\left[-\frac{i}{2} v+C_{0}(V)\right]+h^{\frac{11}{8}} R_{0}^{h}(V) \tag{3.6}
\end{equation*}
$$

where $V=(v, \bar{v})$,

$$
\begin{align*}
Q_{0}(V)= & -\frac{i}{8} \mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right)\left[\left(\operatorname{Op}_{h}\left(\xi|\xi|^{\frac{1}{2}}\right)(v+\bar{v})\right)^{2}+\left(\mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right)(v+\bar{v})\right)^{2}\right] \\
& +\frac{i}{4} \mathrm{Op}_{h}(|\xi|)\left((v-\bar{v}) \mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right)(v+\bar{v})\right)  \tag{3.7}\\
& -\frac{i}{4} \mathrm{Op}_{h}(\xi)\left((v-\bar{v}) \mathrm{Op}_{h}\left(\xi|\xi|^{-\frac{1}{2}}\right)(v+\bar{v})\right),
\end{align*}
$$

$C_{0}(V)$ stands for the cubic contribution

$$
\begin{align*}
C_{0}(V)= & \frac{1}{8} \mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right)\left[\left(\mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right)(v+\bar{v})\right) \mathrm{Op}_{h}(|\xi|)\left((v-\bar{v}) \mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right)(v+\bar{v})\right)\right] \\
& -\frac{1}{8} \mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right)\left[\left(\mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right)(v+\bar{v})\right)\left((v-\bar{v}) \mathrm{Op}_{h}\left(|\xi|^{\frac{3}{2}}\right)(v+\bar{v})\right)\right] \\
& -\frac{1}{8} \mathrm{Op}_{h}(|\xi|)\left[(v-\bar{v}) \mathrm{Op}_{h}(|\xi|)\left((v-\bar{v}) \mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right)(v+\bar{v})\right)\right]  \tag{3.8}\\
& +\frac{1}{16} \mathrm{Op}_{h}(|\xi|)\left[(v-\bar{v})^{2} \mathrm{Op}_{h}\left(|\xi|^{\frac{3}{2}}\right)(v+\bar{v})\right] \\
& +\frac{1}{16} \mathrm{Op}_{h}\left(|\xi|^{2}\right)\left[(v-\bar{v})^{2} \mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right)(v+\bar{v})\right]
\end{align*}
$$

and where the remainder satisfies for $p=2$ or $\infty$ and a small positive number $\theta$, for any $d \in \mathbb{N}, k \in \mathbb{N}$ such that $\left\|\left\langle h D_{x}\right\rangle^{\alpha+d} \mathcal{Z}^{k} v(t, \cdot)\right\|_{L^{2}}$ and $\left\|\left\langle h D_{x}\right\rangle^{\alpha+d} \mathcal{Z}^{k} v(t, \cdot)\right\|_{L^{\infty}}$ are finite, the estimate

$$
\begin{align*}
&\left\|\left\langle h D_{x}\right\rangle^{d} \mathcal{Z}^{k}\left|h D_{x}\right|^{-\frac{1}{2}+\theta} R_{0}^{h}(V)\right\|_{L^{p}}  \tag{3.9}\\
& \leq C_{k}[v] h^{\frac{1}{16}} \sum_{\substack{k_{1}+k_{2}+k_{3} \leq k \\
k_{1}, k_{2} \leq k_{3}}} \prod_{j=1}^{2}\left\|\left\langle h D_{x}\right\rangle^{\alpha+d} \mathcal{Z}^{k_{j}} V\right\|_{L^{\infty}}\left\|\left\langle h D_{x}\right\rangle^{\alpha+d} \mathcal{Z}^{k_{3}} V\right\|_{L^{p}}
\end{align*}
$$

where $C_{k}[v]$ depends on

$$
h^{\frac{1}{16}}\left\|\left\langle h D_{x}\right\rangle^{\alpha+d} \mathcal{Z}^{(k-1)_{+}} v\right\|_{L^{\infty}}
$$

and on a uniform bound for $\|v\|_{L^{2}}^{1-2 \theta^{\prime}} h^{\theta^{\prime}}\|v\|_{L^{\infty}}^{2 \theta^{\prime}}$.
Actually (3.6), (3.7), (3.8) follow from (3.1), (3.2), (3.3). The remainder, estimated by (3.4) and (3.5), being at least quartic, would bring in factor a power $h^{3 / 2}$ in (3.6). We retained only the power $h^{11 / 8}$, to keep the extra $h^{1 / 16}$ factor in the right hand side of (3.9), and to keep also a $h^{1 / 16}$-factor in front of one of the $\left\|\mathcal{Z}^{k_{j}} v\right\|_{C^{d+\alpha}}$ in the right hand side of (3.4), (3.5). In that way, we obtain in (3.9) an estimate in terms of cubic expressions, modulo the indicated multiplicative constant. Notice also that the uniform bound assumed for $\|v\|_{L^{2}}^{1-2 \theta^{\prime}} h^{\theta^{\prime}}\|v\|_{L^{\infty}}^{2 \theta^{\prime}}$ will be satisfied, when $\theta^{\prime}>0$ will have been fixed. Actually, we shall obtain a uniform control of $\|v\|_{L^{\infty}}$, and a bound of $\|v(t, \cdot)\|_{L^{2}}$ in $O\left(t^{\delta}\right)$ for some $\delta>0$ as small as we want. Taking this $\delta$ smaller than $\theta^{\prime}$ will provide the wanted uniformity. Finally, notice also that the fact that order zero pseudo-differential operators are not $L^{\infty}$-bounded is harmless in deriving (3.9) with $p=\infty$ from (3.5), as we may always replace $\alpha$ by some larger value.

Our main task in the following subsections will be to deduce from equation (3.6) the oscillatory behavior of $v$ when $h$ goes to zero. We shall do that expressing $v$ from Lagrangian distributions as those defined in the preceding section. This structure will be uncovered writing from (3.26) an equation for $v$ involving only $D_{t}$ derivatives. Actually, since $D_{t}=-i h Z-\mathrm{Op}_{h}(x \xi)$, we may write

$$
\begin{equation*}
\mathrm{Op}_{h}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) v=-\sqrt{h} Q_{0}(V)+h\left[\frac{i}{2} v-i Z V-C_{0}(V)\right]-h^{\frac{11}{8}} R_{0}^{h}(V) \tag{3.10}
\end{equation*}
$$

From nom on, we consider $v(t, \cdot)$ as a family of functions of $x$ indexed by $\left.\left.h=t^{-1} \in\right] 0,1\right]$. We do not write explicitly the parameter $h$ i.e. we write $v$ instead of $\left(v_{h}\right)_{h}$. Let us introduce the Lagrangian submanifold given by the zero set of the symbol in the left hand side of (3.10) outside $\xi=0$ i.e. set

$$
\begin{align*}
\Lambda & =\left\{(x, \xi) \in T^{*}(\mathbb{R} \backslash\{0\}) ; 2 x \xi+|\xi|^{\frac{1}{2}}=0 \xi \neq 0\right\}  \tag{3.11}\\
& =\left\{(x, \mathrm{~d} \omega(x)) ; x \in \mathbb{R}^{*}\right\}
\end{align*}
$$

where

$$
\omega(x)=\frac{1}{4|x|}
$$

In Section 6, we shall need the exact expressions of $Q_{0}(V), C_{0}(V)$ given by (3.7), (3.8). Before that, we shall use only some less precise informations on the structure of these terms that we describe now.

From now on, we denote by $\mathcal{Z}$ the collection of vector fields $\mathcal{Z}=\left(Z, h \partial_{x}\right)$ and, if $v$ is a distribution on $\mathbb{R}$, we define for any natural integer $k$ the vector valued function

$$
\mathcal{Z}^{k} v=\left(Z^{k_{1}}\left(h \partial_{x}\right)^{k_{2}} v\right)_{k_{1}+k_{2} \leq k}
$$

Lemma 3.1. Let $p$ be in $[1,+\infty]$.
i) Denote by $B_{0}$ the symmetric bilinear form associated to the quadratic form $Q_{0}$. Let $k$ be in $\mathbb{N}^{*}$, and for every couple $\left(k_{1}, k_{2}\right) \in \mathbb{N} \times \mathbb{N}$ with $k_{1}+k_{2}=k$, take $p_{k_{1}}, p_{k_{2}}$ in $[1,+\infty]$ such that $\frac{1}{p_{k_{1}}}+\frac{1}{p_{k_{2}}}=\frac{1}{p}$. Then for any distributions $V_{1}=\left(v_{1}, \overline{v_{1}}\right), V_{2}=\left(v_{2}, \overline{v_{2}}\right)$, any $j_{0}, j_{1}, j_{2}$ in $\mathbb{Z}$,

$$
\begin{align*}
\left\|\Delta_{j_{0}}^{h} \mathcal{Z}^{k} B_{0}\left(\Delta_{j_{1}}^{h} V_{1}, \Delta_{j_{2}}^{h} V_{2}\right)\right\|_{L^{p}} \leq & C 2^{j_{0}+\frac{1}{2} \min \left(j_{1}, j_{2}\right)} \mathbf{1}_{\max \left(j_{1}, j_{2}\right) \geq j_{0}-C} \\
& \times \sum_{k_{1}+k_{2} \leq k}\left\|\mathcal{Z}^{k_{1}} \Delta_{j_{1}}^{h} V_{1}\right\|_{L^{p_{k_{1}}}}\left\|\mathcal{Z}^{k_{2}} \Delta_{j_{2}}^{h} V_{2}\right\|_{L^{p_{k_{2}}}} \tag{3.12}
\end{align*}
$$

for some positive constant C. In the same way

$$
\begin{align*}
& \left\|\mathrm{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right) \mathcal{Z}^{k} B_{0}\left(\Delta_{j_{1}}^{h} V_{1}, \Delta_{j_{2}}^{h} V_{2}\right)\right\|_{L^{p}} \\
& \quad \leq C h^{2(1-\sigma)} 2^{\frac{1}{2} \min \left(j_{1}, j_{2}\right)}  \tag{3.13}\\
& \quad \times \sum_{k_{1}+k_{2}=k}\left\|\mathcal{Z}^{k_{1}} \Delta_{j_{1}}^{h} V_{1}\right\|_{L^{p_{k_{1}}}}\left\|\mathcal{Z}^{k_{2}} \Delta_{j_{2}}^{h} V_{2}\right\|_{L^{p_{k_{2}}}} .
\end{align*}
$$

ii) Let $T_{0}$ be the trilinear symmetric form associated to $C_{0}$. Then for any $k \in \mathbb{N}$, for some constant $C$,

$$
\begin{align*}
& \left\|\Delta_{j_{0}}^{h} \mathcal{Z}^{k} T_{0}\left(\Delta_{j_{1}}^{h} V_{1}, \Delta_{j_{2}}^{h} V_{2}, \Delta_{j_{3}}^{h} V_{3}\right)\right\|_{L^{p}} \\
& \quad \leq  \tag{3.14}\\
& \quad C 2^{j_{0} / 2+2 \max \left(j_{1}, j_{2}, j_{3}\right)} \mathbf{1}_{\max \left(j_{1}, j_{2}, j_{3}\right) \geq j_{0}-C} \\
& \quad \times \sum_{k_{1}+k_{2}+k_{3} \leq k}\left\|\mathcal{Z}^{k_{1}} \Delta_{j_{1}}^{h} V_{1}\right\|_{L^{p_{k_{1}}}}\left\|\mathcal{Z}^{k_{2}} \Delta_{j_{2}}^{h} V_{2}\right\|_{L^{p_{k_{2}}}}\left\|\mathcal{Z}^{k_{3}} \Delta_{j_{3}}^{h} V_{3}\right\|_{L^{p_{k_{3}}}}
\end{align*}
$$

where $\frac{1}{p_{k_{1}}}+\frac{1}{p_{k_{2}}}+\frac{1}{p_{k_{3}}}=\frac{1}{p}$.
In particular, for any $d$ in $\mathbb{R}_{+}$, any $p$ in $[1,+\infty]$, any $\alpha>2$

$$
\begin{align*}
& \left\|\Delta_{j}^{h} \mathcal{Z}^{k} T_{0}\left(V_{1}, V_{2}, V_{3}\right)\right\|_{L^{p}} \\
& \quad \leq C 2^{j / 2-j+d} \sum_{\substack{k_{1}+k_{2}+k_{3} \leq k \\
k_{1}, k_{2} \leq k_{3}}} \prod_{\ell=1}^{2}\left\|\mathcal{Z}^{k_{\ell}}\left\langle h D_{x}\right\rangle^{\alpha+d} V_{\ell}\right\|_{L^{\infty}}\left\|\mathcal{Z}^{k_{3}}\left\langle h D_{x}\right\rangle^{\alpha+d} V_{3}\right\|_{L^{p}} . \tag{3.15}
\end{align*}
$$

If, in the left hand side, one replaces $\Delta_{j}^{h}$ by $\operatorname{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right)$, the same estimates hold with the factor $2^{j / 2-j_{+} d}$ in the right hand side replaced by $h^{1-\sigma}$.
iii) The remainder $R_{0}^{h}(V)$ satisfies for any $d \in \mathbb{R}_{+}$, any $j$ in $\mathbb{Z}$, with $2^{j} \geq c h^{2(1-\sigma)}$ estimates

$$
\begin{align*}
& \left\|\Delta_{j}^{h} \mathcal{Z}^{k} R_{0}^{h}(V)\right\|_{L^{2}} \\
& \quad \leq C 2^{j / 2-j_{+} d} \sum_{\substack{k_{1}+k_{2}+k_{3} \leq k \\
k_{1}, k_{2} \leq k_{3}}} \prod_{\ell=1}^{2}\left\|\mathcal{Z}^{k_{\ell}}\left\langle h D_{x}\right\rangle^{\alpha+d} V\right\|_{L^{\infty}}\left\|\mathcal{Z}^{k_{3}}\left\langle h D_{x}\right\rangle^{\alpha+d} V\right\|_{L^{2}} \tag{3.16}
\end{align*}
$$

and

$$
\begin{equation*}
\left\|\Delta_{j}^{h} \mathcal{Z}^{k} R_{0}^{h}(V)\right\|_{L^{\infty}} \leq C 2^{j / 2-j_{+} d} \sum_{k_{1}+k_{2}+k_{3} \leq k} \prod_{\ell=1}^{3}\left\|\mathcal{Z}^{k_{\ell}}\left\langle h D_{x}\right\rangle^{\alpha+d} V\right\|_{L^{\infty}} \tag{3.17}
\end{equation*}
$$

where $C$ depends only on $h^{1 / 16}\left\|\left\langle h D_{x}\right\rangle^{\alpha+d} \mathcal{Z}^{(k-1)_{+}} V\right\|_{L^{\infty}}$ for some large enough $\alpha>0$.
If, in the left hand side of (3.16), (3.17), $\Delta_{j}^{h}$ is replaced by by $\mathrm{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right)$, similar estimates hold with $2^{j / 2-j_{+} d}$ replaced by $h^{1-\sigma}$.

Proof. i) Consider the contribution to $B_{0}\left(V_{1}, V_{2}\right)$ of the first term in the right hand side of (3.7). Its Fourier transform may be written as the symmetrization of a multiple of

$$
h^{2} \int \frac{|\xi h|^{\frac{1}{2}}}{(h|\xi-\eta|)^{\frac{1}{2}}\left(h|\eta|^{\frac{1}{2}}\right)}((\xi-\eta) \eta+|\xi-\eta||\eta|) \widehat{f}_{1}(\xi-\eta) \widehat{f}_{2}(\eta) d \eta
$$

where $f_{1}=v_{1}+\overline{v_{1}}, f_{2}=v_{2}+\overline{v_{2}}$. On the support of the integrand, $(\xi-\eta) \eta \geq 0$ so that $|\xi|=|\xi-\eta|+|\eta|$. Consequently, the contribution of this term to $\Delta_{j_{0}}^{h} B_{0}\left(\Delta_{j_{1}}^{h} V_{1}, \Delta_{j_{2}}^{h} V_{2}\right)$ will be non zero only when $j_{0} \geq \max \left(j_{1}, j_{2}\right)-C$ for some $C>0$. In the same way, the contribution to $B_{0}\left(V_{1}, V_{2}\right)$ of the sum of the last two terms in (3.7) may be written, after Fourier transform an up to symmetries, as a multiple of

$$
h^{\frac{3}{2}} \int \frac{|\xi||\eta|-\xi \eta}{|\eta|^{\frac{1}{2}}} \widehat{f}_{1}(\xi-\eta) \widehat{f}_{2}(\eta) d \eta
$$

On the support of the integrand $\xi \eta \leq 0$, whence $|\xi-\eta| \geq \max (|\xi|,|\eta|)$ so that the contribution to

$$
\Delta_{j_{0}}^{h} B_{0}\left(\Delta_{j_{1}}^{h} V_{1}, \Delta_{j_{2}}^{h} V_{2}\right)
$$

will be non zero only if $j_{2} \leq j_{1}+C$ for some $C>0$. Using these inequalities and taking into account the distribution of the derivatives on the different factors, we conclude

$$
\begin{equation*}
\left\|\Delta_{j_{0}}^{h} B_{0}\left(\Delta_{j_{1}}^{h} V_{1}, \Delta_{j_{2}}^{h} V_{2}\right)\right\|_{L^{2}} \leq C 2^{j_{0}+\frac{1}{2} \min \left(j_{1}, j_{2}\right)}\left\|\Delta_{j_{1}}^{h} V_{1}\right\|_{L^{p_{1}}}\left\|\Delta_{j_{2}}^{h} V_{2}\right\|_{L^{p_{2}}} \tag{3.18}
\end{equation*}
$$

if $\frac{1}{p_{1}}+\frac{1}{p_{2}}=\frac{1}{2}$. Moreover, by spectral localization, we have always $\max \left(j_{1}, j_{2}\right) \geq j_{0}-C$ for some $C>0$. If one makes act $\mathcal{Z}^{k}$ on $B_{0}\left(V_{1}, V_{2}\right)$, the above properties of spectral localization are
not affected since, if $a(\xi)$ is smooth outside zero, $\left[Z, \mathrm{Op}_{h}(a)\right]=-2 \mathrm{Op}_{h}\left(\xi a^{\prime}(\xi)\right)$. Distributing the $Z$-derivatives on the different factors, one gets (3.12). The proof of (3.13) is similar.
ii) We notice first that in all contributions in (3.8), $\mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right)$ is always in factor. This allows to make appear the $2^{j_{0} / 2}$-factor in (3.14). Since the sum of the powers of $|\xi|$ appearing in each term of (3.8) is equal to $5 / 2$, we get as well the factor $2^{2 \max \left(j_{1}, j_{2}, j_{3}\right)}$ in (3.14). The cut-of for $\max \left(j_{1}, j_{2}, j_{3}\right) \geq j_{0}-C$ follows from the spectral localization of each factor. Finally, making act $\mathcal{Z}^{k}$ on $T_{0}$ and commuting each vector field with $\mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right), \mathrm{Op}_{h}\left(\xi|\xi|^{-\frac{1}{2}}\right), \ldots$ we obtain (3.14).

To deduce (3.15) from (3.14), we decompose in the left hand side of (3.15),

$$
V_{\ell}=\operatorname{Op}_{h}\left(\varphi_{0}(\xi)\right) V_{\ell}+\sum_{j_{\ell} \geq 0} \Delta_{j_{\ell}}^{h} V_{\ell} .
$$

Because of the spectral localization, we get for $j_{\ell} \geq 0$,

$$
\begin{aligned}
& \left\|Z^{k_{\ell}} \Delta_{j_{\ell}}^{h} V_{\ell}\right\|_{L^{p}} \leq C 2^{-j_{\ell}(\alpha+d)} \sum_{k_{\ell}^{\prime} \leq k_{\ell}}\left\|Z^{k_{\ell}^{\prime}}\left\langle h D_{x}\right\rangle^{\alpha+d} V_{\ell}\right\|_{L^{p}}, \\
& \left\|Z^{k_{\ell}} \mathrm{Op}_{h}\left(\varphi_{0}\right) V_{\ell}\right\|_{L^{p}} \leq C \sum_{k_{\ell}^{\prime} \leq k_{\ell}}\left\|Z^{k_{\ell}^{\prime}}\left\langle h D_{x}\right\rangle^{\alpha+d} V_{\ell}\right\|_{L^{p}} .
\end{aligned}
$$

We plug these estimates in (3.14) with $p_{k_{1}}=p_{k_{2}}=\infty, p_{k_{3}}=p$ and in the similar inequality where some $\Delta_{j_{\ell}}^{h} V_{\ell}$ is replaced by $\mathrm{Op}_{h}\left(\varphi_{0}\right) V_{\ell}$. We obtain a bound given by the product of the sum in the right hand side of (3.15) multiplied by

$$
C 2^{j / 2} \sum_{\max \left(j_{1}, j_{2}, j_{3}\right) \geq j-C}^{j_{\ell} \geq 0}<1
$$

Since $\alpha>2$, this is bounded by $C 2^{j / 2-j_{+} d}$ as wanted. The analogous statement, when $\Delta_{j}^{h}$ is replaced by $\mathrm{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right)$ in the left hand side of (3.15) is obtained in the same way.
iii) Inequalities (3.16), (3.17) follows from (3.9) with $p=2$ or $p=\infty$, using that the loss $2^{-j \theta} \leq c h^{-2 \theta(1-\sigma)}$ is absorbed by the extra $h^{1 / 16}$ factor in the right hand side of (3.9), if $\theta$ has been taken small enough.

Let us introduce the following decomposition of a solution $v$ of (3.6). Fix $\sigma, \beta$ some small positive numbers, $\varphi_{0}$ in $C_{0}^{\infty}(\mathbb{R})$ the function equal to one close to zero introduced before Definition 2.7. We decompose the solution $v$ of (3.6) as

$$
\begin{align*}
& v=v_{L}+w+v_{H} \\
& v_{L}=\operatorname{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right) v,\right.  \tag{3.19}\\
& v_{H}=\operatorname{Op}_{h}\left(\left(1-\varphi_{0}\right)\left(h^{-\beta} \xi\right)\right) v .
\end{align*}
$$

We notice that if $C$ is a large enough constant, $w=v-v_{L}-v_{H}$ may be written $\sum_{j \in J(h, C)} \Delta_{j}^{h} w$.

Proposition 3.2. Assume that for some $k \in \mathbb{N}$, some $a>b+\frac{3}{2}+\frac{1}{\beta}+\alpha$, some positive constants $\delta_{k}, \delta_{k}^{\prime}, A_{k}, A_{k}^{\prime}$ a solution $v$ of (3.6) satisfies for any $h$ in an interval $\left.] h^{\prime}, 1\right]$, with $\left.\left.h^{\prime} \in\right] 0,1\right]$ given, the a priori $L^{2}$-bounds

$$
\begin{align*}
& \left\|\mathrm{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right) \mathcal{Z}^{k} v\right\|_{L^{2}} \leq \varepsilon A_{k} h^{-\delta_{k}} \\
& \left\|\Delta_{j}^{h} \mathcal{Z}^{k} v\right\|_{L^{2}} \leq \varepsilon A_{k} h^{-\delta_{k}} 2^{-j_{+} a} \quad \text { for } 2^{j} \geq C^{-1} h^{2(1-\sigma)} \tag{3.20}
\end{align*}
$$

and the a priori $L^{\infty}$-bounds

$$
\begin{align*}
& \left\|\mathrm{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right) \mathcal{Z}^{k} v\right\|_{L^{\infty}} \leq \varepsilon A_{k}^{\prime} h^{-\delta_{k}^{\prime}} \\
& \left\|\Delta_{j}^{h} \mathcal{Z}^{k} v\right\|_{L^{\infty}} \leq \varepsilon A_{k}^{\prime} h^{-\delta_{k}^{\prime}} 2^{-j_{+} b} \quad \text { for } 2^{j} \geq C^{-1} h^{2(1-\sigma)} \tag{3.21}
\end{align*}
$$

Then, if $\delta_{k}$, are small enough, one gets that

$$
\begin{equation*}
h^{-\frac{3}{8}} \underline{v}=h^{-\frac{3}{8}}\left(v_{L}+v_{H}\right) \quad \text { belongs to an } \varepsilon \text {-neighborhood of } 0 \text { in } \mathcal{R}_{\infty}^{b} \tag{3.22}
\end{equation*}
$$

with the notation introduced in Definition 2.8. Moreover, $w=v-\underline{v}$ satisfies

$$
\begin{equation*}
\left(D_{t}-\mathrm{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right)\right) w=\sqrt{h} Q_{0}(W)+h\left[-\frac{i}{2} w+C_{0}(W)\right]+h^{\frac{5}{4}} R_{0}(V) \tag{3.23}
\end{equation*}
$$

where $\mathcal{Z}^{k} R_{0}(V)$ belongs to $\mathcal{R}_{\infty}^{b}$, and is an $\varepsilon$-neighborhood of zero in that space.

Notice, for further reference, that as we did for (3.10), we deduce from (3.23)

$$
\begin{equation*}
\mathrm{Op}_{h}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) w=-\sqrt{h} Q_{0}(W)+h\left[\frac{i}{2} w-i Z w-C_{0}(W)\right]-h^{\frac{5}{4}} R_{0}(V) \tag{3.24}
\end{equation*}
$$

The proposition will be proved using the following lemma.
Lemma 3.3. i) Assume that estimates (3.20), (3.21) hold. Then if $a>b+\frac{3}{2}+\frac{1}{\beta}$, $a>$ $b+\alpha+1+\frac{1}{2 \beta}, b>\alpha>2$,

$$
\begin{align*}
& \left\|\mathrm{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right) \mathcal{Z}^{k} Q_{0}(V)\right\|_{L^{\infty}} \leq c_{k} \varepsilon^{2} h^{\frac{3}{4}} \\
& \left\|\Delta_{j}^{h} \operatorname{Op}_{h}\left(\left(1-\varphi_{0}\right)\left(h^{2 \beta} \xi\right)\right) \mathcal{Z}^{k} Q_{0}(V)\right\|_{L^{\infty}} \leq c_{k} \varepsilon^{2} h^{\frac{3}{4}} 2^{-j_{+} b} \quad \text { for any } j \tag{3.25}
\end{align*}
$$

and

$$
\begin{align*}
& \left\|\mathrm{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right) \mathcal{Z}^{k} C_{0}(V)\right\|_{L^{\infty}} \leq c_{k} \varepsilon^{3} h^{\frac{1}{4}}  \tag{3.26}\\
& \left\|\Delta_{j}^{h} \mathrm{Op}_{h}\left(\left(1-\varphi_{0}\right)\left(h^{2 \beta} \xi\right)\right) \mathcal{Z}^{k} C_{0}(V)\right\|_{L^{\infty}} \leq c_{k} \varepsilon^{3} h^{\frac{1}{4}} 2^{-j_{+} b} \quad \text { for any } j
\end{align*}
$$

if $\delta_{k}, \delta_{k}^{\prime}$ in (3.21) are small enough and $c_{k}$ is a convenient constant.
ii) Assume (3.20) and the same inequalities on $a, b$ as above. Then if $\delta_{k}, \delta_{k}^{\prime}$ are small enough

$$
\begin{align*}
& \left\|\mathrm{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right) \mathcal{Z}^{k} v\right\|_{L^{\infty}} \leq c_{k} \varepsilon h^{\frac{7}{16}-\sigma}, \\
& \left\|\Delta_{j}^{h} \operatorname{Op}_{h}\left(\left(1-\varphi_{0}\right)\left(h^{2 \beta} \xi\right)\right) \mathcal{Z}^{k} v\right\|_{L^{\infty}} \leq c_{k} \varepsilon h^{\frac{7}{8}} 2^{-j_{+} b} \quad \text { for any } j,  \tag{3.27}\\
& \left\|2^{j \ell} \Delta_{j}^{h} \mathrm{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right) \mathcal{Z}^{k} v\right\|_{L^{\infty}} \leq c_{k} \varepsilon h^{\frac{3}{8}-\sigma+2 \ell(1-\sigma)}, \quad \ell \geq 0 .
\end{align*}
$$

Moreover, if we assume (3.20) and (3.21),

$$
\begin{align*}
\| & \mathrm{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right) \mathcal{Z}^{k}\left[Q_{0}(V)-Q_{0}(W)\right] \|_{L^{\infty}}  \tag{3.28}\\
& \quad+\sup _{j \geq j_{0}(h, C)} 2^{j+b}\left\|\Delta_{j}^{h} \mathcal{Z}^{k}\left[Q_{0}(V)-Q_{0}(W)\right]\right\|_{L^{\infty}} \leq c_{k} \varepsilon^{2} h^{\frac{3}{4}}
\end{align*}
$$

and

$$
\begin{align*}
\| & \operatorname{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right) \mathcal{Z}^{k}\left[C_{0}(V)-C_{0}(W)\right] \|_{L^{\infty}}  \tag{3.29}\\
& +\sup _{j \geq j_{0}(h, C)} 2^{j+b}\left\|\Delta_{j}^{h} \mathcal{Z}^{k}\left[C_{0}(V)-C_{0}(W)\right]\right\|_{L^{\infty}} \leq c_{k} \varepsilon^{3} h^{\frac{1}{4}}
\end{align*}
$$

Proof. i) To obtain the first formula in (3.25) we use (3.13) with $p_{k_{1}}=p_{k_{2}}=p=\infty$. Using assumption (3.21) we get a bound of the left hand side by

$$
C \varepsilon^{2} A_{k}^{\prime 2} h^{-2 \delta_{k}^{\prime}+2(1-\sigma)} \sum_{j_{1}, j_{2} \in \mathbb{Z}} 2^{\frac{1}{2} \min \left(j_{1}, j_{2}\right)-j_{1+} b-j_{2}+b}
$$

which gives the conclusion since $\sigma \in] 0,1 / 2\left[\right.$ and we take $\delta_{k}^{\prime}$ small enough. To get the second inequality (3.25) we use (3.12) with $p_{k_{1}}=p_{k_{2}}=p=\infty$, and we estimate the $L^{\infty}$ norms in the right hand side using Sobolev injection and (3.20). We obtain

$$
C \varepsilon^{2} A_{k}^{2} h^{-2 \delta_{k}} 2^{j} \sum_{\max \left(j_{1}, j_{2}\right) \geq j-C} 2^{\frac{1}{2} \min \left(j_{1}, j_{2}\right)-j_{1+} a-j_{2}+a+\frac{j_{1}}{2}+\frac{j_{2}}{2}} h^{-1}
$$

If one uses that by assumption $2^{j} \geq c h^{-2 \beta}$, and the fact that $a>b+\frac{3}{2}+\frac{1}{\beta}$, one gets the wanted estimate (for $\delta_{k}$ small enough).

To obtain (3.26), one substitutes inside (3.15) with $p=\infty, d=0$,

$$
V_{\ell}=\mathrm{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right) V_{\ell}+\sum_{j_{\ell} \in \mathbb{Z}} \Delta_{j_{\ell}}^{h} \mathrm{Op}_{h}\left(\left(1-\varphi_{0}\right)\left(h^{-2(1-\sigma)} \xi\right)\right) V_{\ell}
$$

one uses (3.21) to estimate two of the three factors of the right hand side, (3.20) and Sobolev injection to bound the third one, and one makes similar computations as above, exploiting that for the left hand side of the second estimate (3.26) not to vanish, it is necessary that one of the $j_{\ell}$ be larger than $j-C$, and the assumptions on $a$.
ii) Inequalities (3.27) follow from (3.20) and Sobolev injections, using the assumptions on $a$ and the fact that $\delta_{k}<1 / 16$.

We estimate the contribution to (3.28) corresponding to $j \in J(h, C)$. We write $Q_{0}(V)-Q_{0}(W)$ from $B_{0}(V-W, V)$ and $B_{0}(V-W, W)$. By (3.12), $\left\|\Delta_{j}^{h} \mathcal{Z}^{k} B_{0}(V-W, W)\right\|_{L^{\infty}}$ is smaller than

$$
C 2^{j} \sum_{\max \left(j_{1}, j_{2}\right) \geq j-C} 2^{\frac{1}{2} \min \left(j_{1}, j_{2}\right)}\left\|\mathcal{Z}^{k} \Delta_{j_{1}}^{h}(V-W)\right\|_{L^{\infty}}\left\|\mathcal{Z}^{k} \Delta_{j_{2}}^{h} V\right\|_{L^{\infty}} .
$$

By the definition of $w=v-v_{L}-v_{H}, \Delta_{j_{1}}^{h}(V-W)$ is non zero only if $2^{j_{1}} \lesssim h^{2(1-\sigma)}$ or $2^{j_{1}} \gtrsim h^{-2 \beta}$. In the first case, we bound $\left\|\mathcal{Z}^{k} \Delta_{j_{1}}^{h}(V-W)\right\|_{L^{\infty}} 2^{j_{1} / 4}$ using the third inequality (3.27) with $\ell=$ 1/4. We get a bound in $O\left(\varepsilon h^{\frac{7}{8}-\frac{3 \sigma}{2}}\right)$. In the second case $\left\|\mathcal{Z}^{k} \Delta_{j_{1}}^{h}(V-W)\right\|_{L^{\infty}}$ is $O\left(\varepsilon h^{\frac{7}{8}} 2^{-j_{+} b}\right)$ by the second estimate (3.27). Using assumption (3.21) to estimate $\left\|\mathcal{Z}^{k} \Delta_{j_{2}}^{h} V\right\|_{L^{\infty}}$, we get a bound

$$
C \varepsilon^{2} h^{\frac{7}{8}-\delta_{k}^{\prime}-\frac{3 \sigma}{2}} 2^{j} \sum_{\max \left(j_{1}, j_{2}\right) \geq j-C} 2^{\frac{j}{2} \min \left(j_{1}, j_{2}\right)-\frac{1}{4} j_{1-}} 2^{-\left(j_{1}++j_{2}+\right) b} \leq C \varepsilon^{2} 2^{-j_{+}(b-1)} h^{\frac{7}{8}-\delta_{k}^{\prime}-\frac{3}{2} \sigma} .
$$

Since $2^{j} \leq C h^{-2 \beta}$ with $\beta \ll 1$ and $\sigma \ll 1, \delta_{k}^{\prime} \ll 1$, we obtain the wanted conclusion.
One studies in the same way the contributions of indices $j$ in $J(h, C)$ to (3.29), expressing $C_{0}(V)-C_{0}(W)$ from $T_{0}(V-W, V, V)$ and from similar expressions and using (3.15).

To estimate the first term in the left hand side of (3.28), (3.29), or the contribution of $j \geq j_{1}(h, C)$ to the latter, we just need to apply (3.25), (3.26), and to notice that these inequalities remain true with $V$ replaced by $W$.

Proof of Proposition 3.2. Notice first that (3.22) follows from (3.27) if $\sigma \ll 1$. Denote $\Sigma_{h}=$ $I d-\mathrm{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right)-\mathrm{Op}_{h}\left(\left(1-\varphi_{0}\right)\left(h^{2 \beta} \xi\right)\right)$ so that $w=\Sigma_{h} v$ by definition and $\underline{v}=$ $\left(I d-\Sigma_{h}\right) v$. We notice that $\left[D_{t}-\operatorname{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right), \Sigma_{h}\right]=h \tilde{\Sigma}_{h}$, where $\tilde{\Sigma}_{h}$ may be written as a linear combination of quantities $\mathrm{Op}_{h}\left(\tilde{\varphi_{0}}\left(h^{-2(1-\sigma)} \xi\right)\right), \mathrm{Op}_{h}\left(\tilde{\varphi_{0}}\left(h^{2 \beta} \xi\right)\right)$ for new functions $\tilde{\varphi_{0}}$ in $C_{0}^{\infty}\left(\mathbb{R}^{*}\right)$. We deduce from (3.6)

$$
\begin{aligned}
\left(D_{t}-\mathrm{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right)\right) \underline{v}= & \sqrt{h}\left(I d-\Sigma_{h}\right) Q_{0}(V) \\
& +h\left(I d-\Sigma_{h}\right)\left(-\frac{i}{2}+C_{0}(V)\right) \\
& +h^{\frac{11}{8}}\left(I d-\Sigma_{h}\right) R_{0}^{h}(V) \\
& -h \tilde{\Sigma}_{h} v .
\end{aligned}
$$

By estimates (3.25), (3.26) and (3.27) the first, second and last terms of the right hand side may be written $h^{5 / 4} R(V)$ with $R(V)$ in $\mathcal{R}_{\infty}^{b}$.

To estimate the remainder term $R_{0}^{h}(V)$, we estimate its $L^{\infty}$ norm using (3.17) with $d=b+\frac{1}{2}$.

The factors in the right hand side of (3.17) are estimated in the following way:

$$
\begin{aligned}
\left\|\left\langle h D_{x}\right\rangle^{\alpha+b+\frac{1}{2}} V\right\|_{L^{\infty}} \leq & C\left\|\mathrm{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right) V\right\|_{L^{\infty}} \\
& +C \sum_{j \in J(h, C)}\left\|2^{j+\left(\alpha+b+\frac{1}{2}\right)} \Delta_{j}^{h} V\right\|_{L^{\infty}} \\
& +C \sum_{j \geq j_{1}(h, C)} 2^{j\left(\alpha+b+\frac{1}{2}\right)}\left(\frac{2^{j}}{h}\right)^{\frac{1}{2}}\left\|\Delta_{j}^{h} V\right\|_{L^{2}},
\end{aligned}
$$

where we used the Sobolev injection for the last term. Using assumptions (3.21) and (3.20) for the right hand side, together with the fact that $2^{j} \leq C h^{-2 \beta}$ on the the first sum, $2^{j}>c h^{-2 \beta}$ on the last one, and $a>\alpha+b+1+\frac{1}{\beta}$, we bound this quantity by say $C h^{-\frac{1}{24}}$ (if $\delta_{k}, \delta_{k}^{\prime}, \beta$ are small enough). It follows that

$$
\left(D_{t}-\mathrm{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right)\right) w=\sqrt{h} Q_{0}(V)+h\left[-\frac{i}{2} v+C_{0}(V)\right]+h^{\frac{5}{4}} R(V)
$$

with $R(V)$ in $\mathcal{R}_{\infty}^{b}$. Using (3.27), (3.28), (3.29) we may replace the right-hand side of this equation by the right hand side of (3.23) up to a modification of $R(V)$. If we make act the $\mathcal{Z}$-family of vector fields on (3.23), and use the commutation relations

$$
\begin{align*}
& {\left[t \partial_{t}+x \partial_{x}, D_{t}-\mathrm{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right)\right]=-\left(D_{t}-\mathrm{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right)\right),}  \tag{3.30}\\
& {\left[h \partial_{x}, D_{t}-\mathrm{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right)\right]=0,}
\end{align*}
$$

we obtain in the same way the estimates involving $\mathcal{Z}^{k}$ derivatives.

## 4 Weak $L^{\infty}$ estimates

The goal of this subsection is to show that if $v$ is a solution to equation (3.6), and if we are given an $L^{2}$-control of $Z^{k+1} v$ of type $\left\|Z^{k+1} v\right\|_{L^{2}}=O\left(h^{-\delta_{k+1}}\right)$ for some small $\delta_{k+1}>0$, we can deduce from it and the equation an $L^{\infty}$-bound of the form $\left\|Z^{k} v\right\|_{L^{\infty}}=O\left(h^{-\delta_{k}^{\prime}}\right)$ for some small $\delta_{k}^{\prime} \gg \delta_{k+1}$. Actually, we shall get as well bounds for $\left\langle h D_{x}\right\rangle^{b} v$ instead of $v$ for some given $b>0$.

These bounds are not good enough, but they will be the starting point of the more elaborated reasoning that will be pursued in Sections 5 and 6 . Before stating the main result, we fix some notation.

Assume given integers $s \gg N_{1} \gg N_{0} \gg 1$ and an increasing sequence of positive numbers $\left(\delta_{k}\right)_{0 \leq k \leq s / 2+N_{1}+1}$. We consider another increasing sequence $\left(\delta_{k}^{\prime}\right)_{0 \leq k \leq \frac{s}{2}+N_{1}}$ satisfying the inequalities

$$
\begin{align*}
& \delta_{k}^{\prime}>\sum_{j=0}^{\ell^{\prime}} \delta_{k_{j}}^{\prime}+\sum_{j=\ell^{\prime}+1}^{\ell} \delta_{k_{\ell}+1}
\end{align*} \quad \text { if }\left\{\begin{array}{l}
0 \leq \ell^{\prime} \leq \ell \leq 4, \Sigma_{j=0}^{\ell} k_{j} \leq k,  \tag{4.1}\\
k_{j}<k \text { when } 0 \leq j \leq \ell^{\prime},
\end{array}, \quad \text { if } k \geq 1, ~ \$ \delta_{k}^{\prime}>\delta_{k+1}+2 \delta_{0}+4 \delta_{0}^{\prime} \quad \delta^{\prime},\right.
$$

for $k=0, \ldots, \frac{s}{2}+N_{1}$. Clearly such a sequence $\left(\delta_{k}^{\prime}\right)_{k}$ may always be constructed by induction, and if $\delta_{s}$ is small enough, we may assume moreover that

$$
\begin{equation*}
\delta_{k}<\frac{1}{32} \quad k=0, \ldots, \frac{s}{2}+N_{1}+1, \delta_{k}^{\prime}<\frac{\sigma}{8}<\frac{1}{32} k=0, \ldots, \frac{s}{2}+N_{1} . \tag{4.2}
\end{equation*}
$$

We assume that the positive number $\beta$ introduced in (2.12) is small enough so that $2 \beta\left(\alpha+\frac{1}{2}\right)<$ $\frac{1}{8}$, where $\alpha>2$ is the fixed large enough number introduced in (3.4) and (3.15), and that $\beta<\sigma / 2$. We fix positive numbers $a>b>b^{\prime}>b^{\prime \prime}$ such that

$$
\begin{align*}
& a>b+\frac{3}{2}+\frac{1}{\beta}+\alpha, \quad b>\frac{1}{2}  \tag{4.3}\\
& \left(b-b^{\prime}\right) \beta>2, \quad\left(b^{\prime}-b^{\prime \prime}\right) \beta>2
\end{align*}
$$

In that way, the assumptions of Proposition 3.2 will be fulfilled.
For $k$ a nonnegative integer, we define

$$
\begin{equation*}
\mathcal{E}_{k}(v)=\sum_{k^{\prime}=0}^{k} \max \left(\left\|\mathrm{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right) \mathcal{Z}^{k^{\prime}} v\right\|_{L^{\infty}}, \sup _{j \geq j_{0}(h, C)} 2^{j+b}\left\|\Delta_{j}^{h} \mathcal{Z}^{k^{\prime}} v\right\|_{L^{\infty}}\right) \tag{4.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{F}_{k}(v)=\sum_{k^{\prime}=0}^{k} \max \left(\left\|\mathrm{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right) \mathcal{Z}^{k^{\prime}} v\right\|_{L^{2}}, \sup _{j \geq j_{0}(h, C)} 2^{j+a}\left\|\Delta_{j}^{h} \mathcal{Z}^{k^{\prime}} v\right\|_{L^{2}}\right) \tag{4.5}
\end{equation*}
$$

Let $k \in \mathbb{N}^{*}$. We denote by $\mathcal{T}_{k}^{\infty}$ the set of functions $v \mapsto P_{k}(v)$ satisfying for any $v=\left(v_{h}\right)_{h}$ with $\mathcal{E}_{k-1}(v) \leq h^{-1 / 4}$ a bound of type

$$
\begin{align*}
\left|P_{k}(v)\right| \leq C[ & \mathcal{E}_{k}(v)+\sum_{k_{1}+k_{2} \leq k} \mathcal{E}_{k_{1}}(v) \mathcal{E}_{k_{2}}(v) \\
& \left.+\sum_{k_{1}+k_{2}+k_{3} \leq k} \mathcal{E}_{k_{1}}(v) \mathcal{E}_{k_{2}}(v) \mathcal{E}_{k_{3}}(v)\right] . \tag{4.6}
\end{align*}
$$

In the same way, we define $\mathcal{T}_{k}$ as the set of functions $v \mapsto P_{k}(v)$ admitting for any $v=\left(v_{h}\right)_{h}$ with $\mathcal{E}_{k-1}(v) \leq h^{-1 / 4}$ a bound of type

$$
\begin{equation*}
\left|P_{k}(v)\right| \leq C \sum_{1 \leq \ell \leq 4} \sum_{k_{1}+\cdots+k_{\ell} \leq k} \sum_{\ell^{\prime}=0}^{\ell} \prod_{j=1}^{\ell^{\prime}} \mathcal{E}_{k_{j}}(v) \prod_{j=\ell^{\prime}+1}^{\ell} \mathcal{F}_{k_{j+1}}(v) \tag{4.7}
\end{equation*}
$$

The main result of this section is the following one.
Proposition 4.1. Let $k$ be a positive integer. Assume that we are given constants $\tilde{A}_{0}, A_{0}$, $A_{1}, \ldots, A_{k+1}$ and a solution $v$ of (3.6) such that for $h$ in some interval $\left.] h^{\prime}, 1\right]$

$$
\begin{equation*}
\mathcal{E}_{0}(v) \leq \tilde{A}_{0} h^{-\delta_{0}^{\prime}}, \quad \mathcal{F}_{k^{\prime}}(v) \leq \varepsilon A_{k^{\prime}} h^{-\delta_{k^{\prime}}}, \quad 0 \leq k^{\prime} \leq k+1 . \tag{4.8}
\end{equation*}
$$

Then, there are $h_{0}>0, A_{k^{\prime}}^{\prime}>0, k^{\prime}=1, \ldots, k$, depending only on $\varepsilon \tilde{A}_{0}, A_{1}, \ldots, A_{k+1}$ such that for any $h$ in $\left.] h^{\prime}, 1\right]$

$$
\begin{equation*}
\mathcal{E}_{k^{\prime}}(v) \leq \varepsilon A_{k^{\prime}}^{\prime} h^{-\delta_{k^{\prime}}^{\prime}}, \quad k^{\prime}=1, \ldots, k \tag{4.9}
\end{equation*}
$$

Remark 4.2. - The result of the preceding proposition may be thought of as a "KlainermanSobolev" estimate, that allows one to get $L^{\infty}$-decay from $L^{2}$-bounds (There is no decay involved in (4.9) since the negative power of time $t^{-1 / 2}=\sqrt{h}$ has been factored out when we defined $v$ from $u$ ).

