Global solutions and asymptotic behavior for two dimensional gravity water waves

Thomas Alazard, Jean-Marc Delort

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Thomas Alazard

Département de Mathématiques et Applications
École normale supérieure et CNRS UMR 8553
45, rue d'Ulm
F-75230 Paris, France

Jean-Marc Delort

Université Paris 13
Sorbonne Paris Cité, LAGA, CNRS (UMR 7539)
99, avenue J.-B. Clément
F-93430 Villetaneuse

Abstract

This paper is devoted to the proof of a global existence result for the water waves equation with smooth, small, and decaying at infinity Cauchy data. We obtain moreover an asymptotic description in physical coordinates of the solution, which shows that modified scattering holds.

The proof is based on a bootstrap argument involving L^2 and L^∞ estimates. The L^2 bounds are proved in the companion paper [5] of this article. They rely on a normal forms paradifferential method allowing one to obtain energy estimates on the Eulerian formulation of the water waves equation. We give here the proof of the uniform bounds, interpreting the equation in a semi-classical way, and combining Klainerman vector fields with the description of the solution in terms of semi-classical lagrangian distributions. This, together with the L^2 estimates of [5], allows us to deduce our main global existence result.

Introduction

1 Main result

Consider an homogeneous and incompressible fluid in a gravity field, occupying a time-dependent domain with a free surface. We assume that the motion is the same in every vertical section and consider the two-dimensional motion in one such section. At time t, the fluid domain, denoted by $\Omega(t)$, is therefore a two-dimensional domain. We assume that its boundary is a free surface described by the equation $y=\eta(t,x)$, so that

$$\Omega(t)=\{\,(x,y)\in\mathbb{R}\times\mathbb{R}\,;\,y<\eta(t,x)\,\}\,.$$

The velocity field is assumed to satisfy the incompressible Euler equations. Moreover, the fluid motion is assumed to have been generated from rest by conservative forces and is therefore irrotational in character. It follows that the velocity field $v\colon\Omega\to\mathbb{R}^2$ is given by $v=\nabla_{x,y}\phi$ for some velocity potential $\phi\colon\Omega\to\mathbb{R}$ satisfying

$$\Delta_{x,y}\phi=0,\quad \partial_t\phi+\frac{1}{2}\left|\nabla_{x,y}\phi\right|^2+P+gy=0, \tag{1}$$

where g is the modulus of the acceleration of gravity ($g>0$) and where P is the pressure term. Hereafter, the units of length and time are chosen so that $g=1$.

The problem is then given by two boundary conditions on the free surface:

$$\begin{cases}\partial_t\eta=\sqrt{1+(\partial_x\eta)^2}\,\partial_n\phi & \text{on }\partial\Omega,\\ P=0 & \text{on }\partial\Omega,\end{cases} \tag{2}$$

where ∂_n is the outward normal derivative of Ω, so that $\sqrt{1+(\partial_x\eta)^2}\,\partial_n\phi=\partial_y\phi-(\partial_x\eta)\partial_x\phi$. The former condition expresses that the velocity of the free surface coincides with the one of the fluid particles. The latter condition is a balance of forces across the free surface.

Following Zakharov [68] and Craig and Sulem [27], we work with the trace of ϕ at the free boundary

$$\psi(t,x):=\phi(t,x,\eta(t,x)).$$

To form a system of two evolution equations for η and ψ, one introduces the Dirichlet-Neumann operator $G(\eta)$ that relates ψ to the normal derivative $\partial_n\phi$ of the potential by

$$(G(\eta)\psi)(t,x)=\sqrt{1+(\partial_x\eta)^2}\,\partial_n\phi|_{y=\eta(t,x)}.$$

(This definition is made precise in the first section of the companion paper [5]. See proposition 1.2 below). Then (η,ψ) solves (see [27]) the so-called Craig–Sulem–Zakharov system

$$\begin{cases}\partial_t\eta=G(\eta)\psi,\\ \partial_t\psi+\eta+\dfrac{1}{2}(\partial_x\psi)^2-\dfrac{1}{2(1+(\partial_x\eta)^2)}\big(G(\eta)\psi+(\partial_x\eta)(\partial_x\psi)\big)^2=0.\end{cases} \tag{3}$$

In [4], it is proved that if (η,ψ) is a classical solution of (3), such that (η,ψ) belongs to $C^0([0,T];H^s(\mathbb{R}))$ for some $T>0$ and $s>3/2$, then one can define a velocity potential ϕ and a pressure P satisfying (1) and (2). Thus it is sufficient to solve the Craig–Sulem–Zakharov formulation of the water waves equations.

Our main result is stated in full generality in the first section of this paper. A weaker statement is the following:

Main result. *For small enough initial data of size $\varepsilon \ll 1$, sufficiently decaying at infinity, the Cauchy problem for* (3) *is globally in time well-posed. Moreover, $u = |D_x|^{\frac{1}{2}}\psi + i\eta$ admits the following asymptotic expansion as t goes to $+\infty$: There is a continuous function $\underline{\alpha}\colon \mathbb{R} \to \mathbb{C}$, depending of ε but bounded uniformly in ε, such that*

$$u(t,x) = \frac{\varepsilon}{\sqrt{t}}\underline{\alpha}\Big(\frac{x}{t}\Big)\exp\Big(\frac{it}{4|x/t|} + \frac{i\varepsilon^2}{64}\frac{|\underline{\alpha}(x/t)|^2}{|x/t|^5}\log(t)\Big) + \varepsilon t^{-\frac{1}{2}-\kappa}\rho(t,x)$$

where κ is some positive number and ρ is a function uniformly bounded for $t \geq 1$, $\varepsilon \in]0,\varepsilon_0]$.

As an example of small enough initial data sufficiently decaying at infinity, consider

$$\eta|_{t=1} = \varepsilon\eta_0, \quad \psi|_{t=1} = \varepsilon\psi_0, \tag{4}$$

with η_0, ψ_0 in $C_0^\infty(\mathbb{R})$. Then there exists a unique solution (η,ψ) in $C^\infty([1,+\infty[; H^\infty(\mathbb{R}))$ of (3). In fact, in Theorem 1.4 we allow ψ to be merely in some homogeneous Sobolev space.

The strategy of the proof will be explained in the following sections of this introduction. We discuss at the end of this paragraph some related previous works.

For the equations obtained by neglecting the nonlinear terms, the computation of the asymptotic behavior of the solutions was performed by Cauchy [17] who computed the phase of oscillations. The reader is referred to [31] and [30] for many historical comments on Cauchy's memoir.

Many results have been obtained in the study of the Cauchy problem for the water waves equations, starting from the pioneering work of Nalimov [54] who proved that the Cauchy problem is well-posed locally in time, in the framework of Sobolev spaces, under an additional smallness assumption on the data. We also refer the reader to Shinbrot [59], Yoshihara [67] and Craig [23]. Without smallness assumptions on the data, the well-posedness of the Cauchy problem was first proved by Wu for the case without surface tension (see [63, 64]) and by Beyer-Günther in [11] in the case with surface tension. Several extensions of their results have been obtained and we refer the reader to Córdoba, Córdoba and Gancedo [20], Coutand-Shkoller [22], Lannes [48, 49, 47], Linblad [50], Masmoudi-Rousset [52] and Shatah-Zeng [57, 58] for recent results on the Cauchy problem for the gravity water waves equation.

Our proof of global existence is based on the analysis of the Eulerian formulation of the water waves equations by means of microlocal analysis. In particular, the energy estimates discussed in [5] are influenced by the papers by Lannes [48] and Iooss-Plotnikov [43] and follow the paradifferential analysis introduced in [6] and further developed in [2, 1].

It is worth recalling that the only known coercive quantity for (3) is the hamiltonian, which reads (see [68, 27])

$$\mathcal{H} = \frac{1}{2}\int \eta^2\,dx + \frac{1}{2}\int \psi G(\eta)\psi\,dx. \tag{5}$$

We refer to the paper by Benjamin and Olver [10] for considerations on the conservation laws of the water waves equations. One can compare the hamiltonian with the critical threshold given by the scaling invariance of the equations. Recall (see [10, 18]) that if (η, ψ) solves (3), then the functions $(\eta_\lambda, \psi_\lambda)$ defined by

$$\eta_\lambda(t,x) = \lambda^{-2}\eta\left(\lambda t, \lambda^2 x\right), \quad \psi_\lambda(t,x) = \lambda^{-3}\psi\left(\lambda t, \lambda^2 x\right) \qquad (\lambda > 0) \tag{6}$$

are also solutions of (3). In particular, one notices that the critical space for the scaling corresponds to η_0 in $\dot{H}^{3/2}(\mathbb{R})$. Since the hamiltonian (5) only controls the $L^2(\mathbb{R})$-norm of η, one sees that the hamiltonian is highly supercritical for the water waves equation and hence one cannot use it directly to prove global well-posedness of the Cauchy problem.

Given $\varepsilon \geq 0$, consider the solutions to the water waves system (3) with initial data satisfying (4). In her breakthrough result [65], Wu proved that the maximal time of existence T_ε is larger or equal to $e^{c/\varepsilon}$ for $d = 1$. Then Germain–Masmoudi–Shatah [35] and Wu [66] have shown that the Cauchy problem for three-dimensional waves is globally in time well-posed for ε small enough (with linear scattering in Germain-Masmoudi-Shatah and no assumption about the decay to 0 at spatial infinity of $|D_x|^{\frac{1}{2}}\psi$ in Wu). Germain–Masmoudi–Shatah recently proved global existence for pure capillary waves in dimension $d = 2$ in [34].

There is at least one other case where the global existence of solutions is now understood, namely for the equations with viscosity (see [9], [36] and the references therein). Then global well-posedness is obtained by using the dissipation of energy. Without viscosity, the analysis of global well-posedness is based on dispersive estimates. Our approach follows a variant of the vector fields method introduced by Klainerman in [45, 44] to study the wave and Klein-Gordon equations (see the book by Hörmander in [38] or the Bourbaki seminar by Lannes [46] for an introduction to this method). More precisely, as it is discussed later in this introduction, we shall follow the approach introduced in [32] for the analysis of the Klein-Gordon equation in space dimension one, to cope with the fact that solutions of the equation do not scatter. Results for one dimensional Schrödinger equations, that display the same non scattering behavior, have been proved by Hayashi and Naumkin [37], and global existence for a simplified model of the water waves equation studied by Ionescu and Pusateri in [41].

Let us discuss two other questions related to our analysis : the possible emergence of singularities in finite time and the existence of solitary waves.

An important question is to determine whether the lifespan could be finite. Castro, Córdoba, Fefferman, Gancedo and Gómez-Serrano conjecture (see [15]) that blow-up in finite time is possible for some initial data. It is conjectured in [15] that there exists at least one water-wave solution such that, at time 0, the fluid interface is a graph, at a later time $t_1 > 0$ the fluid interface is not a graph, and, at a later time $t_2 > t_1$, the fluid self-intersects. Notice that, according to this conjecture, one does not expect global well-posedness for arbitrarily large initial data. One can quote several results supporting this conjecture (see [14, 16, 21]). In [14] (see also [21]), the authors prove the following result: there exists an initial data such the free surface is a self-intersecting curve, and such that solving backward in time the Cauchy problem, one obtains for small enough negative times a non self-intersecting curve of $\mathbb{R}^2$. On the other hand, it was conjectured that there is no blow-up in finite time for small enough, sufficiently decaying initial data (see the survey paper by Craig and Wayne [29]).

Our main result precludes the existence of solitary waves sufficiently small and sufficiently decaying at infinity. In this direction, notice that Sun [61] has shown that in infinitely deep water, no two-dimensional solitary water waves exist. For further comments and references on solitary waves, we refer the reader to [25] as well as to [13, 39, 53] for recent results.

We refer the reader to [2, 3, 19] for the study of other dispersive properties of the water waves equations (Strichartz estimates and smoothing effect of surface tension).

Finally, let us mention that Ionescu and Pusateri [40] independently obtained a global existence result very similar to the one we get here. The main difference is that they assume less decay on the initial data, and get asymptotics not for the solution in physical space, with control of the remainders in L^∞, but for its space Fourier transform, with remainders in L^2. These asymptotics, as well as ours, show that solutions do not scatter. To get asymptotics with remainders estimated in L^∞, we shall commute iterated vector field $Z = t\partial_t + 2x\partial_x$ to the water waves equations. This introduces several new difficulties and requires that the initial data be sufficiently decaying at infinity.

2 General strategy of proof

Let us describe our general strategy, the difficulties one has to cope with, and the ideas used to overcome them. The general framework we use is the one of Klainerman vector fields. Consider as a (much) simplified model an equation of the form

$$\begin{aligned}(D_t - P(D_x))u &= N(u)\\ u|_{t=1} &= \varepsilon u_0,\end{aligned} \tag{7}$$

where $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, $P(\xi)$ is a real valued symbol (for the linearized water waves equation, $P(\xi)$ would be $|\xi|^{1/2}$), and $N(u)$ is a nonlinearity vanishing at least at order two at zero. Recall that a Klainerman vector field for $D_t - P(D_x)$ is a space-time vector field Z such that $[Z, D_t - P(D_x)]$ is zero (or a multiple of $D_t - P(D_x)$). For the water waves system, Z will be $t\partial_t + 2x\partial_x$ or D_x. In that way, $(D_t - P(D_x))Z^k u = Z^k N(u)$ for any k, and since $P(\xi)$ is real valued, an easy energy inequality shows that

$$\|Z^k u(t,\cdot)\|_{L^2} \leq \|Z^k u(1,\cdot)\|_{L^2} + \int_1^t \|Z^k N(u)(\tau,\cdot)\|_{L^2}\, d\tau, \tag{8}$$

for any $t \geq 1$. Assume first that $N(u)$ is cubic, so that

$$\|Z^k N(u)\|_{L^2} \leq C\|u\|_{L^\infty}^2 \|Z^k u\|_{L^2} + C \sum_{\substack{k_1+k_2+k_3\leq k\\ k_1,k_2\leq k_3\leq k-1}} \|Z^{k_1}u\|_{L^\infty}\|Z^{k_2}u\|_{L^\infty}\|Z^{k_3}u\|_{L^2}. \tag{9}$$

Assuming an a priori L^∞ bound, one can deduce from (8) an L^2 estimate. More precisely, introduce the following property, where s is a large even integer:

$$\begin{aligned}&\text{For } t \text{ in some interval } [1,T[,\ \ \|u(t,\cdot)\|_{L^\infty} = O(\varepsilon/\sqrt{t})\\ &\text{and for } k = 0,\dots,s/2,\ \ \|Z^k u(t,\cdot)\|_{L^\infty} = O\big(\varepsilon t^{-\frac{1}{2}+\tilde{\delta}'_k}\big),\end{aligned} \tag{A}$$

where $\tilde{\delta}'_k$ are small positive numbers. Plugging these a priori bounds in (8), (9), we get

$$\begin{aligned}\|Z^k u(t,\cdot)\|_{L^2} \leq \|Z^k u(1,\cdot)\|_{L^2} &+ C\varepsilon^2 \int_1^t \|Z^k u(\tau,\cdot)\|_{L^2} \frac{d\tau}{\tau} \\ &+ C\varepsilon^2 \int_1^t \|Z^{k-1} u(\tau,\cdot)\|_{L^2} \tau^{2\tilde{\delta}'_{k/2}-1}\, d\tau.\end{aligned} \tag{10}$$

Gronwall inequality implies then that

$$\|Z^k u(t,\cdot)\|_{L^2} = O(\varepsilon t^{\delta_k}),\ k \leq s, \tag{B}$$

for some small $\delta_k > 0$ ($\delta_k > C\varepsilon^2$ and $\delta_k > 2\tilde{\delta}'_{k/2}$).

The proof of global existence is done classically using a bootstrap argument allowing one to to show that if (A) and (B) are assumed to hold on some interval, they actually hold on the same interval with smaller constants in the estimates.

We have outlined above the way of obtaining (B), assuming (A) for a solution of the model equation (7). In this subsection of the introduction, we shall explain, in a non technical way, the new difficulties that have to be solved to prove (B) for the water waves equation. Actually, the proof of a long time energy inequality for system (3) faces two serious obstacles, that we describe now.

• **Apparent loss of derivatives in energy inequalities**

This difficulty already arises for local existence results, and was solved initially by Nalimov [54] and Wu [63, 64]. For long time existence problems, Wu [66] uses arguments combining the Eulerian and Lagrangian formulations of the system. The approach followed in our companion paper [5] is purely Eulerian. We explain the idea on the model obtained from (3) paralinearizing the equations and keeping only the quadratic terms. If we denote $U = \left[\begin{smallmatrix}\eta \\ |D_x|^{1/2}\psi\end{smallmatrix}\right]$, such a model may be written as

$$\partial_t U = T_A U$$

where T_A is the paradifferential operator with symbol A, and where $A(U,x,\xi)$ is a matrix of symbols $A(U,x,\xi) = A_0(U,x,\xi) + A_1(U,x,\xi)$, with

$$A_0(U,x,\xi) = \begin{bmatrix} -i(\partial_x\psi)\xi & |\xi|^{1/2} \\ -|\xi|^{1/2} & -i(\partial_x\psi)\xi \end{bmatrix},\ A_1(U,x,\xi) = (|D_x|\psi)\,|\xi| \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Because of the A_1 contribution, which is self-adjoint, the eigenvalues of $A(U,x,\xi)$ are not purely imaginary. For large $|\xi|$, there is one eigenvalue with positive real part, which shows that one cannot expect for the solution of $\partial_t U = T_A U$ energy inequalities without derivative losses. A way to circumvent this difficulty is well known, and consists in using the "good unknown" of Alinhac [7]. For our quadratic model, this just means introducing as a new unknown $\tilde{U} = \left[\begin{smallmatrix}\eta \\ |D_x|^{1/2}\omega\end{smallmatrix}\right]$, where $\omega = \psi - T_{|D_x|\psi}\eta$ is the (quadratic approximation of the) good unknown. In that way, ignoring again remainders and terms which are at least cubic, one gets for $\tilde{U}$ an evolution equation $\partial_t \tilde{U} = T_{A_0} \tilde{U}$. Since A_0 is anti-self-adjoint, one gets L^2 or Sobolev energy inequalities for $\tilde{U}$. In particular, if for some s, $\| |D_x|^{1/2} \omega \|_{H^s} + \|\eta\|_{H^s}$ is under control, and if one has also an auxiliary bound for $\| |D_x| \, \psi \|_{L^\infty}$, one gets an estimate for $\| |D_x|^{1/2} \psi \|_{H^{s-1/2}} + \|\eta\|_{H^s}$.

• **Quadratic terms in the nonlinearity**

In the model equation (7) discussed above, we considered a cubic nonlinearity: this played an essential role to make appear in the first integral in the right hand side of (10) the almost integrable factor $1/\tau$. For a quadratic nonlinearity, we would have had instead a $1/\sqrt{\tau}$-factor, which would have given in (B), through Gronwall, a $O(e^{\varepsilon\sqrt{t}})$-bound, instead of $O(\varepsilon t^{\delta_k})$. The way to overcome such a difficulty is well known since the work of Shatah [56] devoted to the non-linear Klein-Gordon equation: it is to use a normal forms method to eliminate the quadratic part of the nonlinearity, up to terms that do not contribute to the Sobolev energy inequality.

In practice, one looks for a local diffeomorphism at 0 in H^s, for s large enough, so that the Sobolev energy inequality written for the equation obtained by conjugation by this diffeomorphim be of the form (10). Nonlinear changes of unknowns, reducing the water waves system to a cubic equation, have been known for quite a time (see Craig [24] or Iooss and Plotnikov [42, Lemma 1]). However, these transformations were losing derivatives, as a consequence of the quasi-linear character of the problem. Nevertheless, one can construct a bona fide change of unknown, without derivatives losses, if one notices that it is not necessary to eliminate the whole quadratic part of the nonlinearity, but only the part of it that would bring non zero contributions in a Sobolev energy inequality. This is what we do in our companion paper [5]. Let us also mention that the analysis of normal forms for the water waves system is motivated by physical considerations, such as the derivations of various equations in asymptotic regimes (see [28, 26, 55, 62]).

Our proof of L^2-estimates of type (B), assuming that a priori inequalities of type (A) hold, is performed in [5] using the ideas that we just outlined. Of course, the models we have discussed so far do not make justice to the full complexity of the water waves system. In particular, the good unknown ω is given by a more involved formula than the one indicated above, and one also needs to define precisely the Dirichlet-Neumann operator. The latter is done in [5]. We recall in section 1 below the main properties of the operator $G(\eta)$ when η belongs to a space of the form $C^\gamma(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\gamma > 2$, and is small enough. Once $G(\eta)\psi$ has been defined, one can introduce functions of (η, ψ), $B = (\partial_y \phi)|_{y=\eta}$, $V = (\partial_x \phi)|_{y=\eta}$, where ϕ is the harmonic potential solving (1). Explicit expressions of these quantities are given by

$$B = \frac{G(\eta)\psi + (\partial_x \eta)(\partial_x \psi)}{1 + (\partial_x \eta)^2}, \ V = \partial_x \psi - B \partial_x \eta.$$

The good unknown for the water waves equation is given by $\omega = \psi - T_B \eta$. Following the analysis in [1, 2, 6], we prove in [5] an expression for $G(\eta)\psi$ in terms of ω:

$$G(\eta)\psi = |D_x|\omega - \partial_x(T_V \eta) + F(\eta)\psi,$$

where $F(\eta)\psi$ is a quadratic *smoothing term*, that belongs to $H^{s+\gamma-4}$ if η is in $C^\gamma \cap H^s$ and $|D_x|^{1/2}\psi$ belongs to $C^{\gamma-1/2} \cap H^{s-1/2}$. This gives a quite explicit expression for the main contributions to $G(\eta)\psi$. Moreover, we prove as well tame estimates, that complement similar results due to Craig, Schanz and Sulem (see [26], [60, Chapter 11] and [8, 43]), and establish bounds for the approximation of $G(\eta)\psi$ (resp. $F(\eta)\psi$) by its Taylor expansion at order two $G_{\leq 2}(\eta)\psi$ (resp. $F_{\leq 2}(\eta)\psi$).

3 Klainerman-Sobolev inequalities

As previously mentioned, the proof of global existence relies on a bootstrap argument on properties (A) and (B). We have indicated in the preceding section how (B) may be deduced from (A). On the other hand, one has to prove that conversely, (A) and (B) imply that (A) holds with smaller constants in the inequalities. The first step is to show that if the L^2-estimate (B) holds for $k \leq s$, then bounds of the form

$$\text{(A')} \qquad \|Z^k u(t,\cdot)\|_{L^\infty} = O(\varepsilon t^{-\frac{1}{2}+\delta'_k}),\ k \leq s-100$$

are true, for small positive δ'_k. This is not (A), since the δ'_k may be larger that the $\tilde{\delta}'_k$ of (A), and since this does not give a *uniform* bound for $\|u(t,\cdot)\|_{L^\infty}$. But this first information will allow us to deduce, in the last step of the proof, estimates of the form (A) from (A') and the equation.

Let us make a change of variables $x \to x/t$ in the water waves system. If $u(t,x)$ is given by $u(t,x) = (|D_x|^{1/2}\psi + i\eta)(t,x)$, we define v by $u(t,x) = \frac{1}{\sqrt{t}}v(t,x/t)$. We set $h = 1/t$ and eventually consider v as a family of functions of x depending on the semi-classical parameter h. Moreover, for $a(x,\xi)$ a function satisfying convenient symbol estimates, and $(v_h)_h$ a family of functions on $\mathbb{R}$, we define

$$\mathrm{Op}_h(a)v_h = a(x,hD)v_h = \frac{1}{2\pi}\int e^{ix\xi}a(x,h\xi)\hat{v}_h(\xi)\,d\xi.$$

Then the water waves system is equivalent to the equation

$$\text{(11)} \qquad \big(D_t - \mathrm{Op}_h\big(x\xi + |\xi|^{1/2}\big)\big)v = \sqrt{h}Q_0(V) + h\left[C_0(V) - \frac{i}{2}v\right] + h^{1+\kappa}R(V),$$

where we used the following notations

• Q_0 (resp. C_0) is a nonlocal quadratic (resp. cubic) form of $V = (v,\bar{v})$ that may be written as a linear combination of expressions $\mathrm{Op}_h(b_0)[\prod_{j=1}^{\ell}\mathrm{Op}_h(b_j)v_\pm]$, $\ell = 2$ (resp. $\ell = 3$), where $b_\ell(\xi)$ are homogeneous functions of degree $d_\ell \geq 0$ with $\sum_0^2 d_\ell = 3/2$ (resp. $\sum_0^3 d_\ell = 5/2$) and $v_+ = v, v_- = \bar{v}$.

• $R(V)$ is a remainder, made of the contributions vanishing at least at order four at $V = 0$.

To simplify the exposition in this introduction, we shall assume that v satisfies $\varphi(hD)v = v$ for some $C_0^\infty(\mathbb{R}-\{0\})$-function φ, equal to one on a large enough compact subset of of $\mathbb{R}-\{0\}$.

Such a property is not satisfied by solutions of (11), but one can essentially reduce to such a situation performing a dyadic decomposition $v = \sum_{j\in\mathbb{Z}} \varphi(2^{-j}hD)v$.

The Klainerman vector field associated to the linearization of the water waves equation may be written, in the new coordinates that we are using, as $Z = t\partial_t + x\partial_x$. Remembering $h = 1/t$ and expressing ∂_t from Z in equation (11), we get

$$\mathrm{Op}_h\big(2x\xi + |\xi|^{1/2}\big)v = -\sqrt{h}Q_0(V) + h\left[\frac{i}{2}v - iZv - C_0(V)\right] - h^{1+\kappa}R(V). \tag{12}$$

Since we factored out the expected decay in $1/\sqrt{t}$, our goal is to deduce from assumptions (A) and (B) estimates of the form $\|Z^k v\|_{L^\infty} = O(\varepsilon h^{-\delta'_k})$ for $k \le s - 100$.

Proposition. *Assume that for t in some interval $[T_0, T[$ (i.e. for h in some interval $]h', h_0]$), one has estimates (A) and (B):*

$$\|Z^k v\|_{L^\infty} = O\big(\varepsilon h^{-\tilde{\delta}'_k}\big),\ k \le s/2,\ \|Z^k v\|_{L^2} = O\big(\varepsilon h^{-\delta_k}\big),\ k \le s. \tag{13}$$

Denote $\Lambda = \{(x, d\omega(x)); x \in \mathbb{R}^*\}$ where $\omega(x) = 1/(4|x|)$. Then, if γ_Λ is smooth, supported close to Λ and equal to one on a neighborhood of Λ, and if $\gamma_\Lambda^c = 1 - \gamma_\Lambda$, we have for $k \le s - 100$

$$\|Z^k \,\mathrm{Op}_h(\gamma_\Lambda^c)v\|_{L^2} = O\big(\varepsilon h^{\frac{1}{2}-\delta'_k}\big), \tag{14}$$

$$\|(hD_x - d\omega)Z^k \,\mathrm{Op}_h(\gamma_\Lambda)v\|_{L^2} = O\big(\varepsilon h^{1-\delta'_k}\big), \tag{15}$$

$$\|Z^k v\|_{L^\infty} = O\big(\varepsilon h^{-\delta'_k}\big). \tag{16}$$

Idea of proof. One applies k vector fields Z to (12) and uses their commutation properties to the linearized equation. In that way, taking into account the assumptions, one gets

$$\mathrm{Op}_h\left(2x\xi + |\xi|^{1/2}\right)Z^k v = O_{L^2}\left(\varepsilon h^{\frac{1}{2}-\delta'_k}\right) \tag{17}$$

for some small $\delta'_k > 0$. One remarks that $2x\xi + |\xi|^{1/2}$ vanishes exactly on Λ. Consequently, this symbol is elliptic on the support of γ_Λ^c, and this allows one to get (14) by ellipticity.

To prove the second inequality, one uses the fact that,

$$\mathrm{Op}_h\left(2x\xi + |\xi|^{1/2}\right)Z^k \,\mathrm{Op}_h(\gamma_\Lambda)v = -\sqrt{h}\,\mathrm{Op}_h(\gamma_\Lambda)Z^k Q_0(V) + O\big(\varepsilon h^{1-\delta'_k}\big). \tag{18}$$

We may decompose $v = v_\Lambda + v_{\Lambda^c}$ where $v_\Lambda = \mathrm{Op}_h(\gamma_\Lambda)v$ and $v_{\Lambda^c} = \mathrm{Op}_h(\gamma_\Lambda^c)v$. We may write $Z^kQ_0(V) - Z^kQ_0(v_\Lambda, \bar{v}_\Lambda) = B(v_\Lambda, Z^k v_{\Lambda^c}) + \cdots$ where B is the polar form of Q_0. By (14), $\|Z^k v_{\Lambda^c}\|_{L^2} = O(\varepsilon h^{\frac{1}{2}-\delta'_k})$, and by assumption $\|v_\Lambda\|_{L^\infty} = O(\varepsilon)$. It follows that $\big\|B(v_\Lambda, Z^k v_{\Lambda^c})\big\|_{L^2} = O\big(\varepsilon h^{\frac{1}{2}-\delta'_k}\big)$. The other contributions to $Z^kQ_0(V) - Z^kQ_0(v_\Lambda, \bar{v}_\Lambda)$ may be estimated in a similar way, up to extra contributions, that we do not write explicitly in this outline, and that may be absorbed in the left hand side of (16) at the end of the reasoning. The right hand side of (18) may thus be written

$$-\sqrt{h}\,\mathrm{Op}_h(\gamma_\Lambda)Z^kQ_0(V_\Lambda)+O_{L^2}\big(\varepsilon h^{1-\delta'_k}\big), \tag{19}$$

where $V_\Lambda=(v_\Lambda,\bar v_\Lambda)$. One notices then that since v_Λ (resp. $\bar v_\Lambda$) is microlocally supported close to Λ (resp. $-\Lambda$), $Q_0(V_\Lambda)$ is microlocally supported close to the union of 2Λ, 0Λ and -2Λ, so far away from the support of the cut-off γ_Λ (where $\ell\Lambda=\{(x,\ell d\omega(x));x\in\mathbb{R}^*\}$).

Consequently, the first term in (19) vanishes, and we get

$$\mathrm{Op}_h\big(2x\xi+|\xi|^{1/2}\big)Z^k\,\mathrm{Op}_h(\gamma_\Lambda)v=O_{L^2}\big(\varepsilon h^{1-\delta'_k}\big).$$

Since $2x\xi+|\xi|^{1/2}$ and $\xi-d\omega(x)$ have the same zero set, namely Λ, one deduces (15) from this estimate using symbolic calculus.

Finally, to obtain (16), we write

$$\|Z^kv_\Lambda\|_{L^\infty}=\|e^{-i\omega/h}Z^kv_\Lambda\|_{L^\infty}\le C\|e^{-i\omega/h}Z^kv_\Lambda\|_{L^2}^{1/2}\|D_x(e^{-i\omega/h}Z^kv_\Lambda)\|_{L^2}^{1/2}.$$

The last factor is $h^{-1/2}\|(hD_x-d\omega)Z^kv_\Lambda\|_{L^2}^{1/2}$, which is $O(\sqrt{\varepsilon}h^{-\delta'_k/2})$ by (15). Moreover, (14) and Sobolev inequality imply that $\|Z^kv_{\Lambda^c}\|_{L^\infty}=O\big(\varepsilon h^{-\delta'_k}\big)$, since we have assumed that v is spectrally localized for $|\xi|\sim 1/h$. This gives (16).

4 Optimal L^∞ bounds

As seen in the preceding section, one can deduce from the L^2-estimates (B) some L^∞-estimates (16), which are *not* the optimal estimates of the form (A) that we need (because the exponents δ'_k are larger than $\tilde\delta'_k$, and because δ'_0 is positive, while we need a uniform estimate when no Z field acts on v). In order to get (A), we deduce from the PDE (11) an ODE satisfied by v.

Proposition. *Under the conclusions of the preceding proposition, we may write*

$$v=v_\Lambda+\sqrt{h}(v_{2\Lambda}+v_{-2\Lambda})+h(v_{3\Lambda}+v_{-\Lambda}+v_{-3\Lambda})+h^{1+\kappa}g, \tag{20}$$

where $\kappa>0$, g satisfies bounds of the form $\|Z^kg\|_{L^\infty}=O(\varepsilon h^{-\delta'_k})$, and $v_{\ell\Lambda}$ is microlocally supported close to $\ell\Lambda$ and is a semi-classical lagrangian distribution along $\ell\Lambda$, as well as $Z^kw_{\ell\Lambda}$ for $k\le s/2$, in the following sense

$$\|Z^kv_{\ell\Lambda}\|_{L^\infty}=O(\varepsilon h^{-\delta'_k}), \tag{21}$$

$$\|\mathrm{Op}_h(e_\ell(x,\xi))Z^kv_{\ell\Lambda}\|_{L^\infty}=O(\varepsilon h^{1-\delta'_k}),\ \ \ell\in\{1,-2,2\}, \tag{22}$$

$$\|\mathrm{Op}_h(e_\ell(x,\xi))Z^kv_{\ell\Lambda}\|_{L^\infty}=O(\varepsilon h^{\frac12-\delta'_k}),\ \ \ell\in\{-3,-1,3\}, \tag{23}$$

if e_ℓ vanishes on $\ell\Lambda$.

Remark. Consider a function $w = \alpha(x)\exp(i\omega(x)/h)$. If α is smooth and bounded as well as its derivatives, we see that $(hD_x - d\omega(x))w = O_{L^\infty}(h)$ i.e. w satisfies the second of the above conditions with $\ell = 1$, where $e_1(x,\xi) = \xi - d\omega(x)$ is an equation of Λ. The conclusion of the proposition thus means that $v_{\ell\Lambda}$ enjoys a weak form of such an oscillatory behavior.

The proposition is proved using equation (12). For instance, the bound (22) for $v_\Lambda = \mathrm{Op}_h(\gamma_\Lambda)v$ is proved in the same way as (15), with L^2-norms replaced by L^∞ ones, using (16) to estimate the right hand side. In the same way, one defines $v_{\pm 2\Lambda}$ as the cut-off of v close to $\pm 2\Lambda$. As in the proof of (14), one shows an $O_{L^\infty}(h^{\frac{1}{2}-\delta'_k})$ bound for $Z^k v_{\Lambda^c}$, which implies that the main contribution to $Q_0(v,\bar{v})$ is $Q_0(v_\Lambda,\bar{v}_\Lambda)$. Localizing (12) close to $\pm 2\Lambda$, one gets an elliptic equation that allows to determine $v_{\pm 2\Lambda}$ as a quadratic function of $v_\Lambda, \bar{v}_\Lambda$. Iterating the argument, one gets the expansion of the proposition. One does not get in the $\sqrt{h}$-terms of the expansion a contribution associated to 0Λ because $Q_0(V)$ may be factored out by a Fourier multiplier vanishing on the zero section. Consequently, non oscillating terms form part of the $O(h^{1+\kappa})$ remainder.

Let us use the result of the preceding proposition to obtain an ODE satisfied by v:

Proposition. *The function v satisfies an ODE of the form*

$$
\begin{aligned}
D_t v = {} & \frac{1}{2}(1-\chi(h^{-\beta}x))|d\omega|^{1/2}v - i\sqrt{h}(1-\chi(h^{-\beta}x))\Big[\Phi_2(x)v^2 + \Phi_{-2}(x)\bar{v}^2\Big] \\
& + h(1-\chi(h^{-\beta}x))\Big[\Phi_3(x)v^3 + \Phi_1(x)|v|^2 v + \Phi_{-1}(x)|v|^2\bar{v} + \Phi_{-3}(x)\bar{v}^3\Big] \\
& + O(\varepsilon h^{1+\kappa}),
\end{aligned}
\tag{24}
$$

where $\kappa > 0, \beta > 0$ are small, Φ_ℓ are real valued functions of x defined on $\mathbb{R}^*$ and χ is in $C_0^\infty(\mathbb{R})$, equal to one close to zero.

To prove the proposition, one plugs expansion (20) in equation (11). The key point is to use (22), (23) to express all (pseudo-)differential terms from multiplication operators and remainders. For instance, if $b(\xi)$ is some symbol, one may write $b(\xi) = b|_{\ell\Lambda} + e_\ell$ where e_ℓ vanishes on $\ell\Lambda = \{\xi = \ell d\omega\}$. Consequently

$$\mathrm{Op}_h(b)v_{\ell\Lambda} = b(\ell d\omega)v_{\ell\Lambda} + \mathrm{Op}_h(e_\ell)v_{\ell\Lambda},$$

and by (22), when $\ell = -2,1,2$, one gets $\|\mathrm{Op}_h(e_\ell)v_{\ell\Lambda}\|_{L^\infty} = O(\varepsilon h^{1-\delta'_0})$. Since $Q_0(v_\Lambda,\bar{v}_\Lambda)$ is made of expressions of type

$$S = \mathrm{Op}_h(b_0)[(\mathrm{Op}_h(b_1)v_\Lambda)(\mathrm{Op}_h(b_2)v_\Lambda)]$$

(and similar ones replacing v_Λ by $\bar{v}_\Lambda$), one gets, using that v_Λ^2 is lagrangian along 2Λ,

$$S = b_0(2d\omega)b_1(d\omega)b_2(d\omega)v_\Lambda^2 + O_{L^\infty}(h^{1-\delta'_0}).$$

One applies a similar procedure to the other pseudo-differential terms of equation (11), namely $\mathrm{Op}_h(x\xi + |\xi|^{1/2})v$ and $C_0(V)$, where v is expressed using (20) in which the $v_{\ell\Lambda}$ are written as explicit quadratic or cubic forms in $(v_\Lambda, \bar{v}_\Lambda)$. This permits to write all those terms as polynomial expressions in $(v_\Lambda, \bar{v}_\Lambda)$ with x-depending coefficients, up to a remainder vanishing like $h^{1+\kappa}$ when h goes to zero. Expressing back v_Λ from v, one gets the ODE (24).

As soon as the preceding proposition has been established, the proof of optimal L^∞-estimates for v is straightforward. Applying a Poincaré normal forms method to (24), one is reduced to an equivalent ODE of the form

$$D_t f = \frac{1}{2}(1-\chi(h^{-\beta}x))|d\omega|^{1/2}\left[1+\frac{|d\omega|^2}{t}|f|^2\right]f + O(\varepsilon t^{-1-\kappa}).$$

This implies that $\partial_t|f|^2$ is integrable in time, whence a uniform bound for f and explicit asymptotics when t goes to infinity. Expressing v in terms of f, and writing $u(t,x) = \frac{1}{\sqrt{t}}v(t,x/t)$, one obtains the uniform $O(t^{-1/2})$ bound for u given in (A) as well as the asymptotics of the statement of the main theorem. Estimates for $Z^k u$ are proved in the same way.

1 Statement of the main result

We have already written in the introduction the water waves equations under the form of the Craig-Sulem-Zakharov system (3). We shall give here the precise definition of the Dirichlet-Neuman operator that is used in that system, and state some of its properties that are used in the rest of this paper, as well as in the companion paper [5]. Theses properties, that are essentially well known, are proved in that reference. Once the Dirichlet-Neuman operator has been properly defined, we give the precise statement of our global existence result. Next, we explain the strategy of proof, which relies on a bootstrap argument on some a priori L^2 and L^∞ estimates. The L^2 bounds are proved in the companion paper [5]. The L^∞ ones, that represent the main novelty of our method, are established in sections 2 to 6 of the present paper.

1.1 Dirichlet-Neumann operator

Let $\eta\colon \mathbb{R} \to \mathbb{R}$ be a smooth enough function and consider the open set

$$\Omega := \{ (x, y) \in \mathbb{R} \times \mathbb{R} \,;\, y < \eta(x) \}.$$

It $\psi\colon \mathbb{R} \to \mathbb{R}$ is another function, and if we call $\phi\colon \Omega \to \mathbb{R}$ the unique solution of $\Delta\phi = 0$ in Ω satisfying $\phi|_{y=\eta(x)} = \psi$ and a convenient vanishing condition at $y \to -\infty$, one defines the Dirichlet-Neumann operator $G(\eta)$ by

$$G(\eta)\psi = \sqrt{1 + (\partial_x\eta)^2}\,\partial_n\phi|_{y=\eta},$$

where ∂_n is the outward normal derivative on $\partial\Omega$, so that

$$G(\eta)\psi = (\partial_y\phi)(x, \eta(x)) - (\partial_x\eta)(\partial_x\phi)(x, \eta(x)).$$

In this subsection, we recall the estimates obtained in [5] for $G(\eta)$.

One may reduce the problem to the negative half-space through the change of coordinates $(x, y) \mapsto (x, z = y - \eta(x))$, which sends Ω on $\{(x, z) \in \mathbb{R}^2 \,;\, z < 0\}$. Then $\phi(x, y)$ solves $\Delta\phi = 0$ if and only if $\varphi(x, z) = \phi(x, z + \eta(x))$ is a solution of $P\varphi = 0$ in $z < 0$, where

$$P = (1 + \eta'^2)\partial_z^2 + \partial_x^2 - 2\eta'\partial_x\partial_z - \eta''\partial_z \tag{1.1}$$

(we denote by η' the derivative $\partial_x\eta$). The boundary condition becomes $\varphi(x, 0) = \psi(x)$ and $G(\eta)$ is given by

$$G(\eta)\psi = \left[(1 + \eta'^2)\partial_z\varphi - \eta'\partial_x\varphi\right]\big|_{z=0}.$$

It is convenient and natural to try to solve the boundary value problem

$$P\varphi = 0, \quad \varphi|_{z=0} = \psi$$

when ψ lies in homogeneous Sobolev spaces. Let us introduce them and fix some notation.

We denote by $\mathcal{S}'_\infty(\mathbb{R})$ (resp. $\mathcal{S}'_1(\mathbb{R})$) the quotient space $\mathcal{S}'(\mathbb{R})/\mathbb{C}[X]$ (resp. $\mathcal{S}'(\mathbb{R})/\mathbb{C}$). If $\mathcal{S}_\infty(\mathbb{R})$ (resp. $\mathcal{S}_1(\mathbb{R})$) is the subspace of $\mathcal{S}(\mathbb{R})$ made of the functions orthogonal to any polynomial (resp. to the constants), $\mathcal{S}'_\infty(\mathbb{R})$ (resp. $\mathcal{S}'_1(\mathbb{R})$) is the dual of $\mathcal{S}_\infty(\mathbb{R})$ (resp. $\mathcal{S}_1(\mathbb{R})$). Since the Fourier transform realizes an isomorphism from $\mathcal{S}_\infty(\mathbb{R})$ (resp. $\mathcal{S}_1(\mathbb{R})$) to

$$\widehat{\mathcal{S}}_\infty(\mathbb{R}) = \{u \in \mathcal{S}(\mathbb{R})\,;\, u^{(k)}(0) = 0 \text{ for any } k \text{ in } \mathbb{N}\}$$

(resp. $\widehat{\mathcal{S}}_1(\mathbb{R}) = \{u \in \mathcal{S}(\mathbb{R})\,;\, u(0) = 0\}$), we get by duality that the Fourier transform defines an isomorphism from $\mathcal{S}'_\infty(\mathbb{R})$ to $(\widehat{\mathcal{S}}_\infty(\mathbb{R}))'$, which is the quotient of $\mathcal{S}'(\mathbb{R})$ by the subspace of distributions supported in $\{0\}$ (resp. from $\mathcal{S}'_1(\mathbb{R})$ to $(\widehat{\mathcal{S}}_1(\mathbb{R}))' = \mathcal{S}'(\mathbb{R})/\mathrm{Vect}\,(\delta_0)$).

Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a function defining a Littlewood-Paley decomposition and set for $j \in \mathbb{Z}$, $\Delta_j = \phi(2^{-j}D)$. Then for any u in $\mathcal{S}'_\infty(\mathbb{R})$, the series $\sum_{j\in\mathbb{Z}} \Delta_j u$ converges to u in $\mathcal{S}'_\infty(\mathbb{R})$ (for the weak-$*$ topology associated to the natural topology on $\mathcal{S}_\infty(\mathbb{R})$). Let us recall (an extension of) the usual definition of homogeneous Sobolev or Hölder spaces.

Definition 1.1. *Let s', s be real numbers. One denotes by $\dot{H}^{s',s}(\mathbb{R})$ (resp. $\dot{C}^{s',s}(\mathbb{R})$) the space of elements u in $\mathcal{S}'_\infty(\mathbb{R})$ such that there is a sequence $(c_j)_{j\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$ (resp. a constant $C > 0$) with for any j in $\mathbb{Z}$,*

$$\|\Delta_j u\|_{L^2} \le c_j 2^{-js' - j_+ s}$$

(resp.

$$\|\Delta_j u\|_{L^\infty} \le C 2^{-js' - j_+ s})$$

where $j_+ = \max(j, 0)$. We set $\dot{H}^{s'}$ (resp. $\dot{C}^{s'}$) when $s = 0$.

The series $\sum_{j=0}^{+\infty} \Delta_j u$ always converges in $\mathcal{S}'(\mathbb{R})$ under the preceding assumptions, but the same is not true for $\sum_{j=-\infty}^{-1} \Delta_j u$. If u is in $\dot{H}^{s',s}(\mathbb{R})$ with $s' < 1/2$ (resp. in $\dot{C}^{s',s}(\mathbb{R})$ with $s' < 0$), then $\sum_{j=-\infty}^{-1} \Delta_j u$ converges normally in L^∞, so in $\mathcal{S}'(\mathbb{R})$, and $u \to \sum_{-\infty}^{+\infty} \Delta_j u$ gives the unique dilation and translation invariant realization of $\dot{H}^{s',s}$ (resp. $\dot{C}^{s',s}(\mathbb{R})$) as a subspace of $\mathcal{S}'(\mathbb{R})$. One the other hand, if $s' \in [1/2, 3/2[$ (resp. $s' \in [0, 1[$), the space $\dot{H}^{s'}(\mathbb{R})$ (resp. $\dot{C}^{s'}(\mathbb{R})$) admits no translation commuting realization as a subspace of $\mathcal{S}'(\mathbb{R})$, but the map $u \to \sum_{-\infty}^{+\infty} \Delta_j u$ defines a dilation and translation commuting realization of these spaces as subspaces of $\mathcal{S}'_1(\mathbb{R})$. We refer to Bourdaud [12] for these properties.

Recall also that if s is in $\mathbb{R}$ (resp. γ is in $\mathbb{R} - \mathbb{N}$), the usual Sobolev space $H^s(\mathbb{R})$ (resp. the space $C^\gamma(\mathbb{R})$) is defined as the space of elements u of $\mathcal{S}'(\mathbb{R})$ satisfying, for any j in $\mathbb{N}$, $\|\Delta_j u\|_{L^2} \le c_j 2^{-js}$ (resp. $\|\Delta_j u\|_{L^\infty} \le C 2^{-js}$) for some $\ell^2(\mathbb{N})$-sequence $(c_j)_j$ (resp. some constant C), and $\chi(D)u \in L^2$ (resp. $\chi(D)u \in L^\infty$) for some $C_0^\infty(\mathbb{R})$-function χ equal to one on a large enough neighborhood of zero. Moreover, if γ is in $\mathbb{N}$, we denote by $C^\gamma(\mathbb{R})$ the space of γ times continuously differentiable functions, which are *bounded* as well as their derivatives (endowed with the natural norms).

The main result about the Dirichlet-Neumann operator that we shell use in that paper is the following proposition, which is proved in the companion paper [5] (see Corollary 1.1.8.):

Proposition 1.2. *Let γ be a real number, $\gamma > 2, \gamma \notin \frac{1}{2}\mathbb{N}$. There is some $\delta > 0$ such that, for any η in $L^2 \cap C^\gamma(\mathbb{R})$ satisfying $\|\eta'\|_{C^{\gamma-1}} + \|\eta'\|_{C^{-1}}^{1/2} \|\eta'\|_{H^{-1}}^{1/2} < \delta$, one may define for ψ*

in $\dot{H}^{1/2}(\mathbb{R})$ the Dirichlet-Neumann operator $G(\eta)$ as a bounded operator from $\dot{H}^{1/2}(\mathbb{R})$ to $\dot{H}^{-1/2}(\mathbb{R})$ that satisfies an estimate

$$\|G(\eta)\psi\|_{\dot{H}^{-1/2}} \leq C\big(\|\eta'\|_{C^{\gamma-1}}\big)\big\||D_x|^{\frac{1}{2}}\psi\big\|_{L^2}. \tag{1.2}$$

In particular, if we define $G_{1/2}(\eta) = |D_x|^{-\frac{1}{2}} G(\eta)$, we obtain a bounded operator from $\dot{H}^{1/2}(\mathbb{R})$ to $L^2(\mathbb{R})$ satisfying

$$\big\|G_{1/2}(\eta)\psi\big\|_{L^2} \leq C\big(\|\eta'\|_{C^{\gamma-1}}\big)\big\||D_x|^{\frac{1}{2}}\psi\big\|_{L^2}. \tag{1.3}$$

Moreover, $G(\eta)$ satisfies when ψ is in $\dot{C}^{\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R})$

$$\|G(\eta)\psi\|_{C^{\gamma-1}} \leq C\big(\|\eta'\|_{C^{\gamma-1}}\big)\big\||D_x|^{\frac{1}{2}}\psi\big\|_{C^{\gamma-\frac{1}{2}}}. \tag{1.4}$$

where $C(\cdot)$ is a non decreasing continuous function of its argument.

If we assume moreover that for some $0 < \theta' < \theta < \frac{1}{2}$, $\big\|\eta'\big\|_{H^{-1}}^{1-2\theta'}\big\|\eta'\big\|_{C^{-1}}^{2\theta'}$ is bounded, then $|D_x|^{-\frac{1}{2}+\theta} G(\eta)$ satisfies

$$\big\||D_x|^{-\frac{1}{2}+\theta} G(\eta)\psi\big\|_{C^{\gamma-\frac{1}{2}-\theta}} \leq C\big(\big\|\eta'\big\|_{C^{\gamma-1}}\big)\big\||D_x|^{\frac{1}{2}}\psi\big\|_{C^{\gamma-\frac{1}{2}}}. \tag{1.5}$$

1.2 Global existence result

The goal of this paper is to prove global existence of small solutions with decaying Cauchy data of the Craig-Sulem-Zakharov system. We thus look for a couple of real valued functions (η,ψ) defined on $\mathbb{R}\times\mathbb{R}$ satisfying for $t \geq 1$ the system

$$\begin{cases} \partial_t\eta = G(\eta)\psi, \\ \partial_t\psi + \eta + \dfrac{1}{2}(\partial_x\psi)^2 - \dfrac{1}{2(1+(\partial_x\eta)^2)}\big(G(\eta)\psi + \partial_x\eta\partial_x\psi\big)^2 = 0, \end{cases} \tag{1.6}$$

with Cauchy data small enough in a convenient space.

The operator $G(\eta)$ in (1.6) being defined as in the preceding subsection, we set, for η,ψ smooth enough and small enough functions

$$B(\eta)\psi = \frac{G(\eta)\psi + \partial_x\eta\partial_x\psi}{1+(\partial_x\eta)^2}\cdot \tag{1.7}$$

Before stating our global existence result, let us recall a known local existence theorem (see [63, 47, 1]).

Proposition 1.3. *Let γ be in $]7/2,+\infty[\backslash\frac{1}{2}\mathbb{N}$, $s\in\mathbb{N}$ with $s>2\gamma-1/2$. There are $\delta_0>0$, $T>1$ such that for any couple (η_0,ψ_0) in $H^s(\mathbb{R})\times\dot{H}^{\frac{1}{2},\gamma}(\mathbb{R})$ satisfying*

$$\psi_0-T_{B(\eta_0)\psi_0}\eta_0\in\dot{H}^{\frac{1}{2},s}(\mathbb{R}),\quad \|\eta_0\|_{C^\gamma}+\big\||D_x|^{\frac{1}{2}}\psi_0\big\|_{C^{\gamma-\frac{1}{2}}}<\delta_0, \tag{1.8}$$

equation (1.6) with Cauchy data $\eta|_{t=1}=\eta_0$, $\psi|_{t=1}=\psi_0$ has a unique solution (η,ψ) which is continuous on $[1,T]$ with values in

$$\Big\{(\eta,\psi)\in H^s(\mathbb{R})\times\dot{H}^{\frac{1}{2},\gamma}(\mathbb{R})\,;\,\psi-T_{B(\eta)\psi}\eta\in\dot{H}^{\frac{1}{2},s}(\mathbb{R})\Big\}. \tag{1.9}$$

Moreover, if the data are $O(\varepsilon)$ on the indicated spaces, then $T\geq c/\varepsilon$.