The proof of the proposition will be made in three steps.
First, we treat the case of small or large frequencies, for which we deduce (4.9) from the $L^{2}$-estimate in (4.8) and Sobolev injection.

Next, we are reduced to intermediate frequencies i.e. to $\Delta_{j}^{h} v$ with $j$ belonging to $J(h, C)$. We write the equation for $\Delta_{j}^{h} v$ coming from (3.10). The operator of symbol $2 x \xi+|\xi|^{\frac{1}{2}}$ is elliptic outside the Lagrangian $\Lambda$ defined in (3.11). Since the right hand side of (3.10) is $O\left(h^{1 / 2-0}\right)$, one will get for the $L^{\infty}$-norm of $\Delta_{j}^{h} \mathcal{Z}^{k} v$ cut-off outside a neighborhood of $\Lambda$ some $O\left(h^{1 / 2-0}\right)$ estimates, that are better than what we want.

In the last step, we decompose in the quadratic part $Q_{0}(V)$ of the right hand side of (3.10), $v$ as the sum of the contribution microlocalized outside $\Lambda$, which by the preceding step will give an $O\left(h^{1-0}\right)$ contribution to (3.10), and a contribution microlocalized close to $\Lambda$. The quadratic interactions between the latter will be microlocally supported close to $2 \cdot \Lambda, 0 \cdot \Lambda$, $-2 \cdot \Lambda$ where

$$
\lambda \cdot \Lambda=\{(x, \lambda \xi) ;(x, \xi) \in \Lambda\}
$$

Consequently, if we microlocalize (3.10) close to $\Lambda$, which does not meet $\pm 2 \cdot \Lambda, 0 \cdot \Lambda$, the $\sqrt{h}$-terms of the right hand side disappear, and we get an $O\left(h^{1-0}\right)$ estimate for the $L^{2}$-norm of the left hand side. This allows to deduce the wanted $L^{\infty}$-estimate from a Sobolev embedding, after reduction of $\Lambda$ to the zero section, through a canonical transformation.

First step: Low and large frequencies
We decompose $v=v_{L}+w+v_{H}$ according to (3.19). By assumption (4.8), estimates (3.20) hold. Then $i i$ ) of Lemma 3.3 implies that $v_{L}, v_{H}$ satisfy the first two inequalities (3.27).

Second step: Elliptic estimates for $w$ outside a neighborhood of $\Lambda$.
We define, for $j \in J(h, C)$ with $C$ large enough

$$
\begin{align*}
& w_{j}=\Theta_{-j}^{*} \Delta_{j}^{h} w, \\
& \mathcal{Z}_{j}=\Theta_{-j}^{*} \mathcal{Z} \Theta_{j}^{*}=\left(Z, 2^{j / 2} h \partial_{x}\right), \quad \mathcal{Z}_{j}^{k}=\left(Z^{k_{1}}\left(2^{j / 2} h \partial_{x}\right)^{k_{2}}\right)_{k_{1}+k_{2} \leq k}, \tag{4.10}
\end{align*}
$$

so that $w=\sum_{j \in j(h, C)} \Theta_{j}^{*} w_{j}$.

Let $\Phi \in C_{0}^{\infty}(\mathbb{R} \backslash\{0\})$ be equal to one on a domain $C^{-1} \leq|\xi| \leq C$ for a large constant $C$ and let $\Gamma$ be in $C_{0}^{\infty}(\mathbb{R})$, with small enough support, equal to one close to zero. We define

$$
\begin{align*}
& \gamma_{\Lambda}(x, \xi)=\Phi(\xi) \Gamma\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)  \tag{4.11}\\
& \gamma_{\Lambda}^{c}(x, \xi)=\Phi(\xi)\left(1-\Gamma\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)\right)
\end{align*}
$$

We obtain two symbols of $S(1, F)$ where $F=\left\{\xi ; C^{\prime-1} \leq|\xi| \leq C^{\prime}\right\}$ for a large enough $C^{\prime}$. Moreover, since $2 x \xi+|\xi|^{1 / 2}=0$ is an equation of $\Lambda$, we see that on the domain where $\Phi \equiv 1$, $\gamma_{\Lambda}$ (resp. $\gamma_{\Lambda}^{c}$ ) cuts-off close to $\Lambda$ (resp. outside a neighborhood of $\Lambda$ ). We shall prove the following estimates.

Proposition 4.3. Let $k \geq 1, N \in \mathbb{N}$. We denote by $\kappa$ some fixed small enough positive number (say $\kappa=1 / 24$ ). There is a constant $C_{k}>0$, an element $P_{k}$ of $\mathcal{T}_{k}$ such that for any $v$ satisfying $\mathcal{E}_{k-1}(v) \leq h^{-1 / 16}$, one has for any $j$ in $j(h, C)$,

$$
\begin{align*}
&\left\|\mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}^{c}\right) \mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{3}} \\
& \leq C_{k}\left[\sqrt{h} 2^{2 j / 3-j_{+}\left(b+\frac{1}{3}(a-b)\right.} \sum_{k_{1}+k_{2} \leq k} \prod_{\ell=1}^{2} \mathcal{F}_{k_{\ell}}(v)^{\frac{1}{3}} \mathcal{E}_{k_{\ell}}(v)^{\frac{2}{3}}\right. \\
&+h 2^{j / 6-j_{+} b} h^{-4 \beta-0} \sum_{k_{1}+k_{2}+k_{3} \leq k}\left[\mathcal{F}_{k_{1}}(v) \mathcal{F}_{k_{2}}(v) \mathcal{E}_{k_{3}}(v)+\mathcal{E}_{k_{1}}(v) \mathcal{E}_{k_{2}}(v) \mathcal{E}_{k_{3}}(v)\right]  \tag{4.12}\\
&+2^{j / 4-j_{+} a} h_{j}^{5 / 6} \mathcal{F}_{k+1}(v)+2^{j / 6-j_{+} b} h^{1+\kappa} P_{k}(v) \\
&\left.+2^{j / 6-j_{+}\left(b+\frac{2}{3}(a-b)\right)} h_{j}^{N} \mathcal{F}_{k}(v)^{2 / 3} \mathcal{E}_{k}(v)^{1 / 3}\right]
\end{align*}
$$

where $h^{-4 \beta-0}$ means a bound in $C_{\theta} h^{-4 \beta-\theta}$ for any $\theta>0$.

To prove the proposition, we need to estimate the action of $\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)$ on $\mathcal{Z}_{j}^{k} w$ in various spaces.

Lemma 4.4. i) Let $k \geq 1$. There are an element $P_{k}$ of $\mathcal{T}_{k}$, a matrix $A\left(h_{j}\right)$ with uniformly bounded coefficients, a constant $C_{k}>0$ such that, for any $v$ satisfying $\mathcal{E}_{k-1}(v) \leq h^{-1 / 16}$ when $h$ stays in some interval $\left.] h^{\prime}, 1\right]$, one gets for any $h$ in that interval, any $j$ in $J(h, C)$,

$$
\begin{align*}
& \left\|\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) \mathcal{Z}_{j}^{k} w_{j}-2^{-j / 2} \sqrt{h} A\left(h_{j}\right) \mathcal{Z}_{j}^{k} \Theta_{-j}^{*} \Delta_{j}^{h} Q_{0}(W)\right\|_{L^{2}} \\
& \leq  \tag{4.13}\\
& C_{k} 2^{j / 4-j+\left(b+\frac{1}{2}\right)}\left[\sum_{k_{1}+k_{2}+k_{3} \leq k} \prod_{\ell=1}^{3} \mathcal{F}_{k_{\ell}}(v)^{1 / 3} \mathcal{E}_{k_{\ell}}(v)^{2 / 3} h^{1-0}\right. \\
& \left.\quad+h^{9 / 8} P_{k}(v)+h^{1-0} 2^{-j / 2} \mathcal{F}_{k+1}(v)\right] .
\end{align*}
$$

ii) Under the preceding assumptions, we get as well

$$
\begin{align*}
& \left\|\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) \mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{3}} \\
& \leq C_{k}\left[\sqrt{h} 2^{2 j / 3-j+\left(b+\frac{1}{3}(a-b)\right)} \sum_{k_{1}+k_{2} \leq k} \prod_{\ell=1}^{2} \mathcal{F}_{k_{\ell}}(v)^{\frac{1}{3}} \mathcal{E}_{k_{\ell}}(v)^{\frac{2}{3}}\right.  \tag{4.14}\\
& \\
& \quad+h 2^{j / 6-j+b} h^{-4 \beta-0} \sum_{k_{1}+k_{2}+k_{3} \leq k}\left[\mathcal{F}_{k_{1}}(v) \mathcal{F}_{k_{2}}(v) \mathcal{E}_{k_{3}}(v)+\mathcal{E}_{k_{1}}(v) \mathcal{E}_{k_{2}}(v) \mathcal{E}_{k_{3}}(v)\right] \\
& \\
& \left.\quad+2^{j / 4-j+a} h_{j}^{5 / 6} \mathcal{F}_{k+1}(v)+2^{j / 6-j+b} h^{1+\kappa} P_{k}(v)\right] .
\end{align*}
$$

iii) Under the preceding assumptions

$$
\begin{align*}
& \left\|\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) \mathcal{Z}_{j}^{k} w_{j}-2^{-j / 2} \sqrt{h} A\left(h_{j}\right) \mathcal{Z}_{j}^{k} \Theta_{-j}^{*} \Delta_{j}^{h} Q_{0}(W)\right\|_{L^{\infty}} \\
& \leq  \tag{4.15}\\
& \quad C_{k} h^{1-0} 2^{-j+(b-2)} \sum_{k_{1}+k_{2}+k_{3} \leq k} \mathcal{E}_{k_{1}}(v) \mathcal{E}_{k_{2}}(v) \mathcal{E}_{k_{3}}(v) \\
& \quad+C_{k} h_{j} 2^{-j_{+} b} \mathcal{E}_{k+1}(v)
\end{align*}
$$

and

$$
\begin{equation*}
\left\|2^{-j / 2} \sqrt{h} A\left(h_{j}\right) \mathcal{Z}_{j}^{k} \Theta_{-j}^{*} \Delta_{j}^{h} Q_{0}(W)\right\|_{L^{\infty}} \leq \sqrt{h} 2^{-j+\left(b-\frac{1}{2}\right)} \sum_{k_{1}+k_{2} \leq k} \mathcal{E}_{k_{1}}(v) \mathcal{E}_{k_{2}}(v) \tag{4.16}
\end{equation*}
$$

Proof. We apply $\Delta_{j}^{h}$ to equation (3.24). Denoting $\widetilde{\Delta}_{j}^{h}=\widetilde{\varphi}\left(2^{-j} h D_{x}\right)$ for a new smooth function $\widetilde{\varphi}$ satisfying $\operatorname{Supp} \widetilde{\varphi} \subset \operatorname{Supp} \varphi$, we get

$$
\begin{aligned}
\mathrm{Op}_{h}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) \Delta_{j}^{h} w= & -\sqrt{h} \Delta_{j}^{h} Q_{0}(W) \\
& +h\left[\frac{i}{2} \Delta_{j}^{h} w-i \widetilde{\Delta}_{j}^{h} w-i Z \Delta_{j}^{h} w-\Delta_{j}^{h} C_{0}(W)\right] \\
& -h^{5 / 4} \Delta_{j}^{h} R(V) .
\end{aligned}
$$

Applying $\Theta_{-j}^{*}$ and using (2.13), we get

$$
\begin{align*}
\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) w_{j}= & -\sqrt{h} 2^{-j / 2} \Theta_{-j}^{*} \Delta_{j}^{h} Q_{0}(W) \\
& +h 2^{-j / 2}\left[\frac{i}{2} \widetilde{w}_{j}-i Z w_{j}-\Theta_{-j}^{*} \Delta_{j}^{h} C_{0}(W)\right]  \tag{4.17}\\
& -2^{-j / 2} h^{5 / 4} \Theta_{-j}^{*} \Delta_{j}^{h} R(V)
\end{align*}
$$

where $\widetilde{w}_{j}=w_{j}-2 \Theta_{-j}^{*} \widetilde{\Delta}_{j}^{h} w$ satisfies the same estimates as $w_{j}$. We commute $\mathcal{Z}_{j}^{k}$ to the equation, using that

$$
\begin{aligned}
& {\left[Z, \mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)\right]=-\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)} \\
& {\left[2^{j / 2} h \partial_{x}, \mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)\right]=-2 i h_{j}\left(2^{j / 2} h \partial_{x}\right) .}
\end{aligned}
$$

We get

$$
\begin{align*}
\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)\left(\mathcal{Z}_{j}^{k} w_{j}\right)= & 2^{-j / 2} \sqrt{h} A\left(h_{j}\right) \mathcal{Z}_{j}^{k} \Theta_{-j}^{*} \Delta_{j}^{h} Q_{0}(W)  \tag{4.18}\\
& +h_{j}\left[\widetilde{B}\left(h_{j}\right) \mathcal{Z}_{j}^{k} \widetilde{w}_{j}+B\left(h_{j}\right) \mathcal{Z}_{j}^{k+1} w_{j}+C\left(h_{j}\right) \mathcal{Z}_{j}^{k} \Theta_{-j}^{*} \Delta_{j}^{h} C_{0}(W)\right] \\
& -h^{1 / 4} h_{j} D\left(h_{j}\right) \mathcal{Z}_{j}^{k} \Theta_{-j}^{*} \Delta_{j}^{h} R_{0}^{h}(V)
\end{align*}
$$

where $A\left(h_{j}\right), B\left(h_{j}\right), \widetilde{B}\left(h_{j}\right), C\left(h_{j}\right), D\left(h_{j}\right)$ are matrices with uniformly bounded coefficients.
Let us control the cubic terms in (4.18). We write $C_{0}(W)$ as $T_{0}(W, W, W)$ as in ii) of Lemma 3.1. We express $\mathcal{Z}_{j}^{k} \Theta_{-j}^{*} \Delta_{j}^{h} C_{0}(W)=\Theta_{-j}^{*} \mathcal{Z}^{k} \Delta_{j}^{h} C_{0}(W)$ from $\Theta_{-j}^{*} \Delta_{j}^{h} \mathcal{Z}^{k^{\prime}} T_{0}(W, W, W)$ for $k^{\prime} \leq k$ (changing eventually the definition of the spectral cut-off $\Delta_{j}^{h}$ ) and decompose each argument $W$ as $\sum_{j_{\ell}} \Delta_{j_{\ell}}^{h} W$. Applying estimate (3.14) with $p_{k_{1}}=p_{k_{2}}=p_{k_{3}}=6$ and writing $\|\cdot\|_{L^{6}} \leq\|\cdot\|_{L^{2}}^{1 / 3}\|\cdot\|_{L^{\infty}}^{2 / 3}$, we obtain

$$
\begin{align*}
& \left\|\Delta_{j}^{h} \mathcal{Z}^{k^{\prime}} T_{0}(W, W, W)\right\|_{L^{2}} \\
& \leq C 2^{j / 2} \sum_{j_{1}, j_{2}, j_{3}} \sum_{k_{1}+k_{2}+k_{3} \leq k^{\prime}} 2^{2 \max \left(j_{1}, j_{2}, j_{3}\right)} \mathbf{1}_{\max \left(j_{1}, j_{2}, j_{3}\right) \geq j-C}  \tag{4.19}\\
& \quad \times \prod_{\ell=1}^{3}\left\|\mathcal{Z}^{k_{\ell}} \Delta_{j_{\ell}}^{h} W\right\|_{L^{2}}^{1 / 3}\left\|\mathcal{Z}^{k_{\ell}} \Delta_{j_{\ell}}^{h} W\right\|_{L^{\infty}}^{2 / 3} .
\end{align*}
$$

Using (4.4), (4.5), we bound the last factor by

$$
2^{-\left(j_{1+}+j_{2+}+j_{3+}\right)\left[b+\frac{1}{3}(a-b)\right]} \prod_{\ell=1}^{3} \mathcal{F}_{k_{\ell}}(W)^{1 / 3} \mathcal{E}_{k_{\ell}}(W)^{2 / 3}
$$

Summing in $j_{1}, j_{2}, j_{3}$ in $J(h, C)$, we obtain

$$
\left\|\Delta_{j}^{h} \mathcal{Z}^{k^{\prime}} C_{0}(W)\right\|_{L^{2}} \leq C|\log h|^{2} 2^{j / 2-j_{+}\left(b-2+\frac{1}{3}(a-b)\right)} \times \sum_{k_{1}+k_{2}+k_{3} \leq k^{\prime}} \prod_{\ell} \mathcal{F}_{k_{\ell}}^{1 / 3} \mathcal{E}_{k_{\ell}}^{2 / 3} .
$$

Remembering that $\left\|\Theta_{-j}^{*}\right\|_{\mathcal{L}\left(L^{2}, L^{2}\right)}=O\left(2^{j / 4}\right)$ we conclude that the $L^{2}$-norm of the cubic term in the right hand side of (4.18) is bounded from above by the first term in right hand side of (4.13) (since $a-b$ is large enough for $\beta \ll 1$, according to (4.3)).

To estimate the $R_{0}$-term in (4.18), we use (3.16). We notice first that the right hand side of this inequality may be controlled from $\mathcal{E}_{k_{\ell}}(v), \mathcal{F}_{k_{\ell}}(v)$ : actually

$$
\begin{aligned}
\left\|\mathcal{Z}^{k_{\ell}}\left\langle h D_{x}\right\rangle^{\alpha+d} V\right\|_{L^{\infty}} \leq C \sum_{k_{\ell}^{\prime} \leq k_{\ell}} & {\left[\left\|\operatorname{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right)\right) \mathcal{Z}^{k_{\ell}^{\prime}} v\right\|_{L^{\infty}}\right.} \\
& \left.+\sum_{j \geq j_{0}(h, C)} 2^{j_{+}(\alpha+d)}\left\|\Delta_{j}^{h} \mathcal{Z}^{k_{\ell}^{\prime}} v\right\|_{L^{\infty}}\right],
\end{aligned}
$$

where the constant $C$ depends only on $h^{\frac{1}{16}}\left\|\left\langle h D_{x}\right\rangle^{\alpha+d} \mathcal{Z}^{(k-1)+v}\right\|_{L^{\infty}}$.
We shall take $d=b-\alpha-0$, so that the bounds (4.4) imply that the $j$-series converges and gives a bound

$$
\begin{equation*}
\left\|\mathcal{Z}^{k_{\ell}}\left\langle h D_{x}\right\rangle^{\alpha+d} V\right\|_{L^{\infty}} \leq C \mathcal{E}_{k_{\ell}}(v)|\log h| \tag{4.20}
\end{equation*}
$$

Similarly, we get

$$
\begin{equation*}
\left\|\mathcal{Z}^{k_{\ell}}\left\langle h D_{x}\right\rangle^{\alpha+d} V\right\|_{L^{2}} \leq C \mathcal{F}_{k_{\ell}}(v)|\log h| . \tag{4.21}
\end{equation*}
$$

Plugging this in (3.16), we obtain, using again that $\left\|\Theta_{-j}^{*}\right\|_{\mathcal{L}\left(L^{2}, L^{2}\right)}=O\left(2^{j / 4}\right)$,

$$
h^{1 / 4} h_{j}\left\|\mathcal{Z}_{j}^{k} \Theta_{-j}^{*} \Delta_{j}^{h} R_{0}^{h}(V)\right\|_{L^{2}} \leq C h^{3 / 4-0} h_{j}^{1 / 2} 2^{j / 2-j_{+}(b-\alpha-0)} \sum_{k_{1}+k_{2}+k_{3} \leq k} \mathcal{E}_{k_{1}}(v) \mathcal{E}_{k_{2}}(v) \mathcal{F}_{k_{3}}(v) .
$$

We notice that since $2^{j}=O\left(h^{-2 \beta}\right)$, we may bound $2^{-j_{+}(b-\alpha-0)}$ by $2^{-j_{+}\left(b+\frac{1}{2}\right)} h^{-2\left(\alpha+\frac{1}{2}+0\right) \beta}$. For $\beta$ small enough, this negative power of $h$ will be compensated consuming a $O\left(h^{1 / 8}\right)$-factor, so that we end up with a bound of the remainder in (4.18) by

$$
C h^{9 / 8} 2^{j / 4-j_{+}\left(b+\frac{1}{2}\right)} \sum_{k_{1}+k_{2}+k_{3} \leq k} \mathcal{E}_{k_{1}}(v) \mathcal{E}_{k_{2}}(v) \mathcal{F}_{k_{3}}(v)
$$

so by the $P_{k}$-term in the right hand side of (4.13).
Finally, the linear terms in the right hand side of (4.18) are bounded by the last contribution in (4.13), remembering that $w_{j}$ may be expressed from $\Delta_{j}^{h} w$ by (4.10) and that $\left\|\Theta_{-j}^{*}\right\|_{\mathcal{L}\left(L^{2}, L^{2}\right)}=$ $O\left(2^{j / 4}\right)$. This concludes the proof of $\left.i\right)$ of the lemma.
ii) To prove (4.14), let us bound the $L^{3}$-norm of the right hand side of (4.18). We express first $\mathcal{Z}_{j}^{k} \Theta_{-j}^{*} \Delta_{j}^{h} Q_{0}(W)$ from $\Theta_{-j}^{*} \Delta_{j}^{h} \mathcal{Z}^{k^{\prime}} Q_{0}(W), k^{\prime} \leq k$ write $Q_{0}(W)=B_{0}(W, W)$, decompose $W=\sum_{j_{\ell}} \Delta_{j_{\ell}}^{h} W$ in each factor and apply (3.12) with $p=3, p_{k_{1}}=p_{k_{2}}=6$. We get

$$
\begin{aligned}
\left\|\Delta_{j}^{h} \mathcal{Z}^{k^{\prime}} Q_{0}(W)\right\|_{L^{3}} \leq C 2^{j} & \sum_{j_{1}, j_{2} \in J(h, C)} \sum_{k_{1}+k_{2} \leq k^{\prime}} 2^{\frac{1}{2} \min \left(j_{1}, j_{2}\right)} \mathbf{1}_{\max \left(j_{1}, j_{2}\right) \geq j-C} \\
& \times \prod_{\ell=1}^{2}\left\|\mathcal{Z}^{k_{\ell}} \Delta_{j_{\ell}}^{h} W\right\|_{L^{2}}^{1 / 3}\left\|\mathcal{Z}^{k_{\ell}} \Delta_{j_{\ell}}^{h} W\right\|_{L^{\infty}}^{2 / 3}
\end{aligned}
$$

Using (4.4), (4.5), we see that this quantity is smaller than

$$
2^{j} \sum_{k_{1}+k_{2} \leq k}\left(\prod_{\ell=1}^{2} \mathcal{F}_{k_{\ell}}(v)^{\frac{1}{3}} \mathcal{E}_{k_{\ell}}(v)^{\frac{2}{3}}\right) 2^{-j_{+}\left(b+\frac{1}{3}(a-b)\right)} .
$$

We thus get the first term in the right hand side of (4.14), using that $\left\|\Theta_{-j}^{*}\right\|_{\mathcal{L}\left(L^{3}, L^{3}\right)}=O\left(2^{j / 6}\right)$.

Let us study next the $L^{3}$-norm of the cubic term in (4.18). We proceed as in the proof of (4.19), applying (3.14) with $p=3, p_{k_{1}}=p_{k_{2}}=6, p_{k_{3}}=\infty$. We obtain

$$
\begin{aligned}
\left\|\Delta_{j}^{h} \mathcal{Z}^{k^{\prime}} C_{0}(W)\right\|_{L^{3}} \leq C 2^{j / 2} & \sum_{j_{1}, j_{2}, j_{3}} \sum_{k_{1}+k_{2}+k_{3} \leq k^{\prime}} 2^{2 \max \left(j_{1}, j_{2}, j_{3}\right)} \mathbf{1}_{\max \left(j_{1}, j_{2}, j_{3}\right) \geq j-C} \\
& \times\left(\prod_{\ell=1}^{2}\left\|\mathcal{Z}^{k_{\ell}} \Delta_{j_{\ell}}^{h} W\right\|_{L^{2}}^{1 / 3}\left\|\mathcal{Z}^{k_{\ell}} \Delta_{j_{\ell}}^{h} W\right\|_{L^{\infty}}^{2 / 3}\right)\left\|\mathcal{Z}^{k_{3}} \Delta_{j_{\ell}}^{h} W\right\|_{L^{\infty}} .
\end{aligned}
$$

We bound the general term of this sum by
$C 2^{2 \max \left(j_{1}, j_{2}, j_{3}\right)} \mathbf{1}_{\max \left(j_{1}, j_{2}, j_{3}\right) \geq j-C} 2^{-\left(j_{1+}+j_{2+}+j_{3+}\right) b} \times\left(\mathcal{F}_{k_{1}}(v) \mathcal{F}_{k_{2}}(v)\right)^{1 / 3}\left(\mathcal{E}_{k_{1}}(v) \mathcal{E}_{k_{2}}(v)\right)^{2 / 3} \mathcal{E}_{k_{3}}(v)$.
As $2^{j} \leq C h^{-2 \beta}$, we conclude, using the convexity inequality $a^{1 / 3} b^{2 / 3} \leq(a+2 b) / 3$

$$
\left\|\Delta_{j}^{h} \mathcal{Z}^{k^{\prime}} C_{0}(W)\right\|_{L^{3}} \leq C h^{-4 \beta-0} 2^{j / 2-j_{+} b} \sum_{k_{1}+k_{2}+k_{3} \leq k}\left[\mathcal{F}_{k_{1}}(v) \mathcal{F}_{k_{2}}(v) \mathcal{E}_{k_{3}}(v)+\mathcal{E}_{k_{1}}(v) \mathcal{E}_{k_{2}}(v) \mathcal{E}_{k_{3}}(v)\right] .
$$

This gives in (4.14) a contribution to the second term in the right hand side, using again that $\left\|\Theta_{-j}^{*}\right\|_{\mathcal{L}\left(L^{3}, L^{3}\right)}$ is $O\left(2^{j / 6}\right)$.

Consider next the remainder. We estimate $\left\|\mathcal{Z}_{j}^{k} \Theta_{-j}^{*} \Delta_{j}^{h} R_{0}^{h}(V)\right\|_{L^{3}}$ from

$$
h_{j}^{-1 / 6}\left\|\mathcal{Z}_{j}^{k} \Theta_{-j}^{*} \Delta_{j}^{h} R_{0}^{h}(V)\right\|_{L^{2}}
$$

using that the expression to be bounded is spectrally supported in a ball of radius $O\left(h_{j}^{-1}\right)$. We apply next estimate (3.16) together with (4.20), (4.21). We obtain, using that $\left\|\Theta_{-j}^{*}\right\|_{\mathcal{L}\left(L^{3}, L^{3}\right)}=$ $O\left(2^{j / 6}\right)$, that the $L^{3}$-norm of the last term in (4.18) is bounded from above by

$$
C h^{1 / 4-0} h_{j}^{5 / 6} 2^{2 j / 3-j_{+} d} \sum_{k_{1}+k_{2}+k_{3} \leq k} \mathcal{E}_{k_{1}}(v) \mathcal{E}_{k_{2}}(v) \mathcal{F}_{k_{3}}(v)
$$

with $d=b-\alpha-0$. We get finally as a coefficient $2^{j / 6-j_{+} b} h^{13 / 12-0-2 \beta(\alpha+1 / 12)}$ if we use again that $2^{j}=O\left(h^{-2 \beta}\right)$. If $\beta$ is small enough, we see that the remainder in (4.18) contributes to the last term in the right hand side of (4.14).

Finally, the contribution of the linear terms in (4.18) is bounded from above by

$$
\begin{aligned}
h_{j}\left\|\mathcal{Z}_{j}^{k} \widetilde{w}_{j}\right\|_{L^{3}}+h_{j}\left\|\mathcal{Z}_{j}^{k+1} w_{j}\right\|_{L^{3}} & \leq C h_{j}^{5 / 6}\left(\left\|\mathcal{Z}_{j}^{k} \widetilde{w}_{j}\right\|_{L^{2}}+\left\|\mathcal{Z}_{j}^{k+1} w_{j}\right\|_{L^{2}}\right) \\
& \leq C 2^{j / 4-j_{+} a} h_{j}^{\frac{5}{6}} \mathcal{F}_{k+1}(v)
\end{aligned}
$$

where we have used Sobolev injection and the fact that $\mathcal{Z}_{j}^{k} \widetilde{w}_{j}, \mathcal{Z}_{j}^{k+1} w_{j}$ is spectrally supported for $h_{j}|\xi| \sim 1$, and where the gain $2^{j / 4}$ comes from $\left\|\Theta_{-j}^{*}\right\|_{\mathcal{L}\left(L^{2}, L^{2}\right)}$ when expressing $w_{j}$ from $\Delta_{j}^{h} w$ in (4.10).
iii) Let us prove (4.15) and (4.16). Applying (3.12) with $p=p_{k_{1}}=p_{k_{2}}=\infty$, we get for $k^{\prime} \leq k$,

$$
\begin{aligned}
\left\|\Delta_{j}^{h} \mathcal{Z}^{k^{\prime}} Q_{0}(W)\right\|_{L^{\infty}} \leq C 2^{j} & \sum_{j_{1}, j_{2} \in J(h, C)} \sum_{k_{1}+k_{2} \leq k^{\prime}} 2^{\frac{1}{2} \min \left(j_{1}, j_{2}\right)} \mathbf{1}_{\max \left(j_{1}, j_{2}\right) \geq j-C} \\
& \times\left\|\mathcal{Z}^{k_{1}} \Delta_{j_{1}}^{h} W\right\|_{L^{\infty}}\left\|\mathcal{Z}^{k_{2}} \Delta_{j_{2}}^{h} W\right\|_{L^{\infty}}
\end{aligned}
$$

which gives (4.16). To get (4.15), we use again (4.18). The cubic term in the right hand side of this expression is bounded using (3.14) with $p=p_{k_{1}}=p_{k_{2}}=p_{k_{3}}=\infty$ and gives the first term in the right hand side of (4.15). To estimate the $L^{\infty}$-norm of the $R_{0}$-term in (4.18) we use (3.17) with $d=b-\alpha-0$ and (4.20). The loss $2^{j+(\alpha+0)} \leq C h^{-2 \beta(\alpha+0)}$ may be absorbed by the extra $h^{1 / 4}$ factor in front of the remainder in (4.18), so that we get again a contribution bounded by the first term in the right hand side of (4.15). Finally, the linear term in (4.18) is controlled by $h_{j} 2^{-j+b} \mathcal{E}_{k+1}(v)$. This concludes the proof.

Proof of Proposition 4.3. We apply corollary A. 3 with the weight $m(x, \xi)=\langle x\rangle$. By the definition (4.11) of $\gamma_{\Lambda}^{c}, 2 x \xi+|\xi|^{1 / 2} \geq c\langle x\rangle$ on the support of $\gamma_{\Lambda}^{c}$. Consequently, for any $N$ in $\mathbb{N}$, we may find symbols $q$ in $S\left(\langle x\rangle^{-1}\right), r$ in $S(1)$ such that $\gamma_{\Lambda}^{c}=q \#\left(2 x \xi+|\xi|^{1 / 2}\right)+h_{j}^{N} r$. It follows that for any $p \geq 1$,

$$
\begin{equation*}
\left\|\mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}^{c}\right) \mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{p}} \leq C\left\|\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) \mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{p}}+h_{j}^{N}\left\|\mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{p}} \tag{4.22}
\end{equation*}
$$

Applying this with $p=3$, we may bound by (4.14) the first term in the right hand side in terms of the right hand side of (4.12). The last contribution is smaller than

$$
h_{j}^{N}\left\|\mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{3}} \leq h_{j}^{N}\left\|\mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{2}}^{2 / 3}\left\|\mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{\infty}}^{1 / 3}
$$

going back to the estimates of $w_{j}=\Theta_{-j}^{*} \Delta_{j}^{h} w$ from $\mathcal{E}_{k}(v), \mathcal{F}_{k}(v)$, we obtain the last term in the right hand side of (4.12). This concludes the proof of the proposition.

The $L^{3}$-estimate we obtained in Proposition 4.3 outside a microlocal neighborhood of $\Lambda$ will be useful as auxiliary bounds in the third step of our proof of Proposition 4.1. We also need $L^{\infty}$-estimates for $w$ cut-off outside $\Lambda$. They are given by the following

Proposition 4.5. Let $k \geq 1$. There is an element $P_{k}$ in $\mathcal{T}_{k}^{\infty}$ such that for any $v$ satisfying $\mathcal{E}_{k-1}(v) \leq h^{-1 / 16}$

$$
\begin{equation*}
\left\|\mathcal{Z}_{j}^{k} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}^{c}\right) w_{j}\right\|_{L^{\infty}} \leq C\left[h^{1 / 4} P_{k}(v)+h^{1 / 2} \mathcal{F}_{k+1}(v)\right] 2^{-j+b} . \tag{4.23}
\end{equation*}
$$

Proof. We notice first the commutation relations

$$
\begin{align*}
& {\left[t D_{t}+x D_{x}, \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}^{c}\right)\right]=\mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda, 1}^{c}\right),}  \tag{4.24}\\
& {\left[h_{j} D_{x}, \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}^{c}\right)\right]=h_{j} \operatorname{Op}_{h_{j}}\left(\gamma_{\Lambda, 2}^{c}\right),}
\end{align*}
$$

where $\gamma_{\Lambda, j}^{c}$ is in $S(1)$ with support contained in $\operatorname{Supp}\left(\gamma_{\Lambda}^{c}\right)$. This shows that, up to a modification of the definition of $\gamma_{\Lambda}^{c}$, it is enough to control $\left\|O p_{h_{j}}\left(\gamma_{\Lambda}^{c}\right) \mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{\infty}}$. Let us show

$$
\begin{equation*}
\left\|\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) \mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{\infty}} \leq C\left[h^{1 / 4} P_{k}(v)+h^{1 / 2} \mathcal{F}_{k+1}(v)\right] 2^{-j_{+} b} \tag{4.25}
\end{equation*}
$$

for some $P_{k}$ in $\mathcal{T}_{k}^{\infty}$. This will imply (4.23) using (4.22) with $p=\infty$ and $N$ large enough, since $h_{j}=O\left(h^{\sigma}\right)$ and we may estimate $\left\|\mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{\infty}} \lesssim h_{j}^{-\frac{1}{2}} \mathcal{F}_{k}(w) 2^{-j_{+} b}$ by Sobolev. We prove (4.25) estimating the $L^{\infty}$-norm of (4.18). We bound $\left\|Z^{k} \Delta_{j}^{h} Q_{0}(W)\right\|_{L^{\infty}}$ using (3.12) with $p=p_{k_{1}}=p_{k_{2}}=\infty$ and $\left\|Z^{k_{\ell}} \Delta_{j_{\ell}}^{h} w\right\|_{L^{\infty}}=O\left(\mathcal{E}_{k_{\ell}}(v) 2^{-j_{\ell+} b}\right)$. We obtain a bound in $2^{j-j_{+} b} P_{k}(v)$ for some $P_{k}$ in $\mathcal{T}_{k}^{\infty}$, which gives a contribution to the first term in the right hand side of (4.25), writing $2^{j / 2} \sqrt{h}=O\left(h^{1 / 4}\right)$ as $2^{j / 2} \leq C h^{-\beta}$. To bound the cubic term in (4.18), we apply (3.15) with $p=\infty, d=b-\alpha-0$, and (4.20), and control the loss $2^{j+(\alpha+0)}$ by a small negative power of $h$ using again $2^{j} \leq C h^{-2 \beta}$. We obtain that the cubic term in (4.18) is $O\left(h^{1 / 4} P_{k}(v)\right)$. The $R_{0}$ term of (4.18) is estimated in the same way, using (3.17), (4.20).

Finally, we must bound the linear contributions in (4.18). Their $L^{2}$-norms are

$$
O\left(h_{j} 2^{j / 4-j+b} \mathcal{F}_{k+1}(v)\right)
$$

according to the definition of $\mathcal{F}_{k+1}(v)$ and the expression $w_{j}=\Theta_{-j}^{*} \Delta_{j}^{h} w$. Moreover, they are spectrally localized at $h_{j}|\xi| \sim 1$, so that by Sobolev injection, the $L^{\infty}$-norms are bounded by the $L^{2}$-norms multiplied by $C h_{j}^{-1 / 2}$. This gives a contribution to the last term in (4.25).

Third step: Estimates on a microlocal neighborhood of $\Lambda$.
We have obtained in Proposition $4.6 L^{\infty}$-estimates for $\mathcal{Z}_{j}^{k} w_{j}$ truncated outside a neighborhood of $\Lambda$. We want here to prove similar $L^{\infty}$-estimates for $\mathcal{Z}_{j}^{k} w_{j}$ truncated close to $\Lambda$. They will be deduced from $L^{2}$-estimates for $\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) \mathcal{Z}_{j}^{k} w_{j}$ that follow from (4.18) truncated close to $\Lambda$. Let us introduce the following decomposition of the function $w=$ $\sum_{j \in J(h, C)} \Theta_{j}^{*} w_{j}$ introduced in (3.19), (3.10): we define, using notation (4.11)

$$
\begin{equation*}
w_{\Lambda}=\sum_{j \in J(h, C)} \Theta_{j}^{*} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) w_{j} \tag{4.26}
\end{equation*}
$$

and denote $W_{\Lambda}=\left(w_{\Lambda}, \bar{w}_{\Lambda}\right)$. We shall prove
Proposition 4.6. Let $k \geq 1$. There are $C>0$, an element $P_{k}$ in $\mathcal{T}_{k}$ such that for any $v$ satisfying $\mathcal{F}_{k}(v)=O\left(\varepsilon h^{-\frac{1}{4}}\right), \mathcal{E}_{k-1}(v) \leq h^{-1 / 16}$, for any $j$ in $J(h, C)$

$$
\begin{align*}
\left\|\mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) \mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{\infty}} \leq C h^{-0}[ & \sum_{k_{1}+k_{2}+k_{3} \leq k} \prod_{\ell=1}^{3} \mathcal{F}_{k_{\ell}}(v)^{\frac{1}{3}} \mathcal{E}_{k_{\ell}}(v)^{\frac{2}{3}}  \tag{4.27}\\
& \left.+h^{\frac{1}{8}} P_{k}(v)+\mathcal{F}_{k+1}(v)\right] 2^{-j_{+} b}
\end{align*}
$$

To prove the proposition, we shall use (4.18) with $Q_{0}(W)$ replaced by $Q_{0}\left(w_{\Lambda}\right)$ in the right hand side. Let us estimate the error that is done.

Lemma 4.7. For any $k \in \mathbb{N}$, there are $C>0$, an element $P_{k}$ in $\mathcal{T}_{k}$ such that for any $v$ with $\mathcal{E}_{k-1}(v) \leq h^{-1 / 16}$

$$
\begin{align*}
\left\|\mathcal{Z}^{k} \Delta_{j}^{h}\left(Q_{0}(W)-Q_{0}\left(W_{\Lambda}\right)\right)\right\|_{L^{2}} \leq & C \sqrt{h} 2^{-j_{+} b} \sum_{k_{1}+k_{2}+k_{3} \leq k} \prod_{\ell=1}^{3} \mathcal{F}_{k_{\ell}}(v)^{\frac{1}{3}} \mathcal{E}_{k_{\ell}}(v)^{\frac{2}{3}}  \tag{4.28}\\
& +h^{\frac{3}{4}} 2^{-j_{+} b} P_{k}(v) .
\end{align*}
$$

Proof. We have to bound for $j$ in $J(h, C)$,

$$
\left\|\mathcal{Z}^{k} \Delta_{j}^{h} B_{0}\left(W, W-W_{\Lambda}\right)\right\|_{L^{2}}+\left\|\mathcal{Z}^{k} \Delta_{j}^{h} B_{0}\left(W_{\Lambda}, W-W_{\Lambda}\right)\right\|_{L^{2}}
$$

Consider for instance the first term. We decompose each argument using

$$
w=\sum_{j_{\ell} \in J(h, C)} \Delta_{j_{\ell}}^{h} w=\sum_{j_{\ell} \in J(h, C)} \Theta_{j_{\ell}}^{*} w_{j_{\ell}}
$$

By using (4.26),

$$
w-w_{\Lambda}=\sum_{j_{\ell} \in J(h, C)} \Theta_{j_{\ell}}^{*} \mathrm{Op}_{h_{j_{\ell}}}\left(\gamma_{\Lambda}^{c}\right) w_{j_{\ell}}
$$

We write for $j_{1} \in J(h, C)$,

$$
\begin{align*}
\left\|\mathcal{Z}^{k_{1}} \Delta_{j_{1}}^{h} w\right\|_{L^{6}} & \leq\left\|\mathcal{Z}^{k_{1}} \Delta_{j_{1}}^{h} w\right\|_{L^{2}}^{\frac{1}{2}}\left\|\mathcal{Z}^{k_{1}} \Delta_{j_{1}}^{h} w\right\|_{L^{\infty}}^{\frac{2}{3}}  \tag{4.29}\\
& \leq 2^{-j_{1+}\left(b+\frac{1}{3}(a-b)\right)} \mathcal{F}_{k_{1}}(v)^{\frac{1}{3}} \mathcal{E}_{k_{1}}(v)^{\frac{2}{3}}
\end{align*}
$$

Moreover $\mathcal{Z}^{k_{2}} \Delta_{j_{2}}^{h}\left(w-w_{\Lambda}\right)$ may be written as

$$
\sum_{j_{2}^{\prime} ;\left|j_{2}^{\prime}-j_{2}\right| \leq N_{0}} \mathcal{Z}^{k_{2}} \Delta_{j_{2}}^{h} \Theta_{j_{2}^{\prime}}^{*} \mathrm{Op}_{h_{j_{2}^{\prime}}}\left(\gamma_{\Lambda}^{c}\right) w_{j_{2}^{\prime}}
$$

for some large enough $N_{0}$, up to a remainder whose $L^{3}$ norm is

$$
O\left(h^{\infty} 2^{-j_{2}+\left(b+\frac{2}{3}(a-b)\right)} \mathcal{F}_{k_{2}}(v)^{\frac{2}{3}} \mathcal{E}_{k_{2}}(v)^{\frac{1}{3}}\right)
$$

This follows from the fact that

$$
\Delta_{j_{2}}^{h} \Theta_{j_{2}^{\prime}}^{*} \mathrm{Op}_{h_{j_{2}^{\prime}}}\left(\gamma_{\Lambda}^{c}\right) w_{j_{2}^{\prime}}=\Theta_{j_{2}^{\prime}}^{*} \mathrm{Op}_{h_{j_{2}^{\prime}}}\left(\varphi\left(2^{j_{2}^{\prime}-j_{2}} \xi\right)\right) \mathrm{Op}_{h_{j_{2}^{\prime}}}\left(\gamma_{\Lambda}^{c}\right) w_{j_{2}^{\prime}}
$$

and that $\gamma_{\Lambda}^{c}$ is supported for $|\xi| \sim 1$.