Remarks. • The assumption $\psi_0\in\dot{H}^{\frac{1}{2},\gamma}$ implies that ψ_0 is in $\dot{C}^{\frac{1}{2},\gamma-\frac{1}{2}}$ so that Proposition 1.2 shows that $G(\eta_0)\psi_0$ whence $B(\eta_0)\psi_0$ is in $C^{\gamma-1}\subset L^\infty$. Consequently, by the first half of (1.8), $|D_x|^{\frac{1}{2}}\psi$ is in $H^{s-\frac{1}{2}}\subset C^{\gamma-\frac{1}{2}}$ as our assumption on s implies that $s>\gamma+1/2$. This gives sense to the second assumption (1.8).

• As already mentioned in the introduction, the difficulty in the analysis of equation (1.6) is that writing energy inequalities on the function $(\eta,|D_x|^{\frac{1}{2}}\psi)$ makes appear an apparent loss of half a derivative. A way to circumvent that difficulty is to bound the energy not of $(\eta,|D_x|^{\frac{1}{2}}\psi)$, but of $(\eta,|D_x|^{\frac{1}{2}}\omega)$, where ω is the "good unknown" of Alinhac, defined by $\omega=\psi-T_{B(\eta)\psi}\eta$ (see subsection 2 of the introduction). This explains why the regularity assumption (1.8) on the Cauchy data concerns $\psi_0-T_{B(\eta_0)\psi_0}\eta_0$ and not ψ_0 itself. Notice that this function is in $\dot{H}^{\frac{1}{2},s}$ while ψ_0 itself, written from $\psi_0=\omega_0+T_{B(\eta_0)\psi_0}\eta_0$ is only in $\dot{H}^{\frac{1}{2},s-\frac{1}{2}}$, because of the H^s-regularity of η_0.

• By (1.4) if ψ is in $\dot{C}^{\frac{1}{2},\gamma-\frac{1}{2}}$ and η is in C^γ, $G(\eta)\psi$ is in $C^{\gamma-1}$, so $B(\eta)\psi$ is also in $C^{\gamma-1}$ with $\|B(\eta)\psi\|_{C^{\gamma-1}}\leq C\big(\|\eta'\|_{C^{\gamma-1}}\big)\big\||D_x|^{\frac{1}{2}}\psi\big\|_{C^{\gamma-\frac{1}{2}}}$. In particular, as a paraproduct with an L^∞-function acts on any Hölder space,

$$\big\||D_x|^{\frac{1}{2}}T_{B(\eta)\psi}\eta\big\|_{C^{\gamma-\frac{1}{2}}}\leq C\big(\|\eta'\|_{C^{\gamma-1}}\big)\,\|\eta\|_{C^\gamma}\,\big\||D_x|^{\frac{1}{2}}\psi\big\|_{C^{\gamma-\frac{1}{2}}}.$$

This shows that for $\|\eta\|_{C^\gamma}$ small enough, $\psi\to\psi-T_{B(\eta)\psi}\eta$ is an isomorphism from $\dot{C}^{\frac{1}{2},\gamma-\frac{1}{2}}$ to itself. In particular, if we are given a small enough ω in $\dot{H}^{\frac{1}{2},s}\subset\dot{C}^{\frac{1}{2},\gamma-\frac{1}{2}}$, we may find a unique ψ in $\dot{C}^{\frac{1}{2},\gamma-\frac{1}{2}}$ such that $\omega=\psi-T_{B(\eta)\psi}\eta$. In other words, when interested only in $C^{\gamma-\frac{1}{2}}$-estimates for $|D_x|^{\frac{1}{2}}\omega$, we may as well establish them on $|D_x|^{\frac{1}{2}}\psi$ instead, as soon as $\|\eta\|_{C^\gamma}$ stays small enough.

Let us state now our main result.

We fix real numbers s,s_1,s_0 satisfying, for some large enough numbers a and γ with $\gamma\notin\frac{1}{2}\mathbb{N}$ and $a\gg\gamma$, the following conditions

$$s,s_0,s_1\in\mathbb{N},\quad s-a\geq s_1\geq s_0\geq\frac{s}{2}+\gamma. \tag{1.10}$$

Theorem 1.4. *There is $\varepsilon_0 > 0$ such that for any $\varepsilon \in]0, \varepsilon_0]$, any couple of functions (η_0, ψ_0) satisfying for any integer $p \leq s_1$*

$$
\begin{aligned}
&(x\partial_x)^p \eta_0 \in H^{s-p}(\mathbb{R}), \quad (x\partial_x)^p \psi_0 \in \dot{H}^{\frac{1}{2}, s-p-\frac{1}{2}}(\mathbb{R}),\\
&(x\partial_x)^p \big(\psi_0 - T_{B(\eta_0)\psi_0} \eta_0\big) \in \dot{H}^{\frac{1}{2}, s-p}(\mathbb{R}),
\end{aligned}
\tag{1.11}
$$

and such that the norm of the above functions in the indicated spaces is smaller than 1, equation (1.6) *with the Cauchy data $\eta|_{t=1} = \varepsilon\eta_0$, $\psi|_{t=1} = \varepsilon\psi_0$ has a unique solution (η, ψ) which is defined and continuous on $[1, +\infty[$ with values in the set* (1.9)*.*

Moreover, $u = |D_x|^{\frac{1}{2}} \psi + i\eta$ admits the following asymptotic expansion as t goes to $+\infty$:

There is a continuous function $\underline{\alpha} \colon \mathbb{R} \to \mathbb{C}$, depending of ε but bounded uniformly in ε, such that

$$
u(t,x) = \frac{\varepsilon}{\sqrt{t}} \underline{\alpha}\Big(\frac{x}{t}\Big) \exp\Big(\frac{it}{4|x/t|} + \frac{i\varepsilon^2}{64} \frac{|\underline{\alpha}(x/t)|^2}{|x/t|^5} \log(t)\Big) + \varepsilon t^{-\frac{1}{2}-\kappa} \rho(t,x)
\tag{1.12}
$$

where κ is some positive number and ρ is a function uniformly bounded for $t \geq 1$, $\varepsilon \in]0, \varepsilon_0]$.

Remark. If the integers s, s_1, s_0 are large enough, we shall see in section 6 that $\underline{\alpha}(x/t)$ vanishes when x/t goes to zero at an order that increases with these integers. Because of that, we see that the singularity of the phase at $x/t = 0$ is quite irrelevant: for $|x/t|$ small enough, the first term in the expansion is not larger than the remainder.

1.3 Strategy of the proof

The proof of the main theorem relies on the simultaneous propagation through a bootstrap of L^∞ and L^2-estimates. We state here these two results. The first one is proved in the companion paper [5]. The proof of the second one is the bulk of the present paper. We show below how these two results together imply Theorem 1.4.

The main point will be to prove L^2 and L^∞-estimates for the action of the vector field

$$
Z = t\partial_t + 2x\partial_x
\tag{1.13}
$$

on the unknown in equation (1.6). We introduce the following notation:

We assume given γ, s, a, s_0, s_1 satisfying (1.10). For (η, ψ) a local smooth enough solution of (1.6), we set $\omega = \psi - T_{B(\eta)\psi}\eta$ and for any integer $k \leq s_1$,

$$
M_s^{(k)}(t) = \sum_{p=0}^{k} \big(\big\|Z^p \eta(t,\cdot)\big\|_{H^{s-p}} + \big\||D_x|^{\frac{1}{2}} Z^p \omega(t,\cdot)\big\|_{H^{s-p}}\big).
\tag{1.14}
$$

In the same way, for ρ a positive number (that will be larger than s_0), we set for $k \leq s_0$,

$$
N_\rho^{(k)}(t) = \sum_{p=0}^{k} \big(\big\|Z^p \eta(t,\cdot)\big\|_{C^{\rho-p}} + \big\||D_x|^{\frac{1}{2}} Z^p \psi(t,\cdot)\big\|_{C^{\rho-p}}\big).
\tag{1.15}
$$

By local existence theory, for any given $T_0 > 1$, there is $\varepsilon_0' > 0$ such that if $\varepsilon < \varepsilon_0'$, equation (1.6) has a solution for $t \in [1, T_0]$. Moreover, assumptions (1.11) remain valid at $t = T_0$ (see Proposition A.4.2. in the companion paper). Consequently, it is enough to prove Theorem 1.4 with Cauchy data at $t = T_0$.

The L^2 estimates that we need are given by the following theorem, that is proved in the companion paper [5] (see Theorem 1.2.2. of that paper).

Theorem 1.5. *There is a constant $B_2 > 0$ such that $M_s^{(s_1)}(T_0) < \frac{1}{4}B_2\varepsilon$, and for any constants $B_\infty > 0$, $B'_\infty > 0$ there is ε_0 such that the following holds: Let $T > T_0$ be a number such that equation* (1.6) *with Cauchy data satisfying* (1.11) *has a solution satisfying the regularity properties of Proposition 1.3 on $[T_0, T[\times\mathbb{R}$ and such that*

i) For any $t \in [T_0, T[$, and any $\varepsilon \in]0, \varepsilon_0]$,

$$\left\| |D_x|^{\frac{1}{2}} \psi(t,\cdot) \right\|_{C^{\gamma-\frac{1}{2}}} + \|\eta(t,\cdot)\|_{C^\gamma} \le B_\infty \varepsilon t^{-\frac{1}{2}}. \tag{1.16}$$

ii) For any $t \in [T_0, T[$, any $\varepsilon \in]0, \varepsilon_0]$

$$N_\rho^{(s_0)}(t) \le B_\infty \varepsilon t^{-\frac{1}{2}+B'_\infty\varepsilon^2}. \tag{1.17}$$

Then, there is an increasing sequence $(\delta_k)_{0\le k\le s_1}$, depending only on B'_∞ and ε with $\delta_{s_1} < 1/32$ such that for any t in $[T_0, T[$, any ε in $]0, \varepsilon_0]$, any $k \le s_1$,

$$M_s^{(k)}(t) \le \frac{1}{2} B_2 \varepsilon t^{\delta_k}. \tag{1.18}$$

Remark. We do not get for the L^2-quantities $M_s^{(k)}(t)$ a uniform estimate when $t \to +\infty$. Actually, the form of the principal term in the expansion (1.12) shows that the action of a Z-vector field on it generates a $\log(t)$-loss, so that one cannot expect (1.18) to hold true with $\delta_k = 0$. For similar reasons, one could not expect that $N_\rho^{(s_0)}(t)$ in (1.17) be $O(t^{-1/2})$ when $t \to +\infty$. Such an estimate can be true only if no Z-derivative acts on the solution, as in (1.16).

Let us write down next the L^∞-estimates.

Theorem 1.6. *Let $T > T_0$ be a number such that the equation* (1.6) *with Cauchy data satisfying* (1.11) *has a solution on $[T_0, T[\times\mathbb{R}$ satisfying the regularity properties of Proposition* 1.3. *Assume that, for some constant $B_2 > 0$, for any $t \in [T_0, T[$, any ε in $]0, 1]$, any $k \le s_1$,*

$$\begin{aligned} &M_s^{(k)}(t) \le B_2 \varepsilon t^{\delta_k}, \\ &N_\rho^{(s_0)}(t) \le \sqrt{\varepsilon} < 1 \end{aligned} \tag{1.19}$$

Then there are constants $B_\infty, B'_\infty > 0$ depending only on B_2 and some $\varepsilon_0' \in]0, 1]$, independent of B_2, such that, for any t in $[T_0, T[$, any ε in $]0, \varepsilon_0']$,

$$\begin{aligned} &N_\rho^{(s_0)}(t) \le \frac{1}{2} B_\infty \varepsilon t^{-\frac{1}{2}+\varepsilon^2 B'_\infty}, \\ &\left\| |D_x|^{\frac{1}{2}} \psi(t,\cdot) \right\|_{C^{\gamma-\frac{1}{2}}} + \|\eta(t,\cdot)\|_{C^\gamma} \le \frac{1}{2} B_\infty \varepsilon t^{-\frac{1}{2}}. \end{aligned} \tag{1.20}$$

We deduce form the above results the global existence statements in Theorem 1.7.

Proof of Theorem 1.4. We take for B_2 the constant given by Theorem 1.5. Then Theorem 1.6 provides constants $B_\infty > 0$, $B'_\infty > 0$, and given these B_∞, B'_∞, Theorem 1.5 brings a small positive number ε_0. We denote by T_* the supremum of those $T > T_0$ such that a solution exists over the interval $[T_0, T[$, satisfies over this interval the regularity conditions of Proposition 1.3 and the estimates

$$\begin{aligned} &M_s^{(k)}(t) \leq B_2 \varepsilon t^{\delta_k} \quad \text{for } k \leq s_1,\\ &N_\rho^{(s_0)}(t) \leq B_\infty \varepsilon t^{-\frac{1}{2}+\varepsilon^2 B'_\infty},\\ &\left\| |D_x|^{\frac{1}{2}} \psi(t,\cdot) \right\|_{C^{\gamma-\frac{1}{2}}} + \|\eta(t,\cdot)\|_{C^\gamma} \leq B_\infty \varepsilon t^{-\frac{1}{2}}. \end{aligned} \tag{1.21}$$

We have $T_* > T_0$: By the choice of B_2 in the statement of Theorem 1.5, the first estimate (1.21) holds at $t = T_0$ with B_2 replaced by $B_2/2$. If B_∞ is chosen from start large enough, we may as well assume that at $t = T_0$, the second and third inequalities in (1.21) hold with B_∞ replaced by $B_\infty/2$. Consequently, the local existence results of Appendix A.4 in the companion paper [5] show that a solution exists on some interval $[T_0, T_0 + \delta[$, and will satisfy (1.21) on that interval if δ is small enough.

If $T_* < +\infty$, and if we take ε_0 small enough so that $B_\infty\sqrt{\varepsilon_0} < 1$, we see that (1.21) implies that assumptions (1.16), (1.17), and (1.19) of Theorem 1.5 and 1.6 are satisfied. Consequently, (1.18) and (1.20) hold on the interval $[T_0, T_*[$ i.e. (1.21) is true on this interval with B_2 (resp. B_∞) replaced by $B_2/2$ (resp. $B_\infty/2$). This contradicts the maximality of T_*. So $T_* = +\infty$ and the solution is global. We postpone the proof of (1.12) to the end of Section 6. □

The rest of this paper will be devoted to the proof of Theorem 1.6. Theorem 1.5 is proved in the companion paper [5].

2 Classes of Lagrangian distributions

We denote by h a semi-classical parameter belonging to $]0,1]$. If $(x,\xi) \mapsto m(x,\xi)$ is an order function from $T^*\mathbb{R}$ to $\mathbb{C}$, as defined in Appendix A, and if a is a symbol in the class $S(m)$ of Definition A.1, we set, for $(u_h)_h$ any family of elements of $\mathcal{S}'(\mathbb{R})$

$$\mathrm{Op}_h(a)u = \frac{1}{2\pi}\int e^{ix\xi} a(x, h\xi, h)\widehat{u}(\xi)\, d\xi. \tag{2.1}$$

It turns out that we shall need extensions of this definition to more general classes of symbols. On the one hand, we notice that if a is a continuous function such that $|a(x,\xi,h)| \leq m(x,\xi)$ and if u is in $L^2(\mathbb{R})$, (2.1) is still meaningful.

We shall also use a formula of type (2.1) when the symbol a is defined only on a subset of $T^*\mathbb{R}$. Denote by $\pi_1 : (x,\xi) \mapsto x$ and $\pi_2 : (x,\xi) \mapsto \xi$ the two projections. For F a closed subset of $T^*\mathbb{R}$, and $r > 0$, we set

$$F_r = \{ (x, \xi) \in T^*\mathbb{R} \,;\, d((x, \xi), F) < r \}$$

where d is the euclidian distance.

Definition 2.1. *Let m be an order function on $T^*\mathbb{R}$, F a closed non empty subset of $T^*\mathbb{R}$ such that $\pi_2(F)$ is compact. We denote by $S(m, F)$ the space of functions $(x, \xi, h) \mapsto a(x, \xi, h)$, defined on $F_{r_0} \times]0, 1]$ for some $r_0 > 0$, and satisfying for any α, β in $\mathbb{N}$, any (x, ξ) in F_{r_0}, any h in $]0, 1]$,*

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi, h) \right| \leq C_{\alpha,\beta} m(x, \xi).$$

We define next the notion of a family of functions microlocally supported close to a subset F as above.

Definition 2.2. *i) Let p be in $[1, +\infty]$. We denote by $E^p_\emptyset$ the space of families of L^p functions $(v_h)_h$ indexed by $h \in]0, 1]$, defined on $\mathbb{R}$ with values in $\mathbb{C}$ such that for any N in $\mathbb{N}$, there is $C_N > 0$ with $\|v_h\|_{L^p} \leq C_N h^N$ for any h in $]0, 1]$.*

ii) Let F be a closed non empty subset of $T^*\mathbb{R}$ such that $\pi_2(F)$ is compact. We denote by E^p_F the space of families of functions $(v_h)_h$ of $L^p(\mathbb{R})$ satisfying

- *There are N_0 in $\mathbb{N}$, $C_0 > 0$ and for any h in $]0, 1]$, $\|v_h\|_{L^p} \leq C_0 h^{-N_0}$.*

- *For any $r > r' > 0$, there is an element ϕ of $S(1)$ supported in F_r, equal to one on $\overline{F_{r'}}$ such that $(\mathrm{Op}_h(\phi)v_h - v_h)_h$ belongs to $E^p_\emptyset$. We say that $(v_h)_h$ is microlocally suported close to F.*

Remarks. • Notice that definition 2.2 is non empty only if there exists at least one function ϕ in $S(1)$ supported in F_r, equal to one on $\overline{F_{r'}}$. This holds if F is not "too wild" when $|x|$ goes to infinity, for instance if F is compact, or if $F = \pi_2^{-1}(K)$ for some compact subset K of $\mathbb{R}$. In the sequel, we shall always implicitly assume that such a property holds for the closed subsets in which are microlocally supported the different classes of distributions we shall define.

• It follows from Theorem A.2 of Appendix A that the last condition in Definition 2.2 will hold for any element ϕ of $S(1)$, supported in F_r, equal to one on $\overline{F_{r'}}$.

We may define the action of operators associated to symbols belonging to the class $S(1, F)$ on functions microlocally supported close to F, modulo elements of $E^p_\emptyset$. Let us notice first that if a is in $S(1)$ and is supported in a domain $\{(x, \xi, h) \,;\, |\xi| \leq C\}$ for some $C > 0$, then (2.1) defines an operator bounded on $L^p(\mathbb{R})$ for any p, uniformly in h. It follows then from the theorem of symbolic calculus A.2 of Appendix A that, if a is in $S(1, F)$, if $\tilde{\phi}$ is in $S(1)$ supported in $F_r \cap (\overline{F_{r'}})^c$, for some $0 < r' < r \ll 1$, then $(\mathrm{Op}_h(a\tilde{\phi})v_h)_h$ is in $E^p_\emptyset$ for any $(v_h)_h$ in E^p_F. We may thus state:

Definition 2.3. *Let F be a closed set as in Definition 2.2, a be an element of $S(1, F)$. For $(v_h)_h$ in E^p_F, we define*

$$\mathrm{Op}_h(a)v_h = \mathrm{Op}_h(a\phi)v_h. \tag{2.2}$$

where the right-hand side is defined by (2.1) *and where* ϕ *is in* $S(1)$*, supported in* $F_r \times]0,1]$ *for small enough* $r > 0$ *and equal to one on* $\overline{F_{r'}} \times]0,1]$*, for some* $r' \in]0, r[$*. The definition is independent of the choice of* ϕ *modulo* $E^p_\emptyset$*, so that* $\mathrm{Op}_h(a)$ *is well defined from* $E^p_F / E^p_\emptyset$ *to itself.*

Let K be a compact subset of $T^*\mathbb{R}$, $K_1 = \pi_1(K)$ and let ω be a real valued function defined on an open neighborhood U of K_1. Denote by χ_ω the canonical transformation

$$\begin{aligned} \chi_\omega \colon T^*U &\to T^*U \\ (x,\xi) &\mapsto (x, \xi - \mathrm{d}\,\omega(x)). \end{aligned}$$

Set $K' = \chi_\omega(K)$.

Lemma 2.4. *i) Let* a *be in* $S(1, K')$*. There is a symbol* b *in* $S(1,K)$ *such that for any* $(v_h)_h$ *in* E^p_K*, we may write*

$$e^{i\omega(x)/h}\,\mathrm{Op}_h(a)\left(e^{-i\omega(x)/h} v_h\right) = \mathrm{Op}_h(a \circ \chi_\omega(x,\xi)) v_h + h\,\mathrm{Op}_h(b) v_h \tag{2.3}$$

modulo $E^p_\emptyset$*.*

ii) If $(v_h)_h$ *is in* E^p_K*, then* $\left(e^{-i\omega(x)/h} v_h\right)_h$ *is in* $E^p_{K'}$*.*

Proof. We shall prove both assertions at the same time. Remark first that since $(\theta v_h - v_h)_h$ is in $E^p_\emptyset$ if θ is in $C_0^\infty(U)$ equal to one on a neighborhood of K_1, we may always assume that v_h is compactly supported in U. By symbolic calculus, and the assumption $a \in S(1,K')$, we may also assume that a is compactly supported and that the first projection of the support is contained in U. Consequently, we may replace in (2.3) ω by a $C_0^\infty(\mathbb{R})$ function, equal to the given phase in a neighborhood of K_1.

We compute

$$e^{i\omega(x)/h}\,\mathrm{Op}_h(a)\left(e^{-i\omega(x)/h} v_h\right) = \frac{1}{2\pi} \int e^{ix\xi} c(x, h\xi, h) \widehat{v_h}(\xi)\, d\xi \tag{2.4}$$

with

$$\begin{aligned} c(x,\xi,h) &= \frac{1}{2\pi h} \int e^{-i[y\eta - (\omega(x) - \omega(x-y))]/h} a(x, \xi - \eta, h)\, dy\, d\eta \\ &= \frac{1}{2\pi h} \int e^{-iy\eta/h} a(x, \xi - \eta - \theta(x,y), h)\, dy\, d\eta \end{aligned} \tag{2.5}$$

where $\theta(x,y) = \dfrac{\omega(x) - \omega(x-y)}{y}$. Let κ be a smooth function supported in a small neighborhood of zero in $\mathbb{R}$ and equal to one close to zero. We insert under the last oscillatory integral in (2.5) a factor $\kappa(y)\kappa(\eta)$. The error introduced in that way is a symbol in $h^\infty S(\langle \xi \rangle^{-\infty})$. The action of the associated operator on v_h gives an element of $E^p_\emptyset$. We have reduced ourselves to

$$\frac{1}{2\pi h} \int e^{-iy\eta/h} a(x, \xi - \eta - \theta(x,y), h) \kappa(y) \kappa(\eta)\, dy\, d\eta. \tag{2.6}$$

The argument of a belongs to K'_r if (x,ξ) is in $K_{r'}$ with $r' \ll r$ and the support of κ has been taken small enough. Moreover, it is given by $(x, \xi - \mathrm{d}\,\omega(x) + O(y) + O(\eta))$, so that an integration by parts shows that (2.6) may be written $a \circ \chi_\omega + hb$ for some symbol b in $S(1,K)$. This gives $i)$.

To check $ii)$ we apply (2.4) with $a = \phi'$ an element of $S(1)$ supported in $K'_{r'_0} \times]0,1]$, equal to one on $K'_{r'_1} \times]0,1]$ for some $0 < r'_1 < r'_0$. We assume that $\|\mathrm{Op}_h(\phi)v_h - v_h\|_{L^p} = O(h^\infty)$ for some ϕ in $S(1)$ supported in $K_{r_0} \times]0,1]$, $\phi \equiv 1$ on $K_{r_1} \times]0,1]$ with $r_1 < r_0 \ll r'_1$. Then if (x,ξ) is in K_{r_0} and $\mathrm{Supp}\,\kappa$ has been taken small enough in (2.6), we see that this integral is equal to

$$\frac{1}{2\pi h} \int e^{-iy\eta/h} \kappa(y)\kappa(\eta)\, dy\, d\eta,$$

which is equal to one modulo $O(h^\infty)$. We conclude from (2.4) where we replaced v_h by $\mathrm{Op}_h(\phi)v_h$ modulo $O(h^\infty)$ and from symbolic calculus that

$$\left\| e^{i\omega/h}\, \mathrm{Op}_h(\phi')\big(e^{-i\omega/h} v_h\big) - v_h \right\|_{L^p} = O(h^\infty),$$

which is the wanted conclusion. □

Lagrangian distributions We consider Λ a Lagrangian submanifold of $T^*(\mathbb{R} \setminus \{0\})$ that is, since we are in a one-dimensional setting, a smooth curve of $T^*(\mathbb{R} \setminus \{0\})$. We shall assume that

$$\Lambda = \{\, (x, \mathrm{d}\,\omega(x))\,;\ w \in \mathbb{R}^* \,\}$$

for ω a smooth function from $\mathbb{R}^*$ to $\mathbb{R}$. We want to define semi-classical lagrangian distributions on Λ i.e. distributions generalizing families of oscillating functions $(\theta(x)e^{i\omega(x)/h})_h$. Since in our applications ω will be homogeneous of degree -1, so will have a singularity at zero, we shall define in a first step these distributions above a compact subset of $\mathbb{R} \setminus \{0\}$. In a second step, the lagrangian distributions along Λ will be defined as sums of conveniently rescaled Lagrangian distributions on a compact set.

We fix σ, β two small positive numbers and consider two Planck constants h and $\hbar$ satisfying the inequalities

$$0 < C_0^{-1} h^{1+\beta} \le \hbar \le C_0 h^\sigma \le 1 \tag{2.7}$$

for some constant $C_0 > 0$. Notice that these inequalities imply that $O(\hbar^\infty)$ remainders will be also $O(h^\infty)$ remainders.

Definition 2.5. *Let F be a closed nonempty subset of $T^*\mathbb{R}$ such that $\pi_2(F)$ is compact. Let ν, μ be in $\mathbb{R}$, $\gamma \in \mathbb{R}_+$, $p \in [1, +\infty]$.*

i) One denotes by $h^\nu \mathcal{B}^{\mu,\gamma}_p[F]$ the space of elements $(v_\hbar)_\hbar$ of $E^p_F / E^p_\emptyset$, indexed by $\hbar$ and depending on h, such that there is $C > 0$ and for any $h, \hbar$ in $]0,1]$ satisfying (2.7)

$$\|v_\hbar\|_{L^p} \le C h^\nu \left(\frac{h}{\hbar}\right)^{\mu + \frac{1}{p}} \left(1 + \frac{h}{\hbar}\right)^{-2\gamma}. \tag{2.8}$$

We denote

$$h^\nu \mathcal{B}_p^{\mu,\gamma} = \bigcup_F h^\nu \mathcal{B}_p^{\mu,\gamma}[F],$$

where the union is taken over all closed non empty subsets F of $T^*\mathbb{R}$ such that $\pi_2(F)$ is compact.

ii) Let K be a compact subset of $T^*(\mathbb{R}\setminus\{0\})$ such that $K\cap\Lambda\neq\emptyset$. Denote by e an equation of Λ defined on a neighborhood of K. One denotes by $h^\nu L^p I_\Lambda^{\mu,\gamma}[K]$ (resp. $h^\nu L^p J_\Lambda^{\mu,\gamma}[K]$) the subspace of $h^\nu \mathcal{B}_p^{\mu,\gamma}[K]$ made of those families of functions $(v_\hbar)_\hbar$ such that there is $C>0$ and for any $h,\hbar$ in $]0,1]$ satisfying (2.7), one has the inequality

$$\|\mathrm{Op}_\hbar(e)v_\hbar\|_{L^p} \le Ch^\nu \left(\frac{h}{\hbar}\right)^{\mu+\frac{1}{p}} \left(1+\frac{h}{\hbar}\right)^{-2\gamma} \left[h^{\frac12}+\hbar\right] \tag{2.9}$$

respectively, the inequality

$$\|\mathrm{Op}_\hbar(e)v_\hbar\|_{L^p} \le Ch^\nu \left(\frac{h}{\hbar}\right)^{\mu+\frac{1}{p}} \left(1+\frac{h}{\hbar}\right)^{-2\gamma} \hbar. \tag{2.10}$$

Notice that by definition $h^\nu L^p J_\Lambda^{\mu,\gamma}[K]$ is included in $h^\nu L^p I_\Lambda^{\mu,\nu}[K]$. If $\tilde e$ is another equation of Λ close to K, we may write $\tilde e = ae$ for some symbol $a\in S(1,K)$ on a neighborhood of K. By Theorem A.2, $\mathrm{Op}_\hbar(\tilde e) = \mathrm{Op}_\hbar(a)\,\mathrm{Op}_\hbar(e) + \hbar\,\mathrm{Op}_\hbar(b)$ for another symbol b in $S(1,K)$. Consequently (2.8) and (2.9) imply that the same estimate holds with e replaced by $\tilde e$, so that the space $h^\nu L^p I_\Lambda^{\mu,\gamma}[K]$ depends only on Λ. The same holds for $h^\nu L^p J_\Lambda^{\mu,\gamma}[K]$. In particular, because of our definition of Λ, we may take $e(x,\xi) = \xi - \mathrm{d}\,\omega(x)$.

Example 2.6. Let θ be in $C_0^\infty(\mathbb{R}\setminus\{0\})$ and set $v_\hbar(x) = \theta(x)e^{i\omega(x)/\hbar}$. Then $(v_\hbar)_\hbar$ is in $L^\infty J_\Lambda^{0,0}[K]$ for any compact subset of $T^*(\mathbb{R}\setminus\{0\})$ meeting Λ such that $\mathrm{Supp}\,\theta\subset\pi_1(K)$. Actually $(v_\hbar)_\hbar$ is microlocally supported close to K and Lemma 2.4 shows that $\mathrm{Op}_\hbar(\xi - \mathrm{d}\,\omega(x))v_\hbar$ satisfies estimate (2.10) with $p=\infty$, $\nu=\mu=\gamma=0$. Notice that in this example, one could apply $\mathrm{Op}_\hbar(\xi-\mathrm{d}\,\omega(x))$ several times to $v_\hbar$, and gain at each step one factor $\hbar$ in the L^∞ estimates. It turns out that the lagrangian distributions we shall have to cope with will not satisfy such a strong statement, but only estimates of type (2.9) or (2.10).

Proposition 2.7. *Let $p\in[1,+\infty]$, μ,γ in $\mathbb{R}$, K a compact subset of $T^*(\mathbb{R}\setminus\{0\})$ with $\Lambda\cap K\neq\emptyset$.*

i) Let a be in $S(1,K)$ and $(v_\hbar)_\hbar$ an element of $L^p I_\Lambda^{\mu,\gamma}[K]$. Then $((\mathrm{Op}_\hbar(a) - a(x,\mathrm{d}\,\omega(x))v_\hbar)_\hbar$ is in $(h^{1/2}+\hbar)\mathcal{B}_p^{\mu,\gamma}[K]$.

Assume we are given a vector field $Z = \alpha(\hbar,x)D_\hbar + \beta(\hbar,x)D_x$ satisfying the following conditions: $\|Z\hbar\|_{L^\infty} = O(\hbar)$ and if e is a symbol in $S(1,K)$ (resp. that vanishes on Λ), then $[Z,\mathrm{Op}_\hbar(e)] = \mathrm{Op}_\hbar(\tilde e)$ for some other symbol $\tilde e$ in $S(1,K)$ (resp. that vanishes on Λ). Assume also that for some integer k, $Z^{k'}v_\hbar$ is in $L^p I_\Lambda^{\mu,\gamma}[K]$ for $0\le k'\le k$. Then $Z^k\,\mathrm{Op}_\hbar(a)v_\hbar$ is in $L^p I_\Lambda^{\mu,\gamma}[K]$ and $Z^k[(\mathrm{Op}_\hbar(a) - a(x,\mathrm{d}\,\omega))v_\hbar]$ is in $(h^{1/2}+\hbar)\mathcal{B}_p^{\mu,\gamma}[K]$.

ii) Denote by Λ_0 the zero section of $T^*\mathbb{R}$. Let $(v_\hbar)_\hbar$ be in $L^p I_\Lambda^{\mu,\gamma}[K]$. Then $(e^{-i\omega/\hbar}v_\hbar)_\hbar$ is $L^p I_{\Lambda_0}^{\mu,\gamma}[K_0]$ where $K_0 = \chi_\omega(K)$. Conversely, if $(v_\hbar)_\hbar$ is in $L^p I_{\Lambda_0}^{\mu,\gamma}[K_0]$, $(e^{i\omega/\hbar}v_\hbar)_\hbar$ is in $L^p I_\Lambda^{\mu,\gamma}[K]$.

iii) Let Λ_1, Λ_2 be two Lagrangian submanifolds of $T^*(\mathbb{R}\setminus\{0\})$ satisfying the same assumptions as Λ, let K_1, K_2 be compact subsets of $T^*(\mathbb{R}\setminus\{0\})$ with $\Lambda_1 \cap K_1 \neq \emptyset$, $\Lambda_2 \cap K_2 \neq \emptyset$. Set

$$\Lambda_1 + \Lambda_2 = \{\, (x, \xi_1 + \xi_2)\, ;\, (x, \xi_1) \in \Lambda_1, (x, \xi_2) \in \Lambda_2 \,\}$$

and define in the same way $K_1 + K_2$. Let p_1, p_2 be in $[1, \infty]$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $\mu_1, \mu_2, \gamma_1, \gamma_2$ in $\mathbb{R}$ with $\mu_1 + \mu_2 = \mu$, $\gamma_1 + \gamma_2 = \gamma$. Let $(v_\hbar^\ell)_\hbar$ be in $L^{p_\ell} I^{\mu_\ell, \gamma_\ell}_{\Lambda_\ell}[K_\ell]$ for $\ell = 1, 2$. Then $(v_\hbar^1 \cdot v_\hbar^2)_\hbar$ is in $L^p I^{\mu,\gamma}_{\Lambda_1+\Lambda_2}[K_1 + K_2]$.

A similar statement holds for the classes $L^p J^{\mu,\gamma}_\Lambda[K]$ and $\mathcal{B}^{\mu,\gamma}_p[K]$.

Proof. *i)* We may always modify ω outside a neighborhood of $\pi_1(K)$ so that it is compactly supported, and this will modify the quantities at hand only by an element of $E^p_\emptyset$. We may find a symbol b in $S(1, K)$ so that

$$a(x, \xi) - a(x, \mathrm{d}\,\omega(x)) = b(x, \xi)(\xi - \mathrm{d}\,\omega(x))$$

in K_r for some small r. By the symbolic calculus of appendix A,

$$\mathrm{Op}_\hbar(a) v_\hbar - a(x, \mathrm{d}\,\omega) v_\hbar = \mathrm{Op}_\hbar(b)\, \mathrm{Op}_\hbar(\xi - \mathrm{d}\,\omega(x)) v_\hbar + \hbar\, \mathrm{Op}_\hbar(c) v_\hbar$$

for a new symbol e in $S(1, K)$. The conclusion follows from estimate (2.9).

If we make act a vector field Z as in the statement on the last equality and use the commutation assumptions, we obtain the last statement of *i)*.

ii) We have seen in Lemma 2.4 that $(e^{-i\omega/\hbar} v_\hbar)_\hbar$ is in $E^p_{K_0}/E^p_\emptyset$. Since $\mathrm{Op}_\hbar(\xi)(e^{-i\omega/\hbar} v_\hbar) = e^{-i\omega/\hbar}\, \mathrm{Op}_\hbar(\xi - \mathrm{d}\,\omega(x)) v_\hbar$, we deduce from (2.9) the statement.

iii) Denote by ω_1, ω_2 two smooth functions, that may be assumed to be compactly supported close to $\pi_1(K_1)$, $\pi_1(K_2)$ respectively, such that $\Lambda_\ell = \{(x, \mathrm{d}\,\omega_\ell(x))\}$ close to K_ℓ, $\ell = 1, 2$. Then $\omega = \omega_1 + \omega_2$ parametrizes $\Lambda = \Lambda_1 + \Lambda_2$ close to $K_1 + K_2$. We define $w_\hbar^\ell = e^{-i\omega^\ell/\hbar} v_\hbar^\ell$. By *ii)*, $(w_\hbar^\ell)_\hbar$ is in $L^{p_\ell} I^{\mu_\ell,\gamma_\ell}_{\Lambda_0}[K_{\ell,0}]$, where $K_{\ell,0} = \chi_{\omega_\ell}(K_\ell)$. Writing a product from the convolution of the Fourier transforms of the factors, we see that $(w_\hbar^1 w_\hbar^2)_\hbar$ is in $E^p_{K_{1,0}+K_{2,0}}/E^p_\emptyset$. Let us check that $w_\hbar^1 w_\hbar^2$ satisfies estimate (2.9) when e is an equation of Λ_0 i.e. $e(x, \xi) = \xi$ so that $\mathrm{Op}_\hbar(e) = \hbar D_x$. We write

$$\left\|\hbar D_x (w_\hbar^1 w_\hbar^2)\right\|_{L^p} \leq \left\|\hbar D_x w_\hbar^1\right\|_{L^{p_1}} \left\|w_\hbar^2\right\|_{L^{p_2}} + \left\|w_\hbar^1\right\|_{L^{p_1}} \left\|\hbar D_x w_\hbar^2\right\|_{L^{p_2}}$$

and use (2.8), (2.9) for each factor to get that $(w_\hbar^1 w_\hbar^2)_\hbar$ is in $L^p I^{\mu,\gamma}_{\Lambda_0}[K_{1,0} + K_{2,0}]$. We just have to apply again *ii)* to $v_\hbar = e^{i\omega/\hbar}(w_\hbar^1 w_\hbar^2)$ to get the conclusion. The proof is similar for classes $L^p J^{\mu,\gamma}_\Lambda[K]$ and $\mathcal{B}^{\mu,\gamma}_p[K]$. □

We have defined, up to now, classes of Lagrangian distributions microlocally supported close to a compact set of the phase space. We introduce next classes of Lagrangian distributions that do not obey such a localization property.

From now on, we consider phase functions $\omega\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ which are smooth, non zero, and positively homogeneous of degree -1. We set

$$\Lambda = \{(x, \mathrm{d}\,\omega(x))\,;\, x \in \mathbb{R}\setminus\{0\}\} \subset T^*(\mathbb{R}\setminus\{0\}) \tag{2.11}$$

so that Λ is invariant under the action of $\mathbb{R}_+^*$ on $T^*(\mathbb{R}\setminus\{0\})$ given by $\lambda\cdot(x,\xi) = (\lambda x, \lambda^{-2}\xi)$. For $h \in]0,1]$, C a positive constant, we introduce the notations

$$\begin{aligned} &J(h,C) = \left\{ j \in \mathbb{Z}\,;\, C^{-1}h^{2(1-\sigma)} \le 2^j \le Ch^{-2\beta} \right\},\\ &h_j = h2^{-j/2} \text{ if } j \in J(h,C),\\ &j_0(h,C) = \min(J(h,C)) - 1, \quad j_1(h,C) = \max(J(h,C)) + 1. \end{aligned} \tag{2.12}$$

We note that (2.7) is satisfied by $\hbar = h_j$ if $j \in J(h,C)$ (for a constant $C = C_0^2$). For $j\in\mathbb{Z}$, v a distribution on $\mathbb{R}$, we set

$$\Theta_j^* v = v(2^{j/2}\cdot),$$

so that in particular, if $p\in[1,\infty]$, $\left\|\Theta_j^*\right\|_{\mathcal{L}(L^p,L^p)} = 2^{-j/(2p)}$. If a belongs to the class of symbols $S(m)$ and if $a_j(x,\xi) = a(2^{-j/2}x, 2^j\xi)$ we notice that for $j \in J(h,C)$

$$\Theta_{-j}^* \operatorname{Op}_h(a)\Theta_j^* = \operatorname{Op}_{h_j}(a_j). \tag{2.13}$$

We fix a function φ in $C_0^\infty(\mathbb{R}^d)$ such that $\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) \equiv 1$. We define

$$\varphi_0(\xi) = \sum_{j=-\infty}^{-1} \varphi(2^{-j}\xi), \quad \Delta_j^h = \operatorname{Op}_h(\varphi(2^{-j}\xi)) = \varphi(2^{-j}hD).$$

Definition 2.8. *Let $\nu\in\mathbb{R}$, $p\in[1,\infty]$, $b\in\mathbb{R}$. One denotes by $h^\nu\mathcal{R}_p^b$ the space of families of L^p-functions $(v_\hbar)_\hbar$ such that there is $C>0$ and*

$$\begin{aligned} &\left\|\Delta_j^h v_h\right\|_{L^p} \le C_0 h^\nu 2^{-j+b} \quad \text{for } j \ge j_0(h,C)\\ &\left\|\operatorname{Op}_h(\varphi_0(2^{-j_0(h,C)}\xi))v_h\right\|_{L^p} \le Ch^\nu, \end{aligned} \tag{2.14}$$

where $j_+ = \max(j,0)$.

Clearly the definition is independent of the choice of φ_0.

Definition 2.9. *Let Λ be a lagrangian submanifold of form (2.11), K a compact subset of $T^*(\mathbb{R}\setminus\{0\})$ meeting Λ. Let ν,μ be in $\mathbb{R}$, $\gamma\in\mathbb{R}_+$, F a closed non empty subset of $T^*\mathbb{R}$ such that $\pi_2(F)$ is compact in $\mathbb{R}$. One denotes by $h^\nu L^p\tilde{I}_\Lambda^{\mu,\gamma}[K]$ (resp. $h^\nu L^p\tilde{J}_\Lambda^{\mu,\gamma}[K]$, resp. $h^\nu\tilde{\mathcal{B}}_p^{\mu,\gamma}[F]$) the space of families of functions $(v_h)_{h\in]0,1]}$ such that*

- *For any $j\in J(h,C)$, there is a family $(v_{h_j}^j)_{h_j}$, indexed by*

$$h_j \in \left]0, \min\left(C_0^{\frac{1}{1-\sigma}} 2^{\frac{j\sigma}{2(1-\sigma)}}, C_0^{\frac{1}{\beta}} 2^{-j\frac{1+\beta}{2\beta}}\right)\right]$$

which is an element of $h^\nu L^p I_\Lambda^{\mu,\gamma}[K]$ *(resp.* $h^\nu L^p J_\Lambda^{\mu,\gamma}[K]$*, resp.* $h^\nu \mathcal{B}_p^{\mu,\gamma}[F]$*) with the constants in* (2.8)*,* (2.9)*,* (2.10) *uniform in* $j \in J(h,C)$*.*

- *For any* $h \in]0,1]$*,* $v_h = \sum_{j\in J(h,C)} \Theta_j^* v_{h_j}^j$*.*

One defines $h^\nu \tilde{\mathcal{B}}_p^{\mu,\gamma} = \bigcup h^\nu \tilde{\mathcal{B}}_p^{\mu,\gamma}[F]$ *where the union is taken over the sets* F *which are closed with* $\pi_2(F)$ *compact in* $\mathbb{R}$*.*

Remark 2.10. • The interval of variation imposed to h_j in the preceding definition is the one deduced from (2.7) with $\hbar = h_j$.

• The building blocks $(v_{h_j}^j)_{h_j}$ in the above definition are defined modulo $O(h_j^\infty)$ so modulo $O(h^\infty)$ since $h_j \leq h^\sigma$. Since the cardinal of $J(h,C)$ is $O(|\log h|)$, we see that the classes introduced in the above definition are well defined modulo $O(h^\infty)$.

• It follows from the above two definitions that $h^\nu \tilde{\mathcal{B}}_p^{0,b} \subset h^\nu \mathcal{R}_p^b$. Moreover, by (2.14) and the fact that the cardinal of $\mathbb{Z}_- \cap \{j \geq j_0(h,C)\}$ is $O(|\log h|)$, we see that if u, v are in $\mathcal{R}_\infty^b$ with $b > 0$, then uv is in $h^{-0}\mathcal{R}_\infty^b := \cap_{\theta>0} h^{-\theta}\mathcal{R}_\infty^b$.

Let us prove a statement similar to $i)$ of Proposition 2.7 for elements of the classes of distributions we just defined.

Proposition 2.11. *We assume that the function* ω *defining* Λ *satisfies either* $\omega(x) \neq 0$ *for all* $x \in \mathbb{R}^*$ *or* $\omega \equiv 0$*. In the first (resp. second) case we denote by* K *a compact subset of* $T^*(\mathbb{R}\setminus\{0\})\setminus 0$ *(resp. of* $T^*(\mathbb{R}\setminus\{0\})$*) such that* $K \cap \Lambda \neq \emptyset$*. Let* $\mu \in \mathbb{R}$*,* $\gamma \in \mathbb{R}_+$*,* $p \in [1,\infty]$*,* $k \in \mathbb{N}$ *be given. Consider a function* $(x,\xi) \mapsto a(x,\xi)$ *smooth on* $\mathbb{R}^* \times \mathbb{R}^*$ *(resp.* $\mathbb{R}^*\times\mathbb{R}$*) satisfying for some real numbers* ℓ, ℓ', d, d' *(resp.* ℓ, ℓ'*,* $d \geq 0$*,* $d' \geq 0$*) and all* α, β *in* $\mathbb{N}$

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha\beta}|x|^{\ell-\alpha}\langle x\rangle^{\ell'}|\xi|^{d-\beta}\langle\xi\rangle^{d'} \tag{2.15}$$

when $(x,\xi) \in \mathbb{R}^*\times\mathbb{R}^*$ *(resp.* (2.15) *when* $\beta \leq d$ *and*

$$\partial_x^\alpha \partial_\xi^\beta a(x,\xi) \equiv 0 \quad \textit{for } \beta > d, \tag{2.16}$$

when $(x,\xi) \in \mathbb{R}^*\times\mathbb{R}$*).*

Denote $Z = -hD_h + xD_x$ *and let* $(v_h)_h$ *satisfy for any* $k' \leq k$*,* $(Z^{k'}v_h)_h \in L^p\tilde{I}_\Lambda^{\mu,\nu}[K]$ *(resp.* $\tilde{\mathcal{B}}_p^{\mu,\nu}[K]$*). Then* $(Z^k(\mathrm{Op}_h(a)v_h))_h$ *belongs to* $L^p\tilde{I}_\Lambda^{\tilde{\mu},\tilde{\gamma}}[K]$ *(resp.* $\tilde{\mathcal{B}}_p^{\tilde{\mu},\tilde{\nu}}[K]$*) where* $\tilde{\mu} = \mu+2d-\ell-\ell'$*,* $\tilde{\gamma} = \gamma - \frac{\ell'}{2} - d'$*.*

Moreover, under the assumption $(Z^{k'}v_h)_h \in L^p\tilde{I}_\Lambda^{\mu,\gamma}[K]$*, if* χ *is in* $C_0^\infty(\mathbb{R})$ *is equal to one close to zero, and has small enough support,* $Z^k((1-\chi)(xh^{-\beta})a(x,\mathrm{d}\,\omega)v_h)$ *is also in* $L^p\tilde{I}_\Lambda^{\tilde{\mu},\tilde{\gamma}}[K]$ *and*

$$Z^k\left[\mathrm{Op}_h(a)v_h - (1-\chi)(xh^{-\beta})a(x,\mathrm{d}\,\omega)v_h\right] \tag{2.17}$$

belongs to $h^{1/2}\tilde{\mathcal{B}}_p^{\tilde{\mu},\tilde{\gamma}}[K] + h\tilde{\mathcal{B}}_p^{\tilde{\mu}-1,\tilde{\gamma}}[K]$*.*

If we assume that $(Z^{k'}v_h)_h$ *is in* $L^p\tilde{J}_\Lambda^{\mu,\gamma}[K]$ *for* $k' \leq k$*, we obtain instead that* $(Z^k \operatorname{Op}_h(a)v_h)_h$ *and* $(Z^k(1-\chi)(xh^{-\beta})a(x, \mathrm{d}\,\omega)v_h)_h$ *belong to* $L^p\tilde{J}_\Lambda^{\tilde{\mu},\tilde{\gamma}}[K]$ *and that* (2.17) *is in* $h\tilde{\mathcal{B}}_p^{\tilde{\mu}-1,\tilde{\gamma}}[K]$*.*

When $\omega \equiv 0$*, if* $Z^{k'}v_h$ *is in* $\tilde{\mathcal{B}}_p^{\mu,\gamma}[K]$*, we obtain that* $(Z^k \operatorname{Op}_h(a)v_h)_h$ *is in* $\tilde{\mathcal{B}}_p^{\tilde{\mu},\tilde{\gamma}}[K]$*.*

The same results hold if we quantize a *by* $\operatorname{Op}_h(\overline{a})^*$ *instead of* $\operatorname{Op}_h(a)$ *i.e. under the same assumptions as above* $(Z^k \operatorname{Op}_h(\overline{a})^*(v_h))$ *belongs to* $L^p\tilde{I}_\Lambda^{\tilde{\mu},\tilde{\gamma}}[K]$ *and*

$$Z^k\left[\operatorname{Op}_h(\overline{a})^*v_h - (1-\chi)(xh^{-\beta})a(x, \mathrm{d}\,\omega)v_h\right] \tag{2.18}$$

belongs to the same spaces as indicated above after (2.17)*. In the same way, when* $\omega \equiv 0$*, and when* $(Z^{k'}v_h)_h$ *is in* $\tilde{\mathcal{B}}_p^{\mu,\gamma}[K]$*, for* $k' \leq k$*,* $(Z^k(\operatorname{Op}_h(\overline{a})^*v_h))_h$ *is in* $\tilde{\mathcal{B}}_p^{\mu,\gamma}[K]$*.*

Proof. According to Definition 2.9, we represent $v_h = \sum_{j\in J(h,C)} \Theta_j^* v_{h_j}^j$ where $(v_{h_j}^j)_{h_j}$ is a bounded sequence of elements of $L^pI_\Lambda^{\mu,\gamma}[K]$, as well as $(Z^{k'}v_{h_j})_{h_j}$ for $k' \leq k$. By (2.13) and (2.15),

$$\operatorname{Op}_h(a)v_h = \sum_{j\in J(h,C)} \Theta_j^* w_{h_j}^j \tag{2.19}$$

with $w_{h_j}^j = \operatorname{Op}_{h_j}(a_j)v_{h_j}^j$ and

$$a_j(x,\xi) = a\left(2^{-j/2}x, 2^j\xi\right) = 2^{j\left(d-\frac{\ell+\ell'}{2}\right)+j_+\left(d'+\frac{\ell'}{2}\right)}b_j(x,\xi).$$

When (x,ξ) stays in a compact subset of $T^*(\mathbb{R}\setminus\{0\})\setminus 0$, (2.15) shows that $\partial_x^\alpha\partial_\xi^\beta b_j = O(1)$ uniformly in j. In the same way when (x,ξ) stays in a compact subset of $T^*(\mathbb{R}\setminus\{0\})$, (2.15) for $\beta \leq d$ and (2.16) show also that $\partial_x^\alpha\partial_\xi^\beta b_j = O(1)$ uniformly in j. Since $2^j = (h/h_j)^2$, it follows from Theorem A.2 of the appendix that $(w_{h_j}^j)_{h_j}$ is a bounded sequence indexed by $j \in J(h,C)$ of elements of $L^pI_\Lambda^{\tilde{\mu},\tilde{\gamma}}[K]$. Moreover, the vector field Z satisfies for any symbol e

$$[Z, \operatorname{Op}_{h_j}(e)] = \operatorname{Op}_{h_j}\left((xD_x - 2\xi D_\xi)e\right).$$

Since either $\Lambda = \{(x, \mathrm{d}\,\omega(x))\}$ with ω homogeneous of degree -1 or $\Lambda = \{(x,0)\}$, we see that $(xD_x - 2\xi D_\xi)e$ vanishes on Λ if e does. Consequently, the assumption of the last statement in $i)$ of Proposition 2.7 is satisfied and we conclude that $(Zw_{h_j}^j)_{h_j}$ is a bounded sequence of elements of $L^pI_\Lambda^{\tilde{\mu},\tilde{d}}[K]$.