If we apply (4.12), we conclude that, since $\left\|\Theta_{j_{2}}^{*}\right\|_{\mathcal{L}\left(L^{3}, L^{3}\right)}=O\left(2^{-j_{2} / 6}\right)$,

$$
\begin{array}{r}
\left\|\mathcal{Z}^{k_{2}} \Delta_{j_{2}}^{h}\left(w-w_{\Lambda}\right)\right\|_{L^{3}} \leq C_{0}\left[\sqrt{h} 2^{j_{2} / 2-j_{2+}\left(b+\frac{1}{3}(a-b)\right)} \sum_{k_{1}+k_{2}^{\prime} \leq k_{2}} \prod_{\ell=1}^{2} \mathcal{F}_{k_{\ell}}(v)^{\frac{1}{3}} \mathcal{E}_{k_{\ell}}(v)^{\frac{2}{3}}\right. \\
+h 2^{-j_{2+} b} h^{-4 \beta-0} \sum_{k_{1}^{\prime}+k_{2}^{\prime}+k_{3}^{\prime} \leq k_{2}}\left[\mathcal{F}_{k_{1}^{\prime}}(v) \mathcal{F}_{k_{2}^{\prime}}(v) \mathcal{E}_{k_{3}^{\prime}}(v)+\mathcal{E}_{k_{1}^{\prime}}(v) \mathcal{E}_{k_{2}^{\prime}}(v) \mathcal{E}_{k_{3}^{\prime}}(v)\right]  \tag{4.30}\\
+2^{j_{2} / 12-j_{2+}+a} h_{j_{2}}^{5 / 6} \mathcal{F}_{k_{2}+1}(v)+2^{-j_{2+}+b} h^{1+\kappa} P_{k_{2}}(v) \\
\left.+2^{-j_{2+}\left(b+\frac{2}{3}(a-b)\right)}\left(h_{j_{2}}^{N}+h^{\infty}\right) \mathcal{F}_{k_{2}}(v)^{\frac{2}{3}} \mathcal{E}_{k_{2}}(v)^{\frac{1}{3}}\right] .
\end{array}
$$

We plug (4.29), (4.30) in (3.12) with $p_{k_{1}}=6, p_{k_{2}}=3$ and we sum for $k_{1}+k_{2} \leq k, j_{1}, j_{2}$ in $J(h, C)$. We obtain that, for some $P$ in $\mathcal{T}^{k}$,

$$
\begin{aligned}
&\left\|\mathcal{Z}^{k} \Delta_{j}^{h} B_{0}\left(W, W-W_{\Lambda}\right)\right\|_{L^{2}} \\
& \leq C_{1} \sqrt{h} 2^{j-j_{+}\left(b+\frac{1}{3}(a-b)-\frac{1}{2}\right)} \sum_{k_{1}+k_{2}+k_{3} \leq k} \prod_{\ell=1}^{3} \mathcal{F}_{k_{\ell}}(v)^{\frac{1}{3}} \mathcal{E}_{k_{\ell}}(v)^{\frac{2}{3}} \\
&+C_{1} h^{1-4 \beta-0} 2^{j-j_{+} b} P(v) \\
&+C_{1} 2^{j-j+\left(b+\frac{1}{3}(a-b)\right)} h^{\frac{5}{6}} P(v) .
\end{aligned}
$$

Since $2^{j} \leq C h^{-2 \beta}$, for $\beta$ small enough and $a-b \gg 1$, we get a quantity bounded from above by (4.28). This concludes the proof.

Let us deduce from Lemma 4.7 a sharp version of (4.13).
Corollary 4.8. Let $k \geq 1$. There are $C>0$, and an element $P_{k}$ of $\mathcal{T}_{k}$ such that for any $j$ in $J(h, C)$, any $v$ with $\mathcal{E}_{k-1}(v) \leq h^{-1 / 16}$

$$
\begin{align*}
& \left\|\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) \mathcal{Z}_{j}^{k} w_{j}-2^{-j / 2} \sqrt{h} A\left(h_{j}\right) \mathcal{Z}_{j}^{k} \Theta_{-j}^{*} \Delta_{j}^{h} Q_{0}\left(W_{\Lambda}\right)\right\|_{L^{2}} \\
& \quad \leq C h^{\frac{1}{2}-0} h_{j}^{\frac{1}{2}} 2^{-j+b}\left[\sum_{k_{1}+k_{2}+k_{3} \leq k} \prod_{\ell=1}^{3} \mathcal{F}_{k_{\ell}}(v)^{\frac{1}{3}} \mathcal{E}_{k_{\ell}}(v)^{\frac{2}{3}}+h^{\frac{1}{8}} P_{k}(v)+\mathcal{F}_{k+1}(v)\right] . \tag{4.31}
\end{align*}
$$

Proof. We start from estimate (4.13). If in the left hand side, we replace $Q_{0}(W)$ by $Q_{0}\left(W_{\Lambda}\right)$, the resulting error is bounded from above by the product of (4.28) and of

$$
\sqrt{h} 2^{-j / 2}\left\|\Theta_{-j}^{*}\right\|_{\mathcal{L}\left(L^{2}, L^{2}\right)}=O\left(h_{j}^{1 / 2}\right) .
$$

We obtain a contribution to the right hand side of (4.31). On the other hand, the right hand side of (4.13) is bounded from above by the right hand side of (4.31) if we write that $2^{j / 4} h^{9 / 8} \leq 2^{j / 2} h_{j}^{1 / 2} h^{5 / 8}$. This concludes the proof.

Proof of Proposition 4.6. Let us prove that for any $j$ in $J(h, C)$

$$
\begin{align*}
& \left\|\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) \mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{2}} \\
& \quad \leq C h^{\frac{1}{2}-0} h_{j}^{\frac{1}{2}} 2^{-j_{+} b}\left[\sum_{k_{1}+k_{2}+k_{3} \leq k} \prod_{\ell=1}^{3} \mathcal{F}_{k_{\ell}}(v)^{\frac{1}{3}} \mathcal{E}_{k_{\ell}}(v)^{\frac{2}{3}}+h^{\frac{1}{8}} P_{k}(v)+\mathcal{F}_{k+1}(v)\right] \tag{4.32}
\end{align*}
$$

for some element $P_{k}$ in $\mathcal{T}_{k}$.
We notice first that since $w_{j}=\Theta_{-j}^{*} \Delta_{j}^{h} w$, the definition of $\mathcal{F}_{k}$ shows that

$$
\begin{equation*}
\left\|\mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{2}} \leq \mathcal{F}_{k}(v) 2^{j / 4-j_{+} a} . \tag{4.33}
\end{equation*}
$$

Consequently, by the gain of one power of $h_{j}$ coming from symbolic calculus, we see that

$$
\left\|\left[\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right), \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right)\right] \mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{2}}
$$

is estimated by the last term in the right hand side of (4.32). We are reduced to estimating $\left\|\mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) \mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) \mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{2}}$ which, according to (4.31) is smaller than the right hand side of (4.32) modulo the quantity

$$
\begin{equation*}
2^{-j / 2} \sqrt{h}\left\|\mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) \mathcal{Z}_{j}^{k} \Theta_{-j}^{*} \Delta_{j}^{h} Q_{0}\left(W_{\Lambda}\right)\right\|_{L^{2}} . \tag{4.34}
\end{equation*}
$$

Since $w_{\Lambda}$ is given by (4.26), we may write for $k^{\prime} \leq k$

$$
\mathcal{Z}^{k^{\prime}} w_{\Lambda}=\sum_{j \in J(h, C)} \Theta_{j}^{*}\left(\mathcal{Z}_{j}^{k^{\prime}} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) w_{j}\right)
$$

and by definition of $\mathcal{F}_{k}(v)$, and the fact that $w_{j}=\Theta_{-j}^{*} \Delta_{j}^{h} w$, we have

$$
\left\|\mathcal{Z}_{j}^{k^{\prime}} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) w_{j}\right\|_{L^{2}} \leq C 2^{j / 4-j_{+} a} \mathcal{F}_{k^{\prime}}(v)
$$

Since $w_{j}$ is microlocally supported for $h_{j}|\xi| \sim 1$, we deduce from that

$$
\left\|\mathcal{Z}_{j}^{k^{\prime}} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) w_{j}\right\|_{L^{\infty}} \leq C h^{-\frac{1}{2}} 2^{j / 2-j_{+} a} \mathcal{F}_{k^{\prime}}(v) .
$$

By Definition 2.5, this shows that the family $\left(\mathcal{Z}_{j}^{k^{\prime}} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) w_{j}\right)_{j}$ is a family of elements of $\left(h^{-\frac{1}{2}} \mathcal{B}_{\infty}^{1, a}[K]\right) \cap\left(\mathcal{B}_{2}^{0, a}[K]\right)$ for some compact $K$ of $T^{*}(\mathbb{R} \backslash\{0\}) \backslash 0$, contained in a small neighborhood of $\Lambda$, and that

$$
\left\|\mathcal{Z}_{j}^{k^{\prime}} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) w_{j}\right\|_{h^{-\frac{1}{2}} \mathcal{B}_{\infty}^{1, a}[K]},
$$

which is by definition the best constant in (2.8), is smaller than $C \mathcal{F}_{k^{\prime}}(v)$. A similar estimate holds for $\left\|\mathcal{Z}_{j}^{k^{\prime}} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) w_{j}\right\|_{\mathcal{B}_{2}^{0, a}[K]}$. We shall now prove that $Q_{0}\left(W_{\Lambda}\right)$ is microlocally supported outside a neighborhood of $\Lambda$.

Let us express $\mathcal{Z}^{k} Q_{0}\left(W_{\Lambda}\right)$ as a combination of terms $B_{0}\left(\mathcal{Z}^{k_{1}} W_{\Lambda}, \mathcal{Z}^{k_{2}} W_{\Lambda}\right), k_{1}+k_{2} \leq k$, so as a combination of expressions deduced from (3.7).

$$
\begin{align*}
& \operatorname{Op}_{h}\left(a_{0}(\xi)\right)\left[\left(\operatorname{Op}_{h}\left(a_{1}(\xi)\right) \mathcal{Z}^{k_{1}} w_{\Lambda}\right)\left(\operatorname{Op}_{h}\left(a_{2}(\xi)\right) \mathcal{Z}^{k_{2}} w_{\Lambda}\right)\right], \\
& \operatorname{Op}_{h}\left(a_{0}(\xi)\right)\left[\left(\operatorname{Op}_{h}\left(a_{1}(\xi)\right) \mathcal{Z}^{k_{1}} w_{\Lambda}\right)\left(\operatorname{Op}_{h}\left(a_{2}(\xi)\right) \mathcal{Z}^{k_{2}} \bar{w}_{\Lambda}\right)\right],  \tag{4.35}\\
& \operatorname{Op}_{h}\left(a_{0}(\xi)\right)\left[\left(\operatorname{Op}_{h}\left(a_{1}(\xi)\right) \mathcal{Z}^{k_{1}} \bar{w}_{\Lambda}\right)\left(\operatorname{Op}_{h}\left(a_{2}(\xi)\right) \mathcal{Z}^{k_{2}} \bar{w}_{\Lambda}\right)\right],
\end{align*}
$$

where $a_{0}, a_{1}, a_{2}$ are homogeneous of non negative order. We have just seen that $\mathcal{Z}^{k^{\prime}} w_{\Lambda}$ is in $\left(h^{-\frac{1}{2}} \widetilde{\mathcal{B}}_{\infty}^{1, a}[K]\right) \cap\left(\widetilde{\mathcal{B}}_{2}^{0, a}[K]\right)$ with norm in that space bounded from above by $C \mathcal{F}_{k^{\prime}}(v)$. It follows from Proposition 2.12 and the fact that $a$ is large enough relatively to $b$ that the first (resp. second, resp. third) expression (4.35) belongs to $h^{-\frac{1}{2}} \widetilde{\mathcal{B}}_{2}^{1, b}\left[K_{2}\right]$ (resp. $h^{-\frac{1}{2}} \widetilde{\mathcal{B}}_{2}^{1, b}\left[K_{0}\right]$, resp. $\left.h^{-\frac{1}{2}} \widetilde{\mathcal{B}}_{2}^{1, a}\left[K_{-2}\right]\right)$ where $K_{2}\left(\right.$ resp. $K_{0}$, resp. $\left.K_{-2}\right)$ is a compact subset of $T^{*}(\mathbb{R} \backslash\{0\})$ contained in a small neighborhood of $2 \cdot \Lambda$ (resp. $0 \cdot \Lambda$, resp. $-2 \cdot \Lambda$ ), and that the norm of these functions in those spaces is $O\left(\mathcal{F}_{k_{1}}(v) \mathcal{F}_{k_{2}}(v)\right)$. Consequently $\mathcal{Z}_{j}^{k} \Theta_{-j}^{*} \Delta_{j}^{h} Q_{0}\left(W_{\Lambda}\right)$ is microlocally supported far away from $\Lambda$. When we apply a $\mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right)$ cut-off as in (4.34), we gain a $O\left(h_{j}^{\infty}\right)=O\left(h^{\infty}\right)$ factor. We conclude that (4.34) is bounded from above $h^{N} \sum_{k_{1}+k_{2} \leq k} \mathcal{F}_{k_{1}}(v) \mathcal{F}_{k_{2}}(v)$ so that (4.34) is controlled by the $h^{1 / 8}$-term in (4.32).

To finish the proof of Proposition 4.6, we are left with showing
Lemma 4.9. Assume that (4.32) holds. Then estimate (4.27) holds as well.

Proof. The definition of $\mathcal{F}_{k}(v)$ and the fact that $w_{j}=\Theta_{-j}^{*} \Delta_{j}^{h} v$ implies that

$$
\begin{equation*}
\left\|\mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) \mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{2}} \leq C 2^{\frac{j}{4}-j_{+} a} \mathcal{F}_{k}(v) . \tag{4.36}
\end{equation*}
$$

Since $\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)=(\xi-\mathrm{d} \omega(x)) g(x, \xi)$ for some elliptic symbol $g$, on a neighborhood of the support of $\gamma_{\Lambda}$, we deduce from (4.32), (4.36) and symbolic calculus that

$$
\begin{equation*}
\left\|\mathrm{Op}_{h_{j}}(\xi-\mathrm{d} \omega(x)) \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) \mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{2}} \leq h_{j}^{\frac{1}{2}} h^{\frac{1}{2}-0} M 2^{-j+b} \tag{4.37}
\end{equation*}
$$

where

$$
M=C\left[\sum_{k_{1}+k_{2}+k_{3} \leq k} \prod_{\ell=1}^{3} \mathcal{F}_{k_{\ell}}(v)^{\frac{1}{3}} \mathcal{E}_{k_{\ell}}(v)^{\frac{2}{3}}+h^{\frac{1}{8}} P_{k}(v)+\mathcal{F}_{k+1}(v)\right]
$$

for some $P_{k}$ in $\mathcal{T}_{k}$. We may rewrite (4.36) and (4.37) as

$$
\begin{aligned}
& \left\|e^{-i \omega(x) / h_{j}} \operatorname{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) \mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{2}} \leq C 2^{\frac{j}{4}-j_{+} a} \mathcal{F}_{k}(v) \\
& \left\|\left(h_{j} D_{x}\right)\left(e^{-i \omega(x) / h_{j}} \operatorname{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) \mathcal{Z}_{j}^{k} w_{j}\right)\right\|_{L^{2}} \leq C h_{j}^{\frac{1}{2}} h^{\frac{1}{2}-0} M 2^{-j_{+} b} .
\end{aligned}
$$

Using that $\|f\|_{L^{\infty}}=O\left(\|f\|_{L^{2}}^{\frac{1}{2}}\left\|D_{x} f\right\|_{L^{2}}^{\frac{1}{2}}\right)$, we get

$$
\begin{aligned}
\left\|\mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) \mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{\infty}} & \leq C h^{-0} 2^{\frac{j}{4}-j_{+} \frac{a+b}{2}}\left(\mathcal{F}_{k}(v) M\right)^{\frac{1}{2}} \\
& \leq C h^{-0} 2^{-j_{+} b} M
\end{aligned}
$$

This implies (4.27).

Proof of Proposition 4.1: We combine estimates (4.23) and (4.27). We obtain

$$
\begin{array}{r}
\left\|\mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{\infty}} \leq C h^{-0}\left[\sum_{k_{1}+k_{2}+k_{3} \leq k \ell=1} \prod_{k_{\ell}}^{3} \mathcal{F}_{k^{\prime}}(v)^{\frac{1}{3}} \mathcal{E}_{k_{\ell}}(v)^{\frac{2}{3}}\right.  \tag{4.38}\\
\left.+h^{\frac{1}{8}} P_{k}(v)+\mathcal{F}_{k+1}(v)\right] 2^{-j_{+} b}
\end{array}
$$

for any $j$ in $J(h, C)$ and some $P_{k}$ in $\mathcal{T}_{k}$, assuming an a priori bound $\mathcal{E}_{k-1}(v) \leq h^{-1 / 16}$.
We assume that (4.8) holds and that (4.9) has been proved up to order $k-1$. Consequently, by (ii) of Lemma 3.3, we know that

$$
\begin{aligned}
\sup _{j}\left(2^{j+b}\left\|\Delta_{j}^{h} \operatorname{Op}_{h}\left(\left(1-\varphi_{0}\right)\left(h^{2 \beta} \xi\right)\right) \mathcal{Z}^{k} v\right\|_{L^{\infty}}\right) \leq C_{k} \varepsilon h^{\frac{7}{8}} \\
\| \operatorname{Op}_{h}\left(\varphi_{0}\left(h^{-2(1-\sigma)} \xi\right) \mathcal{Z}^{k} v \|_{L^{\infty}} \leq C_{k} \varepsilon h^{\frac{7}{16}-\sigma}\right.
\end{aligned}
$$

where $C_{K}$ depends only on $A_{0}, \ldots, A_{k}$.
On the other hand, (4.38) gives a control of $\left\|\Delta_{j}^{h} \mathcal{Z}^{k} v\right\|_{L^{\infty}}$ for $j \in J(h, C)$. Going back to the definition (4.4) of $\mathcal{E}_{k}(v)$ we obtain

$$
\begin{align*}
& \mathcal{E}_{k}(v) \leq C h^{-0}\left[\sum_{k_{1}+k_{2}+k_{3} \leq k} \prod_{\ell=1}^{3} \mathcal{F}_{k_{\ell}}(v)^{\frac{1}{3}} \mathcal{E}_{k_{\ell}}(v)^{\frac{2}{3}}\right]  \tag{4.39}\\
&+h^{\frac{1}{16}}\left[P_{k}(v)+C_{k} \varepsilon\right]+h^{-0} \mathcal{F}_{k+1}(v)
\end{align*}
$$

Let us deduce from that that (4.9) holds at rank $k$. By the assumption (4.8) and the fact that by (4.1) $\delta_{k}^{\prime}>\delta_{k+1}$, we may bound $h^{-0} \mathcal{F}_{k+1}(v)$ by $\varepsilon A_{k}^{\prime} h^{-\delta_{k}^{\prime}}$ for some $A_{k}^{\prime}>0$ depending only on $A_{k+1}$. The same is true for $h^{\frac{1}{16}} C_{k} \varepsilon$, with $A_{k}^{\prime}$ depending only on $A_{0}, \ldots, A_{k+1}$. Consider the $h^{\frac{1}{16}} P_{k}(v)$ contribution. By definition of the class $\mathcal{T}_{k}$ and (4.7), this term has modulus bounded from above by quantities of the form

$$
\begin{equation*}
h^{\frac{1}{16}} \mathcal{E}_{k_{1}}(v) \cdots \mathcal{E}_{k_{\ell^{\prime}}}(v) \mathcal{F}_{k_{\ell^{\prime}+1}}(v) \cdots \mathcal{F}_{k_{\ell+1}}(v) \tag{4.40}
\end{equation*}
$$

where $\ell^{\prime} \leq \ell \leq 4, k_{1}+\cdots+k_{\ell} \leq k$. Assume first that one of the $k_{j}, 1 \leq j \leq \ell^{\prime}$, is equal to $k$, so that the other ones equal 0 . We obtain, according to assumption (4.8) a bound in $C h^{\frac{1}{16}-\left(\ell^{\prime}-1\right) \delta_{0}^{\prime}-\left(\ell-\ell^{\prime}+1\right) \delta_{1}} \times \mathcal{E}_{k}(v)$, with a constant $C$ depending only on $\widetilde{A}_{0}, A_{1}$.

By (4.1) and (4.2) this is smaller than $C \mathcal{E}_{k}(v) h^{\frac{1}{32}}$ with a constant $C$ depending only on $\widetilde{A}_{0}, A_{1}$. On the other hand, if all $k_{j}, 0 \leq j \leq \ell^{\prime}$, are strictly smaller than $k$, we may apply the induction hypothesis to estimate $\mathcal{E}_{k_{j}}(v)$ and (4.8) to control $\mathcal{F}_{k_{j+1}}(v)$. We obtain for (4.40) a bound in $C \varepsilon h^{\frac{1}{16}-\delta_{k}^{\prime}}$, according to the first inequality (4.1), where the constant depends only on $\widetilde{A}_{0}, A_{0}, \ldots, A_{k}$.

Let us study now the first term in the right hand side of (4.39). When $k_{1}<k, k_{2}<k, k_{3}<k$, we write

$$
\begin{align*}
h^{-0} \prod_{\ell=1}^{3} \mathcal{F}_{k_{\ell}}(v)^{\frac{1}{3}} \mathcal{E}_{k_{\ell}}(v)^{\frac{2}{3}} \leq \frac{h^{-0}}{3}( & \mathcal{F}_{k_{1}}(v) \mathcal{E}_{k_{2}}(v) \mathcal{E}_{k_{3}}(v)  \tag{4.41}\\
& \left.+\mathcal{E}_{k_{1}}(v) \mathcal{F}_{k_{2}}(v) \mathcal{E}_{k_{3}}(v)+\mathcal{E}_{k_{1}}(v) \mathcal{E}_{k_{2}}(v) \mathcal{F}_{k_{3}}(v)\right) .
\end{align*}
$$

By (4.8), the fact that (4.9) is assumed to hold for $k_{\ell}<k$ and the first inequality (4.1), we get that (4.41) is $O\left(\varepsilon h^{-\delta_{k}^{\prime}}\right)$ with a constant depending only on $\widetilde{A}_{0}, A_{1}, \ldots, A_{k+1}$. Finally, we are left with studying the first term in the right hand side of (4.39) when one the $k_{j}$ is equal to $k$, i.e.

$$
h^{-0} \mathcal{E}_{k}(v)^{\frac{2}{3}} \mathcal{F}_{k}(v)^{\frac{1}{3}} \mathcal{F}_{0}(v)^{\frac{2}{3}} \mathcal{E}_{0}(v)^{\frac{4}{3}} \leq \frac{2}{3} \delta \mathcal{E}_{k}(v)+\frac{1}{3} \delta^{-2} h^{-0} \mathcal{F}_{k}(v) \mathcal{F}_{0}(v)^{2} \mathcal{E}_{0}(v)^{4}
$$

for any $\delta>0$ (where in the right-hand side, $h^{-0}$ denotes $h^{-3 \theta}$ if in the left hand side $h^{-0}$ stands for $h^{-\theta}$ with $\theta>0$ small). The last term in the above inequality is $O\left(\varepsilon h^{-\delta_{k}^{\prime}}\right)$ according to assumption (4.8) and the second inequality (4.1), with a constant depending only on $\widetilde{A}_{0}, A_{1}, \ldots, A_{k+1}$. Summing up, we have obtained

$$
\mathcal{E}_{k}(v) \leq\left[\frac{2}{3} \delta+C h^{\frac{1}{32}}\right] \mathcal{E}_{k}(v)+\varepsilon A_{k}^{\prime} h^{-\delta_{k}^{\prime}}
$$

from which (4.9) at rank $k$ follows if $h$ and $\delta$ are taken small enough.

## 5 Decomposition of the solution in oscillating terms

The goal of this subsection is to give a description of the component $w$ in the decomposition (3.19) of $v$ in terms of oscillating contributions. More precisely, we expect $w$ to be a sum of a main term, oscillating along the phase $\omega$ (i.e. a term which is a lagrangian distribution along $\Lambda$ ), of $O(\sqrt{h})$ terms, coming from the quadratic part of the nonlinearity, that will oscillate along the phases $\pm 2 \omega$ (so, which are associated to the lagrangians $\pm 2 \Lambda$ ), of $O(h)$ terms, coming from the cubic part of the nonlinearity, oscillating along the phases $\pm 3 \omega$, $\pm \omega$, and a remainder. Moreover, we shall need, in preparation for next subsection, to get an explicit expression for contributions oscillating on $\pm 2 \Lambda$.

We consider a solution $v$ of (3.6) satisfying for $h$ in some interval $] h^{\prime}, h_{0}$ ] the a priori estimate (4.9) for $k^{\prime} \leq \frac{s}{2}+N_{1}$ for some fixed $N_{1} \ll s$. In particular, for $k \leq \frac{s}{2}+N_{1}$,

$$
\begin{equation*}
\left\|\Delta_{j}^{h} \mathcal{Z}^{k} v\right\|_{L^{\infty}} \leq \varepsilon A_{k}^{\prime} h^{-\delta_{k}^{\prime}} 2^{-j_{+} b} \tag{5.1}
\end{equation*}
$$

for $j \in J(h, C)$. In this section, we shall denote by $K$ compact subsets of $T^{*}(\mathbb{R} \backslash\{0\})$ contained in a small neighborhood of one of the lagrangians $\ell \cdot \Lambda, \ell \neq 0$, by $L$ compact subsets of $T^{*}(\mathbb{R} \backslash\{0\})$ and by $F$ closed subsets of $T^{*} \mathbb{R}$ whose second projection is compact in $\mathbb{R} \backslash\{0\}$.

We first obtain a rough decomposition of $v$.
Lemma 5.1. One may write $v=v_{L}+w_{\Lambda}+w_{\Lambda^{c}}+v_{H}$, where $v_{L}, v_{H}$ are defined in (3.19) and, for some compact subset $K$ of $T^{*}(\mathbb{R} \backslash\{0\})$, lying in a small enough neighborhood of $\Lambda$ and intersecting $\Lambda$, some closed set $F$ as above, $\mathcal{Z}^{k} w_{\Lambda}, 0 \leq k \leq \frac{s}{2}+N_{1}$ is an $O(\varepsilon)$ element of $h^{-\delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{\Lambda}^{0, b-2}[K]$ and $\mathcal{Z}^{k} w_{\Lambda^{c}}, 0 \leq k \leq \frac{s}{2}+N_{1}$ is an $O(\varepsilon)$ element of $h^{\frac{1}{2}-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b-2}[F]+$ $h^{1-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{-1, b-2}[F]$. Moreover, $\operatorname{Op}_{h}(x \xi) \mathcal{Z}^{k} w_{\Lambda^{c}}$ is in $h^{\frac{1}{2}-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{1, b-2}[F]+h^{1-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b-2}[F]$.

Proof. We have written in (3.19) $v=v_{L}+w+v_{H}$ and using notations (4.10), we may decompose $w=\sum_{j \in J(h, C)} \Theta_{j}^{*} w_{j}$. Recall definition (4.11) of symbol $\gamma_{\Lambda}$ and set

$$
w_{j, \Lambda}=\mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) w_{j}, \quad j \in J(h, C)
$$

so that $\mathcal{Z}^{k} w_{\Lambda}=\sum_{j \in J(h, C)} \Theta_{j}^{*}\left(\mathcal{Z}_{j}^{k} w_{j, \Lambda}\right)$. Since, when commuting $\mathcal{Z}_{j}^{k}$ to $\operatorname{Op}_{h_{j}}\left(\gamma_{\Lambda}\right)$, we get expressions of the form $\operatorname{Op}_{h_{j}}\left(\tilde{\gamma}_{\Lambda}^{k^{\prime}}\right) \mathcal{Z}_{j}^{k^{\prime}}$ with $k^{\prime} \leq k$ and $\tilde{\gamma}_{\Lambda}^{k^{\prime}}$ a new symbol, we deduce from estimates (4.9) that $\mathcal{Z}^{k} w_{\Lambda}$ belongs to $h^{-\delta_{k}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b}[K]$ for a compact set $K$ satisfying the conditions of the statement if $\gamma_{\Lambda}$ is supported in a small enough neighborhood of $\Lambda$. Moreover, $\mathcal{Z}^{k} w_{\Lambda}$ is $O(\varepsilon)$ in the preceding space. If we use symbolic calculus, estimates (4.15), (4.16) and the assumed a priori estimates (4.9) together with (4.1), we get

$$
\begin{equation*}
\left\|\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) \mathcal{Z}_{j}^{k} w_{j, \Lambda}\right\|_{L^{\infty}} \leq C \varepsilon 2^{-j_{+}(b-2)} h^{-\delta_{k+1}^{\prime}}\left[h^{\frac{1}{2}}+h_{j}\right] \tag{5.2}
\end{equation*}
$$

i.e. $\left(\mathcal{Z}_{j}^{k} w_{j, \Lambda}\right)_{j}$ belongs to $h^{-\delta_{k+1}^{\prime}} L^{\infty} I_{\Lambda}^{0, b-2}$ and is of size $O(\varepsilon)$ in that space. This gives the statement concerning $w_{\Lambda}$ of the Lemma.

Set $w_{j, \Lambda^{c}}=\operatorname{Op}_{h_{j}}\left(\gamma_{\Lambda}^{c}\right) w_{j}$ so that $w_{\Lambda^{c}}=\sum_{j \in J(h, C)} \Theta_{j}^{*} w_{j, \Lambda^{c}}$. We use (4.22) with $p=\infty$. We estimate the first term in the right hand side of this inequality using (4.15), (4.16), the bounds (4.9) together with inequalities (4.1). We get

$$
\begin{equation*}
\left\|\mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}^{c}\right) \mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{\infty}} \leq C \varepsilon 2^{-j_{+}(b-2)} h^{-\delta_{k+1}^{\prime}}\left(h^{\frac{1}{2}}+h_{j}\right)+C_{N} \varepsilon h_{j}^{N} h^{-\delta_{k}^{\prime}} 2^{-j_{+} b}, \tag{5.3}
\end{equation*}
$$

where the last term has been estimated from (5.1).
If $N$ is large enough, since $h_{j} \leq C h^{\sigma}$ we get that $\mathcal{Z}^{k} w_{\Lambda^{c}}$ is in

$$
h^{\frac{1}{2}-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b-2}[F]+h^{1-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{-1, b-2}[F]
$$

and is $O(\varepsilon)$ in that space (where $F$ is a closed set as described before the statement of Lemma 5.1).

To study $\mathrm{Op}_{h}(x \xi) w_{\Lambda^{c}}$, we write

$$
\mathrm{Op}_{h}(x \xi) w_{\Lambda^{c}}=\sum_{j \in J(h, C)} 2^{\frac{j}{2}} \Theta_{j}^{*} \mathrm{Op}_{h_{j}}(x \xi) w_{j, \Lambda^{c}} .
$$

By symbolic calculus, we may write $\mathrm{Op}_{h_{j}}(x \xi) w_{j, \Lambda^{c}}$ from

$$
\mathrm{Op}_{h_{j}}\left(\tilde{\gamma}_{\Lambda^{c}}\right) w_{j}, \quad \operatorname{Op}_{h_{j}}\left(\gamma_{\Lambda^{c}}\right)\left(\mathrm{Op}_{h_{j}}(x \xi) w_{j}\right)
$$

where $\tilde{\gamma}_{\Lambda^{c}}$ is a cut-off with support contained in the one of $\gamma_{\Lambda^{c}}$. The $L^{\infty}$-norm of the action of $\mathcal{Z}_{j}^{k}$ on the first of these expressions is bounded like (5.3). The second expression may be written from

$$
\mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda^{c}}\right) \mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) w_{j}, \quad \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda^{c}}\right) \mathrm{Op}_{h_{j}}\left(|\xi|^{\frac{1}{2}}\right) w_{j} .
$$

The $L^{\infty}$-norm of the action of $\mathcal{Z}_{j}$ on the last term is bounded using (5.3) by the right hand side of this inequality. For the first term, we use again (4.15), (4.16) as in the proof of (5.2) to get a similar upper bound. This concludes the proof.

The decomposition $w=w_{\Lambda}+w_{\Lambda^{c}}$, in terms of a contribution $w_{\Lambda}$ localized close to $\Lambda$ and another one $w_{\Lambda^{c}}$ supported outside a neighborhood of $\Lambda$ is not precise enough for our purposes. We need to refine it, writing $w_{\Lambda^{c}}$ as a sum of terms oscillating on the lagrangians $\pm 2 \Lambda$, of size of order $\sqrt{h}$, and of a remainder that is $O(h)$. Moreover, we need also to check that $w_{\Lambda}$ is in $h^{-2 \delta_{k+1}^{\prime}} L^{\infty} \tilde{J}_{\Lambda}^{0, b^{\prime}}[K]$. This is the goal of next proposition, that will be proved plugging the decomposition of Lemma 5.1 in the equation (3.23) satisfied by $w$, written under the form

$$
\begin{equation*}
\mathrm{Op}_{h}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) w=-\sqrt{h} Q_{0}(W)+h\left[\frac{i}{2} w-i Z w-C_{0}(W)\right]-h^{\frac{5}{4}} R(V) \tag{5.4}
\end{equation*}
$$

Proposition 5.2. Let $b^{\prime}<b-5$ and $N_{0}<N_{1}$ such that $\left(N_{1}-N_{0}-1\right) \sigma \geq 1$. We may write the first decomposition of $w$

$$
\begin{equation*}
w=w_{\Lambda}+\sqrt{h}\left(w_{2 \Lambda}+w_{-2 \Lambda}\right)+h g \tag{5.5}
\end{equation*}
$$

where, for any $k \leq \frac{s}{2}+N_{0}, \mathcal{Z}^{k} w_{ \pm 2 \Lambda}$ is a $O(\varepsilon)$ element of $h^{-2 \delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{ \pm 2 \Lambda}^{2, b^{\prime}+\frac{3}{2}}\left[K_{ \pm 2}\right], \mathcal{Z}^{k} w_{\Lambda}$ is an $O(\varepsilon)$ element of $h^{-\delta_{k+1}^{\prime}} L^{\infty} \tilde{J}_{\Lambda}^{0, b^{\prime}}[K], \mathcal{Z}^{k} g$ is a $O(\varepsilon)$ element of $h^{-3 \delta_{k+1+N_{1}-N_{0}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}}[F]$ and $\mathcal{Z}^{k} \mathrm{Op}_{h}(x \xi) g$ is an $O(\varepsilon)$ element of $h^{-3 \delta_{k+1+N_{1}-N_{0}} \tilde{\mathcal{B}}_{\infty}^{1, b^{\prime}}}[F]$, for some compact subsets $K_{ \pm 2}$ of $T^{*}(\mathbb{R} \backslash\{0\}) \backslash 0$ contained in small neighborhoods of $\pm 2 \Lambda$, some closed subset $F$ of $T^{*} \mathbb{R}$ whose second projection is compact in $\mathbb{R} \backslash\{0\}$. Moreover, $w_{ \pm 2 \Lambda}$ are given by

$$
\begin{gather*}
w_{2 \Lambda}=-i(1-\chi)\left(x h^{-\beta}\right) \frac{1+\sqrt{2}}{4}|\mathrm{~d} \omega(x)| w_{\Lambda}^{2},  \tag{5.6}\\
w_{-2 \Lambda}=-i(1-\chi)\left(x h^{-\beta}\right) \frac{1-\sqrt{2}}{4}|\mathrm{~d} \omega(x)| \bar{w}_{\Lambda}^{2}
\end{gather*}
$$

where $\chi \in C_{0}^{\infty}(\mathbb{R}), \chi \equiv 1$ close to zero has small enough support.

In order to prove the proposition, we shall compute the main contribution to $Q_{0}(W)$ obtained when plugging inside (3.7) the decomposition $w=w_{\Lambda}+w_{\Lambda^{c}}$ obtained in Lemma 5.1. We make at the same time a similar (and more precise) computation when one knows that an expansion of the form (5.5) holds.

Lemma 5.3. $i$ ) Assume that $w=w_{\Lambda}+w_{\Lambda^{c}}$, where for all $k \leq \frac{s}{2}+N_{1}, \mathcal{Z}^{k} w_{\Lambda}$ (resp. $\left.\mathcal{Z}^{k} w_{\Lambda^{c}}\right)$ is an $O(\varepsilon)$ element of $h^{-\delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{\Lambda}^{0, b-2}[K]$ (resp. of $h^{\frac{1}{2}-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b-2}[F]+h^{1-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{-1, b-2}[F]$ such that $\mathcal{Z}^{k} \mathrm{Op}_{h}(x \xi) w_{\Lambda^{c}}$ is in $\left.h^{\frac{1}{2}-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{1, b-2}[F]+h^{1-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b-2}[F]\right)$. Denote by $b^{\prime}$ any number $b^{\prime}<b-5$. Then, there are functions $\widetilde{w}_{ \pm 2 \Lambda}$ such that for $k \leq \frac{s}{2}+N_{1}$

$$
\mathcal{Z}^{k} \widetilde{w}_{ \pm 2 \Lambda} \text { is } O(\varepsilon) \text { in } h^{-2 \delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{ \pm 2 \Lambda}^{3, b^{\prime}+\frac{3}{2}}\left[K_{ \pm 2}\right]
$$

so that

$$
\begin{align*}
\sum_{j \in J(h, C)} \Delta_{j}^{h} Q_{0}(W) & =\widetilde{w}_{2 \Lambda}+\widetilde{w}_{-2 \Lambda}+\sqrt{h} \tilde{g}_{2}, \\
\sum_{j \in J(h, C)} \Delta_{j}^{h} \sqrt{h} C_{0}(W) & =\sqrt{h} \tilde{g}_{3} \tag{5.7}
\end{align*}
$$

where for any $k \leq \frac{s}{2}+N_{1}$,

$$
\mathcal{Z}^{k} \tilde{g}_{2}, \mathcal{Z}^{k} \mathrm{Op}_{h}(x \xi) \tilde{g}_{2} \in h^{-2 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{1, b^{\prime}+\frac{1}{2}}[F], \quad \mathcal{Z}^{k} \tilde{g}_{3}, \mathcal{Z}^{k} \mathrm{Op}_{h}(x \xi) \tilde{g}_{3} \in h^{-3 \delta_{k+1}^{\prime}-0} \tilde{\mathcal{B}}_{\infty}^{1, b^{\prime}+\frac{1}{2}}[F]
$$

for some new closed subset $F$ of $T^{*} \mathbb{R}$. Moreover one may write

$$
\begin{gather*}
\widetilde{w}_{2 \Lambda}=-i(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{3}{2}} \frac{\sqrt{2}}{4} w_{\Lambda}^{2},  \tag{5.8}\\
\widetilde{w}_{-2 \Lambda}=-i(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{3}{2}} \frac{\sqrt{2}}{4} \bar{w}_{\Lambda}^{2},
\end{gather*}
$$

for some $\chi \in C_{0}^{\infty}(\mathbb{R})$, $\chi \equiv 1$ close to zero, with small enough support.
ii) Assume one is given a decomposition of $w$ of the form (5.5) and denote by $b^{\prime}$ any number $b^{\prime}<b-8$. Then there are elements $\widetilde{w}_{ \pm 2 \Lambda}$ such that for $0 \leq k \leq \frac{s}{2}+N_{0}, \mathcal{Z}^{k} \widetilde{w}_{ \pm 2 \Lambda}$ is $O(\varepsilon)$ in $h^{-2 \delta_{k+1}^{\prime}} L^{\infty} \tilde{J}_{ \pm 2 \Lambda}^{3, b^{\prime}+3 / 2}\left[K_{ \pm 2}\right]$, and for $\ell \in\{ \pm 1, \pm 3\}$ elements $\widetilde{w}_{\ell \Lambda}$, such that, for any $0 \leq k \leq \frac{s}{2}+N_{0}, \mathcal{Z}^{k} \widetilde{w}_{\ell \Lambda}$ is $O(\varepsilon)$ in $h^{-3 \delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{\ell \Lambda}^{3, b^{\prime}+3 / 2}\left[K_{\ell}\right]$ so that

$$
\begin{equation*}
\sum_{j \in J(h, C)} \Delta_{j}^{h}\left[Q_{0}(W)+\sqrt{h} C_{0}(W)\right]=\widetilde{w}_{2 \Lambda}+\widetilde{w}_{-2 \Lambda}+\sqrt{h}\left(\widetilde{w}_{3 \Lambda}+\widetilde{w}_{\Lambda}+\widetilde{w}_{-\Lambda}+\widetilde{w}_{-3 \Lambda}\right)+h \tilde{g}, \tag{5.9}
\end{equation*}
$$

where $\mathcal{Z}^{k} \tilde{g}$ is in $h^{-4 \delta_{k+P}^{\prime}} \tilde{\mathcal{B}}^{1, b^{\prime}+\frac{1}{2}}[F]$ and $\mathcal{Z}^{k} \mathrm{Op}_{h}(x \xi) \tilde{g}$ is in $h^{-4 \delta_{k+P}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{1, b^{\prime}}[F]$ with $P=N_{1}-$ $N_{0}+1$.