To prove (2.17), we use that again by $i)$ of Proposition 2.7,

$$w_{h_j}^j = a_j(x, \mathrm{d}\,\omega)v_{h_j}^j + \left(h^{1/2} + h_j\right)r_{h_j}^j$$

where $(r_{h_j}^j)_{h_j}$ is a bounded sequence indexed by $j \in J(h,C)$ of elements of $\mathcal{B}_p^{\tilde{\mu},\tilde{\gamma}}[K]$, that stay in that space if one applies $Z^{k'}$ $(k' \leq k)$ on them. Let χ be as in the statement of the proposition, with small enough support. Then, if x is close to $\pi_1(K)$ and j is in $J(h,C)$, $(1-\chi)(2^{-j/2}xh^{-\beta}) \equiv 1$. Consequently, since $(v^j_{h_j})_{h_j}$, as well as $(Z^{k'}v^j_{h_j})_{h_j}$ is microlocalized close to K, we may write $v^j_{h_j} = v^j_{h_j}(1-\chi)(2^{-j/2}xh^{-\beta})$ modulo a remainder which is $O(h_j^\infty) = O(h^\infty)$ in L^p, as well as its $Z^{k'}$-derivatives, $0 \le k' \le k$. Integrating such a remainder in the $r^j_{h_j}$ contributions, we may write, using that $\mathrm{d}\,\omega$ is homogeneous of degree -2,

$$w^j_{h_j} = (1-\chi)(2^{-j/2}xh^{-\beta})a\left(2^{-j/2}x, \mathrm{d}\,\omega(2^{-j/2}x)\right)v^j_{h_j} + h^{1/2}r^j_{h_j} + h\tilde{r}^j_{h_j}$$

where $(r^j_{h_j})_{h_j}$ is as above and $\tilde{r}^j_{h_j} = 2^{-j/2}r^j_{h_j}$ is such that $(Z^{k'}\tilde{r}^j_{h_j})_{h_j}$ is in $\mathcal{B}^{\tilde{\mu}-1,\tilde{\gamma}}_p[K]$ for $k' \le k$. This gives (2.17) if we plug this expansion in (2.19).

To check that $\left(Z^k((1-\chi)(xh^{-\beta})a(x,\mathrm{d}\,\omega)v_h)\right)_h$ is also in $L^p\tilde{I}^{\tilde{\mu},\tilde{\gamma}}_\Lambda[K]$, we write the function on which acts Z^k as

$$\sum_{j\in J(h,C)} \Theta^*_j\left[a(2^{-j/2}x,\mathrm{d}\,\omega(2^{-j/2}x))(1-\chi)(x2^{-j/2}h^{-\beta})v^j_{h_j}\right]$$

and remark that, as above, the assumptions of microlocal localization of $(v^j_{h_j})_{h_j}$ allow one to remove the cut-off $(1-\chi)$ up to $O(h^\infty)$ remainders. Since $\mathrm{d}\,\omega$ is homogeneous of degree -2,

$$a(2^{-j/2}x,\mathrm{d}\,\omega(2^{-j/2}x)) = O\left(2^{j\left(d-\frac{\ell+\ell'}{2}\right)+j_+\left(d'+\frac{\ell'}{2}\right)}\right)$$

when x stays in a compact subset of $\mathbb{R}^*$, so that the above sum defines an element of $L^p\tilde{I}^{\tilde{\mu},\tilde{\gamma}}_\Lambda[K]$.

The statement of the proposition concerning the case when $(Z^{k'}v_h)_h$ is in $L^p\tilde{J}_\Lambda[K]$ is proved similarly, as well as the one about $\tilde{\mathcal{B}}^{\mu,\gamma}_p[K]$.

Finally, the statements concerning $\mathrm{Op}_h(\overline{a})^*$ instead of $\mathrm{Op}_h(a)$ are proved in the same way: one may write (2.19) with $w^j_{h_j}$ given by $\mathrm{Op}_h(\overline{a_j})^*v^j_{h_j}$. By Theorem A.2 in the appendix, we know that there is a symbol b_j in $S(1,K)$ uniformly in j, such that $\mathrm{Op}_{h_j}(\overline{a_j})^* = \mathrm{Op}_{h_j}(b_j)$. Moreover, $b_j(x,\xi) = a_j(x,\xi) + h_jc_j(x,\xi)$ for some other symbol c_j in $S(1,K)$ uniformly in j. The statements concerning $\mathrm{Op}_h(\overline{a})^*v_h$ thus follows from those we just proved for $\mathrm{Op}_h(a)v_h$. □

Let us study products.

Proposition 2.12. *Let p_1,p_2,p be in $[1,+\infty]$ with $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$, γ_1,γ_2 in $\mathbb{R}$, Λ_1,Λ_2 be two Lagrangian submanifolds of $T^*(\mathbb{R}\setminus\{0\})$ of the form (2.11), defined in terms of phase functions ω_1,ω_2 homogeneous of degree -1, $\omega_1\not\equiv 0$, $\omega_2\not\equiv 0$. Let K_1, K_2 be two compact subsets of $T^*(\mathbb{R}\setminus\{0\})$ with $K_\ell\cap\Lambda_\ell\neq\emptyset$, $\ell=1,2$. Let $(v^\ell_h)_h$ be an element of $\tilde{\mathcal{B}}^{\mu_\ell,\gamma_\ell}_{p_\ell}[K_\ell]$ (resp. $L^{p_\ell}\tilde{I}^{\mu_\ell,\gamma_\ell}_{\Lambda_\ell}[K_\ell]$, resp. $L^{p_\ell}\tilde{J}^{\mu_\ell,\gamma_\ell}_{\Lambda_\ell}[K_\ell]$) $\ell=1,2$.*

There is a compact subset K of $T^*(\mathbb{R}\setminus\{0\})$ with $K\cap(\Lambda_1+\Lambda_2)\neq\emptyset$ such that $(v^1_h\cdot v^2_h)_h$ belongs to $\tilde{\mathcal{B}}^{\mu,\gamma}_p[K]$ (resp. $L^p\tilde{I}^{\mu,\gamma}_{\Lambda_1+\Lambda_2}[K]$, resp. $L^p\tilde{J}^{\mu,\gamma}_{\Lambda_1+\Lambda_2}[K]$) with $\mu=\mu_1+\mu_2$, $\gamma=\gamma_1+\gamma_2$. Moreover, for any neighborhood Ω of $\Lambda_1+\Lambda_2$, any compact subset L of $\mathbb{R}\setminus\{0\}$, there are neighborhoods Ω_ℓ of Λ_ℓ, $\ell=1,2$, such that if $K_\ell\subset\Omega_\ell\cap\pi_1^{-1}(L)$, $\ell=1,2$, then $K\subset\Omega$.

Proof. By Definition 2.9, we may write for $\ell = 1, 2$,

$$v_h^\ell = \sum_{j_\ell \in J(h,C)} \Theta_{j_\ell}^* v_{h_{j_\ell}}^{\ell, j_\ell}$$

where $(v_{h_{j_\ell}}^{\ell, j_\ell})_{h_{j_\ell}}$ is a bounded sequence of $L^{p_\ell} I_{\Lambda_\ell}^{\mu_\ell, \gamma_\ell}[K_\ell]$. We write

$$v_h^1 \cdot v_h^2 = \sum_{j_1 \in J(h,C)} \Theta_{j_1}^* w_{h_{j_1}}^{j_1} \tag{2.20}$$

with

$$w_{h_{j_1}}^{j_1} = v_{h_{j_1}}^{1, j_1} \sum_{j_2 \in J(h,C)} \Theta_{j_2 - j_1}^* v_{h_{j_2}}^{2, j_2}. \tag{2.21}$$

Because of the microlocal localization properties of $(v_{h_{j_\ell}}^{\ell, j_\ell})_{h_{j_\ell}}$ we may, up to an $O(h^\infty)$ remainder in L^{p_ℓ}, replace $(v_{h_{j_\ell}}^{\ell, j_\ell})_{h_{j_\ell}}$ by $(\theta v_{h_{j_\ell}}^{\ell, j_\ell})_{h_{j_\ell}}$ where θ is in $C_0^\infty(\mathbb{R})$ and is equal to one on a large enough compact subset of $\mathbb{R}^*$. This shows that in (2.21), we may limit the summation to those j_2 such that $|j_2 - j_1| \le C_0$ for some large enough C_0, up to remainders which are $O(h^\infty)$ in L^p. Define

$$\tilde{v}_{h_{j_1}}^{1, j_1} = \sum_{\substack{j_2 \in J(h,C) \\ |j_1 - j_2| \le C_0}} \Theta_{j_2 - j_1}^* v_{h_{j_2}}^{2, j_2}.$$

Then $(\tilde{v}_{h_{j_1}}^{1, j_1})_{h_{j_1}}$ is a bounded sequence of $\mathcal{B}_{p_2}^{\mu_2, \gamma_2}[\tilde{K}_2]$ (resp. $L^{p_2} I_{\Lambda_2}^{\mu_2, \gamma_2}[\tilde{K}_2]$, resp. $L^{p_2} J_{\Lambda_2}^{\mu_2, \gamma_2}[\tilde{K}_2]$) for some large enough compact subset $\tilde{K}_2$ of $T^*(\mathbb{R} \setminus \{0\})$, as follows from (2.13) and the homogeneity properties of Λ_2. We just need to apply *iii)* of Proposition 2.7 to conclude that $(w_{h_{j_1}}^{j_1})_{h_{j_1}}$ is a bounded sequence of elements of $\mathcal{B}_p^{\mu, \gamma}[K_1 + \tilde{K}_2]$ (resp. $L^p I_{\Lambda_1 + \Lambda_2}^{\mu, \gamma}[K_1 + \tilde{K}_2]$, resp. $L^p J_{\Lambda_1 + \Lambda_2}^{\mu, \gamma}[K_1 + \tilde{K}_2]$).

The last statement of the proposition follows from the fact that $K = K_1 + \tilde{K}_2$, and that $\tilde{K}_2$ mat be taken in an arbitrary neighborhood of Λ_2 if K_2 is contained in an even smaller neighborhood of that submanifold. □

Proposition 2.13. *Let F_1, F_2 be closed subsets of $T^*\mathbb{R}$ such that $\pi_2(F_\ell)$ is compact, $\ell = 1, 2$. Let (v_h^ℓ) be in $\tilde{\mathcal{B}}_\infty^{\mu_\ell, \gamma}[F_\ell]$ with $\mu_\ell \ge 0$, $\gamma > \mu_\ell$. Then $v_h^1 \cdot v_h^2$ is in $h^{-\theta} \mathcal{B}_\infty^{\mu, \gamma}[F]$ with $\mu = \mu_1 + \mu_2$ for any $\theta > 0$ and some closed subset F of $T^*\mathbb{R}$ whose second projection is compact.*

Proof. We write (2.20)

$$v_h^1 \cdot v_h^2 = \sum_{j_1 \in J(h,C)} \Theta_{j_1}^* w_{h_{j_1}}^1 + \sum_{j_2 \in J(h,C)} \Theta_{j_2}^* w_{h_{j_2}}^2$$

with

$$w_{h_{j_1}}^1 = v_{h_{j_1}}^{1, j_1} \sum_{\substack{j_2 \in J(h,C) \\ j_2 \le j_1}} \Theta_{j_2 - j_1}^* v_{h_{j_2}}^{2, j_2}$$

and a symmetric expression for $w^2_{h_{j_2}}$. Then $w^1_{h_{j_1}}$ is microlocally supported in some closed subset of $T^*\mathbb{R}$, whose ξ projection is compact, as

$$\mathrm{Op}_{h_{j_1}}(\tilde{\varphi})\Big[\Theta^*_{j_2-j_1} v^{2,j_2}_{h_{j_2}}\Big] = \Theta^*_{j_2-j_1} v^{2,j_2}_{h_{j_2}}$$

if $\tilde{\varphi}$ is supported for $|\xi| \leq C$, equal to one on $|\xi| \leq C/2$, for some $C > 0$. The L^∞-norm of $w^1_{h_{j_1}}$ is bounded from above by

$$2^{j_1\mu_1/2 - j_1 + \gamma} \sum_{\substack{j_2 \in J(h,C)\\ j_2 \leq j_1}} 2^{j_2\mu_2/2} 2^{-j_2+\gamma} \leq C|\log h| 2^{j_1\mu/2 - j_1 + \gamma},$$

since $\mu_\ell \geq 0$, $\gamma > \mu_\ell/2$. □

3 The semi-classical water waves equation

Let us recall an equivalent form of the water waves equation that is obtained in the companion paper [5] (see Corollary 4.3.13. in that paper). If (η, ψ) is a solution of the water waves equation, if $\mathcal{Z}$ denotes the collection of vector fields $\mathcal{Z} = (Z, \partial_x)$ and if we assume that $u = |D_x|^{\frac{1}{2}} \psi + i\eta$ satisfies for k smaller than some integer s_0, for $\alpha > 0$ large enough and $d \in \mathbb{N}$,

$$\sup_{[T_0,T]} \big\|\mathcal{Z}^k u(t,\cdot)\big\|_{H^{d+\alpha}} < +\infty, \quad \sup_{[T_0,T]} \big\|\mathcal{Z}^k u(t,\cdot)\big\|_{C^{d+\alpha}} < +\infty$$

on an interval $[T_0, T]$, we may write, using the notation $\mathcal{U} = (u, \overline{u})$,

$$D_t u = |D_x|^{\frac{1}{2}} u + \widetilde{Q}_0(\mathcal{U}) + \widetilde{C}_0(\mathcal{U}) + \widetilde{R}_0(\mathcal{U}), \tag{3.1}$$

where $\widetilde{Q}_0(\mathcal{U})$ denotes the quadratic part of the nonlinearity

$$\begin{aligned}\widetilde{Q}_0(\mathcal{U}) = &-\frac{i}{8}|D_x|^{\frac{1}{2}}\Big[\big(D_x|D_x|^{-\frac{1}{2}}(u+\bar{u})\big)^2 + \big(|D_x|^{\frac{1}{2}}(u+\bar{u})\big)^2\Big]\\ &+\frac{i}{4}|D_x|\big((u-\bar{u})|D_x|^{\frac{1}{2}}(u+\bar{u})\big) - \frac{i}{4}D_x\big((u-\bar{u})D_x|D_x|^{-\frac{1}{2}}(u+\bar{u})\big),\end{aligned} \tag{3.2}$$

$\widetilde{C}_0(\mathcal{U})$ stands for the cubic contribution

$$\begin{aligned}\widetilde{C}_0(\mathcal{U}) = &\frac{1}{8}|D_x|^{\frac{1}{2}}\Big[\big(|D_x|^{\frac{1}{2}}(u+\bar{u})\big)|D_x|\Big((u-\bar{u})|D_x|^{\frac{1}{2}}(u+\bar{u})\Big)\Big]\\ &-\frac{1}{8}|D_x|^{\frac{1}{2}}\Big[\big(|D_x|^{\frac{1}{2}}(u+\bar{u})\Big((u-\bar{u})|D_x|^{\frac{3}{2}}(u+\bar{u})\Big)\Big]\\ &-\frac{1}{8}|D_x|\Big[(u-\bar{u})|D_x|\Big((u-\bar{u})|D_x|^{\frac{1}{2}}(u+\bar{u})\Big)\Big]\\ &+\frac{1}{16}|D_x|\Big[(u-\bar{u})^2|D_x|^{\frac{3}{2}}(u+\bar{u})\Big] + \frac{1}{16}|D_x|^2\Big[(u-\bar{u})^2|D_x|^{\frac{1}{2}}(u+\bar{u})\Big]\end{aligned} \tag{3.3}$$

and where $\widetilde{\mathcal{R}}_0(\mathcal{U})$ is a remainder, vanishing at least at order 4 at $\mathcal{U}=0$, which satisfies for $k \leq s_0$ the following estimates

$$\left\|\mathcal{Z}^k\,|D_x|^{-\frac{1}{2}}\,\widetilde{R}_0(\mathcal{U})\right\|_{H^d} \leq C_k[u] \sum_{\substack{k_1+\cdots+k_4\leq k\\ k_1,k_2,k_3\leq k_4}} \prod_{j=1}^{3} \left\|\mathcal{Z}^{k_j}u\right\|_{C^{d+\alpha}} \left\|\mathcal{Z}^{k_4}u\right\|_{H^{d+\alpha}} \tag{3.4}$$

with a constant $C_k[u]$ depending only on $\left\|\mathcal{Z}^{(k-1)_+}u\right\|_{C^{d+\alpha}}$ and, if $\theta>0$ is small enough,

$$\left\|\mathcal{Z}^k\,|D_x|^{-\frac{1}{2}+\theta}\,\widetilde{R}_0(\mathcal{U})\right\|_{C^d} \leq C_k[u] \sum_{k_1+\cdots+k_4\leq k} \prod_{j=1}^{4} \left\|\mathcal{Z}^{k_j}u\right\|_{C^{d+\alpha}} \tag{3.5}$$

where $C_k[u]$ depends only on $\left\|\mathcal{Z}^{(k-1)_+}u\right\|_{C^{d+\alpha}}$ and on a bound on $\|u\|_{L^2}^{1-2\theta'}\|u\|_{L^\infty}^{2\theta'}$ for some $\theta' \in]0,\theta[$.

We make the change of variables $t=t'$, $x=t'x'$ and set $h=t'^{-1}$, $u(t,x)=h^{1/2}v(t',x')$, so that

$$D_t u = h^{\frac{1}{2}}\Big[(D_{t'}-x'hD_{x'})v + \frac{i}{2}hv\Big].$$

The vector field $Z=t\partial_t+2x\partial_x$ becomes $Z=t'\partial_{t'}+x'\partial_{x'}$. We deduce from (3.1) the following equation for v, in which we write (t,x) instead of (t',x'), since we shall not go back to the old coordinates

$$\big(D_t - \mathrm{Op}_h(x\xi+|\xi|^{\frac{1}{2}})\big)v = \sqrt{h}Q_0(V) + h\Big[-\frac{i}{2}v + C_0(V)\Big] + h^{\frac{11}{8}}R_0^h(V) \tag{3.6}$$

where $V=(v,\bar{v})$,

$$\begin{aligned} Q_0(V) = &-\frac{i}{8}\,\mathrm{Op}_h(|\xi|^{\frac{1}{2}})\Big[\big(\mathrm{Op}_h(\xi|\xi|^{\frac{1}{2}})(v+\bar{v})\big)^2 + \big(\mathrm{Op}_h(|\xi|^{\frac{1}{2}})(v+\bar{v})\big)^2\Big]\\ &+\frac{i}{4}\,\mathrm{Op}_h(|\xi|)\big((v-\bar{v})\,\mathrm{Op}_h(|\xi|^{\frac{1}{2}})(v+\bar{v})\big)\\ &-\frac{i}{4}\,\mathrm{Op}_h(\xi)\big((v-\bar{v})\,\mathrm{Op}_h(\xi|\xi|^{-\frac{1}{2}})(v+\bar{v})\big), \end{aligned} \tag{3.7}$$

$C_0(V)$ stands for the cubic contribution

$$\begin{aligned} C_0(V) = &\ \frac{1}{8}\,\mathrm{Op}_h(|\xi|^{\frac{1}{2}})\Big[\Big(\mathrm{Op}_h\big(|\xi|^{\frac{1}{2}}\big)(v+\bar{v})\Big)\,\mathrm{Op}_h(|\xi|)\Big((v-\bar{v})\,\mathrm{Op}_h\big(|\xi|^{\frac{1}{2}}\big)(v+\bar{v})\Big)\Big]\\ &-\frac{1}{8}\,\mathrm{Op}_h\big(|\xi|^{\frac{1}{2}}\big)\Big[\Big(\mathrm{Op}_h\big(|\xi|^{\frac{1}{2}}\big)(v+\bar{v})\Big)\Big((v-\bar{v})\,\mathrm{Op}_h\big(|\xi|^{\frac{3}{2}}\big)(v+\bar{v})\Big)\Big]\\ &-\frac{1}{8}\,\mathrm{Op}_h(|\xi|)\Big[(v-\bar{v})\,\mathrm{Op}_h(|\xi|)\Big((v-\bar{v})\,\mathrm{Op}_h\big(|\xi|^{\frac{1}{2}}\big)(v+\bar{v})\Big)\Big]\\ &+\frac{1}{16}\,\mathrm{Op}_h(|\xi|)\Big[(v-\bar{v})^2\,\mathrm{Op}_h(|\xi|^{\frac{3}{2}})(v+\bar{v})\Big]\\ &+\frac{1}{16}\,\mathrm{Op}_h\big(|\xi|^2\big)\Big[(v-\bar{v})^2\,\mathrm{Op}_h(|\xi|^{\frac{1}{2}})(v+\bar{v})\Big] \end{aligned} \tag{3.8}$$

and where the remainder satisfies for $p = 2$ or ∞ and a small positive number θ, for any $d \in \mathbb{N}$, $k \in \mathbb{N}$ such that $\left\| \langle hD_x \rangle^{\alpha+d} \mathcal{Z}^k v(t,\cdot) \right\|_{L^2}$ and $\left\| \langle hD_x \rangle^{\alpha+d} \mathcal{Z}^k v(t,\cdot) \right\|_{L^\infty}$ are finite, the estimate

$$
\begin{aligned}
(3.9) \quad & \left\| \langle hD_x \rangle^d \mathcal{Z}^k \, |hD_x|^{-\frac{1}{2}+\theta} \, R_0^h(V) \right\|_{L^p} \\
& \leq C_k[v] h^{\frac{1}{16}} \sum_{\substack{k_1+k_2+k_3 \leq k \\ k_1,k_2 \leq k_3}} \prod_{j=1}^{2} \left\| \langle hD_x \rangle^{\alpha+d} \mathcal{Z}^{k_j} V \right\|_{L^\infty} \left\| \langle hD_x \rangle^{\alpha+d} \mathcal{Z}^{k_3} V \right\|_{L^p}
\end{aligned}
$$

where $C_k[v]$ depends on

$$
h^{\frac{1}{16}} \left\| \langle hD_x \rangle^{\alpha+d} \mathcal{Z}^{(k-1)_+} v \right\|_{L^\infty}
$$

and on a uniform bound for $\|v\|_{L^2}^{1-2\theta'} \, h^{\theta'} \, \|v\|_{L^\infty}^{2\theta'}$.

Actually (3.6), (3.7), (3.8) follow from (3.1), (3.2), (3.3). The remainder, estimated by (3.4) and (3.5), being at least quartic, would bring in factor a power $h^{3/2}$ in (3.6). We retained only the power $h^{11/8}$, to keep the extra $h^{1/16}$ factor in the right hand side of (3.9), and to keep also a $h^{1/16}$-factor in front of one of the $\left\| \mathcal{Z}^{k_j} v \right\|_{C^{d+\alpha}}$ in the right hand side of (3.4), (3.5). In that way, we obtain in (3.9) an estimate in terms of cubic expressions, modulo the indicated multiplicative constant. Notice also that the uniform bound assumed for $\|v\|_{L^2}^{1-2\theta'} \, h^{\theta'} \, \|v\|_{L^\infty}^{2\theta'}$ will be satisfied, when $\theta' > 0$ will have been fixed. Actually, we shall obtain a uniform control of $\|v\|_{L^\infty}$, and a bound of $\|v(t,\cdot)\|_{L^2}$ in $O(t^\delta)$ for some $\delta > 0$ as small as we want. Taking this δ smaller than θ' will provide the wanted uniformity. Finally, notice also that the fact that order zero pseudo-differential operators are not L^∞-bounded is harmless in deriving (3.9) with $p = \infty$ from (3.5), as we may always replace α by some larger value.

Our main task in the following subsections will be to deduce from equation (3.6) the oscillatory behavior of v when h goes to zero. We shall do that expressing v from Lagrangian distributions as those defined in the preceding section. This structure will be uncovered writing from (3.26) an equation for v involving only D_t derivatives. Actually, since $D_t = -ihZ - \mathrm{Op}_h(x\xi)$, we may write

$$
\mathrm{Op}_h\big(2x\xi + |\xi|^{\frac{1}{2}}\big) v = -\sqrt{h} Q_0(V) + h\Big[\frac{i}{2} v - iZV - C_0(V)\Big] - h^{\frac{11}{8}} R_0^h(V). \tag{3.10}
$$

From nom on, we consider $v(t,\cdot)$ as a family of functions of x indexed by $h = t^{-1} \in]0,1]$. We do not write explicitly the parameter h i.e. we write v instead of $(v_h)_h$. Let us introduce the Lagrangian submanifold given by the zero set of the symbol in the left hand side of (3.10) outside $\xi = 0$ i.e. set

$$
\begin{aligned}
\Lambda &= \Big\{ (x,\xi) \in T^*(\mathbb{R} \setminus \{0\}) \, ; \, 2x\xi + |\xi|^{\frac{1}{2}} = 0 \;\; \xi \neq 0 \Big\} \\
&= \big\{ (x, \mathrm{d}\,\omega(x)) \, ; \, x \in \mathbb{R}^* \big\}
\end{aligned} \tag{3.11}
$$

where

$$
\omega(x) = \frac{1}{4|x|}.
$$

In Section 6, we shall need the exact expressions of $Q_0(V)$, $C_0(V)$ given by (3.7), (3.8). Before that, we shall use only some less precise informations on the structure of these terms that we describe now.

From now on, we denote by $\mathcal{Z}$ the collection of vector fields $\mathcal{Z} = (Z, h\partial_x)$ and, if v is a distribution on $\mathbb{R}$, we define for any natural integer k the vector valued function

$$\mathcal{Z}^k v = (Z^{k_1}(h\partial_x)^{k_2} v)_{k_1+k_2\leq k}.$$

Lemma 3.1. *Let p be in $[1,+\infty]$.*

i) Denote by B_0 the symmetric bilinear form associated to the quadratic form Q_0. Let k be in $\mathbb{N}^*$, and for every couple $(k_1,k_2)\in\mathbb{N}\times\mathbb{N}$ with $k_1+k_2=k$, take p_{k_1}, p_{k_2} in $[1,+\infty]$ such that $\frac{1}{p_{k_1}}+\frac{1}{p_{k_2}}=\frac{1}{p}$. Then for any distributions $V_1=(v_1,\bar{v_1})$, $V_2=(v_2,\bar{v_2})$, any j_0,j_1,j_2 in $\mathbb{Z}$,

$$\begin{aligned}\big\|\Delta_{j_0}^h\mathcal{Z}^k B_0\big(\Delta_{j_1}^h V_1,\Delta_{j_2}^h V_2\big)\big\|_{L^p} &\leq C2^{j_0+\frac{1}{2}\min(j_1,j_2)}\mathbf{1}_{\max(j_1,j_2)\geq j_0-C}\\ &\quad\times\sum_{k_1+k_2\leq k}\big\|\mathcal{Z}^{k_1}\Delta_{j_1}^h V_1\big\|_{L^{p_{k_1}}}\big\|\mathcal{Z}^{k_2}\Delta_{j_2}^h V_2\big\|_{L^{p_{k_2}}}\end{aligned}\tag{3.12}$$

for some positive constant C. In the same way

$$\begin{aligned}&\big\|\mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big)\mathcal{Z}^k B_0\big(\Delta_{j_1}^h V_1,\Delta_{j_2}^h V_2\big)\big\|_{L^p}\\ &\qquad\leq Ch^{2(1-\sigma)}2^{\frac{1}{2}\min(j_1,j_2)}\\ &\qquad\quad\times\sum_{k_1+k_2=k}\big\|\mathcal{Z}^{k_1}\Delta_{j_1}^h V_1\big\|_{L^{p_{k_1}}}\big\|\mathcal{Z}^{k_2}\Delta_{j_2}^h V_2\big\|_{L^{p_{k_2}}}.\end{aligned}\tag{3.13}$$

ii) Let T_0 be the trilinear symmetric form associated to C_0. Then for any $k\in\mathbb{N}$, for some constant C,

$$\begin{aligned}&\big\|\Delta_{j_0}^h\mathcal{Z}^k T_0\big(\Delta_{j_1}^h V_1,\Delta_{j_2}^h V_2,\Delta_{j_3}^h V_3\big)\big\|_{L^p}\\ &\qquad\leq C2^{j_0/2+2\max(j_1,j_2,j_3)}\mathbf{1}_{\max(j_1,j_2,j_3)\geq j_0-C}\\ &\qquad\quad\times\sum_{k_1+k_2+k_3\leq k}\big\|\mathcal{Z}^{k_1}\Delta_{j_1}^h V_1\big\|_{L^{p_{k_1}}}\big\|\mathcal{Z}^{k_2}\Delta_{j_2}^h V_2\big\|_{L^{p_{k_2}}}\big\|\mathcal{Z}^{k_3}\Delta_{j_3}^h V_3\big\|_{L^{p_{k_3}}}\end{aligned}\tag{3.14}$$

where $\frac{1}{p_{k_1}}+\frac{1}{p_{k_2}}+\frac{1}{p_{k_3}}=\frac{1}{p}$.

In particular, for any d in $\mathbb{R}_+$, any p in $[1,+\infty]$, any $\alpha>2$

$$\begin{aligned}&\big\|\Delta_j^h\mathcal{Z}^k T_0(V_1,V_2,V_3)\big\|_{L^p}\\ &\qquad\leq C2^{j/2-j_+d}\sum_{\substack{k_1+k_2+k_3\leq k\\ k_1,k_2\leq k_3}}\prod_{\ell=1}^{2}\big\|\mathcal{Z}^{k_\ell}\langle hD_x\rangle^{\alpha+d}V_\ell\big\|_{L^\infty}\big\|\mathcal{Z}^{k_3}\langle hD_x\rangle^{\alpha+d}V_3\big\|_{L^p}.\end{aligned}\tag{3.15}$$

If, in the left hand side, one replaces Δ_j^h *by* $\mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big)$*, the same estimates hold with the factor* $2^{j/2-j_+d}$ *in the right hand side replaced by* $h^{1-\sigma}$*.*

iii) The remainder $R_0^h(V)$ *satisfies for any* $d\in\mathbb{R}_+$*, any* j *in* $\mathbb{Z}$*, with* $2^j \geq ch^{2(1-\sigma)}$ *estimates*

$$\begin{aligned}&\big\|\Delta_j^h \mathcal{Z}^k R_0^h(V)\big\|_{L^2}\\ &\quad\leq C2^{j/2-j_+d}\sum_{\substack{k_1+k_2+k_3\leq k\\ k_1,k_2\leq k_3}}\prod_{\ell=1}^{2}\big\|\mathcal{Z}^{k_\ell}\langle hD_x\rangle^{\alpha+d}V\big\|_{L^\infty}\big\|\mathcal{Z}^{k_3}\langle hD_x\rangle^{\alpha+d}V\big\|_{L^2}\end{aligned}\tag{3.16}$$

and

$$\big\|\Delta_j^h \mathcal{Z}^k R_0^h(V)\big\|_{L^\infty}\leq C2^{j/2-j_+d}\sum_{k_1+k_2+k_3\leq k}\prod_{\ell=1}^{3}\big\|\mathcal{Z}^{k_\ell}\langle hD_x\rangle^{\alpha+d}V\big\|_{L^\infty}\tag{3.17}$$

where C *depends only on* $h^{1/16}\big\|\langle hD_x\rangle^{\alpha+d}\mathcal{Z}^{(k-1)_+}V\big\|_{L^\infty}$ *for some large enough* $\alpha>0$*.*

If, in the left hand side of (3.16), (3.17)*,* Δ_j^h *is replaced by by* $\mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big)$*, similar estimates hold with* $2^{j/2-j_+d}$ *replaced by* $h^{1-\sigma}$*.*

Proof. $i)$ Consider the contribution to $B_0(V_1,V_2)$ of the first term in the right hand side of (3.7). Its Fourier transform may be written as the symmetrization of a multiple of

$$h^2\int\frac{|\xi h|^{\frac12}}{(h|\xi-\eta|)^{\frac12}(h|\eta|^{\frac12})}\big((\xi-\eta)\eta+|\xi-\eta|\,|\eta|\big)\widehat{f}_1(\xi-\eta)\widehat{f}_2(\eta)\,d\eta.$$

where $f_1=v_1+\bar{v}_1$, $f_2=v_2+\bar{v}_2$. On the support of the integrand, $(\xi-\eta)\eta\geq 0$ so that $|\xi|=|\xi-\eta|+|\eta|$. Consequently, the contribution of this term to $\Delta_{j_0}^h B_0\big(\Delta_{j_1}^h V_1,\Delta_{j_2}^h V_2\big)$ will be non zero only when $j_0\geq\max(j_1,j_2)-C$ for some $C>0$. In the same way, the contribution to $B_0(V_1,V_2)$ of the sum of the last two terms in (3.7) may be written, after Fourier transform an up to symmetries, as a multiple of

$$h^{\frac32}\int\frac{|\xi|\,|\eta|-\xi\eta}{|\eta|^{\frac12}}\widehat{f}_1(\xi-\eta)\widehat{f}_2(\eta)\,d\eta.$$

On the support of the integrand $\xi\eta\leq 0$, whence $|\xi-\eta|\geq\max(|\xi|,|\eta|)$ so that the contribution to

$$\Delta_{j_0}^h B_0\big(\Delta_{j_1}^h V_1,\Delta_{j_2}^h V_2\big)$$

will be non zero only if $j_2\leq j_1+C$ for some $C>0$. Using these inequalities and taking into account the distribution of the derivatives on the different factors, we conclude

$$\big\|\Delta_{j_0}^h B_0\big(\Delta_{j_1}^h V_1,\Delta_{j_2}^h V_2\big)\big\|_{L^2}\leq C2^{j_0+\frac12\min(j_1,j_2)}\big\|\Delta_{j_1}^h V_1\big\|_{L^{p_1}}\big\|\Delta_{j_2}^h V_2\big\|_{L^{p_2}}\tag{3.18}$$

if $\frac{1}{p_1}+\frac{1}{p_2}=\frac12$. Moreover, by spectral localization, we have always $\max(j_1,j_2)\geq j_0-C$ for some $C>0$. If one makes act $\mathcal{Z}^k$ on $B_0(V_1,V_2)$, the above properties of spectral localization are not affected since, if $a(\xi)$ is smooth outside zero, $[Z, \mathrm{Op}_h(a)] = -2\,\mathrm{Op}_h(\xi a'(\xi))$. Distributing the Z-derivatives on the different factors, one gets (3.12). The proof of (3.13) is similar.

ii) We notice first that in all contributions in (3.8), $\mathrm{Op}_h(|\xi|^{\frac{1}{2}})$ is always in factor. This allows to make appear the $2^{j_0/2}$-factor in (3.14). Since the sum of the powers of $|\xi|$ appearing in each term of (3.8) is equal to $5/2$, we get as well the factor $2^{2\max(j_1,j_2,j_3)}$ in (3.14). The cut-of for $\max(j_1,j_2,j_3) \geq j_0 - C$ follows from the spectral localization of each factor. Finally, making act $\mathcal{Z}^k$ on T_0 and commuting each vector field with $\mathrm{Op}_h(|\xi|^{\frac{1}{2}})$, $\mathrm{Op}_h(\xi\,|\xi|^{-\frac{1}{2}}), \dots$ we obtain (3.14).

To deduce (3.15) from (3.14), we decompose in the left hand side of (3.15),

$$V_\ell = \mathrm{Op}_h(\varphi_0(\xi))V_\ell + \sum_{j_\ell \geq 0} \Delta^h_{j_\ell} V_\ell.$$

Because of the spectral localization, we get for $j_\ell \geq 0$,

$$\begin{aligned}
\left\|Z^{k_\ell}\Delta^h_{j_\ell}V_\ell\right\|_{L^p} &\leq C2^{-j_\ell(\alpha+d)} \sum_{k'_\ell \leq k_\ell} \left\|Z^{k'_\ell}\langle hD_x\rangle^{\alpha+d}V_\ell\right\|_{L^p},\\
\left\|Z^{k_\ell}\,\mathrm{Op}_h(\varphi_0)V_\ell\right\|_{L^p} &\leq C \sum_{k'_\ell \leq k_\ell} \left\|Z^{k'_\ell}\langle hD_x\rangle^{\alpha+d}V_\ell\right\|_{L^p}.
\end{aligned}$$

We plug these estimates in (3.14) with $p_{k_1} = p_{k_2} = \infty$, $p_{k_3} = p$ and in the similar inequality where some $\Delta^h_{j_\ell}V_\ell$ is replaced by $\mathrm{Op}_h(\varphi_0)V_\ell$. We obtain a bound given by the product of the sum in the right hand side of (3.15) multiplied by

$$C2^{j/2} \sum_{\substack{\max(j_1,j_2,j_3)\geq j-C\\ j_\ell \geq 0}} 2^{2\max(j_1,j_2,j_3)-(j_1+j_2+j_3)(\alpha+d)}.$$

Since $\alpha > 2$, this is bounded by $C2^{j/2 - j_+ d}$ as wanted. The analogous statement, when Δ^h_j is replaced by $\mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big)$ in the left hand side of (3.15) is obtained in the same way.

iii) Inequalities (3.16), (3.17) follows from (3.9) with $p = 2$ or $p = \infty$, using that the loss $2^{-j\theta} \leq ch^{-2\theta(1-\sigma)}$ is absorbed by the extra $h^{1/16}$ factor in the right hand side of (3.9), if θ has been taken small enough. $\square$

Let us introduce the following decomposition of a solution v of (3.6). Fix σ, β some small positive numbers, φ_0 in $C_0^\infty(\mathbb{R})$ the function equal to one close to zero introduced before Definition 2.7. We decompose the solution v of (3.6) as

$$\begin{aligned}
v &= v_L + w + v_H,\\
v_L &= \mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)v,\\
v_H &= \mathrm{Op}_h\big((1-\varphi_0)(h^{-\beta}\xi)\big)v.
\end{aligned} \tag{3.19}$$

We notice that if C is a large enough constant, $w = v - v_L - v_H$ may be written $\sum_{j\in J(h,C)} \Delta^h_j w$.

Proposition 3.2. *Assume that for some $k \in \mathbb{N}$, some $a > b + \frac{3}{2} + \frac{1}{\beta} + \alpha$, some positive constants $\delta_k, \delta'_k, A_k, A'_k$ a solution v of (3.6) satisfies for any h in an interval $]h', 1]$, with $h' \in]0, 1]$ given, the a priori L^2-bounds*

$$\begin{aligned} &\big\|\mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big)\mathcal{Z}^k v\big\|_{L^2} \leq \varepsilon A_k h^{-\delta_k},\\ &\big\|\Delta_j^h \mathcal{Z}^k v\big\|_{L^2} \leq \varepsilon A_k h^{-\delta_k} 2^{-j+a} \quad \text{for } 2^j \geq C^{-1}h^{2(1-\sigma)} \end{aligned} \tag{3.20}$$

and the a priori L^∞-bounds

$$\begin{aligned} &\big\|\mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big)\mathcal{Z}^k v\big\|_{L^\infty} \leq \varepsilon A'_k h^{-\delta'_k},\\ &\big\|\Delta_j^h \mathcal{Z}^k v\big\|_{L^\infty} \leq \varepsilon A'_k h^{-\delta'_k} 2^{-j+b} \quad \text{for } 2^j \geq C^{-1}h^{2(1-\sigma)}. \end{aligned} \tag{3.21}$$

Then, if δ_k, are small enough, one gets that

$$h^{-\frac{3}{8}}\underline{v} = h^{-\frac{3}{8}}(v_L + v_H) \quad \text{belongs to an } \varepsilon\text{-neighborhood of } 0 \text{ in } \mathcal{R}^b_\infty, \tag{3.22}$$

with the notation introduced in Definition 2.8. Moreover, $w = v - \underline{v}$ satisfies

$$\big(D_t - \mathrm{Op}_h\big(x\xi + |\xi|^{\frac{1}{2}}\big)\big)w = \sqrt{h}Q_0(W) + h\Big[-\frac{i}{2}w + C_0(W)\Big] + h^{\frac{5}{4}}R_0(V) \tag{3.23}$$

where $\mathcal{Z}^k R_0(V)$ belongs to $\mathcal{R}^b_\infty$, and is an ε-neighborhood of zero in that space.

Notice, for further reference, that as we did for (3.10), we deduce from (3.23)

$$\mathrm{Op}_h\big(2x\xi + |\xi|^{\frac{1}{2}}\big)w = -\sqrt{h}Q_0(W) + h\Big[\frac{i}{2}w - iZw - C_0(W)\Big] - h^{\frac{5}{4}}R_0(V). \tag{3.24}$$

The proposition will be proved using the following lemma.

Lemma 3.3. *i) Assume that estimates (3.20), (3.21) hold. Then if $a > b + \frac{3}{2} + \frac{1}{\beta}$, $a > b + \alpha + 1 + \frac{1}{2\beta}$, $b > \alpha > 2$,*

$$\begin{aligned} &\big\|\mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big)\mathcal{Z}^k Q_0(V)\big\|_{L^\infty} \leq c_k\varepsilon^2 h^{\frac{3}{4}},\\ &\big\|\Delta_j^h \mathrm{Op}_h\big((1-\varphi_0)\big(h^{2\beta}\xi\big)\big)\mathcal{Z}^k Q_0(V)\big\|_{L^\infty} \leq c_k\varepsilon^2 h^{\frac{3}{4}} 2^{-j+b} \quad \text{for any } j, \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} &\big\|\mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big)\mathcal{Z}^k C_0(V)\big\|_{L^\infty} \leq c_k\varepsilon^3 h^{\frac{1}{4}},\\ &\big\|\Delta_j^h \mathrm{Op}_h\big((1-\varphi_0)\big(h^{2\beta}\xi\big)\big)\mathcal{Z}^k C_0(V)\big\|_{L^\infty} \leq c_k\varepsilon^3 h^{\frac{1}{4}} 2^{-j+b} \quad \text{for any } j, \end{aligned} \tag{3.26}$$

if δ_k, δ'_k in (3.21) are small enough and c_k is a convenient constant.

ii) Assume (3.20) *and the same inequalities on* a, b *as above. Then if* δ_k, δ'_k *are small enough*

$$
\begin{aligned}
&\big\|\mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big)\mathcal{Z}^k v\big\|_{L^\infty} \le c_k\varepsilon h^{\frac{7}{16}-\sigma},\\
&\big\|\Delta_j^h \,\mathrm{Op}_h\big((1-\varphi_0)\big(h^{2\beta}\xi\big)\big)\mathcal{Z}^k v\big\|_{L^\infty} \le c_k\varepsilon h^{\frac{7}{8}}2^{-j+b} \quad \textit{for any } j,\\
&\big\|2^{j\ell}\Delta_j^h \,\mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big)\mathcal{Z}^k v\big\|_{L^\infty} \le c_k\varepsilon h^{\frac{3}{8}-\sigma+2\ell(1-\sigma)}, \quad \ell \ge 0.
\end{aligned}
\tag{3.27}
$$

Moreover, if we assume (3.20) *and* (3.21),

$$
\begin{aligned}
\big\|\mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big)\mathcal{Z}^k\big[Q_0(V)-Q_0(W)\big]\big\|_{L^\infty}&\\
+\sup_{j\ge j_0(h,C)} 2^{j+b}\big\|\Delta_j^h\mathcal{Z}^k\big[Q_0(V)-Q_0(W)\big]\big\|_{L^\infty} &\le c_k\varepsilon^2 h^{\frac{3}{4}}
\end{aligned}
\tag{3.28}
$$

and

$$
\begin{aligned}
\big\|\mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big)\mathcal{Z}^k\big[C_0(V)-C_0(W)\big]\big\|_{L^\infty}&\\
+\sup_{j\ge j_0(h,C)} 2^{j+b}\big\|\Delta_j^h\mathcal{Z}^k\big[C_0(V)-C_0(W)\big]\big\|_{L^\infty} &\le c_k\varepsilon^3 h^{\frac{1}{4}}.
\end{aligned}
\tag{3.29}
$$

Proof. i) To obtain the first formula in (3.25) we use (3.13) with $p_{k_1} = p_{k_2} = p = \infty$. Using assumption (3.21) we get a bound of the left hand side by

$$
C\varepsilon^2 {A'_k}^2 h^{-2\delta'_k+2(1-\sigma)} \sum_{j_1,j_2\in\mathbb{Z}} 2^{\frac{1}{2}\min(j_1,j_2)-j_1+b-j_2+b}
$$

which gives the conclusion since $\sigma \in]0, 1/2[$ and we take δ'_k small enough. To get the second inequality (3.25) we use (3.12) with $p_{k_1} = p_{k_2} = p = \infty$, and we estimate the L^∞ norms in the right hand side using Sobolev injection and (3.20). We obtain

$$
C\varepsilon^2 A_k^2 h^{-2\delta_k} 2^j \sum_{\max(j_1,j_2)\ge j-C} 2^{\frac{1}{2}\min(j_1,j_2)-j_1+a-j_2+a+\frac{j_1}{2}+\frac{j_2}{2}} h^{-1}.
$$

If one uses that by assumption $2^j \ge ch^{-2\beta}$, and the fact that $a > b + \frac{3}{2} + \frac{1}{\beta}$, one gets the wanted estimate (for δ_k small enough).

To obtain (3.26), one substitutes inside (3.15) with $p = \infty$, $d = 0$,

$$
V_\ell = \mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big)V_\ell + \sum_{j_\ell\in\mathbb{Z}} \Delta_{j_\ell}^h \,\mathrm{Op}_h\big((1-\varphi_0)\big(h^{-2(1-\sigma)}\xi\big)\big)V_\ell
$$

one uses (3.21) to estimate two of the three factors of the right hand side, (3.20) and Sobolev injection to bound the third one, and one makes similar computations as above, exploiting that for the left hand side of the second estimate (3.26) not to vanish, it is necessary that one of the j_ℓ be larger than $j - C$, and the assumptions on a.

ii) Inequalities (3.27) follow from (3.20) and Sobolev injections, using the assumptions on a and the fact that $\delta_k < 1/16$.

We estimate the contribution to (3.28) corresponding to $j \in J(h,C)$. We write $Q_0(V) - Q_0(W)$ from $B_0(V-W,V)$ and $B_0(V-W,W)$. By (3.12), $\left\|\Delta_j^h \mathcal{Z}^k B_0(V-W,W)\right\|_{L^\infty}$ is smaller than

$$C2^j \sum\sum_{\max(j_1,j_2)\geq j-C} 2^{\frac{1}{2}\min(j_1,j_2)} \left\|\mathcal{Z}^k \Delta_{j_1}^h (V-W)\right\|_{L^\infty} \left\|\mathcal{Z}^k \Delta_{j_2}^h V\right\|_{L^\infty}.$$

By the definition of $w = v - v_L - v_H$, $\Delta_{j_1}^h(V-W)$ is non zero only if $2^{j_1} \lesssim h^{2(1-\sigma)}$ or $2^{j_1} \gtrsim h^{-2\beta}$. In the first case, we bound $\left\|\mathcal{Z}^k \Delta_{j_1}^h (V-W)\right\|_{L^\infty} 2^{j_1/4}$ using the third inequality (3.27) with $\ell = 1/4$. We get a bound in $O\big(\varepsilon h^{\frac{7}{8}-\frac{3\sigma}{2}}\big)$. In the second case $\left\|\mathcal{Z}^k \Delta_{j_1}^h (V-W)\right\|_{L^\infty}$ is $O\big(\varepsilon h^{\frac{7}{8}} 2^{-j+b}\big)$ by the second estimate (3.27). Using assumption (3.21) to estimate $\left\|\mathcal{Z}^k \Delta_{j_2}^h V\right\|_{L^\infty}$, we get a bound

$$C\varepsilon^2 h^{\frac{7}{8}-\delta_k'-\frac{3\sigma}{2}} 2^j \sum\sum_{\max(j_1,j_2)\geq j-C} 2^{\frac{j}{2}\min(j_1,j_2)-\frac{1}{4}j_1 -} 2^{-(j_{1+}+j_{2+})b} \leq C\varepsilon^2 2^{-j+(b-1)} h^{\frac{7}{8}-\delta_k'-\frac{3}{2}\sigma}.$$

Since $2^j \leq Ch^{-2\beta}$ with $\beta \ll 1$ and $\sigma \ll 1$, $\delta_k' \ll 1$, we obtain the wanted conclusion.

One studies in the same way the contributions of indices j in $J(h,C)$ to (3.29), expressing $C_0(V) - C_0(W)$ from $T_0(V-W,V,V)$ and from similar expressions and using (3.15).

To estimate the first term in the left hand side of (3.28), (3.29), or the contribution of $j \geq j_1(h,C)$ to the latter, we just need to apply (3.25), (3.26), and to notice that these inequalities remain true with V replaced by W. □

Proof of Proposition 3.2. Notice first that (3.22) follows from (3.27) if $\sigma \ll 1$. Denote $\Sigma_h = Id - \mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big) - \mathrm{Op}_h\big((1-\varphi_0)\big(h^{2\beta}\xi\big)\big)$ so that $w = \Sigma_h v$ by definition and $\underline{v} = (Id - \Sigma_h)v$. We notice that $\big[D_t - \mathrm{Op}_h\big(x\xi + |\xi|^{\frac{1}{2}}\big), \Sigma_h\big] = h\tilde{\Sigma}_h$, where $\tilde{\Sigma}_h$ may be written as a linear combination of quantities $\mathrm{Op}_h\big(\tilde{\varphi_0}\big(h^{-2(1-\sigma)}\xi\big)\big)$, $\mathrm{Op}_h\big(\tilde{\varphi_0}\big(h^{2\beta}\xi\big)\big)$ for new functions $\tilde{\varphi_0}$ in $C_0^\infty(\mathbb{R}^*)$. We deduce from (3.6)

$$\begin{aligned}\big(D_t - \mathrm{Op}_h\big(x\xi + |\xi|^{\frac{1}{2}}\big)\big)\underline{v} = \sqrt{h}&(Id - \Sigma_h)Q_0(V)\\ &+ h(Id-\Sigma_h)\Big(-\frac{i}{2} + C_0(V)\Big)\\ &+ h^{\frac{11}{8}}(Id-\Sigma_h)R_0^h(V)\\ &- h\tilde{\Sigma}_h v.\end{aligned}$$

By estimates (3.25), (3.26) and (3.27) the first, second and last terms of the right hand side may be written $h^{5/4}R(V)$ with $R(V)$ in $\mathcal{R}_\infty^b$.

To estimate the remainder term $R_0^h(V)$, we estimate its L^∞ norm using (3.17) with $d = b + \frac{1}{2}$.

The factors in the right hand side of (3.17) are estimated in the following way:

$$\begin{aligned}\big\|\langle hD_x\rangle^{\alpha+b+\frac{1}{2}}V\big\|_{L^\infty} \leq{}& C\big\|\mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big)V\big\|_{L^\infty}\\ &+C\sum_{j\in J(h,C)}\big\|2^{j_+(\alpha+b+\frac{1}{2})}\Delta_j^hV\big\|_{L^\infty}\\ &+C\sum_{j\geq j_1(h,C)}2^{j(\alpha+b+\frac{1}{2})}\left(\frac{2^j}{h}\right)^{\frac{1}{2}}\big\|\Delta_j^hV\big\|_{L^2},\end{aligned}$$

where we used the Sobolev injection for the last term. Using assumptions (3.21) and (3.20) for the right hand side, together with the fact that $2^j\leq Ch^{-2\beta}$ on the the first sum, $2^j>ch^{-2\beta}$ on the last one, and $a>\alpha+b+1+\frac{1}{\beta}$, we bound this quantity by say $Ch^{-\frac{1}{24}}$ (if δ_k,δ_k',β are small enough). It follows that

$$\big(D_t-\mathrm{Op}_h\big(x\xi+|\xi|^{\frac{1}{2}}\big)\big)w=\sqrt{h}Q_0(V)+h\Big[-\frac{i}{2}v+C_0(V)\Big]+h^{\frac{5}{4}}R(V)$$

with $R(V)$ in $\mathcal{R}_\infty^b$. Using (3.27), (3.28), (3.29) we may replace the right-hand side of this equation by the right hand side of (3.23) up to a modification of $R(V)$. If we make act the $\mathcal{Z}$-family of vector fields on (3.23), and use the commutation relations

$$\begin{aligned}&\big[t\partial_t+x\partial_x,D_t-\mathrm{Op}_h\big(x\xi+|\xi|^{\frac{1}{2}}\big)\big]=-\big(D_t-\mathrm{Op}_h\big(x\xi+|\xi|^{\frac{1}{2}}\big)\big),\\ &\big[h\partial_x,D_t-\mathrm{Op}_h\big(x\xi+|\xi|^{\frac{1}{2}}\big)\big]=0,\end{aligned}\tag{3.30}$$

we obtain in the same way the estimates involving $\mathcal{Z}^k$ derivatives. ☐

4 Weak L^∞ estimates

The goal of this subsection is to show that if v is a solution to equation (3.6), and if we are given an L^2-control of $Z^{k+1}v$ of type $\big\|Z^{k+1}v\big\|_{L^2}=O\big(h^{-\delta_{k+1}}\big)$ for some small $\delta_{k+1}>0$, we can deduce from it and the equation an L^∞-bound of the form $\big\|Z^kv\big\|_{L^\infty}=O\big(h^{-\delta_k'}\big)$ for some small $\delta_k'\gg\delta_{k+1}$. Actually, we shall get as well bounds for $\langle hD_x\rangle^bv$ instead of v for some given $b>0$.