Moreover, $\widetilde{w}_{\Lambda}$ is given in terms of $w_{\Lambda}$ and of the functions in (5.6) by

$$
\begin{align*}
\widetilde{w}_{\Lambda}= & \frac{i}{2}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{3}{2}} \bar{w}_{\Lambda}\left(w_{2 \Lambda}-\bar{w}_{-2 \Lambda}\right) \\
& +\frac{1}{4}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega(x)|^{\frac{5}{2}}\left|w_{\Lambda}\right|^{2} w_{\Lambda}, \tag{5.10}
\end{align*}
$$

for some $\chi \in C_{0}^{\infty}(\mathbb{R})$, $\chi \equiv 1$ close to zero, with small enough support, and $\widetilde{w}_{3 \Lambda}, \widetilde{w}_{-\Lambda}, \widetilde{w}_{-3 \Lambda}$ have similar expressions

$$
\begin{align*}
\widetilde{w}_{3 \Lambda}= & \frac{i}{2}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{3}{2}} w_{\Lambda}\left(\mu_{3}^{\prime} w_{2 \Lambda}+\mu_{3}^{\prime \prime} \bar{w}_{-2 \Lambda}\right) \\
& +(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega(x)|^{\frac{5}{2}} \mu_{3}^{\prime \prime \prime} w_{\Lambda}^{3}, \\
\widetilde{w}_{-\Lambda}= & \frac{i}{2}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{3}{2}} w_{\Lambda}\left(\mu_{-1}^{\prime} w_{-2 \Lambda}+\mu_{-1}^{\prime \prime} \bar{w}_{2 \Lambda}\right)  \tag{5.11}\\
& +(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega(x)|^{\frac{5}{2}} \mu_{-1}^{\prime \prime \prime}\left|w_{\Lambda}\right|^{2} \bar{w}_{\Lambda}, \\
\widetilde{w}_{-3 \Lambda}= & \frac{i}{2}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{3}{2}} \bar{w}_{\Lambda}\left(\mu_{-3}^{\prime} w_{-2 \Lambda}+\mu_{-3}^{\prime \prime} \bar{w}_{2 \Lambda}\right) \\
& +(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega(x)|^{\frac{5}{2}} \mu_{-3}^{\prime \prime \prime} \bar{w}_{\Lambda}^{3},
\end{align*}
$$

for some real coefficients $\mu_{\ell}^{\prime}, \mu_{\ell}^{\prime \prime}, \mu_{\ell}^{\prime \prime \prime}, \ell \in\{-3,-1,3\}$.
Before starting the proof, we make the following remark that will be used several times below.
Remark 5.4. Let $\kappa$ be a smooth function on $\mathbb{R}^{*}$, such that for some real numbers $\ell, \ell^{\prime}$ and for any integer $k, \partial_{x}^{k} \kappa=O\left(|x|^{-\ell-k}\langle x\rangle^{-\ell^{\prime}}\right)$. Let $\chi$ be in $C_{0}^{\infty}(\mathbb{R})$, equal to one close to zero and let $r$ be an element of $\tilde{\mathcal{B}}_{\infty}^{\mu, \gamma}[K]$ for some compact subset of $T^{*}(\mathbb{R} \backslash\{0\})$. Then $(1-\chi)\left(x h^{-\beta}\right) \kappa(x) r$ belongs to $\tilde{\mathcal{B}}_{\infty}^{\mu+\left(\ell+\ell^{\prime}\right), \gamma+\ell^{\prime} / 2}[K]$.

Proof. We decompose $r=\sum_{j \in J(h, C)} \Theta_{j}^{*} r_{j}$ where $\left(r_{j}\right)_{j}$ is a bounded family in $\mathcal{B}_{\infty}^{\mu, \gamma}[K]$. Then

$$
(1-\chi)\left(x h^{-\beta}\right) \kappa r=\sum_{j \in J(h, C)} \Theta_{j}^{*} \tilde{r}_{j}
$$

with $\tilde{r}_{j}=(1-\chi)\left(x 2^{-j / 2} h^{-\beta}\right) \kappa\left(2^{-j / 2} x\right) r_{j}$. Since $r_{j}$ is microlocally supported in $K$ we may, modulo a $O\left(h_{j}^{\infty}\right)=O\left(h^{\infty}\right)$ remainder, replace $r_{j}$ by $\theta(x) r_{j}$ for some $\theta \in C_{0}^{\infty}(\mathbb{R})$, equal to one on a large enough compact subset of $\mathbb{R}^{*}$. Since

$$
x \rightarrow(1-\chi)\left(x 2^{-j / 2} h^{-\beta}\right) 2^{-\frac{j}{2}\left(\ell+\ell^{\prime}\right)+j+\frac{\ell^{\prime}}{2}} \kappa\left(2^{-\frac{j}{2}} x\right) \theta(x)
$$

is in $C_{0}^{\infty}\left(\mathbb{R}^{*}\right)$ and has derivatives uniformly estimated in $j$, $h$, we see that $\left(\tilde{r}_{j}\right)_{j}$ is microlocally supported on $K$ and satisfies uniform bounds in $\mathcal{B}_{\infty}^{\mu+\left(\ell+\ell^{\prime}\right), \gamma+\frac{\ell^{\prime}}{2}}[K]$.
Remark 5.5. Let $\chi_{1}, \chi_{2}$ be two $C_{0}^{\infty}(\mathbb{R})$ functions equal to one close to zero and $r$ be in $\tilde{\mathcal{B}}_{\infty}^{\mu, \gamma}[K]$ for some compact set $K$ of $T^{*}(\mathbb{R} \backslash\{0\})$. Then, if Supp $\chi_{1}$ and Supp $\chi_{2}$ are small enough, $\left(1-\chi_{1}\right)\left(x h^{-\beta}\right) r$ and $\left(1-\chi_{2}\right)\left(x h^{-\beta}\right) r$ coincide modulo $O\left(h^{\infty}\right)$ (so that they are identified).

Proof. We write again

$$
\left[\left(1-\chi_{1}\right)\left(x h^{-\beta}\right)-\left(1-\chi_{2}\right)\left(x h^{-\beta}\right)\right] r=\sum_{j \in J(h, C)} \Theta_{j}^{*}\left[\left(\chi_{2}-\chi_{1}\right)\left(x 2^{-j / 2} h^{-\beta}\right) r_{j}\right]
$$

As above, modulo $O\left(h^{\infty}\right)$, we may insert some cut-off $\theta$ against $r_{j}$. We may then notice that $\left(\chi_{2}-\chi_{1}\right)\left(x 2^{-j / 2} h^{-\beta}\right) \theta(x) \equiv 0$ if Supp $\chi_{\ell}$ is small enough, as $2^{-j / 2} h^{-\beta}>c$ for some $c>0$ since $j$ is in $J(h, C)$.

To prove lemma 5.3, it will be necessary to compute explicitly the action of some multilinear operators on functions of the type $w=w_{\Lambda}+w_{\Lambda^{c}}$.

Let us fix some notation. If $p_{1}, p_{2}$ are in $\mathbb{Z}^{*}, K_{p_{1}}, K_{p_{2}}$ are compact subsets contained in small neighborhoods of $p_{1} \cdot \Lambda, p_{2} \cdot \Lambda$ and if $w_{p_{\ell} \cdot \Lambda}^{\ell}$ is an element of $L^{\infty} \tilde{I}_{p_{\ell} \cdot \Lambda}^{\mu_{\ell}, \gamma_{\ell}}\left[K_{p_{\ell}}\right]$, Proposition 2.12 shows that the product $w_{p_{1} \cdot \Lambda}^{1} \cdot w_{p_{2} \cdot \Lambda}^{2}$ belongs to $L^{\infty} \tilde{I}_{\left(p_{1}+p_{2}\right) \cdot \Lambda}^{\mu_{1}+\mu_{2}, \gamma_{1}+\gamma_{2}}\left[K_{p_{1}+p_{2}}\right]$ for some compact subset $K_{p_{1}+p_{2}}$ of $T^{*}(\mathbb{R} \backslash\{0\})$ contained in a small neighborhood of $\left(p_{1}+p_{2}\right) \cdot \Lambda$, if $K_{p_{1}}$ and $K_{p_{2}}$ were contained in small enough neighborhoods of $p_{1} \cdot \Lambda, p_{2} \cdot \Lambda$ respectively. In the sequel, to avoid heavy notations, we shall eventually denote by $K_{p_{\ell}}$ different compact subsets of $T^{*}(\mathbb{R} \backslash\{0\})$ contained in a small enough neighborhood of $p_{\ell} \cdot \Lambda$. All of them will be constructed from a compact subset $K$ of $T^{*}(\mathbb{R} \backslash\{0\})$ contained in a small enough neighborhood of $\Lambda$. To simplify some notations, $p \Lambda$ will sometimes stand for $p \cdot \Lambda$. We shall also denote by $L$ some compact subset of $T^{*}(\mathbb{R} \backslash\{0\})$ which may vary from line to line.

Lemma 5.6. Let $b_{\ell}: \mathbb{R}^{*} \rightarrow \mathbb{C}, \ell=1,2,3$ be smooth functions positively homogeneous of degree $d_{\ell}$ and $a_{0}, a_{1}: \mathbb{R}^{*} \rightarrow \mathbb{C}$ be smooth, positively homogeneous of degree $m_{0}, m_{1}$.

Let $p_{\ell}$ be in $\mathbb{Z}^{*},\left|p_{\ell}\right| \leq 3, \ell=1,2,3$. If $p_{1}+p_{2}=0$ (resp. $p_{1}+p_{2}+p_{3}=0$ ), assume moreover that $a_{1}\left(\right.$ resp. $a_{0}$ ) is an homogeneous polynomial of order $m_{1} \in \mathbb{N}^{*}$ (resp. $m_{0} \in \mathbb{N}^{*}$ ). Let $b^{\prime}$ be a large enough positive number. Let $\chi$ be in $C_{0}^{\infty}(\mathbb{R}), \chi \equiv 1$ close to zero, with small enough support.
i) Assume given for $\ell=1,2,3$ functions $w_{p_{\ell} \Lambda}^{\ell}$ such that for some $N$ and any $k \leq \frac{s}{2}+N$, $\mathcal{Z}^{k} w_{p_{\ell} \Lambda}^{\ell}$ is in $h^{-\delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{p_{\ell} \cdot \Lambda}^{0, b^{\prime}}\left[K_{p_{\ell}}\right]$, for compact subsets $K_{p_{\ell}}$ satisfying the above conditions. Denote $\mu_{2}=2\left(m_{1}+d_{1}+d_{2}\right), \mu_{3}=2\left(m_{0}+m_{1}+d_{1}+d_{2}+d_{3}\right)$. Then,

$$
\begin{equation*}
\mathrm{Op}_{h}\left(a_{1}\right)\left[\left(\mathrm{Op}_{h}\left(b_{1}\right) w_{p_{1} \Lambda}^{1}\right)\left(\mathrm{Op}_{h}\left(b_{2}\right) w_{p_{2} \Lambda}^{2}\right)\right] \tag{5.12}
\end{equation*}
$$

may be written as the sum of

$$
\begin{equation*}
(1-\chi)\left(x h^{-\beta}\right) a_{1}\left(\left(p_{1}+p_{2}\right) \mathrm{d} \omega\right) b_{1}\left(p_{1} \mathrm{~d} \omega\right) b_{2}\left(p_{2} \mathrm{~d} \omega\right) w_{p_{1} \Lambda}^{1} w_{p_{2} \Lambda}^{2}, \tag{5.13}
\end{equation*}
$$

which is an element of $h^{-2 \delta_{1}^{\prime}} L^{\infty} \tilde{I}_{\left(p_{1}+p_{2}\right) \cdot \Lambda}^{\mu_{2}, 2 b^{\prime}}\left[K_{p_{1}+p_{2}}\right]$ such that the action of $\mathcal{Z}^{k}$ on it gives an element of $h^{-2 \delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{\left(p_{1}+p_{2}\right) \cdot \Lambda}^{\mu_{2}, 2 b^{\prime}}\left[K_{p_{1}+p_{2}}\right]$ for $k \leq \frac{s}{2}+N$, and of a remainder $R$ such that, for those $k^{\prime} s, \mathcal{Z}^{k} R$ is in

$$
\begin{equation*}
h^{\frac{1}{2}-2 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{\mu_{2}-1,2 b^{\prime}-\frac{1}{2}}[L]+h^{1-2 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}^{\mu_{2}-2,2 b^{\prime}-1}[L] . \tag{5.14}
\end{equation*}
$$

In the same way, a cubic term

$$
\begin{equation*}
\mathrm{Op}_{h}\left(a_{0}\right)\left[\mathrm{Op}_{h}\left(a_{1}\right)\left\{\left(\mathrm{Op}_{h}\left(b_{1}\right) w_{p_{1} \Lambda}^{1}\right)\left(\mathrm{Op}_{h}\left(b_{2}\right) w_{p_{2} \Lambda}^{2}\right)\right\}\left(\mathrm{Op}_{h}\left(b_{3}\right) w_{p_{3} \Lambda}^{3}\right)\right] \tag{5.15}
\end{equation*}
$$

may be written as the sum of

$$
\begin{equation*}
(1-\chi)\left(x h^{-\beta}\right) a_{0}\left(\left(p_{1}+p_{2}+p_{3}\right) \mathrm{d} \omega\right) a_{1}\left(\left(p_{1}+p_{2}\right) \mathrm{d} \omega\right) \prod_{\ell=1}^{3} b_{\ell}\left(p_{\ell} \mathrm{d} \omega\right) w_{p_{\ell} \Lambda}^{\ell} \tag{5.16}
\end{equation*}
$$

which is a function such that the action of $\mathcal{Z}^{k}$ on it, $k \leq \frac{s}{2}+N$, gives an element of

$$
h^{-3 \delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{\left(p_{1}+p_{2}+p_{3}\right) \cdot \Lambda}^{\mu_{3}, 3 A^{\prime}}\left[K_{p_{1}+p_{2}+p_{3}}\right]
$$

and of a remainder $R$ such that $\mathcal{Z}^{k} R$ is in

$$
\begin{equation*}
\sum_{j=1}^{3} h^{\frac{j}{2}-3 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{\mu_{3}-j, 3 b^{\prime}-j / 2}[L] . \tag{5.17}
\end{equation*}
$$

ii) Assume that we are given a function

$$
\begin{equation*}
w^{\ell}=w_{p_{\ell} \cdot \Lambda}^{\ell}+\sqrt{h}\left[w_{2 p_{\ell} \cdot \Lambda}^{\ell}+w_{-2 p_{\ell} \cdot \Lambda}^{\ell}\right] \tag{5.18}
\end{equation*}
$$

where for $k \leq \frac{s}{2}+N$,

$$
\begin{aligned}
& \mathcal{Z}^{k} w_{ \pm 2 p_{\ell} \cdot \Lambda}^{\ell} \text { is in } h^{-2 \delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{ \pm 2 p_{\ell} \cdot \Lambda}^{0, b^{\prime}}\left[K_{ \pm 2 p_{\ell}}\right], \\
& \mathcal{Z}^{k} w_{p_{\ell} \cdot \Lambda}^{\ell} \quad \text { is in } h^{-\delta_{k+1}^{\prime}} L^{\infty}{\tilde{p_{\ell}} \cdot}_{0, \cdot b^{\prime}}^{0,}\left[K_{p_{\ell}}\right] .
\end{aligned}
$$

Assume also that $p_{1} \pm 2 p_{2} \neq 0, p_{2} \pm 2 p_{1} \neq 0$. Then (5.12) may be written as the sum of a quadratic term, given by (5.13), which is such that the action of $\mathcal{Z}^{k}$ on it gives an element of $h^{-2 \delta_{k+1}^{\prime}} L^{\infty} \tilde{J}_{\left(p_{1}+p_{2}\right) \cdot \Lambda}^{\mu_{2}, 2 b^{\prime}}\left[K_{p_{1}+p_{2}}\right]$, of a cubic term, which may be written as the product of $\sqrt{h}$ and

$$
\begin{align*}
(1-\chi)\left(x h^{-\beta}\right)[ & a_{1}\left(\left(p_{1}+2 p_{2}\right) \mathrm{d} \omega\right) b_{1}\left(p_{1} \mathrm{~d} \omega\right) b_{2}\left(2 p_{2} \mathrm{~d} \omega\right) w_{p_{1} \Lambda}^{1} w_{2 p_{2} \Lambda}^{2} \\
& +a_{1}\left(\left(p_{1}-2 p_{2}\right) \mathrm{d} \omega\right) b_{1}\left(p_{1} \mathrm{~d} \omega\right) b_{2}\left(-2 p_{2} \mathrm{~d} \omega\right) w_{p_{1} \Lambda}^{1} w_{-2 p_{2} \Lambda}^{2}  \tag{5.19}\\
& +a_{1}\left(\left(2 p_{1}+p_{2}\right) \mathrm{d} \omega\right) b_{1}\left(2 p_{1} \mathrm{~d} \omega\right) b_{2}\left(p_{2} \mathrm{~d} \omega\right) w_{2 p_{1} \Lambda}^{1} w_{p_{2} \Lambda}^{2} \\
& \left.+a_{1}\left(\left(-2 p_{1}+p_{2}\right) \mathrm{d} \omega\right) b_{1}\left(-2 p_{1} \mathrm{~d} \omega\right) b_{2}\left(p_{2} \mathrm{~d} \omega\right) w_{-2 p_{1} \Lambda}^{1} w_{p_{2} \Lambda}^{2}\right]
\end{align*}
$$

and of a remainder term $R$. Moreover the action of $\mathcal{Z}^{k}$ on (5.19), $0 \leq k \leq \frac{s}{2}+N$, gives an element of

$$
\begin{aligned}
& \sum_{+,-} h^{-3 \delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{\left(p_{1} \pm 2 p_{2}\right) \cdot \Lambda}^{\mu_{2}, 2 b^{\prime}}\left[K_{p_{1} \pm 2 p_{2}}\right] \\
& \quad+\sum_{+,-} h^{-3 \delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{\left(p_{2} \pm 2 p_{1}\right) \cdot \Lambda}^{\mu_{2}, 2 b^{\prime}}\left[K_{p_{2} \pm 2 p_{1}}\right]
\end{aligned}
$$

and the action of $\mathcal{Z}^{k}$ on $R$ gives an element of

Proof. $i$ ) We use Proposition 2.11. By (2.17) applied with a symbol $a \equiv b_{\ell}(\xi), \ell=1,2$, we may write

$$
\begin{equation*}
\mathrm{Op}_{h}\left(b_{\ell}\right) w_{p_{\ell} \Lambda}^{\ell}=(1-\chi)\left(x h^{-\beta}\right) b_{\ell}\left(p_{\ell} \mathrm{d} \omega\right) w_{p_{\ell} \Lambda}^{\ell}+\tilde{r}^{\ell} \tag{5.21}
\end{equation*}
$$

where the action of $\mathcal{Z}^{k}$ on $\tilde{r}^{\ell}$ (resp. on the left hand side, resp. on the first term in the right hand side) of (5.21) is in $h^{\frac{1}{2}-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{2 d_{\ell}-1, b^{\prime}-\frac{1}{2}}[L]$ (resp. in $h^{-\delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{p_{\ell} \Lambda}^{2 d_{\ell}, b^{\prime}}\left[K_{p_{\ell}}\right]$ ).

By Proposition 2.12,

$$
\begin{equation*}
Z^{k}\left[\left(\mathrm{Op}_{h}\left(b_{1}\right) w_{p_{1} \Lambda}^{1}\right)\left(\mathrm{Op}_{h}\left(b_{2}\right) w_{p_{2} \Lambda}^{2}\right)-(1-\chi)\left(x h^{-\beta}\right)^{2} b_{1}\left(p_{1} \mathrm{~d} \omega\right) b_{2}\left(p_{2} \mathrm{~d} \omega\right) w_{p_{1} \Lambda}^{1} w_{p_{2} \Lambda}^{2}\right] \tag{5.22}
\end{equation*}
$$

is in

$$
h^{\frac{1}{2}-2 \delta_{k+1}^{\prime} \tilde{\mathcal{B}}_{\infty}^{2\left(d_{1}+d_{2}\right)-1,2 b^{\prime}-\frac{1}{2}}[L]+h^{1-2 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{2\left(d_{1}+d_{2}\right)-2,2 b^{\prime}-1}[L], ~}
$$

and the second term in (5.22) belongs to

$$
h^{-2 \delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{\left(p_{1}+p_{2}\right) \cdot \Lambda}^{2\left(d_{1}+d_{2}\right), 2 b^{\prime}}\left[K_{p_{1}+p_{2}}\right] .
$$

We make act $\mathrm{Op}_{h}\left(a_{1}\right)$ on the bracket in (5.22). By Proposition 2.11, this gives a remainder $R$ satisfying the conclusions of the statement. Moreover, the action of $\mathrm{Op}_{h}\left(a_{1}\right)$ on

$$
(1-\chi)\left(x h^{-\beta}\right)^{2} b_{1}\left(p_{1} \mathrm{~d} \omega\right) b_{2}\left(p_{2} \mathrm{~d} \omega\right) w_{p_{1} \Lambda}^{1} w_{p_{2} \Lambda}^{2}
$$

may be written as (5.13) modulo similar remainders. Notice that the second remark after the statement of Lemma 5.3 allows one to replace any power $(1-\chi)^{p}\left(x h^{-\beta}\right)$ by $(1-\chi)\left(x h^{-\beta}\right)$ if Supp $\chi$ is small enough.

One studies the cubic expressions (5.15) in the same way.
ii) We start from the stronger assumption (5.18). By (2.17) and the lines following that formula we may write $\mathrm{Op}_{h}\left(b_{\ell}\right) w^{\ell}$ as

$$
\begin{align*}
(1 & -\chi)\left(x h^{-\beta}\right) b_{\ell}\left(p_{\ell} \mathrm{d} \omega\right) w_{p_{\ell} \Lambda}^{\ell} \\
& +\sqrt{h}(1-\chi)\left(x h^{-\beta}\right)\left[b_{\ell}\left(2 p_{\ell} \mathrm{d} \omega\right) w_{2 p_{\ell} \Lambda}^{\ell}+b_{\ell}\left(-2 p_{\ell} \mathrm{d} \omega\right) w_{-2 p_{\ell} \Lambda}^{\ell}\right]  \tag{5.23}\\
& +\tilde{r}^{\ell}
\end{align*}
$$

where the action of $\mathcal{Z}^{k}$ on $\tilde{r}^{\ell}$ is in $h^{1-2 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{2 d_{\ell}-1, b^{\prime}-\frac{1}{2}}[L]$.
Moreover $\mathcal{Z}^{k}\left[(1-\chi)\left(x h^{-\beta}\right) b_{\ell}\left(p_{\ell} \mathrm{d} \omega\right) w_{p_{\ell} \Lambda}^{\ell}\right]$ (resp. $\left.\mathcal{Z}^{k}\left[(1-\chi)\left(x h^{-\beta}\right) b_{\ell}\left( \pm 2 p_{\ell} \mathrm{d} \omega\right) w_{ \pm 2 p_{\ell} \Lambda}^{\ell}\right]\right)$ belongs to

$$
h^{-\delta_{k+1}^{\prime}} L^{\infty} \tilde{J}_{p_{\ell} \cdot \Lambda}^{2 d_{\ell}, b^{\prime}}\left[K_{p_{\ell}}\right] \quad\left(\text { resp. } h^{-2 \delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{ \pm 2 p_{\ell} \cdot \Lambda}^{2 d_{\ell}, b^{\prime}}\left[K_{ \pm 2 p_{\ell}}\right]\right) .
$$

Applying Proposition 2.12, we obtain that $\left(\mathrm{Op}_{h}\left(b_{1}\right) w^{1}\right)\left(\mathrm{Op}_{h}\left(b_{2}\right) w^{2}\right)$ may be written as the sum of quadratic terms

$$
\begin{equation*}
(1-\chi)\left(x h^{-\beta}\right)^{2} b_{1}\left(p_{1} \mathrm{~d} \omega\right) b_{2}\left(p_{2} \mathrm{~d} \omega\right)\left(w_{p_{1} \Lambda}^{1}\right)\left(w_{p_{2} \Lambda}^{2}\right) \tag{5.24}
\end{equation*}
$$

of cubic terms

$$
\begin{align*}
\sqrt{h}(1-\chi)\left(x h^{-\beta}\right)^{2}[ & b_{1}\left(p_{1} \mathrm{~d} \omega\right) b_{2}\left(2 p_{2} \mathrm{~d} \omega\right)\left(w_{p_{1} \Lambda}^{1}\right)\left(w_{2 p_{2} \Lambda}^{2}\right) \\
& +b_{1}\left(p_{1} \mathrm{~d} \omega\right) b_{2}\left(-2 p_{2} \mathrm{~d} \omega\right)\left(w_{p_{1} \Lambda}^{1}\right)\left(w_{-2 p_{2} \Lambda}^{2}\right) \\
& +b_{1}\left(2 p_{1} \mathrm{~d} \omega\right) b_{2}\left(p_{2} \mathrm{~d} \omega\right)\left(w_{2 p_{1} \Lambda}^{1}\right)\left(w_{p_{2} \Lambda}^{2}\right)  \tag{5.25}\\
& \left.+b_{1}\left(-2 p_{1} \mathrm{~d} \omega\right) b_{2}\left(p_{2} \mathrm{~d} \omega\right)\left(w_{-2 p_{1} \Lambda}^{1}\right)\left(w_{p_{2} \Lambda}^{2}\right)\right]
\end{align*}
$$

and of a remainder $\tilde{R}^{\ell}$ such that $\mathcal{Z}^{k} \tilde{R}^{\ell}$ is in $h^{1-4 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{2\left(d_{1}+d_{2}\right)-1,2 b^{\prime}-\frac{1}{2}}[L]$.
To study (5.12), we make act $\mathrm{Op}_{h}\left(a_{0}\right)$ on (5.24), (5.25) and on the remainder. We know from Proposition 2.12 that $(1-\chi)\left(x h^{-\beta}\right)^{2} w_{p_{1} \Lambda}^{1} w_{p_{2} \Lambda}^{2}$ is in $h^{-2 \delta_{k+1}^{\prime}} L^{\infty} \tilde{J}_{\left(p_{1}+p_{2}\right) \cdot \Lambda}^{0,2 b^{\prime}}\left[K_{p_{1}+p_{2}}\right]$ and that $(1-\chi)\left(x h^{-\beta}\right)^{2} w_{p_{1} \Lambda}^{1} w_{ \pm 2 p_{2} \Lambda}^{2}$ (resp. $\left.(1-\chi)\left(x h^{-\beta}\right)^{2} w_{ \pm 2 p_{1} \Lambda}^{1} w_{p_{2} \Lambda}^{2}\right)$ belongs to

$$
h^{-3 \delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{\left(p_{1} \pm 2 p_{2}\right) \cdot \Lambda}^{0,2 b^{\prime}}\left[K_{p_{1} \pm 2 p_{2}}\right] \quad\left(\text { resp. } h^{-3 \delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{\left( \pm 2 p_{1}+p_{2}\right) \cdot \Lambda}^{0,2 b^{\prime}}\left[K_{ \pm 2 p_{1}+p_{2}}\right]\right)
$$

To study the action of $\mathrm{Op}_{h}\left(a_{0}\right)$ on (5.24), (5.25), we may use (2.18), noticing that, since $\mathrm{d} \omega$ is homogeneous of degree -2 ,

$$
\begin{aligned}
& a_{1}(\xi) b_{1}\left(p_{1} \mathrm{~d} \omega(x)\right) b_{2}\left(p_{2} \mathrm{~d} \omega(x)\right), \\
& a_{1}(\xi) b_{1}\left(p_{1} \mathrm{~d} \omega(x)\right) b_{2}\left( \pm 2 p_{2} \mathrm{~d} \omega(x)\right), \\
& a_{1}(\xi) b_{1}\left( \pm 2 p_{1} \mathrm{~d} \omega(x)\right) b_{2}\left(p_{2} \mathrm{~d} \omega(x)\right)
\end{aligned}
$$

satisfy the assumptions (2.15) with $\left(\ell, \ell^{\prime}, d, d^{\prime}\right)$ replaced by $\left(-2\left(d_{1}+d_{2}\right), 0, m_{1}, 0\right)$. We conclude that (5.12) is given by the sum of (5.13), (5.19) and remainders $R$ such that the action of
 spaces indicated in the statement of the lemma. This concludes the proof.

Proof of Lemma 5.3. $i$ ) Let us prove the first equality (5.7). Recall that we denote $W=$ $(w, \bar{w})$. In the same way, set $W_{\Lambda}=\left(w_{\Lambda}, \bar{w}_{\Lambda}\right), W_{\Lambda^{c}}=\left(w_{\Lambda^{c}}, \bar{w}_{\Lambda^{c}}\right)$. If $B_{0}$ denotes the polar form of $Q_{0}$, we have

$$
Q_{0}(W)=Q_{0}\left(W_{\Lambda}\right)+2 B_{0}\left(W_{\Lambda}, W_{\Lambda^{c}}\right)+Q_{0}\left(W_{\Lambda^{c}}\right) .
$$

For $j$ in $J(h, C)$, we set

$$
\tilde{g}_{2, j}=h^{-\frac{1}{2}} \Theta_{-j}^{*} \Delta_{j}^{h}\left[2 B_{0}\left(W_{\Lambda}, W_{\Lambda^{c}}\right)+Q_{0}\left(W_{\Lambda^{c}}\right)\right] .
$$

Let us show that $\tilde{g}_{2}=\sum_{j \in J(h, C)} \Theta_{j}^{*} \tilde{g}_{2, j}$ satisfies the conclusions of the lemma. By assumption $\mathcal{Z}^{k} w_{\Lambda}$ is in $h^{-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b-2}[K]$ and $\mathcal{Z}^{k} w_{\Lambda^{c}}$ is in $h^{\frac{1}{2}-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b-2}[F]+h^{1-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{-1, b-2}[F]$. We plug these informations inside (3.12). We get

$$
\begin{aligned}
&\left\|\Delta_{j}^{h} \mathcal{Z}^{k} B_{0}\left(W_{\Lambda}, W_{\Lambda^{c}}\right)\right\|_{L^{\infty}} \leq C 2^{j} \sum_{\substack{\max \left(j_{1}, j_{2}\right) \geq j-C \\
j_{1}, j_{2} \in J(h, C)}} 2^{\frac{1}{2} \min \left(j_{1}, j_{2}\right)}\left[h^{\frac{1}{2}-2 \delta_{k+1}^{\prime}}+h^{\left.1-2 \delta_{k+1}^{\prime} 2^{-\frac{j_{2}}{2}}\right]}\right. \\
& \times 2^{-\left(j_{1+}+j_{2+}\right)(b-2)} .
\end{aligned}
$$

Summing using the fact that the number of negative $j_{\ell}$ 's in $J(h, C)$ is $O(|\log (h)|)$, we obtain a bound in $2^{j-j+(b-2)} h^{\frac{1}{2}-2 \delta_{k+1}^{\prime}}$ which is the bound characterizing elements of $h^{\frac{1}{2}-2 \delta_{k+1}^{\prime}} \mathcal{B}_{\infty}^{2, b-2}[F]$ (where $F$ is a closed set of the form $C^{-1} \leq|\xi| \leq C$ ). In the same way

$$
\begin{aligned}
\left\|\Delta_{j}^{h} \mathcal{Z}^{k} B_{0}\left(W_{\Lambda^{c}}, W_{\Lambda^{c}}\right)\right\|_{L^{\infty}} \leq C 2^{j} \sum_{\substack{\max \left(j_{1}, j_{2}\right) \geq j-C \\
j_{1}, j_{2} \in J(h, C)}} 2^{\frac{1}{2} \min \left(j_{1}, j_{2}\right)} & {\left[h^{\frac{1}{2}-\delta_{k+1}^{\prime}}+h^{1-\delta_{k+1}^{\prime}} 2^{-\frac{j_{1}}{2}}\right] } \\
& \times\left[h^{\frac{1}{2}-\delta_{k+1}^{\prime}}+h^{1-\delta_{k+1}^{\prime}} 2^{-\frac{j_{2}}{2}}\right] \\
& \times 2^{-\left(j_{1}+j_{2+}\right)(b-2)} .
\end{aligned}
$$

Summing we get a bound in $C\left[2^{j} h^{1-2 \delta_{k+1}^{\prime}}+2^{\frac{j}{2}} h^{2-0-2 \delta_{k+1}^{\prime}}\right] 2^{-j_{+}(b-2)}$. This characterizes an element of $h^{1-2 \delta_{k+1}^{\prime}} \mathcal{B}_{\infty}^{2, b-2}[F]+h^{2-0-2 \delta_{k+1}^{\prime}} \mathcal{B}_{\infty}^{1, b-2}[F]$. Summarizing, we get finally that $\tilde{g}_{2}$ is in $h^{-2 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{1, b-\frac{5}{2}}[F]$ which is the wanted conclusion since we assume $b^{\prime}<b-3$.

To estimate $\mathrm{Op}_{h}(x \xi) \tilde{g}_{2}$, we have to perform similar estimates replacing $B_{0}\left(W_{\Lambda}, W_{\Lambda^{c}}\right)$ (resp. $\left.B_{0}\left(W_{\Lambda^{c}}, W_{\Lambda^{c}}\right)\right)$ by $\left(x h D_{x}\right) B_{0}\left(W_{\Lambda}, W_{\Lambda^{c}}\right)$ (resp. $\left.\left(x h D_{x}\right) B_{0}\left(W_{\Lambda^{c}}, W_{\Lambda^{c}}\right)\right)$. If $S(\xi)$ is a positively homogeneous function of order $\lambda>0$, smooth outside zero,

$$
\left[\mathrm{Op}_{h}(x \xi), \mathrm{Op}_{h}(S(\xi))\right]=i \lambda h \mathrm{Op}_{h}(S)
$$

Consequently, the expression (3.7) of $Q_{0}$ and Leibniz rule show that $\mathrm{Op}_{h}(x \xi) Q_{0}(V)$ may be expressed from $B_{0}\left(\mathrm{Op}_{h}(x \xi) V, V\right)$ and from $h \tilde{B}_{0}(V, V)$, where $\tilde{B}_{0}$ is a bilinear form satisfying the same estimates (3.12) as $B_{0}$ (Actually, $\tilde{B}_{0}$ is either a multiple of the polar form of the quadratic form in the first line of the right hand side of (3.7), or a multiple of the polar forms of the sum of the second and third lines). The last property stated in Lemma 5.1 implies that

$$
\mathcal{Z}^{k} \mathrm{Op}_{h}(x \xi) w_{\Lambda^{c}} \in h^{\frac{1}{2}-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b-\frac{5}{2}}[F]+h^{1-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{-1, b-\frac{5}{2}}[F]
$$

Moreover, still because of this lemma, $\mathcal{Z}^{k} w_{\Lambda}$ is in $h^{-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b-2}[K]$ for a compact subset $K$ of $T^{*}(\mathbb{R} \backslash\{0\})$. It follows from (2.13) and the fact that $x \xi$ restricted to such a compact set is in the class of symbols $S(1)$, that $\mathcal{Z}^{k} \mathrm{Op}_{h}(x \xi) w_{\Lambda}$ is in $h^{-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{1, b-2}[K] \subset h^{-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b-\frac{5}{2}}[F]$. This shows that to estimate $\operatorname{Op}_{h}(x \xi) B_{0}\left(W_{\Lambda}, W_{\Lambda^{c}}\right), \mathrm{Op}_{h}(x \xi) B_{0}\left(W_{\Lambda^{c}}, W_{\Lambda^{c}}\right)$, it suffices to use the bounds obtained above for $B_{0}\left(W_{\Lambda}, W_{\Lambda^{c}}\right), B_{0}\left(W_{\Lambda^{c}}, W_{\Lambda^{c}}\right)$ replacing $b$ by $b-\frac{1}{2}$. We conclude that $\mathrm{Op}_{h}(x \xi) \tilde{g}_{2}$ is in $h^{-2 \delta_{k+1}^{\prime} \tilde{\mathcal{B}}_{\infty}^{1, b-3}}[F] \subset h^{-2 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{1, b^{\prime}+\frac{1}{2}}[F]$.

We compute next $Q_{0}\left(W_{\Lambda}\right)$ from (3.7). Let us examine first the contributions that are bilinear
in $\left(w_{\Lambda}, \bar{w}_{\Lambda}\right)$ i.e.

$$
\begin{align*}
& -\frac{i}{4} \mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right)\left[\left(\mathrm{Op}_{h}\left(\xi|\xi|^{-\frac{1}{2}}\right) w_{\Lambda}\right)\left(\mathrm{Op}_{h}\left(\xi|\xi|^{-\frac{1}{2}}\right) \bar{w}_{\Lambda}\right)\right] \\
& -\frac{i}{4} \mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right)\left[\left(\mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right) w_{\Lambda}\right)\left(\mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right) \bar{w}_{\Lambda}\right)\right]  \tag{5.26}\\
& +\frac{i}{4} \mathrm{Op}_{h}(|\xi|)\left[w_{\Lambda} \mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right) \bar{w}_{\Lambda}-\bar{w}_{\Lambda} \mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right) w_{\Lambda}\right] \\
& -\frac{i}{4} \mathrm{Op}_{h}(\xi)\left[w_{\Lambda} \mathrm{Op}_{h}\left(\xi|\xi|^{-\frac{1}{2}}\right) \bar{w}_{\Lambda}-\bar{w}_{\Lambda} \mathrm{Op}_{h}\left(\xi|\xi|^{-\frac{1}{2}}\right) w_{\Lambda}\right]
\end{align*}
$$

We use that (5.12) may be computed from (5.13), up to a remainder given by (5.14) with $\mu_{2}=3, b^{\prime}=b-2$, that contributes to $\sqrt{h} \tilde{g}_{2}$ in (5.7) (since $b^{\prime}<b-\frac{5}{2}$ and $b$ is large enough). Notice that the main contribution, computed from (5.13) vanishes. For the terms inside the first two brackets in (5.26), this follows from a two by two cancellation between the two contributions in each bracket. For the last term in (5.26), we remark that the symbol $\xi$ of the outside operator $\mathrm{Op}_{h}(\xi)$ is an homogeneous polynomial, which allows us to make use of expansion (5.13) with $a_{1} \equiv \xi, p_{1}+p_{2}=0$, and implies as well the vanishing of that term.

We are left with studying the quadratic terms in $w_{\Lambda}$ and the quadratic terms in $\bar{w}_{\Lambda}$ in (3.7). We may apply to both of them $i$ ) of Lemma 5.6 with $\left(p_{1}, p_{2}\right)=(1,1)$ or $\left(p_{1}, p_{2}\right)=(-1,-1)$. We get the contribution to $Q_{0}\left(W_{\Lambda}\right)$ given by the sum of the two expressions (5.8).

To study $C_{0}(W)$, we use that the assumptions imply that $\mathcal{Z}^{k} w$ is in $h^{-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b-2}[F]$ (This follows from the fact that $h \tilde{\mathcal{B}}_{\infty}^{-1, b-2}[F] \subset h^{\sigma} \tilde{\mathcal{B}}_{\infty}^{0, b-2}[F]$, as a consequence of the inequality $h_{j}=O\left(h^{\sigma}\right)$ ). To bound $\left\|\Delta_{j}^{h} \mathcal{Z}^{k} C_{0}(W)\right\|_{L^{\infty}}$, we apply (3.15) with $p=\infty, d=b-\alpha-2-0$, $V_{1}=V_{2}=V_{3}=W$. Our assumptions on $w$ and $d$ imply that

$$
\left\|\mathcal{Z}^{k}\left\langle h D_{x}\right\rangle^{\alpha+d} w\right\|_{L^{\infty}}=O\left(h^{-\delta_{k+1}^{\prime}-0}\right)
$$

as is seen from the expansion $w=\sum_{j \in J(h, C)} \Theta_{j}^{*} w_{j}$ and the bounds on the $w_{j}$ 's. It follows from (3.15) that

$$
\begin{equation*}
\left\|\Delta_{j}^{h} \mathcal{Z}^{k} C_{0}(W)\right\|_{L^{\infty}}=O\left(2^{\frac{j}{2}-j_{+}(b-\alpha-2-0)} h^{-3 \delta_{k+1}^{\prime}-0}\right) . \tag{5.27}
\end{equation*}
$$

The conclusion $\mathcal{Z}^{k} g_{3} \in h^{-3 \delta_{k+1}^{\prime}-0} \tilde{\mathcal{B}}_{\infty}^{1, b^{\prime}+\frac{1}{2}}[F]$ follows if we assume $b^{\prime}<b-\frac{9}{2}$ (since $\alpha$ is any number strictly larger than 2 ).

To obtain that $\mathcal{Z}^{k} \mathrm{Op}_{h}(x \xi) \tilde{g_{3}}$ is in $h^{-3 \delta_{k+1}^{\prime}-0} \tilde{\mathcal{B}}_{\infty}^{1, b^{\prime}+\frac{1}{2}}[F]$ we make act $\mathrm{Op}_{h}(x \xi)$ on $C_{0}(W)$ and we argue as in the study of quadratic terms, distributing $x h D_{x}$ on the different factors using Leibniz rule. We have seen that $\mathcal{Z}^{k} \operatorname{Op}_{h}(x \xi) W$ is $h^{-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b-\frac{5}{2}}[F]$. It follows as above that we get for $\left\|\Delta_{j}^{h} \mathcal{Z}^{k} \mathrm{Op}_{h}(x \xi) C_{0}(W)\right\|_{L^{\infty}}$ the same estimate as (5.27), with $b$ replaced by $b-1 / 2$. This gives the wanted bound as $b^{\prime}<b-5$. This concludes the proof of $i$ ) of the lemma.
ii) Let us show first that we may replace in the quadratic (resp. cubic) part of the left hand side of (5.9) $w$ by $w_{p}=w_{\Lambda}+\sqrt{h}\left(w_{2 \Lambda}+w_{-2 \Lambda}\right)$ (resp. by $w_{\Lambda}$ ) up to a contribution to the
$h \tilde{g}$ term in the right hand side. By assumption, $\mathcal{Z}^{k} w_{p}$ is in $h^{-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}[F]$ for some $b^{\prime}<$ $b-\frac{9}{2}$. Set $W_{p}=\left(w_{p}, \bar{w}_{p}\right), G=(g, \bar{g})$ and let us show that the contributions of $\mathcal{Z}^{k} B_{0}\left(W_{p}, G\right)$ and $h \mathcal{Z}^{k} B_{0}(G, G)$ are in $h^{-4 \delta_{k+P}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{2, b^{\prime}}[F]$. We use (3.12) and the assumption that $\mathcal{Z}^{k} g$ is in $h^{-3 \delta^{\prime}{ }_{k+P}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}[F]$ to bound using (3.12)

$$
\left\|\Delta_{j}^{h} \mathcal{Z}^{k} B_{0}\left(W_{p}, G\right)\right\|_{L^{\infty}} \leq C 2^{j} \sum_{\substack{\max \left(j_{1}, j_{2}\right) \geq j-C \\ j_{1}, j_{2} \in J(h, C)}} 2^{\frac{1}{2} \min \left(j_{1}, j_{2}\right)} h^{-\delta_{k+1}^{\prime}-3 \delta_{k+P}^{\prime}} 2^{-\left(j_{1}+j_{2+}\right) b^{\prime}} .
$$

We get an estimate in $O\left(h^{-4 \delta_{k+P}^{\prime}} 2^{j-j_{+} b^{\prime}}\right)$ which shows the wanted conclusion. One argues in the same way for $h B_{0}(G, G)$.

Considering the cubic term, we write $w=w_{\Lambda}+\sqrt{h} g^{\prime}$, where $g^{\prime}=\left(w_{2 \Lambda}+w_{-2 \Lambda}\right)+\sqrt{h} g$ satisfies $\mathcal{Z}^{k} g^{\prime} \in h^{-2 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}[F]$ and where $\mathcal{Z}^{k} w_{\Lambda}$ is in $h^{-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}[F]$. If we set $G^{\prime}=\left(g^{\prime}, \bar{g}^{\prime}\right)$, we have to study $\left\|\Delta_{j}^{h} \mathcal{Z}^{k}\left[C_{0}\left(W_{\Lambda}+\sqrt{h} G^{\prime}\right)-C_{0}\left(W_{\Lambda}\right)\right]\right\|_{L^{\infty}}$ i.e. $\sqrt{h}\left\|\Delta_{j}^{h} \mathcal{Z}^{k} T_{0}\left(W_{\Lambda}, W_{\Lambda}, G^{\prime}\right)\right\|_{L^{\infty}}$, $h\left\|\Delta_{j}^{h} \mathcal{Z}^{k} T_{0}\left(W_{\Lambda}, G^{\prime}, G^{\prime}\right)\right\|_{L^{\infty}}$ and $h^{\frac{3}{2}}\left\|\Delta_{j}^{h} \mathcal{Z}^{k} T_{0}\left(G^{\prime}, G^{\prime}, G^{\prime}\right)\right\|_{L^{\infty}}$. If $d=b^{\prime}-\alpha-0$, we bound

$$
\left\|\mathcal{Z}^{k}\left\langle h D_{x}\right\rangle^{\alpha+d} W_{\Lambda}\right\|_{L^{\infty}}=O\left(h^{-0-\delta_{k+1}^{\prime}}\right), \quad\left\|\mathcal{Z}^{k}\left\langle h D_{x}\right\rangle^{\alpha+d} G^{\prime}\right\|_{L^{\infty}}=O\left(h^{-2 \delta_{k+1}^{\prime}-0}\right) .
$$

Plugging these estimates in (3.15), we get for the quantities under study a bound in terms of $2^{\frac{j}{2}-j+\left(b^{\prime}-\alpha-0\right)} h^{-4 \delta_{k+1}^{\prime}+\frac{1}{2}}$. Let us study as well

$$
\mathrm{Op}_{h}(x \xi) B_{0}\left(W_{\Lambda}, G\right), \quad \mathrm{Op}_{h}(x \xi) B_{0}(G, G), \quad \mathrm{Op}_{h}(x \xi)\left[C_{0}\left(W_{\Lambda}+\sqrt{h} G^{\prime}\right)-C_{0}\left(W_{\Lambda}\right)\right] .
$$

As in the proof of $i$, we may express these quantities from

$$
\begin{aligned}
& B_{0}\left(\mathrm{Op}_{h}(x \xi) W_{p}, G\right), \quad B_{0}\left(W_{p}, \mathrm{Op}_{h}(x \xi) G\right), \\
& \sqrt{h} T_{0}\left(\mathrm{Op}_{h}(x \xi) W_{\Lambda}, W_{\Lambda}, G^{\prime}\right), \quad \sqrt{h} T_{0}\left(W_{\Lambda}, W_{\Lambda}, \mathrm{Op}_{h}(x \xi) G^{\prime}\right) \\
& h T_{0}\left(W_{\Lambda}, \mathrm{Op}_{h}(x \xi) G^{\prime}, G^{\prime}\right), \quad h T_{0}\left(W_{\Lambda}, \mathrm{Op}_{h}(x \xi) G^{\prime}, G^{\prime}\right), \quad h^{\frac{3}{2}} T_{0}\left(\mathrm{Op}_{h}(x \xi) G^{\prime}, G^{\prime}, G^{\prime}\right),
\end{aligned}
$$

and from quadratic and cubic quantities of the form of those already estimated. By assumption, the $g$-term in (5.5) satisfies $\mathcal{Z}^{k} \mathrm{Op}_{h}(x \xi) g \in h^{-3 \delta_{k+P}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{1, b^{\prime}}[F] \subset h^{-3 \delta_{k+P}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}-\frac{1}{2}}[F]$. Moreover, by definition of that quantity, $\mathcal{Z}^{k} \mathrm{Op}_{h}(x \xi) w_{p}$ is in $h^{-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}[F]$. The above estimate of $B_{0}$, with $b^{\prime}$ replaced by $b^{\prime}-\frac{1}{2}$, shows that

$$
\left\|\Delta_{j}^{h} \mathcal{Z}^{k} B_{0}\left(\mathrm{Op}_{h}(x \xi) W_{p}, G\right)\right\|_{L^{\infty}}+\left\|\Delta_{j}^{h} \mathcal{Z}^{k} B_{0}\left(W_{p}, \mathrm{Op}_{h}(x \xi) G\right)\right\|_{L^{\infty}}=O\left(h^{-4 \delta_{k+P}^{\prime} 2^{j-j_{+}\left(b^{\prime}-\frac{1}{2}\right)}}\right)
$$

We obtain a $h^{-4 \delta_{k+P}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{2, b^{\prime}-\frac{1}{2}}[F] \subset h^{-4 \delta_{k+P}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{1, b^{\prime}-\frac{3}{2}}[F]$ contribution to $\mathcal{Z}^{k} \mathrm{Op}_{h}(x \xi) \tilde{g}$ in the action of $\mathrm{Op}_{h}(x \xi)$ on (5.9). The definition of $g^{\prime}$ implies that $\mathcal{Z}^{k} \mathrm{Op}_{h}(x \xi) g^{\prime}$ is in $h^{-2 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}-\frac{1}{2}}[F]$. Bounding, with $d=b^{\prime}-\frac{1}{2}-\alpha-0$

$$
\begin{aligned}
& \left\|\mathcal{Z}^{k}\left\langle h D_{x}\right\rangle^{\alpha+d} \operatorname{Op}_{h}(x \xi) W_{\Lambda}\right\|_{L^{\infty}}=O\left(h^{-0-\delta_{k+1}^{\prime}}\right), \\
& \left\|\mathcal{Z}^{k}\left\langle h D_{x}\right\rangle^{\alpha+d} \operatorname{Op}_{h}(x \xi) G^{\prime}\right\|_{L^{\infty}}=O\left(h^{-0-2 \delta_{k+1}^{\prime}}\right)
\end{aligned}
$$

we get again from (2.15) that

$$
\left\|\Delta_{j}^{h} \mathcal{Z}^{k} \operatorname{Op}_{h}(x \xi)\left[C_{0}\left(W_{\Lambda}+\sqrt{h} G^{\prime}\right)-C_{0}\left(W_{\Lambda}\right)\right]\right\|_{L^{\infty}}=O\left(2^{\frac{j}{2}-j+\left(b^{\prime}-\frac{1}{2}-\alpha-0\right)} h^{-4 \delta_{k+1}^{\prime}+\frac{1}{2}}\right)
$$

If we replace above $b^{\prime}$ by $b^{\prime}+3$ (since $\alpha=2+0$ ), which corresponds to decreasing by 3 units the assumption made on $b^{\prime}-b$ in Proposition 5.2 (i.e. imposing $b^{\prime}<b-8$ ), we obtain finally that the contributions of $Q_{0}(W)-Q_{0}\left(W_{p}\right)$ and $C_{0}(W)-C_{0}\left(W_{\Lambda}\right)$ to (5.9) may be incorporated into the $h \tilde{g}$ term of the right hand side. We are reduced to the study of

$$
Q_{0}\left(W_{\Lambda}+\sqrt{h}\left(W_{2 \Lambda}+W_{-2 \Lambda}\right)\right), \quad C_{0}\left(W_{\Lambda}\right) .
$$

To treat the first expression, we use $i i$ ) of Lemma 5.6, which allows us to compute expressions (3.7) using (3.11). The remainders satisfy bounds of the form (5.20) with $\mu_{2}=3$, so may be incorporated to the $h \tilde{g}$ term in (5.9). We have already seen in the proof of $i$ ) that the $O(1)$ term in (5.9) is given by (5.8).

The $O(\sqrt{h})$ term is computed from (5.19) applied to the different contributions to $Q_{0}$ given by (3.7). We need to compute explicitly only the $\Lambda$-oscillating term i.e. the contributions to (5.19) corresponding to $p_{1} \pm 2 p_{2}=1$ and $p_{2} \pm 2 p_{1}=1\left(p_{1}, p_{2} \in\{-1,1\}\right)$. From the expression (3.7) of $Q_{0}$ and (5.19), we get a contribution

$$
\begin{equation*}
(1-\chi)\left(x h^{-\beta}\right) \frac{i}{2}|\mathrm{~d} \omega|^{\frac{3}{2}} \bar{w}_{\Lambda}\left(w_{2 \Lambda}-\bar{w}_{-2 \Lambda}\right) . \tag{5.28}
\end{equation*}
$$

In the same way, using (5.16), we compute the $\Lambda$-oscillating cubic term coming from the expression (3.8) of $C_{0}\left(W_{\Lambda}\right)$. We obtain a contribution $\frac{1}{4}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega(x)|^{\frac{5}{2}}\left|w_{\Lambda}\right|^{2} w_{\Lambda}$. Summing up, we get

$$
\begin{aligned}
Q_{0}(w, \bar{w})+\sqrt{h} C_{0}(w, \bar{w})= & \widetilde{w}_{2 \Lambda}+\widetilde{w}_{-2 \Lambda} \\
& +\sqrt{h}\left(\widetilde{w}_{3 \Lambda}+\widetilde{w}_{\Lambda}+\widetilde{w}_{-\Lambda}+\widetilde{w}_{-3 \Lambda}\right) \\
& +h \tilde{g}_{1}
\end{aligned}
$$

where according to (5.17) and (5.20),

$$
\mathcal{Z}^{k} \tilde{g}_{1} \in h^{-4 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{2, b^{\prime}+1}[F] \subset h^{\left.-4 \delta_{k+1}^{\prime} \tilde{\mathcal{B}}_{\infty}^{1, b^{\prime}+\frac{1}{2}}[F], ~\right]}
$$

(if $b^{\prime}$ is large enough), and where $\widetilde{w}_{\Lambda}$ is given by (5.10). The contributions $\widetilde{w}_{3 \Lambda}, \widetilde{w}_{-\Lambda}, \widetilde{w}_{-3 \Lambda}$ have expressions (5.11), for which we do not need to compute explicitly the coefficients $\mu_{\ell}^{\prime}$, $\mu_{\ell}^{\prime \prime}$. This concludes the proof of the lemma.

The next step of the proof of Proposition 5.2 will be to deduce from equation (3.10), and from the description provided by Lemma 5.3 of the right hand side of this equation, an expansion of $\mathrm{Op}_{h}\left(\gamma_{\Lambda}^{c}\right) w$, exploiting that $2 x \xi+|\xi|^{\frac{1}{2}}$ is an elliptic symbol on the support of $\gamma_{\Lambda}^{c}$. In a first step, we establish some a priori bounds for the components $w_{j}$ of $w$ cut-off outside a neighborhood of $\Lambda$.

Lemma 5.7. Assume that (4.9) holds for $k \leq \frac{s}{2}+N_{1}$ for some integer $N_{1}$ satisfying ( $N_{1}-$ $\left.N_{0}\right) \sigma \geq 1$. Then for any symbol a in $S(1)$, microlocally supported outside a neighborhood of $\Lambda$, the following estimate holds for $k \leq \frac{s}{2}+N_{0}$, and any $j$ in $J(h, C)$,

$$
\begin{equation*}
\left\|\mathrm{Op}_{h_{j}}(a) \mathcal{Z}_{j}^{k} w_{j}\right\|_{L^{\infty}} \leq C_{k} \varepsilon h^{\frac{1}{2}-\delta_{k+N_{1}-N_{0}}^{\prime}}\left(2^{j / 2}+h^{1 / 2}\right) 2^{-j_{+} b^{\prime}} \tag{5.29}
\end{equation*}
$$

for any $b^{\prime}<b-\alpha<b-2$.

Proof. Let us construct for any $1 \leq \ell \leq N_{1}-N_{0}$, any $k \leq \frac{s}{2}+N_{1}-\ell$, a family of symbols in $S(1),\left(b_{\ell^{\prime}}^{\ell}\right)_{0 \leq \ell^{\prime} \leq \ell}$, vanishing close to $\Lambda$ and a sequence $\left(r_{j}^{\ell, k}\right)_{j \in J(h, C)}$ with

$$
\begin{equation*}
\left\|r_{j}^{\ell, k}\right\|_{L^{\infty}} \leq C \varepsilon h^{\frac{1}{2}-\delta_{k+\ell}^{\prime}}\left(2^{j / 2}+h^{1 / 2}\right) 2^{-j_{+} b^{\prime}} \tag{5.30}
\end{equation*}
$$

such that

$$
\begin{equation*}
\mathcal{Z}_{j}^{k} \mathrm{Op}_{h_{j}}(a) w_{j}=h_{j}^{\ell}\left[\sum_{\ell^{\prime}=0}^{\ell} \mathcal{Z}_{j}^{k+\ell^{\prime}} \mathrm{Op}_{h_{j}}\left(b_{\ell^{\prime}}^{\ell}\right) w_{j}^{\left(\ell, \ell^{\prime}\right)}\right]+r_{j}^{\ell, k} \tag{5.31}
\end{equation*}
$$

where $w_{j}^{\left(\ell, \ell^{\prime}\right)}$ denotes a function defined like $w_{j}=\Theta_{-j}^{*} \Delta_{j}^{h} w$ but with $\Delta_{j}^{h}$ replaced by another cut-off of the same type. Then (5.31) with $\ell=N_{1}-N_{0}$ implies (5.29) since $h_{j}^{N_{1}-N_{0}} \leq$ $h^{\sigma\left(N_{1}-N_{0}\right)} \leq h$ and since $w_{j}^{\left(\ell, \ell^{\prime}\right)}$ satisfies the same $L^{\infty}$ estimates (4.9) as $w_{j}$.

We remark that to prove (5.31), we just need to treat the case $\ell=1$ and iterate the formula. Finally, to obtain (5.31) with $\ell=1$, we use that, by the symbolic calculus of appendix, and since $a$ vanishes close to $\Lambda$, we may find a symbol $q$ in $S\left(\langle x\rangle^{-1}\right) \subset S(1)$, vanishing close to $\Lambda$, a symbol $\rho$ in $S(1)$ such that

$$
\mathrm{Op}_{h_{j}}(a)=\mathrm{Op}_{h_{j}}(q) \mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)+h_{j}^{N} \mathrm{Op}_{h_{j}}(\rho)
$$

for an arbitrary integer $N$. If $N$ is large enough, the fact that (4.9) holds implies that $\mathcal{Z}_{j}^{k}\left(h_{j}^{N} \mathrm{Op}_{h_{j}}(\rho) w_{j}\right)$ satisfies estimate (5.30) with $\ell=1$ for all $j \in J(h, C)$. We are thus reduced to showing that

$$
\begin{equation*}
\mathcal{Z}_{j}^{k} \mathrm{Op}_{h_{j}}(q) \mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) w_{j} \tag{5.32}
\end{equation*}
$$

may be written as the right hand side of (5.31) with $\ell=1$. We use (4.17). On the one hand, we get a contribution to (5.32) of the form

$$
\mathcal{Z}_{j}^{k}\left[h_{j} \mathrm{Op}_{h_{j}}(q)\left(\frac{i}{2} \widetilde{w}_{j}-i Z w_{j}\right)\right]
$$

that forms part of the sum in (5.31) with $\ell=1$. On the other hand, the nonlinear terms in (4.17) bring an expression

$$
\mathcal{Z}_{j}^{k} \mathrm{Op}_{h_{j}}(q)\left[-\sqrt{h} 2^{-\frac{j}{2}} \Theta_{-j}^{*} \Delta_{j}^{h} Q_{0}(W)-h 2^{-\frac{j}{2}} \Theta_{-j}^{*} \Delta_{j}^{h} C_{0}(W)-2^{-\frac{j}{2}} h^{\frac{5}{4}} \Theta_{-j}^{*} \Delta_{j}^{h} R(V)\right] .
$$

Let us check that these terms satisfy estimates (5.30) with $\ell=1$. For the quadratic terms, this follows from (3.12) (with $p=p_{k_{1}}=p_{k_{2}}=\infty$ ) and from the assumption $\delta_{k_{1}}^{\prime}+\delta_{k_{2}}^{\prime} \leq \delta_{k+1}^{\prime}$ if $k_{1}+k_{2} \leq k$, that follows from (4.1). For the cubic term (resp. the remainder) we use (3.15) (resp. (3.17)) and the estimates of $\left\|\mathcal{Z}^{k}\left\langle h D_{x}\right\rangle^{\alpha+d} V\right\|_{L^{\infty}}$ deduced from (4.9) with $d=b-\alpha-0$, $b^{\prime}<b-\alpha-0$. This concludes the proof.

We shall use the preceding lemma to give an asymptotic expansion of $\mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}^{c}\right) w_{j}$, assuming that we know a priori that $Q_{0}(w, \bar{w})$ admits the expansion given by equality (5.7) in Lemma 5.3, or that $Q_{0}(w, \bar{w})+\sqrt{h} C_{0}(w, \bar{w})$ obeys the equality (5.9) of the same lemma.

Lemma 5.8. $i$ ) Assume $b^{\prime}<b-5$ and that $Q_{0}(W)$ satisfies (5.7). Then there are functions $w_{ \pm 2 \Lambda}=\sum_{j \in J(h, C)} \Theta_{j}^{*} w_{ \pm 2 \Lambda, j}$ such that for any $k \leq \frac{s}{2}+N_{0}, \mathcal{Z}^{k} w_{ \pm 2 \Lambda}$ is an $O(\varepsilon)$ element of $h^{-2 \delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{ \pm 2 \Lambda}^{2, b^{\prime}+\frac{3}{2}}\left[K_{ \pm 2}\right]$ and a function $g=\sum_{j \in J(h, C)} \Theta_{j}^{*} g_{j}$ such that $\mathcal{Z}^{k} g$ is an $O(\varepsilon)$ element of $h^{-3 \delta_{k+1+N_{1}-N_{0}}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}[F]$ for some closed subset $F$ of $T^{*} \mathbb{R}$ whose second projection is compact in $\mathbb{R}^{*}$ and $\mathcal{Z}^{k} \mathrm{Op}_{h}(x \xi) g$ is an $O(\varepsilon)$ element of $h^{-3 \delta_{k+1+N_{1}-N_{0}}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{1, b^{\prime}}[F]$, so that

$$
\begin{align*}
w_{2 \Lambda} & =-i(1-\chi)\left(x h^{-\beta}\right) \frac{1+\sqrt{2}}{4}|\mathrm{~d} \omega(x)| w_{\Lambda}^{2}  \tag{5.33}\\
w_{-2 \Lambda} & =-i(1-\chi)\left(x h^{-\beta}\right) \frac{1-\sqrt{2}}{4}|\mathrm{~d} \omega(x)| \bar{w}_{\Lambda}^{2}
\end{align*}
$$

and for any $j$ in $J(h, C)$

$$
\begin{equation*}
\operatorname{Op}_{h_{j}}\left(\gamma_{\Lambda}^{c}\right) w_{j}=\sqrt{h}\left(w_{2 \Lambda, j}+w_{-2 \Lambda, j}\right)+h g_{j} . \tag{5.34}
\end{equation*}
$$

ii) Assume that $b^{\prime}<b-8$ and $Q_{0}(w, \bar{w})+\sqrt{h} C_{0}(w, \bar{w})$ obeys (5.9). Then there are functions $w_{ \pm 2 \Lambda}$ such that for any $k \leq \frac{s}{2}+N_{0}, \mathcal{Z}^{k} w_{ \pm 2 \Lambda}$ is an $O(\varepsilon)$ element of $h^{-2 \delta_{k+1}^{\prime}} L^{\infty} \tilde{J}_{ \pm 2 \Lambda}^{2, b^{\prime}+\frac{3}{2}}\left[K_{ \pm 2}\right]$, there are functions $w_{\ell \Lambda}=\sum_{j \in J(h, C)} \Theta_{j}^{*} w_{\ell \Lambda, j}$ for $\ell=-3,-1,3$, such that for $k \leq \frac{s}{2}+$ $N_{0}, \mathcal{Z}^{k} w_{\ell \Lambda}$ is an $O(\varepsilon)$ element in $h^{-3 \delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{\ell \Lambda}^{2, b^{\prime}+\frac{3}{2}}\left[K_{\ell}\right]$, a function $g=\sum_{j \in J(h, C)} \Theta_{j}^{*} g_{j}$
 $h^{-4 \delta_{k+1+N_{1}-N_{0}}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}-\frac{1}{2}}[F]$ so that

$$
\begin{align*}
\mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}^{c}\right) w_{j}= & \sqrt{h}\left(w_{2 \Lambda, j}+w_{-2 \Lambda, j}\right) \\
& +h\left(w_{3 \Lambda, j}+w_{-\Lambda, j}+w_{-3 \Lambda, j}\right)  \tag{5.35}\\
& +h^{1+\sigma} g_{j} .
\end{align*}
$$

Moreover, $w_{ \pm 2 \Lambda}$ is still given by (5.33) and

$$
\begin{align*}
w_{3 \Lambda} & =(1-\chi)\left(x h^{-\beta}\right) \lambda_{3}|\mathrm{~d} \omega(x)|^{2} w_{\Lambda}^{3} \\
w_{-\Lambda} & =(1-\chi)\left(x h^{-\beta}\right) \lambda_{-1}|\mathrm{~d} \omega(x)|^{2}\left|w_{\Lambda}\right|^{2} \bar{w}_{\Lambda}  \tag{5.36}\\
w_{-3 \Lambda} & =(1-\chi)\left(x h^{-\beta}\right) \lambda_{-3}|\mathrm{~d} \omega(x)|^{2} \bar{w}_{\Lambda}^{3}
\end{align*}
$$

for some real constants $\lambda_{3}, \lambda_{-1}, \lambda_{-3}$.

Proof. i) By Corollary A. 3 of the appendix, we may find symbols $a$ in $S\left(\langle x\rangle^{-1}\right), c$ in $S(1)$, supported in a domain $C_{0}^{-1} \leq|\xi| \leq C_{0}$ and outside a neighborhood of $\Lambda$ such that $\gamma_{\Lambda}^{c}=$ $a \#\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)+h_{j}^{N} c$. Moreover, we may write

$$
\begin{equation*}
a=\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)^{-1} \gamma_{\Lambda}^{c}+h_{j} a_{1} \tag{5.37}
\end{equation*}
$$

for some symbol $a_{1}$ in $S\left(\langle x\rangle^{-1}\right)$. We get

$$
\begin{equation*}
\mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}^{c}\right) w_{j}=\mathrm{Op}_{h_{j}}(a) \mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) w_{j}+h_{j}^{N} \mathrm{Op}_{h_{j}}(c) w_{j} . \tag{5.38}
\end{equation*}
$$

Since $h_{j} \leq h^{\sigma}$, taking $N$ large enough, we see that assumption (4.9) implies that the last term in (5.38) may be written as $h^{\frac{3}{2}} g_{j}$ with $\left\|\mathcal{Z}_{j}^{k} g_{j}\right\|_{L^{\infty}}=O\left(h^{-\delta_{k}^{\prime}} 2^{-j+b^{\prime}}\right)$ for $k \leq \frac{s}{2}+N_{0}$. By construction, $g_{j}$ is microlocally supported in a closed set of the form $C_{0}^{-1} \leq|\xi| \leq C_{0}$.

Moreover, since by Lemma $5.1 \mathcal{Z}^{k} w_{\Lambda}$ is in $h^{-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b-2}[K]$, we get that $\mathcal{Z}^{k} \mathrm{Op}_{h}(x \xi) w_{\Lambda}$ belongs to $h^{-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{1, b-2}[K]$. Since $\mathcal{Z}^{k} \mathrm{Op}_{h}(x \xi) w_{\Lambda^{c}}$ is by the same lemma in $h^{\frac{1}{2}-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{1, b-2}[F]+$ $h^{1-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b-2}[F]$, we get that $\mathcal{Z}^{k} \mathrm{Op}_{h}(x \xi) w$ belongs to $h^{-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{1, b-2}[F]+h^{1-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b-2}[F]$. Since $h 2^{-\frac{j}{2}}=h_{j}=O\left(h^{\sigma}\right)=O(1)$, this implies for $\left\|\mathcal{Z}_{j}^{k} \mathrm{Op}_{h_{j}}(x \xi) w_{j}\right\|_{L^{\infty}}$ a bound in

$$
2^{-\frac{j}{2}} h^{-\delta_{k+1}^{\prime}}\left[2^{\frac{j}{2}-j_{+}(b-2)}+h 2^{-j_{+}(b-2)}\right] \leq C h^{-\delta_{k+1}^{\prime}-j_{+}(b-2)}
$$

 contribution to the $g$ function in the statement of the lemma.

We use expression (4.17) to study the first term in the right hand side of (5.38). The contribution of

$$
\begin{equation*}
\mathcal{Z}_{j}^{k}\left[h_{j} \mathrm{Op}_{h_{j}}(a)\left(\frac{i}{2} \widetilde{w}_{j}-i Z w_{j}\right)\right] \tag{5.39}
\end{equation*}
$$

has according to (5.29) a bound of the form

$$
\begin{equation*}
C_{k} \varepsilon h^{\frac{1}{2}-\delta_{k+1+N_{1}-N_{0}}^{\prime}}\left(h+h_{j} h^{\frac{1}{2}}\right) 2^{-j_{+} b^{\prime}} \leq C_{k} \varepsilon h^{1+\sigma-\delta_{k+1+N_{1}-N_{0}}^{\prime}} 2^{-j_{+} b^{\prime}} \tag{5.40}
\end{equation*}
$$

using $h_{j}=O\left(h^{\sigma}\right)$. This will give a contribution to $g_{j}$ in (5.34) since the action of $\mathrm{Op}_{h_{j}}(x \xi)$ on (5.39) admits similar bounds as $a$ is in $S\left(\langle x\rangle^{-1}\right)$.

Let us examine the contribution of

$$
\begin{align*}
\mathcal{Z}_{j}^{k} & {\left[-\sqrt{h} 2^{-j / 2} \mathrm{Op}_{h_{j}}(a) \Theta_{-j}^{*} \Delta_{j}^{h} Q_{0}(W)-h 2^{-j / 2} \mathrm{Op}_{h_{j}}(a) \Theta_{-j}^{*} \Delta_{j}^{h} C_{0}(W)\right.} \\
& \left.-2^{-\frac{j}{2}} h^{\frac{5}{4}} \mathrm{Op}_{h_{j}}(a) \Theta_{-j}^{*} \Delta_{j}^{h} R(V)\right] \tag{5.41}
\end{align*}
$$

to (5.38). We use expressions (5.7). The contributions of $\tilde{g}_{2}, \tilde{g}_{3}$ to (5.41) induce in (5.34) an expression contributing to $h g_{j}$. Actually, they give terms whose $L^{\infty}$-norm is $O\left(h^{-2 \delta_{k+1}^{\prime}+1} 2^{-j_{+} b^{\prime}}\right)$. Moreover, the action of $\mathrm{Op}_{h_{j}}(x \xi)$ on these terms admit similar bounds, again because (5.41) contains a $\mathrm{Op}_{h_{j}}(a)$ operator in factor, with $a$ in $S\left(\langle x\rangle^{-1}\right)$. In the same way, the $L^{\infty}$-norm of the last term in (5.41) (and of the action of $\mathrm{Op}_{h_{j}}(x \xi)$ on it) may be estimated using (3.17) with $d=b^{\prime}-\alpha-0$ and the fact that $\left\|\mathcal{Z}^{k}\left\langle h D_{x}\right\rangle^{\alpha+d} V\right\|_{L^{\infty}}$ is bounded using (4.9) (The loss in $2^{j_{+}(\alpha+0)}=O\left(h^{-2 \beta(\alpha+0)}\right)$ coming from the right hand side of (3.17) is absorbed by part of the $h^{1 / 4}$-extra factor in the last term in (5.41)). This brings another contribution to $h g_{j}$. Consequently the only contribution to (5.41) that we are left with is

$$
\begin{equation*}
-\mathcal{Z}_{j}^{k}\left[2^{-j / 2} \sqrt{h} \mathrm{Op}_{h_{j}}(a) \Theta_{-j}^{*} \Delta_{j}^{h}\left(\widetilde{w}_{2 \Lambda}+\widetilde{w}_{-2 \Lambda}\right)\right] . \tag{5.42}
\end{equation*}
$$

We shall study this expression in part $i i$ ) of the proof below.
ii) We assume that $Q_{0}+\sqrt{h} C_{0}$ obeys (5.9) and write again (5.38), expressing the right hand side from (4.17). The contribution (5.39) brings, according to (5.40), part of the term $h^{1+\sigma} g$ in (5.35). The same holds for the remainder term in (4.17). We are thus reduced to the study of the quadratic and cubic terms in (5.41). By (5.9), we have an expression for $Q_{0}(W)+\sqrt{h} C_{0}(W)$. The term $\tilde{g}$ in that expansion will bring part of the $g_{j}$ term in (5.35). Consequently, we are reduced to studying

$$
\begin{align*}
& \mathcal{Z}_{j}^{k}\left[\sqrt{h} 2^{-j / 2} \mathrm{Op}_{h_{j}}(a) \Theta_{-j}^{*} \Delta_{j}^{h}\left(\widetilde{w}_{2 \Lambda}+\widetilde{w}_{-2 \Lambda}\right)\right] \\
& \mathcal{Z}_{j}^{k}\left[h 2^{-j / 2} \mathrm{Op}_{h_{j}}(a) \Theta_{-j}^{*} \Delta_{j}^{h}\left(\widetilde{w}_{3 \Lambda}+\widetilde{w}_{\Lambda}+\widetilde{w}_{-\Lambda}+\widetilde{w}_{-3 \Lambda}\right)\right] . \tag{5.43}
\end{align*}
$$

We notice first that $\widetilde{w}_{\Lambda}$ is microlocally supported close to $\Lambda$ while $\mathrm{Op}_{h_{j}}(a)$ cut-offs outside a neighborhood of that set. Consequently, the $\widetilde{w}_{\Lambda}$ term in the second formula (5.43) gives rise to a remainder. For $\ell \in\{-3,-2,-1,2,3\}$ and $j$ in $J(h, C)$, set

$$
\begin{equation*}
w_{\ell \Lambda, j}^{(1)}=-2^{-j / 2} \mathrm{Op}_{h_{j}}(a) \Theta_{-j}^{*} \Delta_{j}^{h} \widetilde{w}_{\ell \Lambda} . \tag{5.44}
\end{equation*}
$$

Expressions (5.41), (5.42), (5.43) show that (5.35) holds with $w_{\ell \Lambda, j}$ replaced by $w_{\ell \Lambda, j}^{(1)}$. We have to show that, up to a modification of $g_{j}$ in (5.35), $w_{\ell \Lambda, j}^{(1)}$ may be replaced by a function $w_{\ell \Lambda, j}$ such that $\sum_{j \in J(h, C)} \Theta_{j}^{*} w_{\ell \Lambda, j}=w_{\ell \Lambda}$ satisfies the conclusions of the lemma.

We write $\widetilde{w}_{\ell \Lambda}=\sum_{j^{\prime} \in J(h, C)} \Theta_{j^{\prime}}^{*} \widetilde{w}_{\ell \Lambda, j^{\prime}}$ and set

$$
\widetilde{w}_{\ell \Lambda, j}^{(1)}:=\Theta_{-j}^{*} \Delta_{j}^{h} \widetilde{w}_{\ell \Lambda}=\sum_{j^{\prime} \in J(h, C)} \Theta_{-j+j^{\prime}}^{*} \mathrm{Op}_{h_{j^{\prime}}}\left(\varphi\left(2^{-j+j^{\prime}} \xi\right)\right) \widetilde{w}_{\ell \Lambda, j^{\prime}} .
$$

Since $\widetilde{w}_{\ell \Lambda, j^{\prime}}=\mathrm{Op}_{h_{j^{\prime}}}(\widetilde{\varphi}(\xi)) \widetilde{w}_{\ell \Lambda, j^{\prime}}$ for some $\widetilde{\varphi}$ in $C_{0}^{\infty}\left(\mathbb{R}^{*}\right)$, we may limit the sum above to those $j^{\prime}$ satisfying $\left|j-j^{\prime}\right| \leq M$ for some $M$. This shows that $\left(\mathcal{Z}_{j}^{k} \widetilde{w}_{\ell \Lambda, j}^{(1)}\right)_{j}$ is a bounded family in

$$
h^{-2 \delta_{k+1}^{\prime}} L^{\infty} J_{ \pm 2 \Lambda}^{3, b^{\prime}+\frac{3}{2}}\left[K_{ \pm 2}\right] \subset h^{-2 \delta_{k+1}^{\prime}} L^{\infty} J_{ \pm 2 \Lambda}^{2, b^{\prime}+1}\left[K_{ \pm 2}\right]
$$

when $\ell= \pm 2$ and in $h^{-3 \delta_{k+1}^{\prime}} L^{\infty} I_{\ell \Lambda}^{3, b^{\prime}+\frac{3}{2}}\left[K_{\ell}\right]$ if $\ell \in\{-3,-1,3\}$, according to the assumptions made on $\widetilde{w}_{\ell \Lambda}$. In the expression (5.44) of

$$
w_{\ell \Lambda, j}^{(1)}=-2^{-j / 2} \mathrm{Op}_{h_{j}}(a) \widetilde{w}_{\ell \Lambda, j}^{(1)},
$$

we insert the decomposition (5.37) of $a$. Since $\gamma_{\Lambda}^{c}$ may be assumed to be equal to one close to $\ell \Lambda, \ell \neq 1$, if the support of $\gamma_{\Lambda}$ is close enough to $\Lambda$, we may write

$$
\left.a\right|_{\ell \Lambda}=\left(|\ell|^{\frac{1}{2}}-\ell\right)^{-1}|\mathrm{~d} \omega(x)|^{-\frac{1}{2}}+\left.h_{j} a_{1}\right|_{\ell \Lambda}
$$

for $\ell \in\{-3,-2,-1,2,3\}$. Consequently, (5.44) may be written as the sum of

$$
\begin{equation*}
w_{\ell \Lambda, j}=-2^{-\frac{j}{2}}\left(|\ell|^{\frac{1}{2}}-\ell\right)^{-1}|\mathrm{~d} \omega(x)|^{-\frac{1}{2}} \Theta_{-j}^{*} \Delta_{j}^{h} \widetilde{w}_{\ell \Lambda} \tag{5.45}
\end{equation*}
$$

and of

$$
\begin{equation*}
-2^{-\frac{j}{2}} \mathrm{Op}_{h_{j}}\left(c^{\ell}+h_{j} d^{\ell}\right) \widetilde{w}_{\ell \Lambda, j}^{(1)} \tag{5.46}
\end{equation*}
$$

where $c^{\ell}, d^{\ell}$ are symbols, with $c^{\ell}$ vanishing on $\ell \cdot \Lambda$.
Let us show first that (5.46) multiplied by $\sqrt{h}$ when $\ell= \pm 2$ and by $h$ when $\ell$ belongs to $\{-3,-1,3\}$ provides a contribution to $h^{1+\sigma} g_{j}$ in (5.35). Since $\left(\mathcal{Z}_{j}^{k} \widetilde{w}_{ \pm 2 \Lambda, j}^{(1)}\right)_{j}$ is a $O(\varepsilon)$ family in $h^{-2 \delta_{k+1}^{\prime}} L^{\infty} J_{ \pm 2 \Lambda}^{2, b^{\prime}+1}\left[K_{ \pm 2}\right]$ and $c^{ \pm 2}$ vanishes on $\pm 2 \Lambda$, we see that the $L^{\infty}$-norm of the action of $\mathcal{Z}_{j}^{k}$ on (5.46) with $\ell= \pm 2$ is bounded from above by

$$
C \varepsilon 2^{-\frac{j}{2}} h^{-2 \delta_{k+1}^{\prime} 2^{j-j_{+}\left(b^{\prime}+1\right)} h_{j} \leq C \varepsilon h^{1-2 \delta_{k+1}^{\prime}} 2^{-j_{+}\left(b^{\prime}+1\right)} . . ~}
$$

Consequently, when $\ell= \pm 2$, if we make act $\mathcal{Z}_{j}^{k}$ on (5.46) multiplied by $\sqrt{h}$, we obtain an element of $h^{\frac{3}{2}-2 \delta_{k+1}^{\prime}} \mathcal{B}_{\infty}^{0, b^{\prime}}[L]$ i.e. a contribution to $h^{1+\sigma} g_{j}$ in (5.35). In the same way, when $\ell \in\{-3,-1,3\}$, using that $\left(\mathcal{Z}_{j}^{k} \widetilde{w}_{\ell \Lambda, j}^{(1)}\right)_{j}$ is in $h^{-3 \delta_{k+1}^{\prime}} L^{\infty} I_{\ell \Lambda}^{2, b^{\prime}+1}\left[K_{\ell}\right]$, we may estimate the $L^{\infty}$-norm of (5.46) on which acts $\mathcal{Z}_{j}^{k}$ by

$$
C \varepsilon 2^{-\frac{j}{2}} h^{-3 \delta_{k+1}^{\prime}} 2^{j-j_{+}\left(b^{\prime}+1\right)}\left[h^{\frac{1}{2}}+h_{j}\right] \leq C \varepsilon h^{\frac{1}{2}-3 \delta_{k+1}^{\prime}} 2^{-j_{+}\left(b^{\prime}+\frac{1}{2}\right)} .
$$

Again, after multiplication by $h$, this gives a contribution to $h^{1+\sigma} g_{j}$ in (5.35). Notice that the fact that $\mathrm{Op}_{h}(x \xi) g=\sum_{j \in J(h, C)} 2^{\frac{j}{2}} \Theta_{j}^{*} \mathrm{Op}_{h_{j}}(x \xi) g_{j}$ satisfies the same estimates as $g$, with $b^{\prime}$ replaced by $b^{\prime}-\frac{1}{2}$, follows from the above bounds since $\widetilde{w}_{\ell \Lambda, j}^{(1)}$ is microlocally supported in a compact set of $T^{*}(\mathbb{R} \backslash\{0\})$.

We have thus shown that $\operatorname{Op}_{h_{j}}\left(\gamma_{\Lambda}^{c}\right) w_{j}$ is given by the right hand side of (5.35), with $w_{\ell \Lambda, j}$ given by (5.45). In particular since $|\mathrm{d} \omega|$ is positively homogeneous of degree -2 , we get

$$
w_{\ell \Lambda}=\sum_{j \in J(h, C)} \Theta_{j}^{*} w_{\ell \Lambda, j}=-\left(|\ell|^{\frac{1}{2}}-\ell\right)^{-1}|\mathrm{~d} \omega(x)|^{-\frac{1}{2}} \widetilde{w}_{\ell \Lambda} .
$$

Combining this with (5.8), (5.11), we obtain (5.33) and (5.36). Moreover, expressions (5.45) and the properties of $\widetilde{w}_{\ell \Lambda}$ obtained in $i i$ ) if Lemma 5.3 show that $\mathcal{Z}^{k} w_{ \pm 2 \Lambda}$ is in the space $h^{-2 \delta_{k+1}^{\prime}} L^{\infty} \tilde{J}_{ \pm 2 \Lambda}^{2, b^{\prime}+\frac{3}{2}}\left[K_{ \pm 2}\right]$ and that $w_{\ell \Lambda}, \ell \in\{-3,-1,3\}$ belongs to $L^{\infty} \tilde{I}_{\ell \Lambda}^{2, b^{\prime}+\frac{3}{2}}\left[K_{\ell}\right]$, and are $O(\varepsilon)$ in these spaces.

Proof of Proposition 5.2. Let $b^{\prime}$ satisfying the assumption of the proposition.
By Lemma 5.1, $\mathcal{Z}^{k} w_{\Lambda}$ is an $O(\varepsilon)$ element of $h^{-\delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{\Lambda}^{0, b-2}[K]$ and $\mathcal{Z}^{k} w_{\Lambda^{c}}$ is an $O(\varepsilon)$ element of $h^{\frac{1}{2}-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b-2}[F]+h^{1-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{-1, b-2}[F]$, for $0 \leq k \leq \frac{s}{2}+N_{1}$. We may therefore apply $i$ ) of Lemma 5.3 which shows that (5.7) holds. This allows us to use $i$ ) of Lemma 5.8. In that way, we obtain functions $w_{ \pm 2 \Lambda}$, in the spaces indicated in the statement of that lemma, such that (5.6) holds. Writing

$$
w=w_{\Lambda}+\sum_{j \in J(h, C)} \Theta_{j}^{*} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}^{c}\right) w_{j}
$$

and using (5.34), we obtain equality (5.5).
We still have to check that $\mathcal{Z}^{k} w_{\Lambda}$ is an $O(\varepsilon)$ element in $h^{-2 \delta_{k+1}^{\prime}} L^{\infty} \tilde{J}_{\Lambda}^{0, b^{\prime}}[K]$ since Lemma 5.1 was only ensuring that this function is $O(\varepsilon)$ in the space $h^{-\delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{\Lambda}^{0, b^{\prime}}[K]$. To do so, we must show that

$$
\begin{equation*}
\left\|\mathcal{Z}_{j}^{k} \mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) w_{\Lambda, j}\right\|_{L^{\infty}} \leq C \varepsilon h^{-2 \delta_{k+1}^{\prime}} h_{j} 2^{-j_{+} b^{\prime}} \tag{5.47}
\end{equation*}
$$

We notice that, by symbolic calculus and assumption (4.9)

$$
\left\|\mathcal{Z}_{j}^{k}\left[\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right), \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right)\right] w_{j}\right\|_{L^{\infty}}
$$

satisfies the wanted bound, since the commutators between the vector fields and $\mathrm{Op}_{h_{j}}(e)$, for a symbol $e$, are of the form $\mathrm{Op}_{h_{j}}(\tilde{e})$ for another symbol $\tilde{e}$. We may therefore study

$$
\left\|\mathcal{Z}_{j}^{k} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) \mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) w_{j}\right\|_{L^{\infty}} .
$$

Using the commutation relation

$$
\left[t D_{t}+x D_{x}, \mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)\right]=i \mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)
$$

we see that the above quantity may be estimated from

$$
\left\|\mathrm{Op}_{h_{j}}\left(\widetilde{\gamma}_{\Lambda}\right) \mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) \mathcal{Z}_{j}^{k^{\prime}} w_{j}\right\|_{L^{\infty}}
$$

for $k^{\prime} \leq k$ and $\widetilde{\gamma}_{\Lambda}$ a symbol with $\operatorname{Supp} \widetilde{\gamma}_{\Lambda} \subset \operatorname{Supp} \gamma_{\Lambda}$. We use now (4.15), which provides the wanted bound of type (5.47), up to a similar estimate for

$$
2^{-\frac{j}{2}} \sqrt{h}\left\|\mathrm{Op}_{h_{j}}\left(\widetilde{\gamma}_{\Lambda}\right) \mathcal{Z}_{j}^{k^{\prime}} \Theta_{-j}^{*} \Delta_{j}^{h} Q_{0}(W)\right\|_{L^{\infty}}
$$

To study this quantity, we need to exploit the structure of $Q_{0}(W)$ given by (5.7). The remainder in the first equation (5.7) gives a contribution bounded by the right hand side of (5.47). On the other hand,

$$
\left\|\operatorname{Op}_{h_{j}}\left(\widetilde{\gamma}_{\Lambda}\right) \mathcal{Z}_{j}^{k^{\prime}} \Theta_{-j}^{*} \Delta_{j}^{h}\left(\widetilde{w}_{2 \Lambda}+\widetilde{w}_{-2 \Lambda}\right)\right\|_{L^{\infty}}
$$

is $O\left(\varepsilon h^{\infty}\right)$ since $\widetilde{\gamma}_{\Lambda}$ cuts-off on a neighborhood of $\Lambda$ while $\widetilde{w}_{ \pm 2 \Lambda}$ are supported close to $\pm 2 \Lambda$, so outside such a neighborhood. This concludes the proof of (5.47), whence the proposition.

Let us deduce from expansion (5.5) of $w$ a second refined decomposition of $w$ in oscillating factors.

Corollary 5.9. Under the assumptions of Proposition 5.2 with moreover $b^{\prime}<b-8$, we may write

$$
\begin{equation*}
w=w_{\Lambda}+\sqrt{h}\left(w_{2 \Lambda}+w_{-2 \Lambda}\right)+h\left(w_{3 \Lambda}+w_{-\Lambda}+w_{-3 \Lambda}\right)+h^{1+\sigma} g \tag{5.48}
\end{equation*}
$$

where $w_{ \pm 2 \Lambda}, w_{ \pm 3 \Lambda}, w_{-\Lambda}$ satisfy the conclusions of ii) of Lemma 5.8 and are given in terms of $w_{\Lambda}$ by (5.33), (5.36), and where for $k \leq \frac{s}{2}+N_{0}, \mathcal{Z}^{k} g$ is $O(\varepsilon)$ in $h^{-4 \delta_{k+1+N_{1}-N_{0}}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}[F]$ and


Proof. By Proposition 5.2, w may be written as (5.5). Consequently, the assumptions of $i i$ ) of Lemma 5.3 hold and this lemma implies a decomposition (5.9) for

$$
\sum_{j \in J(h, C)} \Delta_{j}^{h}\left[Q_{0}(W)+\sqrt{h} C_{0}(W)\right] .
$$

This shows that the assumptions of $i i$ ) of Lemma 5.8 hold. According to this lemma, $\mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}^{c}\right) w_{j}$ is given by (5.35). We define $w_{\ell \Lambda}=\sum_{j \in J(h, C)} \Theta_{j}^{*} w_{\ell \Lambda, j}$ and get (5.48), remembering that we defined $w_{\Lambda}=\sum_{j \in J(h, C)} \Theta_{j}^{*} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) w_{j}$ if $w=\sum_{j \in J(h, C)} \Theta_{j}^{*} w_{j}$. The expansion in terms of $w_{\Lambda}, \bar{w}_{\Lambda}$ follow from (5.33), (5.36).