These bounds are not good enough, but they will be the starting point of the more elaborated reasoning that will be pursued in Sections 5 and 6. Before stating the main result, we fix some notation.

Assume given integers $s\gg N_1\gg N_0\gg 1$ and an increasing sequence of positive numbers $(\delta_k)_{0\leq k\leq s/2+N_1+1}$. We consider another increasing sequence $(\delta_k')_{0\leq k\leq\frac{s}{2}+N_1}$ satisfying the inequalities

$$\begin{aligned}&\delta_k'>\sum_{j=0}^{\ell'}\delta_{k_j}'+\sum_{j=\ell'+1}^{\ell}\delta_{k_\ell+1} &&\text{if }\begin{cases}0\leq\ell'\leq\ell\leq 4,\ \Sigma_{j=0}^{\ell}k_j\leq k,\\ k_j<k\text{ when }0\leq j\leq\ell',\end{cases}\\ &\delta_k'>\delta_{k+1}+2\delta_0+4\delta_0' &&\text{if }k\geq 1,\end{aligned}\tag{4.1}$$

for $k = 0, \dots, \frac{s}{2} + N_1$. Clearly such a sequence $(\delta'_k)_k$ may always be constructed by induction, and if δ_s is small enough, we may assume moreover that

$$\delta_k < \frac{1}{32} \quad k = 0, \dots, \frac{s}{2} + N_1 + 1,\ \delta'_k < \frac{\sigma}{8} < \frac{1}{32}\ k = 0, \dots, \frac{s}{2} + N_1. \tag{4.2}$$

We assume that the positive number β introduced in (2.12) is small enough so that $2\beta(\alpha+\frac{1}{2}) < \frac{1}{8}$, where $\alpha > 2$ is the fixed large enough number introduced in (3.4) and (3.15), and that $\beta < \sigma/2$. We fix positive numbers $a > b > b' > b''$ such that

$$\begin{aligned} &a > b + \frac{3}{2} + \frac{1}{\beta} + \alpha, \quad b > \frac{1}{2}, \\ &(b - b')\beta > 2, \quad (b' - b'')\beta > 2. \end{aligned} \tag{4.3}$$

In that way, the assumptions of Proposition 3.2 will be fulfilled.

For k a nonnegative integer, we define

$$\mathcal{E}_k(v) = \sum_{k'=0}^{k} \max\Big(\big\|\mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big)\mathcal{Z}^{k'}v\big\|_{L^\infty}, \sup_{j \geq j_0(h,C)} 2^{j+b}\big\|\Delta_j^h \mathcal{Z}^{k'}v\big\|_{L^\infty}\Big) \tag{4.4}$$

and

$$\mathcal{F}_k(v) = \sum_{k'=0}^{k} \max\Big(\big\|\mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big)\mathcal{Z}^{k'}v\big\|_{L^2}, \sup_{j \geq j_0(h,C)} 2^{j+a}\big\|\Delta_j^h \mathcal{Z}^{k'}v\big\|_{L^2}\Big). \tag{4.5}$$

Let $k \in \mathbb{N}^*$. We denote by $\mathcal{T}_k^\infty$ the set of functions $v \mapsto P_k(v)$ satisfying for any $v = (v_h)_h$ with $\mathcal{E}_{k-1}(v) \leq h^{-1/4}$ a bound of type

$$\begin{aligned} |P_k(v)| \leq C\Big[&\mathcal{E}_k(v) + \sum_{k_1+k_2 \leq k} \mathcal{E}_{k_1}(v)\mathcal{E}_{k_2}(v) \\ &+ \sum_{k_1+k_2+k_3 \leq k} \mathcal{E}_{k_1}(v)\mathcal{E}_{k_2}(v)\mathcal{E}_{k_3}(v)\Big]. \end{aligned} \tag{4.6}$$

In the same way, we define $\mathcal{T}_k$ as the set of functions $v \mapsto P_k(v)$ admitting for any $v = (v_h)_h$ with $\mathcal{E}_{k-1}(v) \leq h^{-1/4}$ a bound of type

$$|P_k(v)| \leq C \sum_{1 \leq \ell \leq 4} \sum_{k_1+\cdots+k_\ell \leq k} \sum_{\ell'=0}^{\ell} \prod_{j=1}^{\ell'} \mathcal{E}_{k_j}(v) \prod_{j=\ell'+1}^{\ell} \mathcal{F}_{k_{j+1}}(v). \tag{4.7}$$

The main result of this section is the following one.

Proposition 4.1. *Let k be a positive integer. Assume that we are given constants $\tilde{A}_0$, A_0, $A_1, \dots, A_{k+1}$ and a solution v of (3.6) such that for h in some interval $]h', 1]$*

$$\mathcal{E}_0(v) \leq \tilde{A}_0 h^{-\delta'_0}, \quad \mathcal{F}_{k'}(v) \leq \varepsilon A_{k'} h^{-\delta_{k'}}, \quad 0 \leq k' \leq k+1. \tag{4.8}$$

Then, there are $h_0 > 0$, $A'_{k'} > 0$, $k' = 1, \dots, k$, depending only on $\varepsilon \tilde{A}_0, A_1, \dots, A_{k+1}$ such that for any h in $]h', 1]$

$$\mathcal{E}_{k'}(v) \leq \varepsilon A'_{k'} h^{-\delta'_{k'}}, \quad k' = 1, \dots, k. \tag{4.9}$$

Remark 4.2. • The result of the preceding proposition may be thought of as a "Klainerman-Sobolev" estimate, that allows one to get L^∞-decay from L^2-bounds (There is no decay involved in (4.9) since the negative power of time $t^{-1/2} = \sqrt{h}$ has been factored out when we defined v from u).

The proof of the proposition will be made in three steps.

First, we treat the case of small or large frequencies, for which we deduce (4.9) from the L^2-estimate in (4.8) and Sobolev injection.

Next, we are reduced to intermediate frequencies i.e. to $\Delta_j^h v$ with j belonging to $J(h, C)$. We write the equation for $\Delta_j^h v$ coming from (3.10). The operator of symbol $2x\xi + |\xi|^{\frac{1}{2}}$ is elliptic outside the Lagrangian Λ defined in (3.11). Since the right hand side of (3.10) is $O(h^{1/2-0})$, one will get for the L^∞-norm of $\Delta_j^h \mathcal{Z}^k v$ cut-off outside a neighborhood of Λ some $O(h^{1/2-0})$ estimates, that are better than what we want.

In the last step, we decompose in the quadratic part $Q_0(V)$ of the right hand side of (3.10), v as the sum of the contribution microlocalized outside Λ, which by the preceding step will give an $O(h^{1-0})$ contribution to (3.10), and a contribution microlocalized close to Λ. The quadratic interactions between the latter will be microlocally supported close to $2 \cdot \Lambda$, $0 \cdot \Lambda$, $-2 \cdot \Lambda$ where

$$\lambda \cdot \Lambda = \big\{(x, \lambda\xi)\,;\,(x, \xi) \in \Lambda\big\}.$$

Consequently, if we microlocalize (3.10) close to Λ, which does not meet $\pm 2 \cdot \Lambda$, $0 \cdot \Lambda$, the $\sqrt{h}$-terms of the right hand side disappear, and we get an $O(h^{1-0})$ estimate for the L^2-norm of the left hand side. This allows to deduce the wanted L^∞-estimate from a Sobolev embedding, after reduction of Λ to the zero section, through a canonical transformation.

First step: Low and large frequencies

We decompose $v = v_L + w + v_H$ according to (3.19). By assumption (4.8), estimates (3.20) hold. Then $ii)$ of Lemma 3.3 implies that v_L, v_H satisfy the first two inequalities (3.27).

Second step: Elliptic estimates for w outside a neighborhood of Λ.

We define, for $j \in J(h, C)$ with C large enough

$$\begin{aligned} &w_j = \Theta^*_{-j} \Delta_j^h w, \\ &\mathcal{Z}_j = \Theta^*_{-j} \mathcal{Z} \Theta^*_j = (Z, 2^{j/2} h \partial_x), \quad \mathcal{Z}_j^k = (Z^{k_1} (2^{j/2} h \partial_x)^{k_2})_{k_1 + k_2 \leq k}, \end{aligned} \tag{4.10}$$

so that $w = \sum_{j \in j(h,C)} \Theta^*_j w_j$.

Let $\Phi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ be equal to one on a domain $C^{-1} \leq |\xi| \leq C$ for a large constant C and let Γ be in $C_0^\infty(\mathbb{R})$, with small enough support, equal to one close to zero. We define

$$\begin{aligned}\gamma_\Lambda(x,\xi) &= \Phi(\xi)\Gamma\big(2x\xi + |\xi|^{\frac{1}{2}}\big),\\ \gamma_\Lambda^c(x,\xi) &= \Phi(\xi)\big(1 - \Gamma\big(2x\xi + |\xi|^{\frac{1}{2}}\big)\big).\end{aligned} \tag{4.11}$$

We obtain two symbols of $S(1,F)$ where $F = \{\xi\,;\, C'^{-1} \leq |\xi| \leq C'\}$ for a large enough C'. Moreover, since $2x\xi + |\xi|^{1/2} = 0$ is an equation of Λ, we see that on the domain where $\Phi \equiv 1$, γ_Λ (resp. γ_Λ^c) cuts-off close to Λ (resp. outside a neighborhood of Λ). We shall prove the following estimates.

Proposition 4.3. *Let $k \geq 1$, $N \in \mathbb{N}$. We denote by κ some fixed small enough positive number (say $\kappa = 1/24$). There is a constant $C_k > 0$, an element P_k of $\mathcal{T}_k$ such that for any v satisfying $\mathcal{E}_{k-1}(v) \leq h^{-1/16}$, one has for any j in $j(h,C)$,*

$$\begin{aligned}&\big\|\mathrm{Op}_{h_j}(\gamma_\Lambda^c)\mathcal{Z}_j^k w_j\big\|_{L^3}\\ &\leq C_k\Big[\sqrt{h}2^{2j/3-j_+(b+\frac{1}{3}(a-b)}\sum_{k_1+k_2\leq k}\prod_{\ell=1}^{2}\mathcal{F}_{k_\ell}(v)^{\frac{1}{3}}\mathcal{E}_{k_\ell}(v)^{\frac{2}{3}}\\ &\quad + h2^{j/6-j_+b}h^{-4\beta-0}\sum_{k_1+k_2+k_3\leq k}[\mathcal{F}_{k_1}(v)\mathcal{F}_{k_2}(v)\mathcal{E}_{k_3}(v) + \mathcal{E}_{k_1}(v)\mathcal{E}_{k_2}(v)\mathcal{E}_{k_3}(v)]\\ &\quad + 2^{j/4-j_+a}h_j^{5/6}\mathcal{F}_{k+1}(v) + 2^{j/6-j_+b}h^{1+\kappa}P_k(v)\\ &\quad + 2^{j/6-j_+(b+\frac{2}{3}(a-b))}h_j^N\mathcal{F}_k(v)^{2/3}\mathcal{E}_k(v)^{1/3}\Big]\end{aligned} \tag{4.12}$$

where $h^{-4\beta-0}$ means a bound in $C_\theta h^{-4\beta-\theta}$ for any $\theta > 0$.

To prove the proposition, we need to estimate the action of $\mathrm{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big)$ on $\mathcal{Z}_j^k w$ in various spaces.

Lemma 4.4. *i) Let $k \geq 1$. There are an element P_k of $\mathcal{T}_k$, a matrix $A(h_j)$ with uniformly bounded coefficients, a constant $C_k > 0$ such that, for any v satisfying $\mathcal{E}_{k-1}(v) \leq h^{-1/16}$ when h stays in some interval $]h',1]$, one gets for any h in that interval, any j in $J(h,C)$,*

$$\begin{aligned}&\big\|\mathrm{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big)\mathcal{Z}_j^k w_j - 2^{-j/2}\sqrt{h}A(h_j)\mathcal{Z}_j^k\Theta_{-j}^*\Delta_j^h Q_0(W)\big\|_{L^2}\\ &\leq C_k 2^{j/4-j_+(b+\frac{1}{2})}\Big[\sum_{k_1+k_2+k_3\leq k}\prod_{\ell=1}^{3}\mathcal{F}_{k_\ell}(v)^{1/3}\mathcal{E}_{k_\ell}(v)^{2/3}h^{1-0}\\ &\quad + h^{9/8}P_k(v) + h^{1-0}2^{-j/2}\mathcal{F}_{k+1}(v)\Big].\end{aligned} \tag{4.13}$$

ii) Under the preceding assumptions, we get as well

$$
\begin{aligned}
&\big\|\mathrm{Op}_{h_j}\big(2x\xi+|\xi|^{\frac12}\big)\mathcal{Z}_j^k w_j\big\|_{L^3}\\
&\quad\le C_k\Big[\sqrt{h}2^{2j/3-j_+(b+\frac13(a-b))}\sum_{k_1+k_2\le k}\prod_{\ell=1}^{2}\mathcal{F}_{k_\ell}(v)^{\frac13}\mathcal{E}_{k_\ell}(v)^{\frac23}\\
&\qquad+h2^{j/6-j_+b}h^{-4\beta-0}\sum_{k_1+k_2+k_3\le k}\big[\mathcal{F}_{k_1}(v)\mathcal{F}_{k_2}(v)\mathcal{E}_{k_3}(v)+\mathcal{E}_{k_1}(v)\mathcal{E}_{k_2}(v)\mathcal{E}_{k_3}(v)\big]\\
&\qquad+2^{j/4-j_+a}h_j^{5/6}\mathcal{F}_{k+1}(v)+2^{j/6-j_+b}h^{1+\kappa}P_k(v)\Big].
\end{aligned}\tag{4.14}
$$

iii) Under the preceding assumptions

$$
\begin{aligned}
&\big\|\mathrm{Op}_{h_j}\big(2x\xi+|\xi|^{\frac12}\big)\mathcal{Z}_j^k w_j-2^{-j/2}\sqrt{h}A(h_j)\mathcal{Z}_j^k\Theta_{-j}^*\Delta_j^hQ_0(W)\big\|_{L^\infty}\\
&\qquad\le C_kh^{1-0}2^{-j_+(b-2)}\sum_{k_1+k_2+k_3\le k}\mathcal{E}_{k_1}(v)\mathcal{E}_{k_2}(v)\mathcal{E}_{k_3}(v)\\
&\qquad\quad+C_kh_j2^{-j_+b}\mathcal{E}_{k+1}(v)
\end{aligned}\tag{4.15}
$$

and

$$
\big\|2^{-j/2}\sqrt{h}A(h_j)\mathcal{Z}_j^k\Theta_{-j}^*\Delta_j^hQ_0(W)\big\|_{L^\infty}\le\sqrt{h}2^{-j_+(b-\frac12)}\sum_{k_1+k_2\le k}\mathcal{E}_{k_1}(v)\mathcal{E}_{k_2}(v).\tag{4.16}
$$

Proof. We apply Δ_j^h to equation (3.24). Denoting $\widetilde{\Delta}_j^h=\widetilde{\varphi}(2^{-j}hD_x)$ for a new smooth function $\widetilde{\varphi}$ satisfying $\operatorname{Supp}\widetilde{\varphi}\subset\operatorname{Supp}\varphi$, we get

$$
\begin{aligned}
\mathrm{Op}_h\big(2x\xi+|\xi|^{\frac12}\big)\Delta_j^hw&=-\sqrt{h}\Delta_j^hQ_0(W)\\
&\quad+h\Big[\frac{i}{2}\Delta_j^hw-i\widetilde{\Delta}_j^hw-iZ\Delta_j^hw-\Delta_j^hC_0(W)\Big]\\
&\quad-h^{5/4}\Delta_j^hR(V).
\end{aligned}
$$

Applying Θ_{-j}^* and using (2.13), we get

$$
\begin{aligned}
\mathrm{Op}_{h_j}\big(2x\xi+|\xi|^{\frac12}\big)w_j&=-\sqrt{h}2^{-j/2}\Theta_{-j}^*\Delta_j^hQ_0(W)\\
&\quad+h2^{-j/2}\Big[\frac{i}{2}\widetilde{w}_j-iZw_j-\Theta_{-j}^*\Delta_j^hC_0(W)\Big]\\
&\quad-2^{-j/2}h^{5/4}\Theta_{-j}^*\Delta_j^hR(V)
\end{aligned}\tag{4.17}
$$

where $\widetilde{w}_j=w_j-2\Theta_{-j}^*\widetilde{\Delta}_j^hw$ satisfies the same estimates as w_j. We commute $\mathcal{Z}_j^k$ to the equation, using that

$$
\begin{aligned}
\big[Z,\mathrm{Op}_{h_j}\big(2x\xi+|\xi|^{\frac12}\big)\big]&=-\mathrm{Op}_{h_j}\big(2x\xi+|\xi|^{\frac12}\big),\\
\big[2^{j/2}h\partial_x,\mathrm{Op}_{h_j}\big(2x\xi+|\xi|^{\frac12}\big)\big]&=-2ih_j\big(2^{j/2}h\partial_x\big).
\end{aligned}
$$

We get
(4.18)

$$\mathrm{Op}_{h_j}\big(2x\xi+|\xi|^{\frac12}\big)\big(\mathcal{Z}_j^k w_j\big) = 2^{-j/2}\sqrt{h}A(h_j)\mathcal{Z}_j^k\Theta_{-j}^*\Delta_j^h Q_0(W) + h_j\Big[\widetilde{B}(h_j)\mathcal{Z}_j^k\widetilde{w}_j + B(h_j)\mathcal{Z}_j^{k+1}w_j + C(h_j)\mathcal{Z}_j^k\Theta_{-j}^*\Delta_j^h C_0(W)\Big] - h^{1/4}h_j D(h_j)\mathcal{Z}_j^k\Theta_{-j}^*\Delta_j^h R_0^h(V)$$

where $A(h_j), B(h_j), \widetilde{B}(h_j), C(h_j), D(h_j)$ are matrices with uniformly bounded coefficients.

Let us control the cubic terms in (4.18). We write $C_0(W)$ as $T_0(W,W,W)$ as in *ii*) of Lemma 3.1. We express $\mathcal{Z}_j^k\Theta_{-j}^*\Delta_j^h C_0(W) = \Theta_{-j}^*\mathcal{Z}^k\Delta_j^h C_0(W)$ from $\Theta_{-j}^*\Delta_j^h\mathcal{Z}^{k'}T_0(W,W,W)$ for $k'\leq k$ (changing eventually the definition of the spectral cut-off Δ_j^h) and decompose each argument W as $\sum_{j_\ell}\Delta_{j_\ell}^h W$. Applying estimate (3.14) with $p_{k_1}=p_{k_2}=p_{k_3}=6$ and writing $\|\cdot\|_{L^6}\leq\|\cdot\|_{L^2}^{1/3}\|\cdot\|_{L^\infty}^{2/3}$, we obtain

$$\begin{aligned}\big\|\Delta_j^h\mathcal{Z}^{k'}T_0(W,W,W)\big\|_{L^2} &\leq C2^{j/2}\sum_{j_1,j_2,j_3}\sum_{k_1+k_2+k_3\leq k'}2^{2\max(j_1,j_2,j_3)}\mathbf{1}_{\max(j_1,j_2,j_3)\geq j-C}\\ &\quad\times\prod_{\ell=1}^3\big\|\mathcal{Z}^{k_\ell}\Delta_{j_\ell}^h W\big\|_{L^2}^{1/3}\big\|\mathcal{Z}^{k_\ell}\Delta_{j_\ell}^h W\big\|_{L^\infty}^{2/3}.\end{aligned}\tag{4.19}$$

Using (4.4), (4.5), we bound the last factor by

$$2^{-(j_{1+}+j_{2+}+j_{3+})[b+\frac13(a-b)]}\prod_{\ell=1}^3\mathcal{F}_{k_\ell}(W)^{1/3}\mathcal{E}_{k_\ell}(W)^{2/3}.$$

Summing in j_1,j_2,j_3 in $J(h,C)$, we obtain

$$\big\|\Delta_j^h\mathcal{Z}^{k'}C_0(W)\big\|_{L^2}\leq C|\log h|^2 2^{j/2-j_+(b-2+\frac13(a-b))}\times\sum_{k_1+k_2+k_3\leq k'}\prod_\ell\mathcal{F}_{k_\ell}^{1/3}\mathcal{E}_{k_\ell}^{2/3}.$$

Remembering that $\big\|\Theta_{-j}^*\big\|_{\mathcal{L}(L^2,L^2)}=O(2^{j/4})$ we conclude that the L^2-norm of the cubic term in the right hand side of (4.18) is bounded from above by the first term in right hand side of (4.13) (since $a-b$ is large enough for $\beta\ll 1$, according to (4.3)).

To estimate the R_0-term in (4.18), we use (3.16). We notice first that the right hand side of this inequality may be controlled from $\mathcal{E}_{k_\ell}(v)$, $\mathcal{F}_{k_\ell}(v)$: actually

$$\big\|\mathcal{Z}^{k_\ell}\langle hD_x\rangle^{\alpha+d}V\big\|_{L^\infty}\leq C\sum_{k_\ell'\leq k_\ell}\Big[\big\|\mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\big)\mathcal{Z}^{k_\ell'}v\big\|_{L^\infty} + \sum_{j\geq j_0(h,C)}2^{j_+(\alpha+d)}\big\|\Delta_j^h\mathcal{Z}^{k_\ell'}v\big\|_{L^\infty}\Big],$$

where the constant C depends only on $h^{\frac{1}{16}}\big\|\langle hD_x\rangle^{\alpha+d}\mathcal{Z}^{(k-1)_+v}\big\|_{L^\infty}$.

We shall take $d=b-\alpha-0$, so that the bounds (4.4) imply that the j-series converges and gives a bound

$$\big\|\mathcal{Z}^{k_\ell}\langle hD_x\rangle^{\alpha+d}V\big\|_{L^\infty}\le C\mathcal{E}_{k_\ell}(v)|\log h|. \tag{4.20}$$

Similarly, we get

$$\big\|\mathcal{Z}^{k_\ell}\langle hD_x\rangle^{\alpha+d}V\big\|_{L^2}\le C\mathcal{F}_{k_\ell}(v)|\log h|. \tag{4.21}$$

Plugging this in (3.16), we obtain, using again that $\big\|\Theta^*_{-j}\big\|_{\mathcal{L}(L^2,L^2)}=O(2^{j/4})$,

$$h^{1/4}h_j\big\|\mathcal{Z}^k_j\Theta^*_{-j}\Delta^h_jR^h_0(V)\big\|_{L^2}\le Ch^{3/4-0}h_j^{1/2}2^{j/2-j_+(b-\alpha-0)}\sum_{k_1+k_2+k_3\le k}\mathcal{E}_{k_1}(v)\mathcal{E}_{k_2}(v)\mathcal{F}_{k_3}(v).$$

We notice that since $2^j=O(h^{-2\beta})$, we may bound $2^{-j_+(b-\alpha-0)}$ by $2^{-j_+(b+\frac12)}h^{-2(\alpha+\frac12+0)\beta}$. For β small enough, this negative power of h will be compensated consuming a $O(h^{1/8})$-factor, so that we end up with a bound of the remainder in (4.18) by

$$Ch^{9/8}2^{j/4-j_+(b+\frac12)}\sum_{k_1+k_2+k_3\le k}\mathcal{E}_{k_1}(v)\mathcal{E}_{k_2}(v)\mathcal{F}_{k_3}(v)$$

so by the P_k-term in the right hand side of (4.13).

Finally, the linear terms in the right hand side of (4.18) are bounded by the last contribution in (4.13), remembering that w_j may be expressed from Δ^h_jw by (4.10) and that $\big\|\Theta^*_{-j}\big\|_{\mathcal{L}(L^2,L^2)}=O(2^{j/4})$. This concludes the proof of $i)$ of the lemma.

$ii)$ To prove (4.14), let us bound the L^3-norm of the right hand side of (4.18). We express first $\mathcal{Z}^k_j\Theta^*_{-j}\Delta^h_jQ_0(W)$ from $\Theta^*_{-j}\Delta^h_j\mathcal{Z}^{k'}Q_0(W)$, $k'\le k$ write $Q_0(W)=B_0(W,W)$, decompose $W=\sum_{j_\ell}\Delta^h_{j_\ell}W$ in each factor and apply (3.12) with $p=3$, $p_{k_1}=p_{k_2}=6$. We get

$$\begin{aligned}\big\|\Delta^h_j\mathcal{Z}^{k'}Q_0(W)\big\|_{L^3}\le C2^j\sum_{j_1,j_2\in J(h,C)}\sum_{k_1+k_2\le k'}&2^{\frac12\min(j_1,j_2)}\mathbf{1}_{\max(j_1,j_2)\ge j-C}\\&\times\prod_{\ell=1}^2\big\|\mathcal{Z}^{k_\ell}\Delta^h_{j_\ell}W\big\|_{L^2}^{1/3}\big\|\mathcal{Z}^{k_\ell}\Delta^h_{j_\ell}W\big\|_{L^\infty}^{2/3}.\end{aligned}$$

Using (4.4), (4.5), we see that this quantity is smaller than

$$2^j\sum_{k_1+k_2\le k}\Big(\prod_{\ell=1}^2\mathcal{F}_{k_\ell}(v)^{\frac13}\mathcal{E}_{k_\ell}(v)^{\frac23}\Big)2^{-j_+(b+\frac13(a-b))}.$$

We thus get the first term in the right hand side of (4.14), using that $\big\|\Theta^*_{-j}\big\|_{\mathcal{L}(L^3,L^3)}=O(2^{j/6})$.

Let us study next the L^3-norm of the cubic term in (4.18). We proceed as in the proof of (4.19), applying (3.14) with $p=3$, $p_{k_1}=p_{k_2}=6$, $p_{k_3}=\infty$. We obtain

$$\begin{aligned}\big\|\Delta_j^h \mathcal{Z}^{k'} C_0(W)\big\|_{L^3} \leq C 2^{j/2} \sum_{j_1,j_2,j_3} \sum_{k_1+k_2+k_3\leq k'} & 2^{2\max(j_1,j_2,j_3)} \mathbf{1}_{\max(j_1,j_2,j_3)\geq j-C} \\ & \times \Big(\prod_{\ell=1}^{2} \big\|\mathcal{Z}^{k_\ell}\Delta_{j_\ell}^h W\big\|_{L^2}^{1/3} \big\|\mathcal{Z}^{k_\ell}\Delta_{j_\ell}^h W\big\|_{L^\infty}^{2/3}\Big) \big\|\mathcal{Z}^{k_3}\Delta_{j_\ell}^h W\big\|_{L^\infty}.\end{aligned}$$

We bound the general term of this sum by

$$C 2^{2\max(j_1,j_2,j_3)} \mathbf{1}_{\max(j_1,j_2,j_3)\geq j-C} 2^{-(j_{1+}+j_{2+}+j_{3+})b} \times \Big(\mathcal{F}_{k_1}(v)\mathcal{F}_{k_2}(v)\Big)^{1/3} \Big(\mathcal{E}_{k_1}(v)\mathcal{E}_{k_2}(v)\Big)^{2/3} \mathcal{E}_{k_3}(v).$$

As $2^j \leq Ch^{-2\beta}$, we conclude, using the convexity inequality $a^{1/3}b^{2/3} \leq (a+2b)/3$

$$\big\|\Delta_j^h \mathcal{Z}^{k'} C_0(W)\big\|_{L^3} \leq C h^{-4\beta-0} 2^{j/2-j_+b} \sum_{k_1+k_2+k_3\leq k} [\mathcal{F}_{k_1}(v)\mathcal{F}_{k_2}(v)\mathcal{E}_{k_3}(v) + \mathcal{E}_{k_1}(v)\mathcal{E}_{k_2}(v)\mathcal{E}_{k_3}(v)].$$

This gives in (4.14) a contribution to the second term in the right hand side, using again that $\big\|\Theta_{-j}^*\big\|_{\mathcal{L}(L^3,L^3)}$ is $O(2^{j/6})$.

Consider next the remainder. We estimate $\big\|\mathcal{Z}_j^k \Theta_{-j}^* \Delta_j^h R_0^h(V)\big\|_{L^3}$ from

$$h_j^{-1/6} \big\|\mathcal{Z}_j^k \Theta_{-j}^* \Delta_j^h R_0^h(V)\big\|_{L^2}$$

using that the expression to be bounded is spectrally supported in a ball of radius $O(h_j^{-1})$. We apply next estimate (3.16) together with (4.20), (4.21). We obtain, using that $\big\|\Theta_{-j}^*\big\|_{\mathcal{L}(L^3,L^3)} = O(2^{j/6})$, that the L^3-norm of the last term in (4.18) is bounded from above by

$$C h^{1/4-0} h_j^{5/6} 2^{2j/3-j_+d} \sum_{k_1+k_2+k_3\leq k} \mathcal{E}_{k_1}(v)\mathcal{E}_{k_2}(v)\mathcal{F}_{k_3}(v)$$

with $d=b-\alpha-0$. We get finally as a coefficient $2^{j/6-j_+b} h^{13/12-0-2\beta(\alpha+1/12)}$ if we use again that $2^j = O(h^{-2\beta})$. If β is small enough, we see that the remainder in (4.18) contributes to the last term in the right hand side of (4.14).

Finally, the contribution of the linear terms in (4.18) is bounded from above by

$$\begin{aligned} h_j\big\|\mathcal{Z}_j^k \widetilde{w}_j\big\|_{L^3} + h_j\big\|\mathcal{Z}_j^{k+1} w_j\big\|_{L^3} &\leq C h_j^{5/6}\Big(\big\|\mathcal{Z}_j^k \widetilde{w}_j\big\|_{L^2} + \big\|\mathcal{Z}_j^{k+1} w_j\big\|_{L^2}\Big) \\ &\leq C 2^{j/4-j_+a} h_j^{\frac{5}{6}} \mathcal{F}_{k+1}(v)\end{aligned}$$

where we have used Sobolev injection and the fact that $\mathcal{Z}_j^k \widetilde{w}_j$, $\mathcal{Z}_j^{k+1} w_j$ is spectrally supported for $h_j\,|\xi| \sim 1$, and where the gain $2^{j/4}$ comes from $\big\|\Theta_{-j}^*\big\|_{\mathcal{L}(L^2,L^2)}$ when expressing w_j from $\Delta_j^h w$ in (4.10).

iii) Let us prove (4.15) and (4.16). Applying (3.12) with $p = p_{k_1} = p_{k_2} = \infty$, we get for $k' \leq k$,

$$\big\|\Delta_j^h \mathcal{Z}^{k'} Q_0(W)\big\|_{L^\infty} \leq C2^j \sum_{j_1,j_2 \in J(h,C)} \sum_{k_1+k_2 \leq k'} 2^{\frac{1}{2}\min(j_1,j_2)} \mathbf{1}_{\max(j_1,j_2)\geq j-C} \times \big\|\mathcal{Z}^{k_1}\Delta_{j_1}^h W\big\|_{L^\infty}\big\|\mathcal{Z}^{k_2}\Delta_{j_2}^h W\big\|_{L^\infty}$$

which gives (4.16). To get (4.15), we use again (4.18). The cubic term in the right hand side of this expression is bounded using (3.14) with $p = p_{k_1} = p_{k_2} = p_{k_3} = \infty$ and gives the first term in the right hand side of (4.15). To estimate the L^∞-norm of the R_0-term in (4.18) we use (3.17) with $d = b - \alpha - 0$ and (4.20). The loss $2^{j+(\alpha+0)} \leq Ch^{-2\beta(\alpha+0)}$ may be absorbed by the extra $h^{1/4}$ factor in front of the remainder in (4.18), so that we get again a contribution bounded by the first term in the right hand side of (4.15). Finally, the linear term in (4.18) is controlled by $h_j 2^{-j+b}\mathcal{E}_{k+1}(v)$. This concludes the proof. □

Proof of Proposition 4.3. We apply corollary A.3 with the weight $m(x,\xi) = \langle x\rangle$. By the definition (4.11) of γ_Λ^c, $2x\xi + |\xi|^{1/2} \geq c\langle x\rangle$ on the support of γ_Λ^c. Consequently, for any N in $\mathbb{N}$, we may find symbols q in $S(\langle x\rangle^{-1}), r$ in $S(1)$ such that $\gamma_\Lambda^c = q\#(2x\xi + |\xi|^{1/2}) + h_j^N r$. It follows that for any $p \geq 1$,

$$\big\|\mathrm{Op}_{h_j}(\gamma_\Lambda^c)\mathcal{Z}_j^k w_j\big\|_{L^p} \leq C\big\|\mathrm{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big)\mathcal{Z}_j^k w_j\big\|_{L^p} + h_j^N\big\|\mathcal{Z}_j^k w_j\big\|_{L^p}. \tag{4.22}$$

Applying this with $p = 3$, we may bound by (4.14) the first term in the right hand side in terms of the right hand side of (4.12). The last contribution is smaller than

$$h_j^N\big\|\mathcal{Z}_j^k w_j\big\|_{L^3} \leq h_j^N\big\|\mathcal{Z}_j^k w_j\big\|_{L^2}^{2/3}\big\|\mathcal{Z}_j^k w_j\big\|_{L^\infty}^{1/3}$$

going back to the estimates of $w_j = \Theta_{-j}^*\Delta_j^h w$ from $\mathcal{E}_k(v)$, $\mathcal{F}_k(v)$, we obtain the last term in the right hand side of (4.12). This concludes the proof of the proposition. □

The L^3-estimate we obtained in Proposition 4.3 outside a microlocal neighborhood of Λ will be useful as auxiliary bounds in the third step of our proof of Proposition 4.1. We also need L^∞-estimates for w cut-off outside Λ. They are given by the following

Proposition 4.5. *Let $k \geq 1$. There is an element P_k in $\mathcal{T}_k^\infty$ such that for any v satisfying $\mathcal{E}_{k-1}(v) \leq h^{-1/16}$*

$$\big\|\mathcal{Z}_j^k \,\mathrm{Op}_{h_j}(\gamma_\Lambda^c)w_j\big\|_{L^\infty} \leq C\Big[h^{1/4}P_k(v) + h^{1/2}\mathcal{F}_{k+1}(v)\Big]2^{-j+b}. \tag{4.23}$$

Proof. We notice first the commutation relations

$$\begin{aligned}
&\big[tD_t + xD_x, \mathrm{Op}_{h_j}(\gamma_\Lambda^c)\big] = \mathrm{Op}_{h_j}\big(\gamma_{\Lambda,1}^c\big),\\
&\big[h_jD_x, \mathrm{Op}_{h_j}(\gamma_\Lambda^c)\big] = h_j\,\mathrm{Op}_{h_j}\big(\gamma_{\Lambda,2}^c\big),
\end{aligned} \tag{4.24}$$

where $\gamma^c_{\Lambda,j}$ is in $S(1)$ with support contained in $\mathrm{Supp}\,(\gamma^c_\Lambda)$. This shows that, up to a modification of the definition of γ^c_Λ, it is enough to control $\left\|\mathrm{Op}_{h_j}(\gamma^c_\Lambda)\mathcal{Z}^k_j w_j\right\|_{L^\infty}$. Let us show

$$\left\|\mathrm{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big)\mathcal{Z}^k_j w_j\right\|_{L^\infty} \leq C\Big[h^{1/4}P_k(v) + h^{1/2}\mathcal{F}_{k+1}(v)\Big]2^{-j_+b} \tag{4.25}$$

for some P_k in $\mathcal{T}^\infty_k$. This will imply (4.23) using (4.22) with $p = \infty$ and N large enough, since $h_j = O(h^\sigma)$ and we may estimate $\left\|\mathcal{Z}^k_j w_j\right\|_{L^\infty} \lesssim h_j^{-\frac{1}{2}}\mathcal{F}_k(w)2^{-j_+b}$ by Sobolev. We prove (4.25) estimating the L^∞-norm of (4.18). We bound $\left\|Z^k\Delta^h_j Q_0(W)\right\|_{L^\infty}$ using (3.12) with $p = p_{k_1} = p_{k_2} = \infty$ and $\left\|Z^{k_\ell}\Delta^h_{j_\ell} w\right\|_{L^\infty} = O(\mathcal{E}_{k_\ell}(v)2^{-j_{\ell+}b})$. We obtain a bound in $2^{j-j_+b}P_k(v)$ for some P_k in $\mathcal{T}^\infty_k$, which gives a contribution to the first term in the right hand side of (4.25), writing $2^{j/2}\sqrt{h} = O(h^{1/4})$ as $2^{j/2} \leq Ch^{-\beta}$. To bound the cubic term in (4.18), we apply (3.15) with $p = \infty$, $d = b - \alpha - 0$, and (4.20), and control the loss $2^{j_+(\alpha+0)}$ by a small negative power of h using again $2^j \leq Ch^{-2\beta}$. We obtain that the cubic term in (4.18) is $O(h^{1/4}P_k(v))$. The R_0 term of (4.18) is estimated in the same way, using (3.17), (4.20).

Finally, we must bound the linear contributions in (4.18). Their L^2-norms are

$$O(h_j 2^{j/4-j_+b}\mathcal{F}_{k+1}(v))$$

according to the definition of $\mathcal{F}_{k+1}(v)$ and the expression $w_j = \Theta^*_{-j}\Delta^h_j w$. Moreover, they are spectrally localized at $h_j|\xi| \sim 1$, so that by Sobolev injection, the L^∞-norms are bounded by the L^2-norms multiplied by $Ch_j^{-1/2}$. This gives a contribution to the last term in (4.25). □

Third step: Estimates on a microlocal neighborhood of Λ.

We have obtained in Proposition 4.6 L^∞-estimates for $\mathcal{Z}^k_j w_j$ truncated outside a neighborhood of Λ. We want here to prove similar L^∞-estimates for $\mathcal{Z}^k_j w_j$ truncated close to Λ. They will be deduced from L^2-estimates for $\mathrm{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big)\,\mathrm{Op}_{h_j}(\gamma_\Lambda)\mathcal{Z}^k_j w_j$ that follow from (4.18) truncated close to Λ. Let us introduce the following decomposition of the function $w = \sum_{j\in J(h,C)}\Theta^*_j w_j$ introduced in (3.19), (3.10): we define, using notation (4.11)

$$w_\Lambda = \sum_{j\in J(h,C)}\Theta^*_j\,\mathrm{Op}_{h_j}(\gamma_\Lambda)w_j \tag{4.26}$$

and denote $W_\Lambda = (w_\Lambda, \overline{w}_\Lambda)$. We shall prove

Proposition 4.6. *Let $k \geq 1$. There are $C > 0$, an element P_k in $\mathcal{T}_k$ such that for any v satisfying $\mathcal{F}_k(v) = O(\varepsilon h^{-\frac{1}{4}})$, $\mathcal{E}_{k-1}(v) \leq h^{-1/16}$, for any j in $J(h,C)$*

$$\begin{aligned}\left\|\mathrm{Op}_{h_j}(\gamma_\Lambda)\mathcal{Z}^k_j w_j\right\|_{L^\infty} \leq Ch^{-0}\Bigg[&\sum_{k_1+k_2+k_3\leq k}\prod_{\ell=1}^{3}\mathcal{F}_{k_\ell}(v)^{\frac{1}{3}}\mathcal{E}_{k_\ell}(v)^{\frac{2}{3}}\\ &+ h^{\frac{1}{8}}P_k(v) + \mathcal{F}_{k+1}(v)\Bigg]2^{-j_+b}.\end{aligned} \tag{4.27}$$

To prove the proposition, we shall use (4.18) with $Q_0(W)$ replaced by $Q_0(w_\Lambda)$ in the right hand side. Let us estimate the error that is done.

Lemma 4.7. *For any $k \in \mathbb{N}$, there are $C > 0$, an element P_k in $\mathcal{T}_k$ such that for any v with $\mathcal{E}_{k-1}(v) \le h^{-1/16}$*

$$\begin{aligned}\big\|\mathcal{Z}^k \Delta_j^h\big(Q_0(W) - Q_0(W_\Lambda)\big)\big\|_{L^2} \le C\sqrt{h}2^{-j+b} \sum_{k_1+k_2+k_3\le k} \prod_{\ell=1}^{3} \mathcal{F}_{k_\ell}(v)^{\frac13}\mathcal{E}_{k_\ell}(v)^{\frac23} \\ + h^{\frac34}2^{-j+b}P_k(v).\end{aligned} \tag{4.28}$$

Proof. We have to bound for j in $J(h,C)$,

$$\big\|\mathcal{Z}^k\Delta_j^h B_0(W, W - W_\Lambda)\big\|_{L^2} + \big\|\mathcal{Z}^k\Delta_j^h B_0(W_\Lambda, W - W_\Lambda)\big\|_{L^2}.$$

Consider for instance the first term. We decompose each argument using

$$w = \sum_{j_\ell \in J(h,C)} \Delta_{j_\ell}^h w = \sum_{j_\ell\in J(h,C)} \Theta_{j_\ell}^* w_{j_\ell}.$$

By using (4.26),

$$w - w_\Lambda = \sum_{j_\ell \in J(h,C)} \Theta_{j_\ell}^* \operatorname{Op}_{h_{j_\ell}}(\gamma_\Lambda^c) w_{j_\ell}.$$

We write for $j_1 \in J(h,C)$,

$$\begin{aligned}\big\|\mathcal{Z}^{k_1}\Delta_{j_1}^h w\big\|_{L^6} &\le \big\|\mathcal{Z}^{k_1}\Delta_{j_1}^h w\big\|_{L^2}^{\frac13}\big\|\mathcal{Z}^{k_1}\Delta_{j_1}^h w\big\|_{L^\infty}^{\frac23} \\ &\le 2^{-j_1+(b+\frac13(a-b))}\mathcal{F}_{k_1}(v)^{\frac13}\mathcal{E}_{k_1}(v)^{\frac23}.\end{aligned} \tag{4.29}$$

Moreover $\mathcal{Z}^{k_2}\Delta_{j_2}^h(w - w_\Lambda)$ may be written as

$$\sum_{j_2'\,;\,|j_2'-j_2|\le N_0} \mathcal{Z}^{k_2}\Delta_{j_2}^h\Theta_{j_2'}^* \operatorname{Op}_{h_{j_2'}}\big(\gamma_\Lambda^c\big) w_{j_2'}$$

for some large enough N_0, up to a remainder whose L^3 norm is

$$O\Big(h^\infty 2^{-j_2+(b+\frac23(a-b))}\mathcal{F}_{k_2}(v)^{\frac23}\mathcal{E}_{k_2}(v)^{\frac13}\Big).$$

This follows from the fact that

$$\Delta_{j_2}^h\Theta_{j_2'}^*\operatorname{Op}_{h_{j_2'}}\big(\gamma_\Lambda^c\big)w_{j_2'} = \Theta_{j_2'}^*\operatorname{Op}_{h_{j_2'}}\Big(\varphi\big(2^{j_2'-j_2}\xi\big)\Big)\operatorname{Op}_{h_{j_2'}}\big(\gamma_\Lambda^c\big)w_{j_2'}$$

and that γ_Λ^c is supported for $|\xi| \sim 1$.

If we apply (4.12), we conclude that, since $\|\Theta^*_{j_2}\|_{\mathcal{L}(L^3,L^3)} = O(2^{-j_2/6})$,

$$
\begin{aligned}
\|\mathcal{Z}^{k_2}\Delta^h_{j_2}(w-w_\Lambda)\|_{L^3} \leq C_0\Big[\sqrt{h}2^{j_2/2-j_2+(b+\frac{1}{3}(a-b))}\sum_{k_1+k_2'\leq k_2}\prod_{\ell=1}^{2}\mathcal{F}_{k_\ell}(v)^{\frac{1}{3}}\mathcal{E}_{k_\ell}(v)^{\frac{2}{3}} \\
+h2^{-j_2+b}h^{-4\beta-0}\sum_{k_1'+k_2'+k_3'\leq k_2}[\mathcal{F}_{k_1'}(v)\mathcal{F}_{k_2'}(v)\mathcal{E}_{k_3'}(v)+\mathcal{E}_{k_1'}(v)\mathcal{E}_{k_2'}(v)\mathcal{E}_{k_3'}(v)] \\
+2^{j_2/12-j_2+a}h_{j_2}^{5/6}\mathcal{F}_{k_2+1}(v)+2^{-j_2+b}h^{1+\kappa}P_{k_2}(v) \\
+2^{-j_2+(b+\frac{2}{3}(a-b))}\Big(h_{j_2}^N+h^\infty\Big)\mathcal{F}_{k_2}(v)^{\frac{2}{3}}\mathcal{E}_{k_2}(v)^{\frac{1}{3}}\Big].
\end{aligned}
\tag{4.30}
$$

We plug (4.29), (4.30) in (3.12) with $p_{k_1}=6$, $p_{k_2}=3$ and we sum for $k_1+k_2\leq k$, j_1, j_2 in $J(h,C)$. We obtain that, for some P in $\mathcal{T}^k$,

$$
\begin{aligned}
&\|\mathcal{Z}^k\Delta^h_j B_0(W, W-W_\Lambda)\|_{L^2} \\
&\qquad \leq C_1\sqrt{h}2^{j-j+(b+\frac{1}{3}(a-b)-\frac{1}{2})}\sum_{k_1+k_2+k_3\leq k}\prod_{\ell=1}^{3}\mathcal{F}_{k_\ell}(v)^{\frac{1}{3}}\mathcal{E}_{k_\ell}(v)^{\frac{2}{3}} \\
&\qquad + C_1h^{1-4\beta-0}2^{j-j+b}P(v) \\
&\qquad + C_1 2^{j-j+(b+\frac{1}{3}(a-b))}h^{\frac{5}{6}}P(v).
\end{aligned}
$$

Since $2^j\leq Ch^{-2\beta}$, for β small enough and $a-b\gg 1$, we get a quantity bounded from above by (4.28). This concludes the proof. □

Let us deduce from Lemma 4.7 a sharp version of (4.13).

Corollary 4.8. *Let $k\geq 1$. There are $C>0$, and an element P_k of $\mathcal{T}_k$ such that for any j in $J(h,C)$, any v with $\mathcal{E}_{k-1}(v)\leq h^{-1/16}$*

$$
\begin{aligned}
&\|\mathrm{Op}_{h_j}(2x\xi+|\xi|^{\frac{1}{2}})\mathcal{Z}_j^k w_j - 2^{-j/2}\sqrt{h}A(h_j)\mathcal{Z}_j^k\Theta^*_{-j}\Delta^h_j Q_0(W_\Lambda)\|_{L^2} \\
&\qquad \leq Ch^{\frac{1}{2}-0}h_j^{\frac{1}{2}}2^{-j+b}\Big[\sum_{k_1+k_2+k_3\leq k}\prod_{\ell=1}^{3}\mathcal{F}_{k_\ell}(v)^{\frac{1}{3}}\mathcal{E}_{k_\ell}(v)^{\frac{2}{3}}+h^{\frac{1}{8}}P_k(v)+\mathcal{F}_{k+1}(v)\Big].
\end{aligned}
\tag{4.31}
$$

Proof. We start from estimate (4.13). If in the left hand side, we replace $Q_0(W)$ by $Q_0(W_\Lambda)$, the resulting error is bounded from above by the product of (4.28) and of

$$\sqrt{h}2^{-j/2}\|\Theta^*_{-j}\|_{\mathcal{L}(L^2,L^2)} = O(h_j^{1/2}).$$

We obtain a contribution to the right hand side of (4.31). On the other hand, the right hand side of (4.13) is bounded from above by the right hand side of (4.31) if we write that $2^{j/4}h^{9/8}\leq 2^{j/2}h_j^{1/2}h^{5/8}$. This concludes the proof. □

Proof of Proposition 4.6. Let us prove that for any j in $J(h,C)$

$$
\begin{aligned}
&\big\|\mathrm{Op}_{h_j}\big(2x\xi+|\xi|^{\frac12}\big)\,\mathrm{Op}_{h_j}(\gamma_\Lambda)\mathcal{Z}_j^k w_j\big\|_{L^2}\\
&\qquad\le Ch^{\frac12-0}h_j^{\frac12}2^{-j_+b}\left[\sum_{k_1+k_2+k_3\le k}\prod_{\ell=1}^{3}\mathcal{F}_{k_\ell}(v)^{\frac13}\mathcal{E}_{k_\ell}(v)^{\frac23}+h^{\frac18}P_k(v)+\mathcal{F}_{k+1}(v)\right]
\end{aligned}
\tag{4.32}
$$

for some element P_k in $\mathcal{T}_k$.

We notice first that since $w_j=\Theta^*_{-j}\Delta^h_j w$, the definition of $\mathcal{F}_k$ shows that

$$
\big\|\mathcal{Z}_j^k w_j\big\|_{L^2}\le \mathcal{F}_k(v)2^{j/4-j_+a}.
\tag{4.33}
$$

Consequently, by the gain of one power of h_j coming from symbolic calculus, we see that

$$
\big\|\big[\mathrm{Op}_{h_j}\big(2x\xi+|\xi|^{\frac12}\big),\mathrm{Op}_{h_j}(\gamma_\Lambda)\big]\,\mathcal{Z}_j^k w_j\big\|_{L^2}
$$

is estimated by the last term in the right hand side of (4.32). We are reduced to estimating $\big\|\mathrm{Op}_{h_j}(\gamma_\Lambda)\,\mathrm{Op}_{h_j}\big(2x\xi+|\xi|^{\frac12}\big)\mathcal{Z}_j^k w_j\big\|_{L^2}$ which, according to (4.31) is smaller than the right hand side of (4.32) modulo the quantity

$$
2^{-j/2}\sqrt{h}\big\|\mathrm{Op}_{h_j}(\gamma_\Lambda)\mathcal{Z}_j^k\Theta^*_{-j}\Delta^h_j Q_0(W_\Lambda)\big\|_{L^2}.
\tag{4.34}
$$

Since w_Λ is given by (4.26), we may write for $k'\le k$

$$
\mathcal{Z}^{k'}w_\Lambda=\sum_{j\in J(h,C)}\Theta^*_j\big(\mathcal{Z}_j^{k'}\,\mathrm{Op}_{h_j}(\gamma_\Lambda)w_j\big)
$$

and by definition of $\mathcal{F}_k(v)$, and the fact that $w_j=\Theta^*_{-j}\Delta^h_j w$, we have

$$
\big\|\mathcal{Z}_j^{k'}\,\mathrm{Op}_{h_j}(\gamma_\Lambda)w_j\big\|_{L^2}\le C2^{j/4-j_+a}\mathcal{F}_{k'}(v).
$$

Since w_j is microlocally supported for $h_j|\xi|\sim 1$, we deduce from that

$$
\big\|\mathcal{Z}_j^{k'}\,\mathrm{Op}_{h_j}(\gamma_\Lambda)w_j\big\|_{L^\infty}\le Ch^{-\frac12}2^{j/2-j_+a}\mathcal{F}_{k'}(v).
$$

By Definition 2.5, this shows that the family $\big(\mathcal{Z}_j^{k'}\,\mathrm{Op}_{h_j}(\gamma_\Lambda)w_j\big)_j$ is a family of elements of $\big(h^{-\frac12}\mathcal{B}^{1,a}_\infty[K]\big)\cap\big(\mathcal{B}^{0,a}_2[K]\big)$ for some compact K of $T^*(\mathbb{R}\setminus\{0\})\setminus 0$, contained in a small neighborhood of Λ, and that

$$
\big\|\mathcal{Z}_j^{k'}\,\mathrm{Op}_{h_j}(\gamma_\Lambda)w_j\big\|_{h^{-\frac12}\mathcal{B}^{1,a}_\infty[K]},
$$

which is by definition the best constant in (2.8), is smaller than $C\mathcal{F}_{k'}(v)$. A similar estimate holds for $\big\|\mathcal{Z}_j^{k'}\,\mathrm{Op}_{h_j}(\gamma_\Lambda)w_j\big\|_{\mathcal{B}^{0,a}_2[K]}$. We shall now prove that $Q_0(W_\Lambda)$ is microlocally supported outside a neighborhood of Λ.

Let us express $\mathcal{Z}^k Q_0(W_\Lambda)$ as a combination of terms $B_0\big(\mathcal{Z}^{k_1}W_\