We have seen in Proposition 5.2, that $\mathcal{Z}^{k} w_{\Lambda}$ is an $O(\varepsilon)$ element in $h^{-\delta_{k+1}^{\prime}} L^{\infty} \tilde{J}_{\Lambda}^{0, b^{\prime}}[K]$. We need a more precise description of this quantity.

Proposition 5.10. Let $w_{\Lambda}=\sum_{j \in J(h, C)} \Theta_{j}^{*} w_{\Lambda, j}$ be the function introduced in Proposition 5.2. There are elements $f=\sum_{j \in J(h, C)} \Theta_{j}^{*} f_{j}$ where $\mathcal{Z}^{k} f$ is $O(\varepsilon)$ in $h^{-3 \delta_{k+2}^{\prime}} L^{\infty} \tilde{I}_{\Lambda}^{0, b^{\prime}}[K]$ for $k \leq$ $\frac{s}{2}+N_{0}$ and $r=\sum_{j \in J(h, C)} \Theta_{j}^{*} r_{j}$, with $\mathcal{Z}^{k} r$ of size $O(\varepsilon)$ in $h^{-4 \delta_{k+N_{1}-N_{0}+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}[F]$ such that

$$
\begin{equation*}
\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) w_{\Lambda, j}=h_{j}\left[f_{j}+h^{\frac{1}{4}} r_{j}\right] . \tag{5.49}
\end{equation*}
$$

Proof. We use the definition of $w_{j, \Lambda}=\operatorname{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) w_{j}$ and (4.17) to write

$$
\begin{align*}
\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) w_{j, \Lambda}= & {\left[\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right), \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right)\right] w_{j} } \\
& -\sqrt{h} 2^{-\frac{j}{2}} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) \Theta_{-j}^{*} \Delta_{j}^{h} Q_{0}(W) \\
& +h 2^{-\frac{j}{2}} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right)\left[\frac{i}{2} \widetilde{w}_{j}-i Z w_{j}-\Theta_{-j}^{*} \Delta_{j}^{h} C_{0}(W)\right]  \tag{5.50}\\
& -2^{-\frac{j}{2}} h^{\frac{5}{4}} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) \Theta_{-j}^{*} \Delta_{j}^{h} R(V) .
\end{align*}
$$

The commutator term may be written $h_{j} \mathrm{Op}_{h_{j}}(e) w_{j}$ for some symbol $e$ in $S(1)$, with support contained in Supp $\gamma_{\Lambda}$ (up to a $O\left(h_{j}^{\infty}\right)=O\left(h^{\infty}\right)$ remainder). Consequently $\mathrm{Op}_{h_{j}}(e) w_{j}$ will satisfy the same type of properties as $w_{\Lambda, j}$ i.e. by Lemma 5.1, $\left(\mathcal{Z}_{j}^{k} \mathrm{Op}_{h_{j}}(e) w_{j}\right)_{j}$ will be a $O(\varepsilon)$ family in $h^{-\delta_{k+1}^{\prime}} L^{\infty} I_{\Lambda}^{0, b^{\prime}}[K]$ so that the first term in the right hand side of (5.50) contributes to $h_{j} f_{j}$ in (5.49).

To study the quadratic and cubic terms in (5.50), we use expression (5.9) for $Q_{0}(W)+$ $\sqrt{h} C_{0}(W)$. The remainder $\tilde{g}$ in (5.9) will bring a contribution to $h_{j} h^{\frac{1}{2}} r_{j}$ in (5.49). The contribution

$$
\sqrt{h} 2^{-\frac{j}{2}} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) \Theta_{-j}^{*} \Delta_{j}^{h}\left(\widetilde{w}_{2 \Lambda}+\widetilde{w}_{-2 \Lambda}+\sqrt{h}\left(\widetilde{w}_{3 \Lambda}+\widetilde{w}_{-\Lambda}+\widetilde{w}_{-3 \Lambda}\right)\right)
$$

and its $\mathcal{Z}$-derivatives are $O\left(\varepsilon h^{\infty}\right)$ since $\gamma_{\Lambda}$ cuts-off close to $\Lambda$, while the terms on which it acts are supported close to $\ell \cdot \Lambda,|\ell| \leq 3, \ell \neq 1$. Consequently, the only remaining term coming from (5.9) is

$$
-h 2^{-\frac{j}{2}} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) \Theta_{-j}^{*} \Delta_{j}^{h} \widetilde{w}_{\Lambda}
$$

Since

$$
\mathcal{Z}^{k} \widetilde{w}_{\Lambda}=\sum_{j^{\prime} \in J(h, C)} \mathcal{Z}^{k} \Theta_{j^{\prime}}^{*} \widetilde{w}_{\Lambda, j^{\prime}} \quad \text { is } O(\varepsilon) \text { in } h^{-3 \delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{\Lambda}^{0, b^{\prime}}[K]
$$

we get a contribution to $h_{j} f_{j}$ in (5.49).
The remainder term $2^{-\frac{j}{2}} h^{\frac{5}{4}} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right) \Theta_{-j}^{*} \Delta_{j}^{h} R(V)$ will contribute to the last term in (5.36) as it has been seen in the estimate of the last term in (5.41).

Finally, we are left with studying

$$
\begin{equation*}
h 2^{-\frac{j}{2}} \mathrm{Op}_{h_{j}}\left(\gamma_{\Lambda}\right)\left[\frac{i}{2} \widetilde{w}_{j}-i Z w_{j}\right] \tag{5.51}
\end{equation*}
$$

We use $i i i$ ) of Lemma 4.4 to bound the action of $\mathcal{Z}_{j}^{k} \mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)$ on (5.51). We obtain an $L^{\infty}$ bound of the form

$$
C h_{j} 2^{-j+b^{\prime}}\left[h^{\frac{1}{2}} \sum_{k_{1}+k_{2} \leq k+1} \mathcal{E}_{k_{1}}(v) \mathcal{E}_{k_{2}}(v)+h^{\frac{1}{2}} \sum_{k_{1}+k_{2}+k_{3} \leq k+1} \mathcal{E}_{k_{1}}(v) \mathcal{E}_{k_{2}}(v) \mathcal{E}_{k_{3}}(v)+h_{j} \mathcal{E}_{k+2}(v)\right] .
$$

Using (4.1) we bound this by $C h_{j}\left(h^{\frac{1}{2}}+h_{j}\right) \mathcal{E}_{k+2}(v)$ so, according to (4.9), by

$$
C h_{j} h^{-\delta_{k+2}^{\prime}}\left(h^{\frac{1}{2}}+h_{j}\right) 2^{-j+b^{\prime}} .
$$

We obtain in that way a contribution to $h_{j} f_{j}$ in (5.49).

In the following section, we shall need estimates not only for $w$, but also for

$$
w^{(\ell)}=\mathrm{Op}_{h}\left(\langle\xi\rangle^{\ell}\right) w, \quad w_{\Lambda}^{(\ell)}=\mathrm{Op}_{h}\left(\langle\xi\rangle^{\ell}\right) w_{\Lambda}
$$

where $\ell$ is an integer $0 \leq \ell \leq \frac{s}{2}+N_{0}+b$. Let us deduce from Corollary 5.9 an expansion for $w^{(\ell)}$ in terms of $w_{\Lambda}^{(\ell)}, \ell=0, \ldots, \frac{s}{2}+N_{0}+b^{\prime}$.
Corollary 5.11. Under the assumptions of Proposition 5.2, for $\ell=0, \ldots, \frac{s}{2}+N_{0}+b^{\prime}$ we may write

$$
\begin{equation*}
w^{(\ell)}=w_{\Lambda}^{(\ell)}+\sqrt{h}\left(w_{2 \Lambda}^{(\ell)}+w_{-2 \Lambda}^{(\ell)}\right)+h\left(w_{3 \Lambda}^{(\ell)}+w_{-\Lambda}^{(\ell)}+w_{-3 \Lambda}^{(\ell)}\right)+h^{1+\sigma} g^{(\ell)} \tag{5.52}
\end{equation*}
$$

where for any $k \leq \min \left(\frac{s}{2}+N_{0}, \frac{s}{2}+N_{0}+b^{\prime}-\ell\right), \mathcal{Z}^{k} g^{(\ell)}$ is in $h^{-4 \delta_{k+1+N_{1}-N_{0}}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}-\ell}[F]$,


$$
\begin{align*}
w_{2 \Lambda}^{(\ell)} & =-i(1-\chi)\left(x h^{-\beta}\right)\langle 2 \mathrm{~d} \omega\rangle^{\ell}\langle\mathrm{d} \omega\rangle^{-2 \ell}|\mathrm{~d} \omega| \frac{1+\sqrt{2}}{4}\left(w_{\Lambda}^{(\ell)}\right)^{2} \\
w_{-2 \Lambda}^{(\ell)} & =-i(1-\chi)\left(x h^{-\beta}\right)\langle 2 \mathrm{~d} \omega\rangle^{\ell}\langle\mathrm{d} \omega\rangle^{-2 \ell}|\mathrm{~d} \omega| \frac{1-\sqrt{2}}{4}\left(\bar{w}_{\Lambda}^{(\ell)}\right)^{2} \\
w_{3 \Lambda}^{(\ell)} & =(1-\chi)\left(x h^{-\beta}\right)\langle 3 \mathrm{~d} \omega\rangle^{\ell}\langle\mathrm{d} \omega\rangle^{-3 \ell}|\mathrm{~d} \omega|^{2} \lambda_{3}^{\ell}\left(w_{\Lambda}^{(\ell)}\right)^{3}  \tag{5.53}\\
w_{-\Lambda}^{(\ell)} & =(1-\chi)\left(x h^{-\beta}\right)\langle\mathrm{d} \omega\rangle^{-2 \ell}|\mathrm{~d} \omega|^{2} \lambda_{-1}^{\ell}\left|w_{\Lambda}^{(\ell)}\right|^{2} \bar{w}_{\Lambda}^{(\ell)} \\
w_{-3 \Lambda}^{(\ell)} & =(1-\chi)\left(x h^{-\beta}\right)\langle 3 \mathrm{~d} \omega\rangle^{\ell}\langle\mathrm{d} \omega\rangle^{-3 \ell}|\mathrm{~d} \omega|^{2} \lambda_{-3}^{\ell}\left(\bar{w}_{\Lambda}^{(\ell)}\right)^{3}
\end{align*}
$$

where $\lambda_{3}^{\ell}, \lambda_{-1}^{\ell}, \lambda_{-3}^{\ell}$ are real constants, $\chi \in C_{0}^{\infty}(\mathbb{R}), \chi \equiv 1$ close to zero, with small enough support.

Finally, in the decomposition

$$
w_{\Lambda}^{(\ell)}=\sum_{j \in J(h, C)} \Theta_{j}^{*} w_{\Lambda, j}^{(\ell)}
$$

of $w_{\Lambda}^{(\ell)}$ deduced from the one of $w_{\Lambda}$, we may write

$$
\begin{equation*}
\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) w_{\Lambda, j}^{(\ell)}=h_{j}\left(f_{j}^{(\ell)}+h^{\frac{1}{4}} r_{j}^{(\ell)}\right) \tag{5.54}
\end{equation*}
$$

where

$$
\begin{aligned}
& \left(\mathcal{Z}_{j}^{k} f_{j}^{(\ell)}\right)_{j} \text { is a } O(\varepsilon) \text { family in } h^{-3 \delta_{k+2}^{\prime}} L^{\infty} I_{\Lambda}^{0, b^{\prime}-\ell}[K] \\
& \left(\mathcal{Z}_{j}^{k} r_{j}^{(\ell)}\right)_{j} \text { is a } O(\varepsilon) \text { family in } h^{-4 \delta_{k+1+N_{1}-N_{0}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}-\ell}}[L] .
\end{aligned}
$$

Proof. By definition, $w_{\Lambda}=\mathrm{Op}_{h}\left(\langle\xi\rangle^{-\ell}\right) w_{\Lambda}^{(\ell)}$ and $\mathcal{Z}^{k} w_{\Lambda}^{(\ell)}$ belongs to $h^{-\delta_{k+1}^{\prime}} L^{\infty} \tilde{J}_{\Lambda}^{0, b^{\prime}-\ell}[K]$. By Proposition 2.11, we may write

$$
w_{\Lambda}=(1-\chi)\left(x h^{-\beta}\right)\langle\mathrm{d} \omega\rangle^{-\ell} w_{\Lambda}^{(\ell)}+h r_{1}
$$

where $\mathcal{Z}^{k} r_{1}$ is in $h^{-\delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{-1, b^{\prime}}[K]$ and the action of $\mathcal{Z}^{k}$ on $(1-\chi)\left(x h^{-\beta}\right)\langle\mathrm{d} \omega\rangle^{-\ell} w_{\Lambda}^{(\ell)}$ is in $h^{-\delta_{k+1}^{\prime}} L^{\infty} \tilde{J}_{\Lambda}^{0, b^{\prime}}[K]$. We apply Proposition 2.12 to compute powers of $w_{\Lambda}$

$$
\begin{align*}
\left(w_{\Lambda}\right)^{2} & =(1-\chi)\left(x h^{-\beta}\right)^{2}\langle\mathrm{~d} \omega\rangle^{-2 \ell}\left(w_{\Lambda}^{(\ell)}\right)^{2}+h r_{2} \\
\left(\bar{w}_{\Lambda}\right)^{2} & =(1-\chi)\left(x h^{-\beta}\right)^{2}\langle\mathrm{~d} \omega\rangle^{-2 \ell}\left(\bar{w}_{\Lambda}^{(\ell)}\right)^{2}+h r_{-2} \\
\left(w_{\Lambda}\right)^{3} & =(1-\chi)\left(x h^{-\beta}\right)^{3}\langle\mathrm{~d} \omega\rangle^{-3 \ell}\left(w_{\Lambda}^{(\ell)}\right)^{3}+h r_{3}  \tag{5.55}\\
\left|w_{\Lambda}\right|^{2} w_{\Lambda}^{2} & =(1-\chi)\left(x h^{-\beta}\right)^{3}\langle\mathrm{~d} \omega\rangle^{-3 \ell}\left|w_{\Lambda}^{(\ell)}\right|^{2} w_{\Lambda}^{(\ell)}+h r_{1} \\
\left|w_{\Lambda}\right|^{2} \bar{w}_{\Lambda}^{2} & =(1-\chi)\left(x h^{-\beta}\right)^{3}\langle\mathrm{~d} \omega\rangle^{-3 \ell}\left|w_{\Lambda}^{(\ell)}\right|^{2} \bar{w}_{\Lambda}^{(\ell)}+h r_{-1} \\
\left(\bar{w}_{\Lambda}\right)^{3} & =(1-\chi)\left(x h^{-\beta}\right)^{3}\langle\mathrm{~d} \omega\rangle^{-3 \ell}\left(\bar{w}_{\Lambda}^{(\ell)}\right)^{3}+h r_{-3}
\end{align*}
$$

where the action of $\mathcal{Z}^{k}$ on $r_{q}$ gives an element of $h^{-2 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{-2,2 b^{\prime}-\frac{1}{2}}\left[K_{q}\right]$ if $q= \pm 2$ and of $h^{-3 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{-3,3 b^{\prime}-1}\left[K_{q}\right]$ if $q=3,1,-1,-3$.

On the other hand, consider the contributions $w_{q \Lambda},|q| \leq 3, q \neq 1$, to the expansion (5.48) and define $w_{q \Lambda}^{(\ell), 1}=\mathrm{Op}_{h}\left(\langle\xi\rangle^{\ell}\right) w_{q \Lambda}$ so that (5.48) may be written

$$
\begin{equation*}
w^{(\ell)}=w_{\Lambda}^{(\ell)}+\sqrt{h}\left(w_{2 \Lambda}^{(\ell), 1}+w_{-2 \Lambda}^{(\ell), 1}\right)+h\left(w_{3 \Lambda}^{(\ell), 1}+w_{-\Lambda}^{(\ell), 1}+w_{-3 \Lambda}^{(\ell), 1}\right)+h^{1+\sigma} g^{(\ell)} \tag{5.56}
\end{equation*}
$$

where $g^{(\ell)}$ satisfies the bounds of the remainder in (5.52). We apply again Proposition 2.11 to get an expansion of $w_{q \Lambda}^{(\ell), 1}$. Since by $\left.i i\right)$ of Lemma 5.8,

$$
\begin{aligned}
& \mathcal{Z}^{k} w_{ \pm 2 \Lambda} \text { is in } h^{-2 \delta_{k+1}^{\prime}} L^{\infty} \tilde{J}_{\Lambda}^{2, b^{\prime}+\frac{3}{2}}\left[K_{ \pm 2}\right], \\
& \mathcal{Z}^{k} w_{q \Lambda} \text { is in } h^{-3 \delta_{k+1}^{\prime}} L^{\infty} \tilde{I}_{\Lambda}^{2, b^{\prime}+\frac{3}{2}}\left[K_{q}\right], \text { for } q=-3,-1,3,
\end{aligned}
$$

we obtain that

$$
\begin{align*}
& w_{ \pm 2 \Lambda}^{(\ell), 1}=(1-\chi)\left(x h^{-\beta}\right)\langle 2 \mathrm{~d} \omega\rangle^{\ell} w_{ \pm 2 \Lambda}+h r_{ \pm 2}^{(\ell), 1}  \tag{5.57}\\
& w_{q \Lambda}^{(\ell), 1}=(1-\chi)\left(x h^{-\beta}\right)\langle q \mathrm{~d} \omega\rangle^{\ell} w_{q \Lambda}+h^{\frac{1}{2}} r_{q}^{(\ell), 1}, \quad q=-3,-1,3,
\end{align*}
$$

where $\mathcal{Z}^{k} r_{ \pm 2}^{(\ell), 1}$ is in $h^{-2 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{1, b^{\prime}+\frac{3}{2}-\ell}\left[K_{2}\right]$ and for $q=-3,-1,3$,

$$
\mathcal{Z}^{k} r_{q}^{(\ell), 1} \text { is in } h^{-3 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{1, b^{\prime}+1-\ell}\left[K_{q}\right] .
$$

We deduce from (5.56) that

$$
\begin{align*}
w^{(\ell)}= & w_{\Lambda}^{(\ell)}+\sqrt{h}\langle 2 \mathrm{~d} \omega\rangle^{\ell}(1-\chi)\left(x h^{-\beta}\right)\left(w_{2 \Lambda}+w_{-2 \Lambda}\right) \\
& +h(1-\chi)\left(x h^{-\beta}\right)\left(\langle 3 \mathrm{~d} \omega\rangle^{\ell} w_{3 \Lambda}+\langle\mathrm{d} \omega\rangle^{\ell} w_{-\Lambda}+\langle 3 \mathrm{~d} \omega\rangle^{\ell} w_{-3 \Lambda}\right)  \tag{5.58}\\
& +h^{1+\sigma} g^{(\ell)}
\end{align*}
$$

with a new remainder $g^{(\ell)}$ as in (5.52). We use next (5.33), (5.36) to express $w_{q \Lambda}$ from $w_{\Lambda}$, $\bar{w}_{\Lambda}$ and (5.55) to compute the resulting quantities from $w_{\Lambda}^{(\ell)}, \bar{w}_{\Lambda}^{(\ell)}$. We get expressions (5.53), with $(1-\chi)$ replaced eventually by some of its powers. As already seen, these powers may be replaced by $(1-\chi)$, up to $O\left(h^{\infty}\right)$ remainders.

The remainders coming from the ones in (5.55) may be expressed as the product of $h^{\frac{3}{2}}$ (resp. $h^{2}$ ) with $(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|\langle 2 \mathrm{~d} \omega\rangle^{\ell} r_{ \pm 2}$ (resp. $(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{2}\langle q \mathrm{~d} \omega\rangle^{\ell} r_{\ell}, \ell=$ $-3,-1,3$ ). By Proposition 2.11, and since $|\mathrm{d} \omega|\langle 2 \mathrm{~d} \omega\rangle^{\ell}$ (resp. $|\mathrm{d} \omega|^{2}\langle q \mathrm{~d} \omega\rangle^{\ell}$ ) satisfies (2.15) with $\left(\ell, \ell^{\prime}, d, d^{\prime}\right)$ replaced by $(-2 \ell-2,2 \ell, 0,0)$ (resp. $(-2 \ell-4,2 \ell, 0,0)$ ), we obtain that the action of $\mathcal{Z}^{k}$ on these functions gives elements belonging to

$$
h^{-2 \delta_{k+1}^{\prime} \tilde{\mathcal{B}}_{\infty}^{0,2 b^{\prime}-\frac{1}{2}-\ell}\left[K_{\ell}\right] \subset h^{-2 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}-\ell}\left[K_{\ell}\right], ~}
$$

(resp. $h^{-3 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{1,3 b^{\prime}-1-\ell}\left[K_{\ell}\right] \subset h^{-3 \delta_{k+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}-\ell}\left[K_{\ell}\right]$ ) for $\ell \in\{-3, \ldots, 3\}$ so that we obtain again a contribution to $g^{\ell}$. (Notice that the action of $\mathrm{Op}_{h}(x \xi)$ on these remainders give elements of the same spaces with $b$ replaced by $b-1 / 2$, since they are microlocally supported in a compact subset of $\left.T^{*}(\mathbb{R} \backslash\{0\})\right)$. This concludes the proof of (5.52).

To prove (5.54), we first write, according to the definition of $w_{\Lambda}^{(\ell)}$ and (2.13), that $w_{\Lambda, j}^{(\ell)}=$ $\mathrm{Op}_{h_{j}}\left(\left\langle 2^{j} \xi\right\rangle^{\ell}\right) w_{\Lambda, j}$. Making act $\mathrm{Op}_{h_{j}}\left(\left\langle 2^{j} \xi\right\rangle^{\ell}\right)$ on (5.49), we get for the left hand side of (5.54) an expression given by its right hand side, modulo a term

$$
\left[\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right), \mathrm{Op}_{h_{j}}\left(\left\langle 2^{j} \xi\right\rangle^{\ell}\right)\right] w_{\Lambda, j}
$$

Since $\left(\mathcal{Z}^{k} w_{\Lambda, j}\right)_{j}$ is a bounded family in $h^{-3 \delta_{k+1}^{\prime}} L^{\infty} J_{\Lambda}^{0, b^{\prime}}[K]$, we see using symbolic calculus, that this expression contributes to the $h_{j} f_{j}^{(\ell)}$ term in (5.54).

## 6 Ordinary differential equation for $w_{\Lambda}$

We consider a solution $v$ of (3.6), satisfying for $h$ in some interval $\left.] h^{\prime}, 1\right]$ the a priori estimate (4.8) for $k^{\prime} \leq k+1$, with $k \leq s-a-1$. By Proposition 4.1, we know that $v$ satisfies then (4.9), and by Corollary 5.9, that $v=v_{L}+w+v_{H}$, where $w$ has an expansion (5.48). Our goal here is to deduce from that and from the equation satisfied by $w$, a uniform estimate for $\left\|\mathrm{Op}_{h}\left(\langle\xi\rangle^{\ell}\right) w(t, \cdot)\right\|_{L^{\infty}}$, and estimates for $\left\|\mathcal{Z}^{k} \mathrm{Op}_{h}\left(\langle\xi\rangle^{\ell}\right) w(t, \cdot)\right\|_{L^{\infty}}$ which are not uniform, but which are better than (4.9) (i.e. that involve exponents closer to zero than the $\delta_{k^{\prime}}^{\prime}$ ).

For $0 \leq \ell \leq \frac{s}{2}+N_{0}$ and $0 \leq k \leq \frac{s}{2}+N_{0}-\ell$ we define

$$
\begin{equation*}
W^{k,(\ell)}=\left(Z^{k^{\prime}} w^{(\ell)}\right)_{0 \leq k^{\prime} \leq k} \tag{6.1}
\end{equation*}
$$

where

$$
w^{(\ell)}=\mathrm{Op}_{h}\left(\langle\xi\rangle^{\ell}\right) w,
$$

as in the preceding section. The estimates we are looking for will follow from an ordinary differential equation satisfied by $W^{k,(\ell)}$.

Proposition 6.1. Under the preceding assumptions, the function $w^{(\ell)}$ satisfies the equation (6.2)

$$
\begin{aligned}
D_{t} w^{(\ell)}= & \frac{1}{2}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega(x)|^{\frac{1}{2}} w^{(\ell)} \\
& -i \frac{\sqrt{h}}{8}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{3}{2}}\langle\mathrm{~d} \omega\rangle^{-2 \ell}\langle 2 \mathrm{~d} \omega\rangle^{\ell}\left[(1+\sqrt{2})\left(w^{(\ell)}\right)^{2}-3(1-\sqrt{2})\left(\bar{w}^{(\ell)}\right)^{2}\right] \\
& +h\left[\Phi_{3}^{(\ell)}\left(w^{(\ell)}\right)^{3}+\Phi_{1}^{(\ell)}\left|w^{(\ell)}\right|^{2} w^{(\ell)}+\Phi_{-1}^{(\ell)}\left|w^{(\ell)}\right|^{2} \bar{w}^{(\ell)}+\Phi_{-3}^{(\ell)}\left(\bar{w}^{(\ell)}\right)^{3}\right] \\
& +h^{1+\kappa} r^{(\ell)}(t, x)
\end{aligned}
$$

where $\chi$ is in $C_{0}^{\infty}(\mathbb{R})$, equal to one close to zero, with small enough support, where $\kappa$ is a small positive number, where $\Phi_{j}^{(\ell)},-3 \leq j \leq 3$ are given by

$$
\begin{align*}
& \Phi_{1}^{(\ell)}(x)=(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{5}{2}}\langle\mathrm{~d} \omega\rangle^{-2 \ell}\left[\frac{\langle 2 \mathrm{~d} \omega\rangle^{2 \ell}}{\langle\mathrm{~d} \omega\rangle^{2 \ell}} \frac{3(3-2 \sqrt{2})}{16}+\frac{1}{2}\right]  \tag{6.3}\\
& \Phi_{j}^{(\ell)}(x)=(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{5}{2}} \Gamma_{j}^{(\ell)}(\mathrm{d} \omega) \quad \ell \neq 1
\end{align*}
$$

for some real valued symbols of order $-2 \ell, \Gamma_{j}^{(\ell)}$, and where $\left\|\left(h D_{x}\right)^{p} \mathcal{Z}^{k} r^{(\ell)}(t, x)\right\|_{L^{\infty}}$ is $O(\varepsilon)$ for any integers $k, p, \ell$ with $k \leq \frac{s}{2}+N_{0}-\ell, 0 \leq p \leq b^{\prime}-1$. Moreover, $W^{k,(\ell)}$ defined by (6.1) satisfies a system of the form

$$
\begin{align*}
D_{t} W^{k,(\ell)}= & \frac{1}{2}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega(x)|^{\frac{1}{2}} W^{k,(\ell)} \\
& +\sqrt{h} Q^{k,(\ell)}\left[x, h ; W^{k,(\ell)}, \bar{W}^{k,(\ell)}\right]  \tag{6.4}\\
& +h \mathcal{C}^{k,(\ell)}\left[x, h ; W^{k,(\ell)}, \bar{W}^{k,(\ell)}\right]+h^{1+\kappa} R^{k,(\ell)}(t, x)
\end{align*}
$$

- where $\left\|\left(h D_{x}\right)^{p} Z^{k^{\prime}} R^{k,(\ell)}(t, \cdot)\right\|_{L^{\infty}}=O(\varepsilon)$ for $0 \leq p \leq b^{\prime \prime} \leq b^{\prime}-2 / \beta$ and $k^{\prime} \leq \frac{s}{2}+N_{0}-\ell-k$;
- where $Q^{k,(\ell)}$ is a vector valued quadratic map in $\left(W^{k,(\ell)}, \bar{W}^{k,(\ell)}\right)$ whose components are linear combination of functions of the form

$$
\begin{align*}
& \theta\left(x h^{-\beta}\right) \Phi(x)\left(Z^{k_{1}} w^{(\ell)}\right)\left(Z^{k_{2}} w^{(\ell)}\right) \\
& \theta\left(x h^{-\beta}\right) \Phi(x)\left(Z^{k_{1}} w^{(\ell)}\right)\left(Z^{k_{2}} \bar{w}^{(\ell)}\right)  \tag{6.5}\\
& \theta\left(x h^{-\beta}\right) \Phi(x)\left(Z^{k_{1}} \bar{w}^{(\ell)}\right)\left(Z^{k_{2}} \bar{w}^{(\ell)}\right)
\end{align*}
$$

for $k_{1}+k_{2} \leq k$, with smooth functions $\theta$ bounded as well as their derivatives, $\theta \equiv 0$ close to zero and $\Phi$ satisfying $\left|Z^{k} \Phi(x)\right| \leq C|x|^{2 \ell-3}\langle x\rangle^{-2 \ell}$ for any $k$;

- where $\mathcal{C}^{k,(\ell)}$ is a vector valued cubic map, whose components are linear combination of quantities

$$
\begin{align*}
& \theta\left(x h^{-\beta}\right) \Phi(x)\left(Z^{k_{1}} w^{(\ell)}\right)\left(Z^{k_{2}} w^{(\ell)}\right)\left(Z^{k_{3}} w^{(\ell)}\right) \\
& \theta\left(x h^{-\beta}\right) \Phi(x)\left(Z^{k_{1}} w^{(\ell)}\right)\left(Z^{k_{2}} w^{(\ell)}\right)\left(Z^{k_{3}} \bar{w}^{(\ell)}\right)  \tag{6.6}\\
& \theta\left(x h^{-\beta}\right) \Phi(x)\left(Z^{k_{1}} w^{(\ell)}\right)\left(Z^{k_{2}} \bar{w}^{(\ell)}\right)\left(Z^{k_{3}} \bar{w}^{(\ell)}\right) \\
& \theta\left(x h^{-\beta}\right) \Phi(x)\left(Z^{k_{1}} \bar{w}^{(\ell)}\right)\left(Z^{k_{2}} \bar{w}^{(\ell)}\right)\left(Z^{k_{3}} \bar{w}^{(\ell)}\right)
\end{align*}
$$

for $k_{1}+k_{2}+k_{3} \leq k,\left|Z^{k} \Phi(x)\right| \leq C|x|^{4 \ell-5}\langle x\rangle^{-4 \ell}$ for any $k$.
We shall prove first (6.2), deducing it from (3.23) on which we make act $\operatorname{Op}_{h}\left(\langle\xi\rangle^{\ell}\right)$. Let us study first the action of this operator on the nonlinearity.

As in the preceding section, we shall call $K$ or $K_{\ell}$ compact subsets of $T^{*}(\mathbb{R} \backslash\{0\})$ contained in a small neighborhood of $\ell \Lambda$ for $\ell \in\{ \pm 3, \pm 2, \pm 1\}$, by $L$ compact subsets of $T^{*}(\mathbb{R} \backslash\{0\})$ and by $F$ closed subsets of $T^{*} \mathbb{R}$ whose second projection is compact in $\mathbb{R} \backslash\{0\}$.
Lemma 6.2. Under the assumptions of the proposition, we may write for $\ell \leq \frac{s}{2}+N_{0}$,

$$
\begin{align*}
\sum_{j \in J(h, C)} \Delta_{j}^{h} & {\left[\mathrm{Op}_{h}\left(\langle\xi\rangle^{\ell}\right)\left[\sqrt{h} Q_{0}(W)+h C_{0}(W)\right]\right] } \\
= & -i \sqrt{h}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{3}{2}} \frac{\sqrt{2}}{4}\langle 2 \mathrm{~d} \omega\rangle^{\ell}\langle\mathrm{d} \omega\rangle^{-2 \ell}\left[\left(w_{\Lambda}^{(\ell)}\right)^{2}+\left(\bar{w}_{\Lambda}^{(\ell)}\right)^{2}\right] \\
& +\frac{h}{2}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{5}{2}}\langle\mathrm{~d} \omega\rangle^{-3 \ell}\langle 3 \mathrm{~d} \omega\rangle^{\ell}\left[\lambda_{3}^{(\ell)}\left(w_{\Lambda}^{(\ell)}\right)^{3}+\lambda_{-3}^{(\ell)}\left(\bar{w}_{\Lambda}^{(\ell)}\right)^{3}\right]  \tag{6.7}\\
& +\frac{h}{2}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{5}{2}}\langle\mathrm{~d} \omega\rangle^{-2 \ell}\left[\left|w_{\Lambda}^{(\ell)}\right|^{2} w_{\Lambda}^{(\ell)}+\lambda_{-1}^{(\ell)}\left|w_{\Lambda}^{(\ell)}\right|^{2} \bar{w}_{\Lambda}^{(\ell)}\right] \\
& +h^{\frac{3}{2}} r^{(\ell)}
\end{align*}
$$

where for $k \leq \frac{s}{2}+N_{0}-\ell Z^{k} r^{(\ell)}$ belongs to $h^{-3 \delta_{k+1+\ell}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}[F]$ and $\lambda_{ \pm 3}^{(\ell)}, \lambda_{-1}^{(\ell)}$ are real constants.
Proof. We apply $\operatorname{Op}_{h}\left(\langle\xi\rangle^{\ell}\right)$ to (5.9) and write the resulting right hand side as in (6.7). By ii) if Lemma 5.3, we know that $\mathcal{Z}^{k+\ell} \tilde{w}_{ \pm 2 \Lambda}$ is $O(\varepsilon)$ in $h^{-2 \delta_{k+\ell+1}^{\prime}} L^{\infty} \tilde{J}_{ \pm 2 \Lambda}^{3, b^{\prime}+\frac{3}{2}}\left[K_{ \pm 2}\right]$, so $Z^{k} \tilde{w}_{ \pm 2 \Lambda}$ is $O(\varepsilon)$ in $h^{-2 \delta_{k+\ell+1}^{\prime}} L^{\infty} \tilde{J}_{ \pm 2 \Lambda}^{3, b^{\prime}+\frac{3}{2}+\ell}\left[K_{ \pm 2}\right]$.

In the same way $Z^{k} \tilde{w}_{q \Lambda}$ is $O(\varepsilon)$ in $h^{-3 \delta_{k+\ell+1}^{\prime}} L^{\infty} \tilde{I}_{q \Lambda}^{3, b^{\prime}+\frac{3}{2}+\ell}\left[K_{q}\right]$, for $q \in\{ \pm 1, \pm 3\}$. Consequently, Proposition 2.11 shows that

$$
\begin{aligned}
\mathrm{Op}_{h}\left(\langle\xi\rangle^{\ell}\right) \tilde{w}_{ \pm 2 \Lambda} & =(1-\chi)\left(x h^{-\beta}\right)\langle 2 \mathrm{~d} \omega\rangle^{\ell} \tilde{w}_{ \pm 2 \Lambda}+h \tilde{r}_{ \pm 2}^{(\ell)} \\
\mathrm{Op}_{h}\left(\langle\xi\rangle^{\ell}\right) \tilde{w}_{q \Lambda} & =(1-\chi)\left(x h^{-\beta}\right)\langle q \mathrm{~d} \omega\rangle^{\ell} \tilde{w}_{q \Lambda}+h^{\frac{1}{2}} \tilde{r}_{q}^{(\ell)},
\end{aligned}
$$

where $Z^{k} \tilde{r}_{ \pm 2}^{(\ell)}\left(\right.$ resp. $\left.Z^{k} \tilde{r}_{q}^{(\ell)}\right)$ is $O(\varepsilon)$ in $h^{-2 \delta_{k+\ell+1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{2, b^{\prime}+\frac{3}{2}}\left[K_{ \pm 2}\right]$ (resp. $h^{-3 \delta_{k+1+\ell}} \tilde{\mathcal{B}}_{\infty}^{2, b^{\prime}+1}\left[K_{q}\right]$ ). We combine this with the expressions (5.8), (5.10), (5.11) of $\tilde{w}_{q \Lambda}$ in terms of $w_{\Lambda}, w_{ \pm 2 \Lambda}$, and with the formulas (5.33) expressing $w_{ \pm 2 \Lambda}$ in terms of $w_{\Lambda}$. If moreover we compute the powers of $w_{\Lambda}, \bar{w}_{\Lambda}$ from $w_{\Lambda}^{(\ell)}, \bar{w}_{\Lambda}^{(\ell)}$ using (5.55), we get

$$
\begin{gathered}
\mathrm{Op}_{h}\left(\langle\xi\rangle^{\ell}\right) \tilde{w}_{2 \Lambda}=-i(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{3}{2}} \frac{\sqrt{2}}{4}\langle 2 \mathrm{~d} \omega\rangle^{\ell}\langle\mathrm{d} \omega\rangle^{-2 \ell}\left(w_{\Lambda}^{(\ell)}\right)^{2}+h r_{2}^{(\ell)}, \\
\mathrm{Op}_{h}\left(\langle\xi\rangle^{\ell}\right) \tilde{w}_{-2 \Lambda}=-i(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{3}{2}} \frac{\sqrt{2}}{4}\langle 2 \mathrm{~d} \omega\rangle^{\ell}\langle\mathrm{d} \omega\rangle^{-2 \ell}\left(\bar{w}_{\Lambda}^{(\ell)}\right)^{2}+h r_{-2}^{(\ell)}
\end{gathered}
$$

with remainders $r_{ \pm 2}^{(\ell)}$ satisfying again, because of Proposition 2.11, that $Z^{k} r_{ \pm 2}^{(\ell)}$ is $O(\varepsilon)$ in $h^{-2 \delta_{k+1+\ell}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}\left[K_{ \pm 2}\right]$. In the same way

$$
\mathrm{Op}_{h}\left(\langle\xi\rangle^{\ell}\right) \tilde{w}_{q \Lambda}=(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{5}{2}} \lambda_{q}^{(\ell)}\langle q \mathrm{~d} \omega\rangle^{\ell}\langle\mathrm{d} \omega\rangle^{-3 \ell} P_{q}\left(w_{\Lambda}^{(\ell)}, \bar{w}_{\Lambda}^{(\ell)}\right)+h^{\frac{1}{2}} r_{q}^{(\ell)},
$$

where $P_{q}\left(w_{\Lambda}^{(\ell)}, \bar{w}_{\Lambda}^{(\ell)}\right)$ is equal to $\left(w_{\Lambda}^{(\ell)}\right)^{3}$ (resp. $\left|w_{\Lambda}^{(\ell)}\right|^{2} w_{\Lambda}^{(\ell)}$, resp. $\left|w_{\Lambda}^{(\ell)}\right|^{2} \bar{w}_{\Lambda}^{(\ell)}$, resp. $\left.\left(\bar{w}_{\Lambda}^{(\ell)}\right)^{3}\right)$ if $q=3$ (resp. $q=1$, resp. $q=-1$, resp. $q=-3$ ), where $\lambda_{q}^{(\ell)}$ are real constants with $\lambda_{1}^{(\ell)}=\frac{1}{2}$, and where $Z^{k} r_{q}^{(\ell)}$ is $O(\varepsilon)$ in $h^{-3 \delta_{k+1+\ell}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}\left[K_{q}\right]$. (We used again remark 5.5 to replace different powers of $(1-\chi)\left(x h^{-\beta}\right)$ by 1.)
This concludes the proof of the lemma.

Let us study next the action of $\mathrm{Op}_{h}\left(\langle\xi\rangle^{\ell}\right)$ on the linear term $\mathrm{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right) w$ of (3.23), writing $w=\mathrm{Op}_{h}\left(\langle\xi\rangle^{-\ell}\right) w^{(\ell)}$.

Lemma 6.3. One may write, for $\ell \leq \frac{s}{2}+N_{0}$

$$
\begin{align*}
\mathrm{Op}_{h}\left(\langle\xi\rangle^{\ell}\right) \mathrm{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right) & \mathrm{Op}_{h}\left(\langle\xi\rangle^{-\ell}\right) w^{(\ell)}+i \ell h \mathrm{Op}_{h}\left(\frac{\xi^{2}}{\langle\xi\rangle^{2}}\right) w^{(\ell)} \\
=(1-\chi)\left(x h^{-\beta}\right) & {\left[\left(\frac{1}{2}|\mathrm{~d} \omega|^{\frac{1}{2}}+\frac{i}{2} h\right) w_{\Lambda}^{(\ell)}\right.} \\
& -i \frac{\sqrt{h}}{4}|\mathrm{~d} \omega|^{\frac{3}{2}}\langle 2 \mathrm{~d} \omega\rangle^{\ell}\langle\mathrm{d} \omega\rangle^{-2 \ell}\left[\left(w_{\Lambda}^{(\ell)}\right)^{2}-\left(\bar{w}_{\Lambda}^{(\ell)}\right)^{2}\right]  \tag{6.8}\\
& +h|\mathrm{~d} \omega|^{\frac{5}{2}}\langle\mathrm{~d} \omega\rangle^{-3 \ell}\left[\langle 3 \mathrm{~d} \omega\rangle^{\ell}\left(\mu_{3}^{(\ell)}\left(w_{\Lambda}^{(\ell)}\right)^{3}+\mu_{-3}^{(\ell)}\left(\bar{w}_{\Lambda}^{(\ell)}\right)^{3}\right)\right. \\
& \left.\left.+\langle\mathrm{d} \omega\rangle^{\ell} \mu_{-1}^{(\ell)}\left|w_{\Lambda}^{(\ell)}\right|^{2} \bar{w}_{\Lambda}^{(\ell)}\right]\right]
\end{align*}
$$

for some real constants $\mu_{3}^{(\ell)}, \mu_{-1}^{(\ell)}, \mu_{-3}^{(\ell)}$ and where for $k \leq \frac{s}{2}+N_{0}-\ell, Z^{k} r^{(\ell)}$ belongs to $h^{-4 \delta_{k+\ell+1+N_{1}-N_{0}}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}-\frac{1}{2}}[F]$.

The proof of the above lemma will use
Lemma 6.4. We may write for $\ell \leq \frac{s}{2}+N_{0}$

$$
\begin{equation*}
\mathrm{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right) w_{\Lambda}^{(\ell)}=\frac{1}{2}(1-\chi)\left(x h^{-\beta}\right)\left[|\mathrm{d} \omega|^{\frac{1}{2}} w_{\Lambda}^{(\ell)}+i h w_{\Lambda}^{(\ell)}\right]+h^{1+\sigma} r_{1}^{(\ell)} \tag{6.9}
\end{equation*}
$$

where for $k \leq \frac{s}{2}+N_{0}-\ell, Z^{k} r_{1}^{(\ell)}$ is in $h^{-4 \delta_{k+2+\ell}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}[L]$.

Proof. We write remembering (2.13)

$$
\begin{equation*}
\mathrm{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right) w_{\Lambda}^{(\ell)}=\sum_{j \in J(h, C)} 2^{\frac{j}{2}} \Theta_{j}^{*} \mathrm{Op}_{h_{j}}\left(x \xi+|\xi|^{\frac{1}{2}}\right) w_{\Lambda, j}^{(\ell)} . \tag{6.10}
\end{equation*}
$$

Let us show that we may write

$$
\begin{align*}
\mathrm{Op}_{h_{j}}\left(x \xi+|\xi|^{\frac{1}{2}}\right)= & \frac{1}{2}|\mathrm{~d} \omega(x)|^{\frac{1}{2}}+\mathrm{Op}_{h_{j}}\left(e_{1}\right)\left(\mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)\right)^{2} \\
& +i h_{j} \mathrm{Op}_{h_{j}}\left(e_{2}\right) \mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)  \tag{6.11}\\
& -i h_{j} \mathrm{Op}_{h_{j}}\left(e_{1}\right) \mathrm{Op}_{h_{j}}\left(|\xi|^{\frac{1}{2}}\right)+h_{j}^{2} \mathrm{Op}_{h_{j}}\left(e_{3}\right)
\end{align*}
$$

where $e_{j}$ are symbols in $S(1, K)$, for some large enough compact subset $K$ of $T^{*}(\mathbb{R} \backslash\{0\})$, satisfying

$$
\begin{equation*}
\left.e_{1}\right|_{\Lambda}=-\frac{1}{2}|\mathrm{~d} \omega(x)|^{-\frac{1}{2}} . \tag{6.12}
\end{equation*}
$$

Denote $a(x, \xi)=x \xi+|\xi|^{\frac{1}{2}}$ and take

$$
e_{1}(x, \xi)=\frac{a(x, \xi)-a(x, \mathrm{~d} \omega)}{\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)^{2}}
$$

A direct computation shows that the numerator vanishes at second order on $\Lambda$, so that the quotient is smooth, and that its restriction to $\Lambda$ is given by (6.12). If we set $e(x, \xi)=2 x \xi+|\xi|^{\frac{1}{2}}$, we obtain by symbolic calculus

$$
e \# e=e^{2}-i h_{j} \partial_{\xi} e \partial_{x} e+h_{j}^{2} \tilde{e}
$$

for some symbol $\tilde{e}$, so that by an immediate computation

$$
\begin{equation*}
e^{2}=e \# e+2 i h_{j} e-i h_{j}|\xi|^{\frac{1}{2}}-h_{j}^{2} \tilde{e} . \tag{6.13}
\end{equation*}
$$

On the other hand, by symbolic calculus $e_{1} e^{2}=e_{1} \# e^{2}+h_{j} e_{2}^{\prime} \# e+h_{j}^{2} \tilde{e}^{\prime}$ for some symbols $e_{2}^{\prime}, \tilde{e}^{\prime}$ in $S(1, K)$, so that taking (6.13) into account

$$
e_{1} e^{2}=e_{1} \# e \# e-i h_{j} e_{1} \#|\xi|^{\frac{1}{2}}+i h_{j} e_{2} \# e+h_{j}^{2} e_{3}
$$

for new symbols $e_{2}, e_{3}$. Since $e_{1} e^{2}=a(x, \xi)-a(x, \mathrm{~d} \omega)=x \xi+|\xi|^{\frac{1}{2}}-\frac{1}{2}|\mathrm{~d} \omega(x)|^{\frac{1}{2}}$, we obtain (6.11) by quantification.

Let us use (6.11) to show that (6.9) holds. Actually, the contribution of the first term in the right hand side of (6.11) to (6.9) gives the $|\mathrm{d} \omega|^{\frac{1}{2}}$ term in the right hand side of (6.9) (Again, we may insert a cut-off $(1-\chi)\left(x h^{-\beta}\right)$ as $w_{\Lambda, j}$ is microlocally supported on a compact subset of $T^{*}(\mathbb{R} \backslash\{0\})$ and $j$ stays in $J(h, C)$, if we accept some $O\left(h^{\infty}\right)$ remainder $)$. The contribution of the last but one term in (6.11) to (6.10) may be written

$$
-i h \sum_{j \in J(h, C)} \Theta_{j}^{*} \mathrm{Op}_{h_{j}}\left(e_{1}\right) \mathrm{Op}_{h_{j}}\left(|\xi|^{\frac{1}{2}}\right) w_{\Lambda, j}^{(\ell)}=-i h \mathrm{Op}_{h}\left(e_{1}\right) \mathrm{Op}_{h}\left(|\xi|^{\frac{1}{2}}\right) w_{\Lambda}^{(\ell)}
$$

By Proposition 2.11 and (6.12), this is equal to

$$
\frac{i}{2} h(1-\chi)\left(x h^{-\beta}\right) w_{\Lambda}^{(\ell)}+h^{2} r_{1,1}^{(\ell)}
$$

where $Z^{k} r_{1,1}^{(\ell)}$ belongs to $h^{-\delta_{k+1+\ell}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{-1, b^{\prime}}[K]$. Actually noticing that $e_{1}(x, \xi)=|x| e_{1}\left(\frac{x}{|x|},|x|^{2} \xi\right)$, and that $\left(\partial_{x}^{\alpha} \partial_{\eta}^{\beta} e_{1}\right)( \pm 1, \eta)=O\left(|\eta|^{-1-|\beta|}\right),|\eta| \rightarrow 0$ and $|\eta| \rightarrow+\infty$, one checks that $e_{1}(x, \xi)$ satisfies (2.15) with $\left(\ell, \ell^{\prime}, d, d^{\prime}\right)=(-1,0,-1,0)$ so that $e_{1}(x, \xi)|\xi|^{\frac{1}{2}}$ obeys these estimates for $\left(\ell, \ell^{\prime}, d, d^{\prime}\right)=(-1,0,-1 / 2,0)$. Since $Z^{k} w_{\Lambda}^{(\ell)}$ is in $h^{-\delta_{k+1+\ell}^{\prime}} L^{\infty} \tilde{J}_{\Lambda}^{0, b^{\prime}}[K]$, the above statement holds. Since for $j \in J(h, C), 2^{j} \geq h^{2(1-\sigma)}$, the remainder may be rewritten as the product of $h^{1+\sigma}$ with an element whose $\mathcal{Z}^{k}$-derivatives are in $h^{-\delta_{k+1+\ell}^{\prime} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}}[K]$ i.e. contributes to $h^{1+\sigma} r_{1}^{(\ell)}$ in (6.9).

We are reduced to showing that the contributions of the second, third and last terms in the right hand side of (6.11) provide remainders. This is evident for the last term as $2^{\frac{j}{2}} h_{j}^{2}=$ $h h_{j}=O\left(h^{1+\sigma}\right)$. Using (5.54), we may write the sum of the two remaining terms

$$
\begin{equation*}
h_{j} \mathrm{Op}_{h_{j}}\left(e_{1}\right) \mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right)\left(f_{j}^{(\ell)}+h^{\frac{1}{4}} r_{j}^{(\ell)}\right)+i h_{j}^{2} \mathrm{Op}_{h_{j}}\left(e_{2}\right)\left(f_{j}^{(\ell)}+h^{\frac{1}{4}} r_{j}^{(\ell)}\right) . \tag{6.14}
\end{equation*}
$$

Since $\left(Z^{k} f_{j}^{(\ell)}\right)_{j}$ is $O(\varepsilon)$ in $h^{-3 \delta_{k+2+\ell}^{\prime}} L^{\infty} I_{\Lambda}^{0, b^{\prime}}[K]$ and $\left(Z^{k} r_{j}^{(\ell)}\right)_{j}$ is $O(\varepsilon)$ in $h^{-4 \delta_{k+1+\ell}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}[L]$, the last term as well as the $r_{j}^{(\ell)}$ contribution to the first one induce in (6.10) a contribution that may be included in the $h^{1+\sigma} r_{1}^{(\ell)}$ remainder term of (6.9). On the other hand, the fact that $\left(Z^{k} f_{j}^{(\ell)}\right)_{j}$ is $O(\varepsilon)$ in $h^{-3 \delta_{k+2+\ell}^{\prime}} L^{\infty} I_{\Lambda}^{0, b^{\prime}}[K]$ implies that $Z^{k} \mathrm{Op}_{h_{j}}\left(2 x \xi+|\xi|^{\frac{1}{2}}\right) f_{j}^{(\ell)}$ belongs to a $\varepsilon$-neighborhood of zero in $h^{-3 \delta_{k+2+\ell}^{\prime}}\left(h^{\frac{1}{2}}+h_{j}\right) \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}[L]$. Consequently, the first term in (6.14) induces also in (6.10) a contribution forming part to the $h^{1+\sigma} r_{1}^{(\ell)}$ term in (6.9). This concludes the proof of the lemma.

Proof of Lemma 6.3. We notice first that

$$
\begin{equation*}
\mathrm{Op}_{h}\left(\langle\xi\rangle^{\ell}\right) \mathrm{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right) \mathrm{Op}_{h}\left(\langle\xi\rangle^{-\ell}\right)+i \ell h \mathrm{Op}_{h}\left(\frac{\xi^{2}}{\langle\xi\rangle^{2}}\right)=\mathrm{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right) \tag{6.15}
\end{equation*}
$$

We make act this operator on the expression of $w^{(\ell)}$ from $w_{\Lambda}^{(\ell)}$ given in (5.52). The action of $\mathrm{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right)$ on $w_{\Lambda}^{(\ell)}$ has been computed in Lemma 6.4. Let us study

$$
\mathrm{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right)\left(\sqrt{h}\left(w_{2 \Lambda}^{(\ell)}+w_{-2 \Lambda}^{(\ell)}\right)\right) .
$$

One may express $w_{ \pm 2 \Lambda}^{(\ell)}$ from $w_{\Lambda}^{(\ell)}, \bar{w}_{\Lambda}^{(\ell)}$ by (5.53). Since, according to Proposition 5.2, $Z^{k} w_{\Lambda}^{(\ell)}$ is in $h^{-\delta_{k+1+\ell}^{\prime}} L^{\infty} \tilde{J}_{\Lambda}^{0, b^{\prime}}[K]$, it follows from Proposition 2.12 that $Z^{k}\left(w_{\Lambda}^{(\ell)}\right)^{2}\left(\right.$ resp. $\left.Z^{k}\left(\bar{w}_{\Lambda}^{(\ell)}\right)^{2}\right)$ belongs to $h^{-2 \delta_{k+1+\ell}^{\prime}} L^{\infty} \tilde{J}_{2 \Lambda}^{0,2 b^{\prime}}\left[K_{2}\right]$ (resp. $h^{-2 \delta_{k+1+\ell}^{\prime}} L^{\infty} \tilde{J}_{-2 \Lambda}^{0,2 b^{\prime}}\left[K_{-2}\right]$ ).

We apply next Proposition 2.11, with $a$ replaced by $\left(x \xi+|\xi|^{\frac{1}{2}}\right)\langle 2 \mathrm{~d} \omega\rangle^{\ell}\langle\mathrm{d} \omega\rangle^{-2 \ell}|\mathrm{~d} \omega|$. Since, because of the fact that $w_{ \pm 2 \Lambda}^{(\ell)}$ is microlocally supported close to $\pm 2 \Lambda$, we may assume that $x \xi+|\xi|^{\frac{1}{2}}$ is cut-off close to this manifold, we see that the above symbol satisfies (2.15) with $\left(\ell, \ell^{\prime}, d, d^{\prime}\right)=(-2+2 \ell,-2 \ell, 1 / 2,0)$ or $(-1+2 \ell,-2 \ell, 1,0)$. It follows from (2.17) that

$$
\begin{align*}
\sqrt{h} \mathrm{Op}_{h}(x \xi & \left.+|\xi|^{\frac{1}{2}}\right)\left(w_{2 \Lambda}^{(\ell)}+w_{-2 \Lambda}^{(\ell)}\right) \\
= & -\frac{i}{4} \sqrt{h}(1-\chi)\left(x h^{-\beta}\right)\langle 2 \mathrm{~d} \omega\rangle^{\ell}\langle\mathrm{d} \omega\rangle^{-2 \ell}|\mathrm{~d} \omega|^{\frac{3}{2}}\left[\left(w_{\Lambda}^{(\ell)}\right)^{2}-\left(\bar{w}_{\Lambda}^{(\ell)}\right)^{2}\right]  \tag{6.16}\\
& +h^{\frac{3}{2}} r^{(\ell)}
\end{align*}
$$

where $Z^{k} r^{(\ell)}$ is in $h^{-2 \delta_{k+1+\ell}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{2,2 b^{\prime}}[L] \subset h^{-2 \delta_{k+1+\ell}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}[L]$.
In the same way

$$
\begin{align*}
& h \mathrm{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right)\left(w_{3 \Lambda}^{(\ell)}+w_{-\Lambda}^{(\ell)}+w_{-3 \Lambda}^{(\ell)}\right) \\
& =h|\mathrm{~d} \omega|^{\frac{5}{2}}\langle\mathrm{~d} \omega\rangle^{-3 \ell}\left[\langle 3 \mathrm{~d} \omega\rangle^{\ell}\left(\mu_{3}^{(\ell)}\left(w_{\Lambda}^{(\ell)}\right)^{3}+\mu_{-3}^{(\ell)}\left(\bar{w}_{\Lambda}^{(\ell)}\right)^{3}\right)+\langle\mathrm{d} \omega\rangle^{\ell} \mu_{-1}^{(\ell)}\left|w_{\Lambda}^{(\ell)}\right| \bar{w}_{\Lambda}^{(\ell)}\right]  \tag{6.17}\\
& \quad+h^{2} r^{(\ell)}
\end{align*}
$$

with $Z^{k} r^{(\ell)}$ in $h^{-3 \delta_{k+1+\ell}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{4,3 b^{\prime}}[L] \subset h^{-3 \delta_{k+1+\ell}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}[L]$ and some real constants $\mu_{ \pm 3}^{(\ell)}, \mu_{-1}^{(\ell)}$.
Finally, since the action of $\mathrm{Op}_{h}\left(x \xi+|\xi|^{\frac{1}{2}}\right)$ on the remainder $g^{(\ell)}$ of (5.52) gives a function $r^{(\ell)}$ such that $Z^{k} r^{(\ell)}$ is in $h^{-4 \delta_{k+N_{1}-N_{0}+1+\ell}^{\mathcal{B}}} \tilde{\infty}^{0, b^{\prime}-\frac{1}{2}}[F]$ we conclude, summing (6.9), (6.16), (6.17) that (6.15) is given by formula (6.8).

We may now prove Proposition 6.1.

Proof of Proposition 6.1. Let us compute

$$
D_{t} w^{(\ell)}=D_{t} \mathrm{Op}_{h}\left(\langle\xi\rangle^{\ell}\right) w=i \ell h \mathrm{Op}_{h}\left(\frac{\xi^{2}}{\langle\xi\rangle^{2}}\right) w^{(\ell)}+\mathrm{Op}_{h}\left(\langle\xi\rangle^{\ell}\right) D_{t} w
$$

According to (3.23) this is the sum of $-\frac{i}{2} h w^{(\ell)}$, of (6.7), of (6.8) and of a remainder $h^{\frac{5}{4}} R^{(\ell)}(V)$ where $\mathcal{Z}^{k} R^{(\ell)}(V)$ is in $\mathcal{R}_{\infty}^{b}$. (Notice that by definition of $w$, we may always insert on the left hand side of (3.23) a cut-off $\sum_{j \in J(h, C)} \Delta_{j}^{h}$ for some large enough $C$, so that the sum of quadratic and cubic contributions is really given by (6.7)). Remembering the expression (5.52) of $w^{(\ell)}$ in terms of $w_{\Lambda}^{(\ell)}$, we may write

$$
-\frac{i}{2} h w^{(\ell)}=-\frac{i}{2} h(1-\chi)\left(x h^{-\beta}\right) w_{\Lambda}^{(\ell)}+h^{\frac{3}{2}} r^{(\ell)}
$$

with $Z^{k} r^{(\ell)}$ in $h^{-4 \delta_{k+N_{1}-N_{0}+1+\ell}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}[F]$ (We used again that the microlocal support properties of $w_{\Lambda}^{(\ell)}$ allow to multiply it by some cut-off $(1-\chi)\left(x h^{-\beta}\right)$ up to a $O\left(h^{\infty}\right)$-remainder). We obtain

$$
\begin{align*}
D_{t} w^{(\ell)}= & \frac{1}{2}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{1}{2}} w_{\Lambda}^{(\ell)}  \tag{6.18}\\
& -i \frac{\sqrt{h}}{4}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{3}{2}}\langle 2 \mathrm{~d} \omega\rangle^{\ell}\langle\mathrm{d} \omega\rangle^{-2 \ell}\left[(1+\sqrt{2})\left(w_{\Lambda}^{(\ell)}\right)^{2}+(\sqrt{2}-1)\left(\bar{w}_{\Lambda}^{(\ell)}\right)^{2}\right] \\
& +\frac{h}{2}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{5}{2}}\langle\mathrm{~d} \omega\rangle^{-3 \ell}\left[\langle 3 \mathrm{~d} \omega\rangle^{\ell}\left(\mu_{3}^{(\ell)}\left(w_{\Lambda}^{(\ell)}\right)^{3}+\mu_{-3}^{(\ell)}\left(\bar{w}_{\Lambda}^{(\ell)}\right)^{3}\right)\right. \\
& \left.\quad+\langle\mathrm{d} \omega\rangle^{\ell}\left|w_{\Lambda}^{(\ell)}\right|^{2} w_{\Lambda}^{(\ell)}+\langle\mathrm{d} \omega\rangle^{\ell} \mu_{-1}^{(\ell)}\left|w_{\Lambda}^{(\ell)}\right| \bar{w}_{\Lambda}^{(\ell)}\right] \\
& +h^{1+\sigma} R(V)
\end{align*}
$$


We express next $w_{\Lambda}^{(\ell)}$ from $w^{(\ell)}$ inverting relation (5.52) i.e. writing, taking (5.53) into account, (6.19)

$$
\begin{aligned}
& w_{\Lambda}^{(\ell)}= w^{(\ell)}+i \frac{\sqrt{h}}{4}(1-\chi)\left(x h^{-\beta}\right)\langle 2 \mathrm{~d} \omega\rangle^{\ell}\langle\mathrm{d} \omega\rangle^{-2 \ell}|\mathrm{~d} \omega|\left[(1+\sqrt{2})\left(w^{(\ell)}\right)^{2}+(1-\sqrt{2})\left(\bar{w}^{(\ell)}\right)^{2}\right] \\
&+h|\mathrm{~d} \omega|^{2}(1-\chi)\left(x h^{-\beta}\right)\left[\Gamma_{3}^{(\ell)}(\mathrm{d} \omega)\left(w^{(\ell)}\right)^{3}+\Gamma_{1}^{(\ell)}(\mathrm{d} \omega)\left|w^{(\ell)}\right|^{2} w^{(\ell)}\right. \\
&\left.+\Gamma_{-1}^{(\ell)}(\mathrm{d} \omega)\left|w^{(\ell)}\right|^{2} \bar{w}^{(\ell)}+\Gamma_{-3}^{(\ell)}(\mathrm{d} \omega)\left(\bar{w}^{(\ell)}\right)^{3}\right] \\
&+h^{1+\sigma} g^{(\ell)},
\end{aligned}
$$

where $Z^{k} g^{(\ell)}$ is in $h^{-4 \delta_{k+N_{1}-N_{0}+1+\ell}^{\prime}} \mathcal{R}_{\infty}^{b^{\prime}}$, and where $\Gamma_{q}^{(\ell)}(\zeta)$ is a symbol of order $-2 \ell$, with

$$
\Gamma_{1}^{(\ell)}(\zeta)=\langle 2 \zeta\rangle^{2 \ell}\langle\zeta\rangle^{-4 \ell}\left(\frac{3-2 \sqrt{2}}{8}\right)
$$

We plug this expansion in (6.18) to get (6.2). The remainder satisfies the conditions of the statement of the proposition if we assume that $4 \delta_{k+N_{1}-N_{0}+1+\ell}^{\prime}<\frac{\sigma}{2}$, so that we may take $\kappa=\sigma / 2$.

We prove now (6.4) by induction from (6.2).
To deduce (6.4) at order $k$ from the similar equality at order $k-1$, we notice first that the action of $Z$ on the quadratic (resp. cubic, resp. remainder) terms of (6.4) at order $k-1$ gives contributions to $Q^{k,(\ell)}$ (resp. $\mathcal{C}^{k,(\ell)}$, resp. $\left.\mathcal{R}^{k,(\ell)}\right)$. Moreover,

$$
\begin{aligned}
{\left[Z, D_{t}-\right.} & \left.\frac{1}{2}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega(x)|^{\frac{1}{2}}\right] \\
= & -\left(D_{t}-\frac{1}{2}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega(x)|^{\frac{1}{2}}\right) \\
& +\frac{1+\beta}{2}\left(x h^{-\beta}\right) \chi^{\prime}\left(x h^{-\beta}\right)|\mathrm{d} \omega(x)|^{\frac{1}{2}} .
\end{aligned}
$$

The product of the last term with $W^{k-1,(\ell)}$ may be computed from expressions of the form $h^{-\beta} \Gamma\left(x h^{-\beta}\right) Z^{k^{\prime}} w^{(\ell)}$ where $k^{\prime} \leq k-1$ and $\Gamma$ is in $C_{0}^{\infty}\left(\mathbb{R}^{*}\right)$. We just have to check that such terms contribute to the remainder in (6.4). Because of the expression (5.52) of $w^{(\ell)}$ in terms of $w_{\Lambda}^{(\ell)}$, we see that we need to check that for $p \leq b^{\prime \prime}$

$$
\begin{equation*}
\left\|\left(h D_{x}\right)^{p}\left[\Gamma\left(x h^{-\beta}\right) Z^{k^{\prime}} w_{\Lambda}^{(\ell)}\right]\right\|_{L^{\infty}}=O\left(h^{1+\kappa+\beta}\right) . \tag{6.20}
\end{equation*}
$$

(The contribution coming from the remainder in (5.52) satisfies the wanted bound as we assume after (4.2) that $\beta<\sigma / 2$.) We remember that $w_{\Lambda}^{(\ell)}=\sum_{j \in J(h, C)} \Theta_{j}^{*} w_{\Lambda, j}^{(\ell)}$, where $w_{\Lambda, j}^{(\ell)}$ is microlocally supported for $x$ in a compact subset of $\mathbb{R}^{*}$, so that

$$
\Gamma\left(x h^{-\beta}\right) Z^{k^{\prime}} w_{\Lambda}^{(\ell)}=\sum_{j \in J(h, C)} \Gamma\left(x h^{-\beta}\right) Z^{k^{\prime}}\left(\Gamma_{1}\left(2^{\frac{j}{2}} x\right) \Theta_{j}^{*} w_{\Lambda, j}^{(\ell)}\right)
$$

for some $\Gamma_{1}$ in $C_{0}^{\infty}(\mathbb{R})$. This shows that the sum is limited to those $j$ for which $2^{j} \sim h^{-2 \beta}$.
Since $Z^{k^{\prime}} w_{\Lambda}^{(\ell)}$ is in $h^{-\delta_{k^{\prime}+\ell+1}^{\prime}} L^{\infty} \tilde{J}_{\Lambda}^{0, b^{\prime}}[K]$ according to Proposition 5.2,

$$
\left\|\Theta_{j}^{*} Z^{k^{\prime}} w_{\Lambda, j}^{(\ell)}\right\|_{L^{\infty}}=O\left(h^{-\delta_{k^{\prime}+\ell+1}^{\prime} 2^{-j+b^{\prime}}}\right) .
$$

Using that $2^{j} \sim h^{-2 \beta}$, we bound $2^{-j_{+} b^{\prime}} \leq 2^{-j+b^{\prime \prime}} h^{\delta_{k^{\prime}+\ell+1}^{\prime}+\beta+\kappa+2}$ since the assumption on $b^{\prime \prime}$
 Consequently, we get (6.20) for all integers $p \leq b^{\prime \prime}$.

This concludes the proof of the proposition.
Proposition 6.5. Let $T_{0}$ be a large enough positive number, $\kappa$ a small positive constant, $C_{0}>0$.

Let $\ell$ be an integer, with $\ell \leq \frac{s}{2}+N_{0}$. Assume given a function $(t, x) \rightarrow r^{(\ell)}(t, x)$ from a domain $\left[T_{0}, T\left[\times \mathbb{R}\right.\right.$ to $\mathbb{C}$, satisfying for $p \leq b^{\prime \prime}$,

$$
\sup _{x}\left|\left(h D_{x}\right)^{p} r^{(\ell)}(t, x)\right| \leq C_{0} \varepsilon
$$

for any $t \in\left[T_{0}, T\left[\right.\right.$. Assume given a solution $w^{(\ell)}:\left[T_{0}, T[\times \mathbb{R} \rightarrow \mathbb{C}\right.$ of equation (6.2), such that $\left|w^{(\ell)}\left(T_{0}, x\right)\right| \leq C_{0} \varepsilon$ for any $x$.

Then there are $\varepsilon_{0}>0, C_{1}>0$, depending only on $C_{0}$, such that for any $\left.\varepsilon \in\right] 0, \varepsilon_{0}[$,

$$
\begin{equation*}
\sup _{\left[T_{0}, T[ \right.}\left\|w^{(\ell)}(t, \cdot)\right\|_{L^{\infty}} \leq C_{1} \varepsilon . \tag{6.21}
\end{equation*}
$$

Moreover, if we assume that $r^{(\ell)}$ is defined and satisfies the above assumption on $\left[T_{0},+\infty[\times \mathbb{R}\right.$, then $w^{(\ell)}$ is defined on $\left[T_{0},+\infty[\times \mathbb{R}\right.$ and there are a continuous bounded function $\alpha: \mathbb{R} \rightarrow \mathbb{C}$, vanishing like $|x|^{2 b^{\prime \prime}}$ when $x$ goes to zero, $(t, x) \rightarrow \rho(t, x)$ a bounded function on $\left[T_{0},+\infty[\times \mathbb{R}\right.$ with values in $\mathbb{C}$ and $\kappa>0$ such that

$$
\begin{equation*}
w(t, x)=\varepsilon \alpha(x) \exp \left[\frac{i}{4|x|} \int_{T_{0}}^{t}(1-\chi)\left(\tau^{\beta} x\right) d \tau+\frac{i}{64} \varepsilon^{2} \frac{|\alpha(x)|^{2}}{|x|^{5}} \log t\right]+\varepsilon t^{-\kappa} \rho(t, x) \tag{6.22}
\end{equation*}
$$

where $\chi \in C_{0}^{\infty}(\mathbb{R}), \chi \equiv 1$ close to zero.
Remark. We may write (6.22) on a more explicit fashion. Assume that $b^{\prime \prime}>\kappa /(2 \beta)$. The contribution of the first term in expansion (6.22) localized for $|x|<C t^{-\kappa /\left(2 b^{\prime \prime}\right)}$ may be incorporated to the remainder, because of the vanishing of $\alpha$ at order $2 b^{\prime \prime}$ at $x=0$. On the other hand, if $|x|>C t^{-\kappa /\left(2 b^{\prime \prime}\right)}$, our assumption on $b^{\prime \prime}$ implies that $|x|^{-1 / \beta}<C^{-1 / \beta} t$, so that, if $C$ is large enough,

$$
\int_{T_{0}}^{t} \chi\left(\tau^{\beta} x\right) d \tau=\int_{T_{0}}^{+\infty} \chi\left(\tau^{\beta} x\right) d \tau
$$

If we define

$$
\underline{\alpha}(x)=\alpha(x) \exp \left[-\frac{i}{4|x|} T_{0}-\frac{i}{4|x|} \int_{T_{0}}^{+\infty} \chi\left(\tau^{\beta} x\right) d \tau\right]
$$

we obtain

$$
\begin{equation*}
w(t, x)=\varepsilon \underline{\alpha}(x) \exp \left[\frac{i t}{4|x|}+\frac{i}{64} \varepsilon^{2} \frac{|\underline{\alpha}(x)|^{2}}{|x|^{5}} \log t\right]+\varepsilon t^{-\kappa} \underline{\rho}(t, x) \tag{6.23}
\end{equation*}
$$

for a new bounded remainder $\underline{\rho}$.

Proof. We shall establish the proposition performing a normal form transform on equation (6.2). Denote by $\mathcal{G}$ the space of continuous bounded functions on $\left[T_{0}, T\left[\times \mathbb{R}\right.\right.$. Let $\chi_{0}$ be in $C_{0}^{\infty}(\mathbb{R}), \chi_{0} \equiv 1$ close to zero, Supp $\chi_{0} \subset\{x ; \chi(x) \equiv 1\}$ and set

$$
\begin{align*}
& f^{(\ell)}= w^{(\ell)}  \tag{6.24}\\
&+i \frac{\sqrt{h}}{4}\left(1-\chi_{0}\right)\left(x h^{-\beta}\right)|\mathrm{d} \omega|\langle 2 \mathrm{~d} \omega\rangle^{\ell}\langle\mathrm{d} \omega\rangle^{-2 \ell}\left[(1+\sqrt{2})\left(w^{(\ell)}\right)^{2}+(1-\sqrt{2})\left(\bar{w}^{(\ell)}\right)^{2}\right] \\
&+h\left(1-\chi_{0}\right)\left(x h^{-\beta}\right)^{2}|\mathrm{~d} \omega|^{2}[ M_{3}^{(\ell)}(\mathrm{d} \omega)\left(w^{(\ell)}\right)^{3}+M_{-3}^{(\ell)}(\mathrm{d} \omega)\left(\bar{w}^{(\ell)}\right)^{3} \\
&\left.+M_{-1}^{(\ell)}(\mathrm{d} \omega)\left|w^{(\ell)}\right|^{2} \bar{w}^{(\ell)}\right]
\end{align*}
$$

where $M_{p}^{(\ell)}(\zeta)$ are symbols of order $-2 \ell$ in $\zeta$ to be chosen, $p=-3,-1,3$.
We consider the polynomial map $\Phi:\binom{w^{(\ell)}}{\bar{w}^{(\ell)}} \rightarrow\binom{f^{(\ell)}}{\bar{f}^{(\ell)}}$ defined on $\mathcal{G}$. For $h=t^{-1}$ small enough (i.e. $t \geq T_{0}$ large enough), this is a local diffeomorphism at zero in $\mathcal{G}$. The inverse $\Phi^{-1}$ sends $\binom{f^{(\ell)}}{\bar{f}^{(\ell)}}$ to $\binom{w^{(\ell)}}{\bar{w}^{(\ell)}}$, where $w^{(\ell)}$ may be expressed explicitly as

$$
\begin{align*}
& w^{(\ell)}=f^{(\ell)}  \tag{6.25}\\
& \begin{aligned}
& \quad i \frac{\sqrt{h}}{4}\left(1-\chi_{0}\right)\left(x h^{-\beta}\right)|\mathrm{d} \omega|\langle 2 \mathrm{~d} \omega\rangle^{\ell}\langle\mathrm{d} \omega\rangle^{-2 \ell}\left[(1+\sqrt{2})\left(f^{(\ell)}\right)^{2}+(1-\sqrt{2})\left(\bar{f}^{(\ell)}\right)^{2}\right] \\
& +h|\mathrm{~d} \omega|^{2}\left(1-\chi_{0}\right)^{2}\left(x h^{-\beta}\right)\left[\widetilde{M}_{3}^{\ell}(\mathrm{d} \omega)\left(f^{(\ell)}\right)^{3}\right. \\
& +\frac{3-2 \sqrt{2}}{8}\langle 2 \mathrm{~d} \omega\rangle^{2 \ell}\langle\mathrm{~d} \omega\rangle^{-4 \ell}\left|f^{(\ell)}\right|^{2} f^{(\ell)} \\
& +\widetilde{M}_{-1}^{\ell}(\mathrm{d} \omega)\left|f^{(\ell)}\right|^{2} \bar{f}^{(\ell)} \\
& \left.+\widetilde{M}_{-3}^{\ell}(\mathrm{d} \omega)\left(\bar{f}^{(\ell)}\right)^{3}\right]
\end{aligned} \\
& \quad+h^{1+\kappa} R_{4}\left(x, h ; f^{(\ell)}, \bar{f}^{(\ell)}\right)
\end{align*}
$$

where $\kappa$ is some positive constant, where $R_{4}\left(x, h ; f^{(\ell)}, \bar{f}^{(\ell)}\right)$ is some analytic function of $\left(f^{(\ell)}, \bar{f}^{(\ell)}\right)$, vanishing at order four at zero, with bounds uniform in $(x, h)$, and $\widetilde{M}_{p}^{(\ell)}(\zeta)=$ $-M_{p}^{(\ell)}(\zeta)+\Gamma_{p}^{\ell}(\zeta), p=-3,-1,1$ for symbols $\Gamma_{p}^{\ell}$ of order $-2 \ell$, independent of $M_{p}^{\ell}$.

We compute $D_{t} f^{(\ell)}$ from (6.24), expressing in the right hand side $D_{t} w^{(\ell)}, D_{t} \bar{w}^{(\ell)}$ using (6.2). We get

$$
\begin{align*}
D_{t} f^{(\ell)}= & \frac{1}{2}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega(x)|^{\frac{1}{2}} w^{(\ell)}  \tag{6.26}\\
& +i \frac{\sqrt{h}}{8}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega(x)|^{\frac{3}{2}}\langle 2 \mathrm{~d} \omega\rangle^{\ell}\langle\mathrm{d} \omega\rangle^{-2 \ell}\left[(1+\sqrt{2})\left(w^{(\ell)}\right)^{2}+(1-\sqrt{2})\left(\bar{w}^{(\ell)}\right)^{2}\right] \\
& +h(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega(x)|^{\frac{5}{2}}\left[\frac{1}{2}\langle\mathrm{~d} \omega\rangle^{-2 \ell}\left|w^{(\ell)}\right|^{2} w^{(\ell)}\right. \\
& +\left(\frac{3}{2} M_{3}^{(\ell)}(\mathrm{d} \omega)+\widetilde{\Gamma}_{3}^{(\ell)}(\mathrm{d} \omega)\right)\left(w^{(\ell)}\right)^{3}+\left(-\frac{1}{2} M_{-1}^{(\ell)}(\mathrm{d} \omega)+\widetilde{\Gamma}_{-1}^{(\ell)}(\mathrm{d} \omega)\right)\left|w^{(\ell)}\right|^{2} \bar{w}^{(\ell)} \\
& \left.+\left(-\frac{3}{2} M_{-3}^{(\ell)}(\mathrm{d} \omega)+\widetilde{\Gamma}_{-3}^{(\ell)}(\mathrm{d} \omega)\right)\left(\bar{w}^{(\ell)}\right)^{3}\right] \\
& +h^{1+\kappa}\left(r^{(\ell)}(t, x)+R_{2}\left(x, h ; w^{(\ell)}, \bar{w}^{(\ell)}\right)\right)
\end{align*}
$$

where $\widetilde{\Gamma}_{p}^{(\ell)}(\zeta)$ are symbols of order $-2 \ell$ in $\zeta$, that depend only on the coefficients of $\left(w^{(\ell)}\right)^{2}$, $\left(w^{(\ell)}\right)^{3}, \ldots$ in the right hand side of (6.2), where $r^{(\ell)}$ is the remainder in (6.2) and where $R_{2}\left(x, h ; w^{(\ell)}, \bar{w}^{(\ell)}\right)$ is some polynomial in $\left(w^{(\ell)}, \bar{w}^{(\ell)}\right)$, vanishing at order 2 at zero, with uniform bounds in $(x, h)$. We express $w^{(\ell)}$ in the right hand side of (6.26) using formula (6.25). The quadratic terms in the definition (6.24) of $f^{(\ell)}$ have been chosen in such a way that the quadratic contributions in the right hand side of the resulting expression for $D_{t} f^{(\ell)}$ vanish. We get

$$
\begin{align*}
D_{t} f^{(\ell)}= & \frac{1}{2}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega(x)|^{\frac{1}{2}} f^{(\ell)} \\
& +h(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega(x)|^{\frac{5}{2}}\left[\frac{1}{2}\langle\mathrm{~d} \omega\rangle^{-2 \ell}\left|f^{(\ell)}\right|^{2} f^{(\ell)}\right. \\
& +\left(M_{3}^{(\ell)}(\mathrm{d} \omega)-\widetilde{\Gamma}_{3}^{(\ell)}(\mathrm{d} \omega)\right)\left(f^{(\ell)}\right)^{3}  \tag{6.27}\\
& +\left(-M_{-1}^{(\ell)}(\mathrm{d} \omega)-{\widetilde{\Gamma^{\prime}}}_{-1}^{(\ell)}(\mathrm{d} \omega)\right)\left|f^{(\ell)}\right|^{2} \bar{f}^{(\ell)} \\
& \left.+\left(-2 M_{-3}^{(\ell)}(\mathrm{d} \omega)-\widetilde{\Gamma}_{-3}^{(\ell)}(\mathrm{d} \omega)\right)\left(\bar{f}^{(\ell)}\right)^{3}\right] \\
& +h^{1+\kappa}\left(r^{(\ell)}(t, x)+R_{2}\left(x, h ; f^{(\ell)}, \bar{f}^{(\ell)}\right)\right)
\end{align*}
$$

where $\widetilde{\Gamma}_{p}^{(\ell)}(\zeta)$ are new symbols of order $-2 \ell$ that do not depend on $M_{p}^{(\ell)}$, and where $R_{2}$ is a new analytic function of $\left(f^{(\ell)}, \bar{f}^{(\ell)}\right)$ vanishing at order 2 at zero, with uniform bounds in $(x, h)$.

We choose now the free symbols $M_{p}^{(\ell)}, p=3,-1,-3$ introduced in the definition (6.24) of $f^{(\ell)}$ so that the coefficients of $\left(f^{(\ell)}\right)^{3},\left|f^{(\ell)}\right|^{2} \bar{f}^{\ell}$ and $\left(\bar{f}^{(\ell)}\right)^{3}$ vanish. In that way, we are reduced to

$$
\begin{align*}
D_{t} f^{(\ell)}= & \frac{1}{2}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega(x)|^{\frac{1}{2}}\left[1+\frac{|\mathrm{d} \omega|^{2}}{t}\langle\mathrm{~d} \omega\rangle^{-2 \ell}\left|f^{(\ell)}\right|^{2}\right] f^{(\ell)}  \tag{6.28}\\
& +t^{-1-\kappa} r^{(\ell)}(t, x)+t^{-1-\kappa} R_{2}\left(x, h ; f^{(\ell)}, \bar{f}^{(\ell)}\right)
\end{align*}
$$

where $\left\|\left(h D_{x}\right)^{p} Z^{k} r^{(\ell)}(t, x)\right\|_{L^{\infty}}$ is $O(\varepsilon)$ for any $p \leq b^{\prime \prime}, k+\ell \leq \frac{s}{2}+N_{0}$. It follows from (6.28) that $\left.\left|\partial_{t}\right| f^{(\ell)}\right|^{2} \mid \leq\left(C_{0} \varepsilon+C_{0}^{\prime}\left|f^{(\ell)}\right|^{2}\right) t^{-1-\kappa}$ as long as $\left|f^{(\ell)}\right|$ stays smaller than 1 . Since at time $t=T_{0},\left|f^{(\ell)}\right|=O(\varepsilon)$, we obtain that $\left|f^{(\ell)}(t, x)\right| \leq C_{1}^{\prime} \varepsilon$ for some constant $C_{1}^{\prime}>0$, as long as the solution exists. Using expression (6.25) for $w^{(\ell)}$ in terms of $f^{(\ell)}$, we get (6.21). If $r_{\ell}$ is defined for $t \in\left[T_{0},+\infty\left[\right.\right.$, we get that $f^{(\ell)}$ and thus $w^{(\ell)}$ is defined on $\left[T_{0},+\infty[\times \mathbb{R}\right.$.

Let us prove the asymptotic expansion for $w$. If $\ell \leq \frac{s}{2}+N_{0}$, we define

$$
\Theta_{\ell}(t, x)=\frac{1}{2}|\mathrm{~d} \omega(x)|^{\frac{1}{2}} \int_{T_{0}}^{t}(1-\chi)\left(\tau^{\beta} x\right)\left[1+\frac{|\mathrm{d} \omega(x)|^{2}}{\tau}\langle\mathrm{~d} \omega\rangle^{-2 \ell}\left|f^{(\ell)}(\tau, x)\right|^{2}\right] d \tau .
$$

Then (6.28) and the uniform a priori bound just obtained for $f^{(\ell)}$ show that

$$
\frac{d}{d t}\left[f^{(\ell)}(t, x) \exp \left[-i \Theta_{\ell}(t, x)\right]\right]=O\left(\varepsilon t^{-1-\kappa}\right)
$$

uniformly for $x \in \mathbb{R}$. It follows that the uniform limit when $t$ goes to $+\infty$ of

$$
f^{(\ell)}(t, x) \exp \left[-i \Theta_{\ell}(t, x)\right]
$$

exists and defines a continuous function $\varepsilon \alpha_{\ell}(x)$ on $\mathbb{R}$, which is $O(\varepsilon)$ in $L^{\infty}(\mathbb{R})$. Moreover

$$
\begin{equation*}
\left\|f^{(\ell)}(t, x)-\varepsilon \alpha_{\ell}(x) \exp \left(i \Theta_{\ell}(t, x)\right)\right\|_{L^{\infty}}=O\left(\varepsilon t^{-\kappa}\right), \quad t \rightarrow+\infty . \tag{6.29}
\end{equation*}
$$

We write

$$
\begin{align*}
\Theta_{\ell}(t, x)= & \frac{1}{2}|\mathrm{~d} \omega(x)|^{\frac{1}{2}} \int_{T_{0}}^{t}(1-\chi)\left(\tau^{\beta} x\right) d \tau \\
& +\frac{\varepsilon^{2}}{2}(1-\chi)\left(t^{\beta} x\right)|\mathrm{d} \omega(x)|^{\frac{5}{2}}\langle\mathrm{~d} \omega(x)\rangle^{-2 \ell}\left|\alpha_{\ell}(x)\right|^{2} \log t \\
& -\frac{\varepsilon^{2}}{2}(1-\chi)\left(T_{0}^{\beta} x\right)|\mathrm{d} \omega(x)|^{\frac{5}{2}}\langle\mathrm{~d} \omega(x)\rangle^{-2 \ell}\left|\alpha_{\ell}(x)\right|^{2} \log T_{0}  \tag{6.30}\\
& +\frac{\varepsilon^{2}}{2} \int_{T_{0}}^{t} \beta \tau^{\beta} x \chi^{\prime}\left(\tau^{\beta} x\right)|\mathrm{d} \omega(x)|^{\frac{5}{2}}\langle\mathrm{~d} \omega(x)\rangle^{-2 \ell}\left|\alpha_{\ell}(x)\right|^{2} \frac{\log \tau}{\tau} d \tau \\
& +\frac{1}{2} \int_{T_{0}}^{t}(1-\chi)\left(\tau^{\beta} x\right)|\mathrm{d} \omega(x)|^{\frac{5}{2}}\langle\mathrm{~d} \omega(x)\rangle^{-2 \ell}\left(\left|f^{(\ell)}\right|^{2}-\left|\varepsilon \alpha_{\ell}(x)\right|^{2}\right) \frac{d \tau}{\tau} .
\end{align*}
$$

We notice that $|\mathrm{d} \omega(x)|^{\frac{5}{2}}\langle\mathrm{~d} \omega\rangle^{-2 \ell}$ is $O\left(\langle x\rangle^{-5}\right)$ if $\ell \geq 5 / 4$ and $\tau^{\beta} x \chi^{\prime}\left(\tau^{\beta} x\right)|\mathrm{d} \omega(x)|^{\frac{5}{2}}\langle\mathrm{~d} \omega(x)\rangle^{-2 \ell}$ is $O\left(\tau^{-\beta}\langle x\rangle^{-6}\right)$ if $\ell \geq 3 / 2$.

These bounds and the estimate $\left\|\left|f^{(\ell)}\right|^{2}-\varepsilon^{2}\left|\alpha_{\ell}\right|^{2}\right\|_{L^{\infty}}=O\left(\varepsilon^{2} t^{-\kappa}\right)$ that follows from (6.29) imply that the last three terms in (6.30) may be written as $\varepsilon^{2} \Gamma_{\ell}(x)+\varepsilon^{2} R_{\ell}(t, x)$ for some continuous function $\Gamma_{\ell}(x)$, which is $O\left(\langle x\rangle^{-5}\right)$ and some remainder $R(t, x)$ satisfying $|R(t, x)|=$ $O\left(t^{-\kappa}\langle x\rangle^{-5}\right)$ (assuming $\left.0<\kappa<\beta\right)$. Modifying the definition of that remainder, we get finally

$$
\begin{aligned}
\Theta_{\ell}(t, x)= & \frac{1}{2}|\mathrm{~d} \omega(x)|^{\frac{1}{2}} \int_{T_{0}}^{t}(1-\chi)\left(\tau^{\beta} x\right) d \tau \\
& +\frac{\varepsilon^{2}}{2}|\mathrm{~d} \omega(x)|^{\frac{5}{2}}\langle\mathrm{~d} \omega\rangle^{-2 \ell}\left|\alpha_{\ell}(x)\right|^{2} \log t \\
& +\varepsilon^{2} \Gamma(x)+\varepsilon^{2} R(t, x)
\end{aligned}
$$

when $\ell>3 / 2$. It follows from this and from (6.29) that

$$
\begin{align*}
f^{(\ell)}(t, x)= & \varepsilon \widetilde{\alpha}_{\ell}(x) \exp \left[\frac{i}{4|x|} \int_{T_{0}}^{t}(1-\chi)\left(\tau^{\beta} x\right) d \tau+\frac{i}{64} \varepsilon^{2} \frac{\left|\widetilde{\alpha}_{\ell}(x)\right|^{2}}{|x|^{5}}\langle\mathrm{~d} \omega\rangle^{-2 \ell} \log t\right]  \tag{6.31}\\
& +\varepsilon t^{-\kappa} \rho(t, x)
\end{align*}
$$

where $\widetilde{\alpha}_{\ell}(x)=e^{i \varepsilon^{2} \Gamma(x)} \alpha_{\ell}(x)$ and where $\rho(t, x)$ is uniformly bounded. If we express $w^{(\ell)}$ from $f^{(\ell)}$ using (6.25), we conclude that the same expansion (6.31) holds for $w^{(\ell)}$ (with a different remainder). Let us compute $w(t, \cdot)=\mathrm{Op}_{h}\left(\langle\xi\rangle^{-\ell}\right) w^{(\ell)}$. The action of $\mathrm{Op}_{h}\left(\langle\xi\rangle^{-\ell}\right)$ on the remainder gives a term of the same type, if $\ell$ is large enough. On the other hand, by the expression (5.52), (5.53) of $w^{(\ell)}$ from $w_{\Lambda}^{(\ell)}$ (and the converse expression), we get that $\mathrm{Op}_{h}\left(\langle\xi\rangle^{-\ell}\right) w=\mathrm{Op}_{h}\left(\langle\xi\rangle^{-\ell}\right) w_{\Lambda}$ up to a remainder bounded in $L^{\infty}(d x)$ by $C \varepsilon t^{-\kappa}$. As $w_{\Lambda}$ is in $h^{-\delta_{1}^{\prime}} L^{\infty} \tilde{J}_{\Lambda}^{0, b^{\prime}}[K]$, Proposition 2.11 applies and shows that $\mathrm{Op}_{h}\left(\langle\xi\rangle^{-\ell}\right) w_{\Lambda}$ may be written as $\langle\mathrm{d} \omega\rangle^{-\ell} w_{\Lambda}$ modulo a remainder in $h^{1-\delta_{1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{-1, b^{\prime}}[K] \subset h^{\sigma-\delta_{1}^{\prime}} \tilde{\mathcal{B}}_{\infty}^{0, b^{\prime}}[K]$, which is $O\left(\varepsilon t^{-\kappa}\right)$ in $L^{\infty}$ for small enough $\kappa>0$ since by (4.2) $\delta_{1}^{\prime}<\sigma / 8$. Using again the expression of $w_{\Lambda}$ from $w$ deduced from (5.52), we deduce that

$$
\left\|w(t, \cdot)-\langle\mathrm{d} \omega\rangle^{-\ell} w^{(\ell)}(t, \cdot)\right\|_{L^{\infty}}=O\left(\varepsilon t^{-\kappa}\right) .
$$

If we define $\alpha(x)=\langle\mathrm{d} \omega\rangle^{-\ell} \widetilde{\alpha}_{\ell}(x)$ with $\ell$ equal to $b^{\prime \prime}$, we obtain a function continuous and bounded on $\mathbb{R}$, vanishing like $|x|^{2 b^{\prime \prime}}$ when $x$ goes to zero and such that $w(t, x)$ is given by the asymptotic expansion (6.22). This concludes the proof.

We prove now a statement concerning the $Z$-derivatives of $w^{(\ell)}$. Let $\left(A_{k}^{\prime \prime}\right)_{k \geq 1}$ be a sequence of positive numbers satisfying $A_{k}^{\prime \prime} \geq A_{k_{1}}^{\prime \prime}+A_{k_{2}}^{\prime \prime}+A_{k_{3}}^{\prime \prime}$ if $k_{1}+k_{2}+k_{3}=k, k_{j}<k, j=1,2,3$ and $A_{k}^{\prime \prime}$ large enough relatively to the constant $C_{1}$ in (6.21).

Proposition 6.6. There is a constant $C_{2}>0$ such that, if we set $\widetilde{\delta}_{k}^{\prime}=A_{k}^{\prime \prime} \varepsilon^{2}$, for any $k, \ell$ with $k+\ell \leq \frac{s}{2}+N_{0}-2$, the solution $w^{(\ell)}$ of (6.2) satisfies

$$
\begin{equation*}
\left\|Z^{k} w^{(\ell)}(t, \cdot)\right\|_{L^{\infty}} \leq C_{2} \varepsilon t^{\tilde{\delta}_{k}^{\prime}} . \tag{6.32}
\end{equation*}
$$

Remark. The gain in (6.32), in comparison with (4.9), is that the exponents $\widetilde{\delta}_{k}^{\prime}$ depend only on the size $\varepsilon$ of the Cauchy data and not on the exponents $\delta_{k}$ that are used in the $L^{2}$-estimates. In particular, taking $\varepsilon$ small enough, we may arrange so that $\widetilde{\delta}_{k}^{\prime} \ll \delta_{k}$.

Proof. We apply a normal forms method to remove the quadratic terms in (6.4). For ( $k, \ell$ ) satisfying $k+\ell \leq \frac{s}{2}+N_{0}$, we define a new quadratic map $\widetilde{Q}^{k,(\ell)}\left[x, h ; W^{k,(\ell)}, \bar{W}^{k,(\ell)}\right]$ in the following way: The components of this map are defined taking the same linear combinations as those used to define the components of $Q^{k,(\ell)}$ from (6.5) of the quantities

$$
\begin{align*}
& \frac{2 \theta\left(x h^{-\beta}\right) \Phi(x)}{(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{1}{2}}}\left(Z^{k_{1}} w^{(\ell)}\right)\left(Z^{k_{2}} w^{(\ell)}\right) \\
& \frac{-2 \theta\left(x h^{-\beta}\right) \Phi(x)}{(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{1}{2}}}\left(Z^{k_{1}} w^{(\ell)}\right)\left(Z^{k_{2}} \bar{w}^{(\ell)}\right)  \tag{6.33}\\
& \frac{-2 \theta\left(x h^{-\beta}\right) \Phi(x)}{3(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{1}{2}}}\left(Z^{k_{1}} \bar{w}^{(\ell)}\right)\left(Z^{k_{2}} \bar{w}^{(\ell)}\right) .
\end{align*}
$$

If we make act $D_{t}-\frac{1}{2}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega(x)|^{\frac{1}{2}}$ on each line of (6.33), we see that we obtain, using (6.4), the corresponding line of (6.5) and the following contributions

- Quantities of the form $\sqrt{h} \widetilde{\mathcal{C}}^{k,(\ell)}\left[x, h ; W^{k,(\ell)}, \bar{W}^{k,(\ell)}\right]$ for cubic forms $\widetilde{\mathcal{C}}^{k,(\ell)}$ which have the same structure (6.6) as $\mathcal{C}^{k,(\ell)}$.
- Quantities given by the product of $h$ and of homogeneous expressions of order 4 in

$$
\left(Z^{k_{j}} w^{(\ell)}, Z^{k_{j}} \bar{w}^{(\ell)}\right), \quad k_{1}+\cdots+k_{4} \leq k
$$

with coefficients depending on $x$ which are $O\left(h^{-7 \beta}\right)$. If we use that $Z^{k_{j}} w^{(\ell)}$ satisfies the a priori estimates (4.9), we see that these contributions may be written as $h^{\frac{1}{2}+\kappa} R^{k,(\ell)}$ for some $\kappa>0$ and a bounded function $R^{k,(\ell)}$.

- Contributions coming from the remainder in (6.4) or from the action of $D_{t}$ on the cut-offs in (6.33), that may be written also as $h^{\frac{1}{2}+\kappa} R^{k,(\ell)}$.

Consequently, if we set for $k+\ell \leq \frac{s}{2}+N_{0}$,

$$
\widetilde{W}^{k,(\ell)}=W^{k,(\ell)}-\sqrt{h} \widetilde{Q}^{k,(\ell)}\left[x, h ; W^{k,(\ell)}, \bar{W}^{k,(\ell)}\right],
$$

we obtain that $\widetilde{W}^{k,(\ell)}$ satisfies bounds of the form (4.9) and solves the equation

$$
\begin{equation*}
\left(D_{t}-\frac{1}{2}(1-\chi)\left(x h^{-\beta}\right)|\mathrm{d} \omega|^{\frac{1}{2}}\right) \widetilde{W}^{k,(\ell)}=h \mathcal{C}^{k,(\ell)}\left[x, h ; W^{k,(\ell)}, \bar{W}^{k,(\ell)}\right]+h^{1+\kappa} \mathcal{R}^{k,(\ell)} \tag{6.34}
\end{equation*}
$$

where $\mathcal{C}^{k,(\ell)}$ is a new cubic map given in terms of monomials of the form (6.6) and $\mathcal{R}^{k,(\ell)}$ a uniformly bounded remainder. Notice that, up to a modification of $\mathcal{R}^{k,(\ell)}$, we may replace $W^{k,(\ell)}$ by $\widetilde{W}^{k,(\ell)}$ in the argument of $\mathcal{C}^{k,(\ell)}$.

Assume by induction that for given $k, \ell$ with $k+\ell \leq \frac{s}{2}+N_{0}, \ell \geq 2$, (6.32) has been established with $k$ replaced by $k-1$. Then $W^{k-1,(\ell)}$ and $\widetilde{W}^{k-1,(\ell)}$ are under control, and we need to obtain (6.32) for the last component $Z^{k} w^{(\ell)}$ of $W^{k,(\ell)}$, or equivalently, for the last component $\widetilde{W}_{k}^{k,(\ell)}$ of $\widetilde{W}^{k,(\ell)}$. We sort the different contributions to

$$
\begin{equation*}
\mathcal{C}^{k,(\ell)}\left[x, h ; \widetilde{W}^{k,(\ell)}, \widetilde{\widetilde{W}}^{k,(\ell)}\right] \tag{6.35}
\end{equation*}
$$

On the one hand, we get terms given by expressions of the form

$$
\begin{align*}
& \theta\left(x h^{-\beta}\right) \Phi(x) \widetilde{W}_{k_{1}}^{k,(\ell)} \widetilde{W}_{k_{2}}^{k,(\ell)} \widetilde{W}_{k_{3}}^{k,(\ell)} \\
& \theta\left(x h^{-\beta}\right) \Phi(x) \widetilde{W}_{k_{1}}^{k,(\ell)} \widetilde{W}_{k_{2}}^{k,(\ell)} \widetilde{W}_{k_{3}}^{k,(\ell)} \\
& \theta\left(x h^{-\beta}\right) \Phi(x) \widetilde{W}_{k_{1}}^{k,(\ell)} \widetilde{W}_{k_{2}}^{,(\ell)} \widetilde{W}_{k_{3}}^{k,(\ell)}  \tag{6.36}\\
& \theta\left(x h^{-\beta}\right) \Phi(x) \widetilde{W}_{k_{1}}^{k,(\ell)} \widetilde{W}_{k_{2}}^{k,(\ell)} \widetilde{W}_{k_{3}}^{k,(\ell)}
\end{align*}
$$

where $\theta, \Phi$ satisfy the same conditions as in (6.6), so are bounded since $\ell \geq 2$, and where two among $k_{1}, k_{2}, k_{3}$ are zero and the other one is equal to $k$. We call $F$ the sum of contributions of that type, so that

$$
|F(t, x)| \leq C\left\|\widetilde{W}_{0}^{k,(\ell)}(t, \cdot)\right\|_{L^{\infty}}^{2}\left|\widetilde{W}_{k}^{k,(\ell)}(t, x)\right| .
$$

Proposition 6.5 gives a uniform estimate for $\left\|w^{(\ell)}(t, \cdot)\right\|_{L^{\infty}}$, so also for

$$
\widetilde{W}_{0}^{k,(\ell)}=w^{(\ell)}-\sqrt{h} \widetilde{Q}_{0}^{k,(\ell)}\left[x, h ; W^{k,(\ell)}, \bar{W}^{k,(\ell)}\right] .
$$

We conclude that for some constant $B>0$, depending only on the constant $C_{1}$ in (6.21),

$$
\begin{equation*}
|F(t, x)| \leq B \varepsilon^{2}\left|\widetilde{W}_{k}^{k,(\ell)}(t, x)\right| . \tag{6.37}
\end{equation*}
$$

On the other hand, (6.35) is also made of terms of the form (6.36) with

$$
k_{1}+k_{2}+k_{3} \leq k, \quad k_{1}, k_{2}, k_{3}<k .
$$

The assumption of induction, together with the inequality between the constants $A_{k}$ made in the statement of the proposition, imply that the contribution $G$ of these terms satisfies

$$
\begin{equation*}
|G(t, x)| \leq C \varepsilon^{3} t^{\tilde{\delta}_{k}^{\prime}} \tag{6.38}
\end{equation*}
$$

We deduce from the equation for the last component $\widetilde{W}_{k}^{k,(\ell)}$ of $\widetilde{W}^{k,(\ell)}$ given by (6.34)

$$
\begin{aligned}
\left|\widetilde{W}_{k}^{k,(\ell)}(t, x)\right|^{2} \leq & \left|\widetilde{W}_{k}^{k,(\ell)}\left(T_{0}, x\right)\right|^{2}+\int_{T_{0}}^{t}|F(\tau, x)|\left|\widetilde{W}_{k}^{k,(\ell)}(\tau, x)\right| \frac{d \tau}{\tau} \\
& +\int_{T_{0}}^{t}|G(\tau, x)|\left|\widetilde{W}_{k}^{k,(\ell)}(\tau, x)\right| \frac{d \tau}{\tau} \\
& +\int_{T_{0}}^{t}\left|R_{k}^{k,(\ell)}(\tau, x)\right|\left|\widetilde{W}_{k}^{k,(\ell)}(\tau, x)\right| \frac{d \tau}{\tau^{1+\kappa}} .
\end{aligned}
$$

Using (6.37), (6.38), and the fact that at $t=T_{0}, \widetilde{W}_{k}^{k,(\ell)}\left(T_{0}, \cdot\right)$ is $O(\varepsilon)$ we deduce that

$$
\begin{align*}
\left|\widetilde{W}_{k}^{k,(\ell)}(t, x)\right| \leq & C \varepsilon+B \varepsilon^{2} \int_{T_{0}}^{t}\left|\widetilde{W}_{k}^{k,(\ell)}(\tau, x)\right| \frac{d \tau}{\tau} \\
& +C \varepsilon^{2} \int_{T_{0}}^{t} \tau_{\widetilde{\delta}_{k}^{\prime}-1} d \tau+C \varepsilon \int_{T_{0}}^{t} \frac{d \tau}{\tau^{1+\kappa}} . \tag{6.39}
\end{align*}
$$

If we use Gronwall inequality, and assume that the constant $A_{k}$ in the definition $\widetilde{\delta}_{k}^{\prime}=A_{k} \varepsilon^{2}$ of $\widetilde{\delta}_{k}^{\prime}$ is large enough relatively to $B$, we deduce from (6.39) that

$$
\left|\widetilde{W}_{k}^{k,(\ell)}(t, x)\right| \leq C^{\prime} \varepsilon t^{\widetilde{\delta}_{k}^{\prime}}
$$

when $k+\ell \leq \frac{s}{2}+N_{0}, \ell \geq 2$. By definition of $\widetilde{W}_{k}^{k,(\ell)}$, the same inequality holds for $Z^{k} w^{(\ell)}$. Since $w^{(\ell-2)}=\mathrm{Op}_{h}\left(\langle\xi\rangle^{-2}\right) w^{(\ell)}$, we conclude that $\left\|Z^{k} w^{(\ell)}(t, \cdot)\right\|_{L^{\infty}}$ is $O\left(\varepsilon \tau_{\tilde{t}_{k}^{\prime}}\right)$ when $k+\ell \leq \frac{\mathrm{s}}{2}+N_{0}-2$. This concludes the proof of the proposition.

To finish this section, we deduce from the results established so far the proof of Theorem 1.6. This will conclude the demonstration of our main theorem.

Proof of Theorem 1.6. We notice first that it is enough to prove the following apparently weaker statement: Assume that for some constants $B_{2}>0, \tilde{A}_{0}>0$, any $t \in\left[T_{0}, T[\right.$, any $\epsilon \in] 0,1]$, any $k \leq s_{1}$

$$
\begin{array}{r}
M_{s}^{\left(k_{1}\right)}(t) \leq B_{2} \epsilon t^{\delta_{k}}, \quad N_{\rho}^{\left(s_{0}\right)}(t) \leq \sqrt{\epsilon}<1, \\
\|\eta(t, \cdot)\|_{C^{\gamma}}+\left\|\left|D_{x}\right|^{1 / 2} \psi(t, \cdot)\right\|_{C^{\gamma-\frac{1}{2}}} \leq \tilde{A}_{0} t^{-\frac{1}{2}+\delta_{0}^{\prime}} \tag{6.40}
\end{array}
$$

Then, (1.20) holds.
Actually, if the preceding implication is proved with $\rho \geq \gamma$, and if we assume only (1.19), then (6.40) holds true on some interval $\left[T_{0}, T^{\prime}\right], T^{\prime}>T_{0}$ taking $\tilde{A}_{0}$ large enough in function of $T_{0}$ (because the last condition in (6.40) follows then from the second one, taking $T^{\prime}$ close enough to $T_{0}$ ). We conclude that (1.20) holds on $\left[T_{0}, T^{\prime}\right]$, and taking $\epsilon<\epsilon_{0}^{\prime}$ small enough so that $\epsilon B_{\infty}<\tilde{A}_{0}$, we see that, by continuity, (6.40) holds on some interval [ $\left.T_{0}, T^{\prime \prime}\right]$ with $T^{\prime \prime}>T^{\prime}$. By bootstrap, we conclude that (1.20) will then be true on the whole interval $\left[T_{0}, T\right]$.

Consequently, we have reduced ourselves to the proof of the fact that (6.40) implies (1.20).
Recall that we have fixed in (4.3) large enough numbers $a, b$. We introduced also at the beginning of Section 4 integers $N_{0}, N_{1}$ and we assumed in Proposition 5.2 that $\left(N_{1}-N_{0}-1\right) \sigma \geq$ 1. Let us fix $\gamma \in] \max (7 / 2, b),+\infty\left[\backslash \frac{1}{2} \mathbb{N}\right.$, and assume that $N_{0}$ is taken large enough so that $N_{0} \geq 2 \gamma+\frac{13}{2}$. We define

$$
\begin{equation*}
s_{1}=\frac{S}{2}+N_{1}+1, \quad s_{0}=\frac{S}{2}+N_{0}-3-[\gamma] \tag{6.41}
\end{equation*}
$$

where $s$ is an even integer taken large enough so that the following conditions hold

$$
\begin{equation*}
s \geq s_{1} \geq s_{0} \geq \frac{1}{2}(s+2 \gamma) \tag{6.42}
\end{equation*}
$$

and that moreover

$$
\begin{equation*}
s_{1} \leq s-a-\frac{1}{2} . \tag{6.43}
\end{equation*}
$$

We set $\rho=s_{0}+\gamma$. It follows from equation (5.2.157) of the companion paper [5] that if $\mathcal{C}_{s_{0}} N_{\rho}^{\left(s_{0}\right)}=C\left(N_{\rho}^{\left(s_{0}\right)}\right) N_{\rho}^{\left(s_{0}\right)}$ is small enough, we have for any $k \leq s_{1}$

$$
\left\|Z^{k} \eta(t)\right\|_{H^{s-k}}+\left\|\left|D_{x}\right|^{\frac{1}{2}} Z^{k} \psi(t)\right\|_{H^{s-k-\frac{1}{2}}} \leq B_{2} \varepsilon t^{\delta_{k}}
$$

for a new value of the constant $B_{2}$. The smallness condition above is satisfied for $\varepsilon<\varepsilon_{0}^{\prime} \ll 1$ using the second estimate (6.40). Since we have set at the beginning of Section 3

$$
u(t, x)=\left|D_{x}\right|^{\frac{1}{2}} \psi+i \eta \text { and } u(t, x)=\frac{1}{\sqrt{t}} v\left(t, \frac{x}{t}\right)
$$

it follows, denoting by the same notation $Z$ the vector field in $(t, x)$ and in $\left(t, \frac{x}{t}\right)$-coordinates, that

$$
\left\|\left(h D_{x}\right)^{\ell} Z^{k^{\prime}} v(t, \cdot)\right\|_{L^{2}} \leq B_{2} \varepsilon t^{\delta_{k}}
$$

for $k^{\prime} \leq s_{1}, \ell \leq a$ since $s_{1}+a \leq s-\frac{1}{2}$. This, together with the definition (4.5) of $\mathcal{F}_{k}$, shows that the second condition in (4.8) holds with $k=s_{1}-1$. The first condition (4.8) holds because of the second estimate (6.40) and the fact that $\rho \geq \gamma>b$. Consequently, Proposition 4.1 implies that (4.9) holds for any $k \leq s_{1}-1=\frac{s}{2}+N_{1}$, with constants $A_{k^{\prime}}^{\prime}$ depending only on $B_{2}$ in (6.40). The assumption (5.1) is thus satisfied, and since we assumed $\left(N_{1}-N_{0}-1\right) \sigma \geq 1$, we may apply Proposition 5.2 and Corollary 5.9 which provides development (5.48). This development is the assumption that allows one to apply the results of Section 6: in particular inequality (6.32) will hold, with a constant $C_{2}$ depending only on the constant $B_{2}$ of (6.40) (and of universal quantities). If $B_{\infty}$ is taken large enough relatively to $B_{2}$ and if $B_{\infty}^{\prime}$ is larger than the constant $A_{s_{0}+[\gamma]+1}^{\prime \prime}$ introduced in Proposition 6.6, we deduce from (6.32)

$$
\left\|\left(h \partial_{x}\right)^{\ell} Z^{k} w(t, \cdot)\right\|_{L^{\infty}} \leq \frac{1}{2} B_{\infty} \varepsilon t^{B_{\infty}^{\prime} \varepsilon^{2}}
$$

for $k+\ell \leq s_{0}+[\gamma]+1$ (since $\frac{s}{2}+N_{0}-2=s_{0}+[\gamma]+1$ by our choice (6.41) of $s_{0}$ ). Coming back to the expression of $u=\left|D_{x}\right|^{\frac{1}{2}} \psi+i \eta$ from $t^{-\frac{1}{2}} v$, and using that by definition $\rho=s_{0}+\gamma$ this will give the bound

$$
\begin{equation*}
N_{\rho}^{\left(s_{0}\right)}(t) \leq \frac{1}{2} B_{\infty} \varepsilon t^{-\frac{1}{2}+B_{\infty}^{\prime} \varepsilon^{2}} \tag{6.44}
\end{equation*}
$$

if we prove that in the decomposition $v=v_{L}+w+v_{H}$, the contributions $v_{L}$ and $v_{H}$ satisfy also a bound of the form

$$
\begin{equation*}
\left\|Z^{k} v_{H}(t, \cdot)\right\|_{C^{\rho-k}}+\left\|Z^{k} v_{L}(t, \cdot)\right\|_{C^{\rho-k}} \leq \frac{1}{4} B_{\infty} \varepsilon t^{B_{\infty}^{\prime} \varepsilon^{2}} \tag{6.45}
\end{equation*}
$$

if $k \leq s_{0}$. Since our assumption (6.40) implies that (4.8) holds (with constants $A_{k^{\prime}}$ depending only on $B_{2}$ ), for $k^{\prime} \leq s_{1}$, we deduce from (4.5) and the definition (3.19) of $v_{L}$ that

$$
\left\|Z^{k} v_{L}(t, \cdot)\right\|_{L^{2}} \leq \varepsilon A_{k} h^{-\delta_{k}}, \quad k \leq s_{1} .
$$

Since $v_{L}$ is spectrally supported for $h|\xi|=O\left(h^{2(1-\sigma)}\right)$, we deduce from that by Sobolev injection that

$$
\begin{equation*}
\left\|Z^{k} v_{L}(t, \cdot)\right\|_{L^{\infty}}=O\left(\varepsilon h^{-\delta_{k}+\frac{1}{2}-\frac{\sigma}{2}}\right) \tag{6.46}
\end{equation*}
$$

with constants depending only on $B_{2}$, which gives for $v_{L}$ a better estimate than the one (6.45) we are looking for (since $v_{L}$ is spectrally supported for small frequencies, estimating $L^{\infty}$ or $C^{\rho-k}$ norms is equivalent).

Consider next the $v_{H}$-contribution. As (4.9) holds for $k=s_{1}-1$ with constants depending only on $B_{2}$, we may write for any $j \geq j_{0}(h, C)$, any $k, \ell$ with $k+\ell \leq s_{1}-1$,

$$
\left\|\Delta_{j}^{h}(h D)^{\ell} Z^{k} v_{H}(t, \cdot)\right\|_{L^{\infty}} \leq C \varepsilon 2^{-j_{+} b} h^{-\delta_{k}^{\prime}} .
$$

This holds in particular for $k+\ell \leq s_{0}+\gamma+1$ as $s_{0}+\gamma+1 \leq s_{1}-1$ by (6.41). Since $v_{H}$ is spectrally supported for $|h \xi| \geq c h^{-\beta}$, we conclude that

$$
\begin{equation*}
\left\|Z^{k} v_{H}(t, \cdot)\right\|_{C^{\rho-k}} \leq C \varepsilon h^{b \beta-\delta_{k}^{\prime}} \leq C \varepsilon h^{2-\delta_{k}^{\prime}} \tag{6.47}
\end{equation*}
$$

with a constant $C$ depending only on $B_{2}$, as we assumed in (4.3) that $b \beta>2$. This largely implies estimate (6.45) for $v_{H}$, and so concludes the proof of (6.44).

We thus have obtained the first inequality (1.20). We are left with showing the second estimate. This follows from (6.21) that holds for $\ell \leq \frac{s}{2}+N_{0}$, so for $\ell \leq s_{0}+\gamma+1$. This concludes the proof of Theorem 1.6.

Final remark on the proof of Theorem 1.4: In Section 1.3, we did not justify the asymptotic expansion (1.12) of $u(t, x)=\frac{1}{\sqrt{t}} v\left(t, \frac{x}{t}\right)$. This follows from (6.23), since we have seen in the proof above that in the decomposition $v=v_{L}+w+v_{H}, v_{L}$ and $v_{H}$ are $O\left(\varepsilon t^{-\kappa}\right)$ for some $\kappa>0$ (see (6.46) and (6.47)).

## A Appendix: Semi-classical pseudo-differential operators

We recall here some definitions and results concerning semi-classical pseudo-differential operators in one dimension. We refer to the books of Dimassi-Sjöstrand [33] Martinez [51] and Zworski [69].

Let $h$ be a parameter in $] 0,1]$. An order function $m$ is a function $m:(x, \xi) \mapsto m(x, \xi)$ from $T^{*} \mathbb{R}$ (identified with $\mathbb{R} \times \mathbb{R}$ ) to $\mathbb{R}_{+}$, smooth, such that there are constants $N_{0} \in \mathbb{N}, C_{0}>0$ with

$$
m(x, \xi) \leq C_{0}(1+|x-y|+|\xi-\eta|)^{N_{0}} m(y, \eta)
$$

for any $(x, \xi),(y, \eta)$ in $T^{*} \mathbb{R}$.
Definition A.1. Let $m$ be an order function on $T^{*} \mathbb{R}$. One denotes by $S(m)$ the set of functions $\left.\left.a: T^{*} \mathbb{R} \times\right] 0,1\right] \rightarrow \mathbb{C},(x, \xi, h) \mapsto a(x, \xi, h)$ such that for any $(\alpha, \beta)$ in $\mathbb{N} \times \mathbb{N}$, there is $C_{\alpha \beta}>0$, and for any $(x, \xi) \in T^{*} \mathbb{R}$, any $h$ in $\left.] 0,1\right]$

$$
\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} a(x, \xi, h)\right| \leq C_{\alpha \beta} m(x, \xi) .
$$

If $\left(u_{h}\right)_{h}$ is a family indexed by $\left.\left.h \in\right] 0,1\right]$ of elements of $\mathcal{S}^{\prime}(\mathbb{R})$, and $a \in S(m)$, we define a family of elements of $\mathcal{S}^{\prime}(\mathbb{R})$ by

$$
\begin{equation*}
\mathrm{Op}_{h}(a) u_{h}=\frac{1}{2 \pi} \int_{\mathbb{R}} e^{i x \xi} a(x, h \xi, h) \widehat{u_{h}}(\xi) d \xi . \tag{A.1}
\end{equation*}
$$

If $m \equiv 1, \mathrm{Op}_{h}(a)$ is a bounded family indexed by $\left.\left.h \in\right] 0,1\right]$ of bounded operators in $L^{2}(\mathbb{R})$. If moreover $\xi \mapsto a(x, \xi, h)$ is supported in a compact subset independent of $(x, h)$, the kernel of $\mathrm{Op}_{h}(a)$, is

$$
K_{h}(x, y)=\frac{1}{h} k_{h}\left(x, \frac{x-y}{h}\right)
$$

where $k_{h}(x, z)=\left(\mathcal{F}_{\xi}^{-1} a\right)(x, z, h)$ is a smooth function satisfying estimates $\left|\partial_{x}^{\alpha} \partial_{z}^{\beta} k_{h}(x, z)\right| \leq$ $C_{\alpha \beta N}(1+|z|)^{-N}$ for any $\alpha, \beta, N$ so that $\mathrm{Op}_{h}(a)$ is uniformly bounded on any $L^{p}$-space, $p \in[1, \infty]$.

Let us recall the main result of symbolic calculus (Theorem 7.9, Proposition 7.7, formulas (7.16) and (7.3) in [33]).

Theorem A.2. Let $m_{1}, m_{2}$ be two order functions, $a_{j}$ an element of $S\left(m_{j}\right), j=1,2$. There is an element $a_{1} \# a_{2}$ of $S\left(m_{1} m_{2}\right)$ such that $\operatorname{Op}_{h}\left(a_{1} \# a_{2}\right)=\operatorname{Op}_{h}\left(a_{1}\right) \mathrm{Op}_{h}\left(a_{2}\right)$. Moreover, one has the expansion

$$
\begin{equation*}
a_{1} \# a_{2}-\sum_{j=0}^{N} \frac{1}{j!}\left(\frac{h}{i}\right)^{j}\left(\partial_{\xi}^{j} a_{1}\right)\left(\partial_{x}^{j} a_{2}\right) \in h^{N+1} S\left(m_{1} m_{2}\right) . \tag{A.2}
\end{equation*}
$$

Let $m$ be an order function, a an element of $S(m)$. There is $b$ in $S(m)$ such that $\operatorname{Op}_{h}(a)^{*}=$ $\mathrm{Op}_{h}(b)$. Moreover, $b=\bar{a}+h b_{1}$ with $b_{1}$ in $S(m)$.

Corollary A.3. Let $m$ be an order function such that $m^{-1}$ is also an order function. Let a be in $S(1)$, $e$ be in $S(m)$ and assume that $e \geq c m$ for some $c>0$ on a neighborhood of the support of $a$. Then for any $N \in \mathbb{N}$, there are $q \in S\left(m^{-1}\right), r \in S(1)$ such that $a=e \# q+h^{N} r$ (resp. $a=q \# e+h^{N} r$ ). Moreover, we may write $q=q_{0}+h q_{1}$ where $q_{0}, q_{1}$ are in $S\left(m^{-1}\right)$ and $q_{0}=\frac{a}{e}$.

Proof. We define $q_{0}=\frac{a}{e}$, which is an element of $S\left(\mathrm{~m}^{-1}\right)$ by assumption. Then Theorem A. 2 shows that $a-e \# q_{0}\left(\right.$ resp. $\left.a-q_{0} \# e\right)$ may be written $h a_{1}+h^{N} r_{0}$ with $a_{1}$ in $S(1), \operatorname{Supp} a_{1} \subset$ Supp $a$. We iterate the construction to get the result.

## References

[1] T. Alazard, N. Burq, and C. Zuily. On the Cauchy problem for gravity water waves. arXiv:1212.0626.
[2] T. Alazard, N. Burq, and C. Zuily. On the water-wave equations with surface tension. Duke Math. J., 158(3):413-499, 2011.
[3] T. Alazard, N. Burq, and C. Zuily. Strichartz estimates for water waves. Ann. Sci. Éc. Norm. Supér. (4), 44(5):855-903, 2011.
[4] T. Alazard, N. Burq, and C. Zuily. The water-wave equations: from Zakharov to Euler. In Studies in Phase Space Analysis with Applications to PDEs, pages 1-20. Springer, 2013.
[5] T. Alazard and J.-M. Delort. Sobolev estimates for two dimensional gravity water waves. Preprint, 2013.
[6] T. Alazard and G. Métivier. Paralinearization of the Dirichlet to Neumann operator, and regularity of three-dimensional water waves. Comm. Partial Differential Equations, 34(10-12):1632-1704, 2009.
[7] S. Alinhac. Paracomposition et opérateurs paradifférentiels. Comm. Partial Differential Equations, 11(1):87-121, 1986.
[8] B. Alvarez-Samaniego and D. Lannes. Large time existence for 3D water-waves and asymptotics. Invent. Math., 171(3):485-541, 2008.
[9] J. T. Beale. Large-time regularity of viscous surface waves. Arch. Rational Mech. Anal., 84(4):307-352, 1983/84.
[10] T. B. Benjamin and P. J. Olver. Hamiltonian structure, symmetries and conservation laws for water waves. J. Fluid Mech., 125:137-185, 1982.
[11] K. Beyer and M. Günther. On the Cauchy problem for a capillary drop. I. Irrotational motion. Math. Methods Appl. Sci., 21(12):1149-1183, 1998.
[12] G. Bourdaud. Realizations of homogeneous Sobolev spaces. Complex Var. Elliptic Equ., 56(10-11):857-874, 2011.
[13] B. Buffoni, M. D. Groves, S.-M. Sun, and E. Wahlén. Existence and conditional energetic stability of three-dimensional fully localised solitary gravity-capillary water waves. J. Differential Equations, 254(3):1006-1096, 2013.
[14] A. Castro, D. Córdoba, C. Fefferman, F. Gancedo, and J. Gómez-Serrano. Finite time singularities for the free boundary incompressible Euler equations. arXiv:1112.2170.
[15] A. Castro, D. Córdoba, C. L. Fefferman, F. Gancedo, and J. Gómez-Serrano. Splash singularity for water waves. Proc. Natl. Acad. Sci. USA, 109(3):733-738, 2012.
[16] A. Castro, D. Córdoba, C. L. Fefferman, F. Gancedo, and M. López-Fernández. Turning waves and breakdown for incompressible flows. Proc. Natl. Acad. Sci. USA, 108(12):47544759, 2011.
[17] A. L. Cauchy. Théorie de la propagation des ondes à la surface d'un fluide pesant d'une profondeur indéfinie. p.5-318. Mémoires présentés par divers savants à l'Académie royale des sciences de l'Institut de France et imprimés par son ordre. Sciences mathématiques et physiques. Tome I, imprimé par autorisation du Roi à l'Imprimerie royale; 1827. Disponible sur le site http://mathdoc.emath.fr/.
[18] R. M. Chen, J. L. Marzuola, D. Spirn, and J. D. Wright. On the regularity of the flow map for the gravity-capillary equations. J. Funct. Anal., 264(3):752-782, 2013.
[19] H. Christianson, V. M. Hur, and G. Staffilani. Strichartz estimates for the water-wave problem with surface tension. Comm. Partial Differential Equations, 35(12):2195-2252, 2010.
[20] A. Córdoba, D. Córdoba, and F. Gancedo. Interface evolution: water waves in 2-D. Adv. Math., 223(1):120-173, 2010.
[21] D. Coutand and S. Shkoller. On the finite-time splash and splat singularities for the 3-d free-surface Euler equations. arXiv:1201.4919.
[22] D. Coutand and S. Shkoller. Well-posedness of the free-surface incompressible Euler equations with or without surface tension. J. Amer. Math. Soc., 20(3):829-930 (electronic), 2007.
[23] W. Craig. An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits. Comm. Partial Differential Equations, 10(8):787-1003, 1985.
[24] W. Craig. Birkhoff normal forms for water waves. In Mathematical problems in the theory of water waves (Luminy, 1995), volume 200 of Contemp. Math., pages 57-74. Amer. Math. Soc., Providence, RI, 1996.
[25] W. Craig. Non-existence of solitary water waves in three dimensions. R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci., 360(1799):2127-2135, 2002. Recent developments in the mathematical theory of water waves (Oberwolfach, 2001).
[26] W. Craig, U. Schanz, and C. Sulem. The modulational regime of three-dimensional water waves and the Davey-Stewartson system. Ann. Inst. H. Poincaré Anal. Non Linéaire, 14(5):615-667, 1997.
[27] W. Craig and C. Sulem. Numerical simulation of gravity waves. J. Comput. Phys., 108(1):73-83, 1993.
[28] W. Craig, C. Sulem, and P.-L. Sulem. Nonlinear modulation of gravity waves: a rigorous approach. Nonlinearity, 5(2):497-522, 1992.
[29] W. Craig and C. E. Wayne. Mathematical aspects of surface waves on water. Uspekhi Mat. Nauk, 62(3(375)):95-116, 2007.
[30] A. D. D. Craik. Wave interactions and fluid flows. Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge University Press, Cambridge, 1988.
[31] O. Darrigol. Worlds of flow. Oxford University Press, New York, 2005. A history of hydrodynamics from the Bernoullis to Prandtl.
[32] J.-M. Delort. Existence globale et comportement asymptotique pour l'équation de KleinGordon quasi linéaire à données petites en dimension 1. Ann. Sci. École Norm. Sup. (4), 34(1), 2001. Erratum : Ann. Sci. École Norm. Sup. (4) 39 (2006), no. 2, 335-345.
[33] M. Dimassi and J. Sjöstrand. Spectral asymptotics in the semi-classical limit, volume 268 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1999.
[34] P. Germain, N. Masmoudi, and J. Shatah. Global existence for capillary water waves. arXiv:1210.1601.
[35] P. Germain, N. Masmoudi, and J. Shatah. Global solutions for the gravity water waves equation in dimension 3. Ann. of Math. (2), 175(2):691-754, 2012.
[36] Y. Guo and I. Tice. Decay of viscous surface waves without surface tension. arXiv:1011.5179.
[37] N. Hayashi and P. I. Naumkin. Asymptotics of small solutions to nonlinear Schrödinger equations with cubic nonlinearities. Int. J. Pure Appl. Math., 3(3):255-273, 2002.
[38] L. Hörmander. Lectures on nonlinear hyperbolic differential equations, volume 26 of Mathématiques \& Applications (Berlin) [Mathematics \& Applications]. Springer-Verlag, Berlin, 1997.
[39] V. M. Hur. No solitary waves exist on 2d deep water. arXiv:1209.1926.
[40] A. Ionescu and F. Pusateri. Global solutions for the gravity water waves system in 2d. arXiv:1303.5357.
[41] A. Ionescu and F. Pusateri. Nonlinear fractional Schrödinger equations in one dimension. arXiv:1209.4943.
[42] G. Iooss and P. Plotnikov. Multimodal standing gravity waves: a completely resonant system. J. Math. Fluid Mech., 7(suppl. 1):S110-S126, 2005.
[43] G. Iooss and P. I. Plotnikov. Small divisor problem in the theory of three-dimensional water gravity waves. Mem. Amer. Math. Soc., 200(940):viii+128, 2009.
[44] S. Klainerman. Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions. Comm. Pure Appl. Math., 38(5):631-641, 1985.
[45] S. Klainerman. Uniform decay estimates and the Lorentz invariance of the classical wave equation. Comm. Pure Appl. Math., 38(3):321-332, 1985.
[46] D. Lannes. Space time resonances [after Germain, Masmoudi, Shatah]. Séminaire BOURBAKI 64ème année, 2011-2012, no 1053.
[47] D. Lannes. Water waves: mathematical analysis and asymptotics. To appear.
[48] D. Lannes. Well-posedness of the water-waves equations. J. Amer. Math. Soc., 18(3):605654 (electronic), 2005.
[49] D. Lannes. A Stability Criterion for Two-Fluid Interfaces and Applications. Arch. Ration. Mech. Anal., 208(2):481-567, 2013.
[50] H. Lindblad. Well-posedness for the motion of an incompressible liquid with free surface boundary. Ann. of Math. (2), 162(1):109-194, 2005.
[51] A. Martinez. An introduction to semiclassical and microlocal analysis. Universitext. Springer-Verlag, New York, 2002.
[52] N. Masmoudi and F. Rousset. Uniform regularity and vanishing viscosity limit for the free surface Navier-Stokes equations. arXiv:1202.0657.
[53] M. Ming, F. Rousset, and N. Tzvetkov. Multi-solitons and related solutions for the water-waves system. arXiv:1304.5263.
[54] V. I. Nalimov. The Cauchy-Poisson problem. Dinamika Splošn. Sredy, (Vyp. 18 Dinamika Zidkost. so Svobod. Granicami):104-210, 254, 1974.
[55] G. Schneider and C. E. Wayne. The rigorous approximation of long-wavelength capillarygravity waves. Arch. Ration. Mech. Anal., 162(3):247-285, 2002.
[56] J. Shatah. Normal forms and quadratic nonlinear Klein-Gordon equations. Comm. Pure Appl. Math., 38(5):685-696, 1985.
[57] J. Shatah and C. Zeng. Geometry and a priori estimates for free boundary problems of the Euler equation. Comm. Pure Appl. Math., 61(5):698-744, 2008.
[58] J. Shatah and C. Zeng. A priori estimates for fluid interface problems. Comm. Pure Appl. Math., 61(6):848-876, 2008.
[59] M. Shinbrot. The initial value problem for surface waves under gravity. I. The simplest case. Indiana Univ. Math. J., 25(3):281-300, 1976.
[60] C. Sulem and P.-L. Sulem. The nonlinear Schrödinger equation, volume 139 of Applied Mathematical Sciences. Springer-Verlag, New York, 1999. Self-focusing and wave collapse.
[61] S.-M. Sun. Some analytical properties of capillary-gravity waves in two-fluid flows of infinite depth. Proc. Roy. Soc. London Ser. A, 453(1961):1153-1175, 1997.
[62] N. Totz and S. Wu. A rigorous justification of the modulation approximation to the 2D full water wave problem. Comm. Math. Phys., 310(3):817-883, 2012.
[63] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2-D. Invent. Math., 130(1):39-72, 1997.
[64] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 3-D. J. Amer. Math. Soc., 12(2):445-495, 1999.
[65] S. Wu. Almost global wellposedness of the 2-D full water wave problem. Invent. Math., 177(1):45-135, 2009.
[66] S. Wu. Global wellposedness of the 3-D full water wave problem. Invent. Math., 184(1):125-220, 2011.
[67] H. Yosihara. Gravity waves on the free surface of an incompressible perfect fluid of finite depth. Publ. Res. Inst. Math. Sci., 18(1):49-96, 1982.
[68] V. E. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. Journal of Applied Mechanics and Technical Physics, 9(2):190-194, 1968.
[69] M. Zworski. Semiclassical analysis, volume 138 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2012.

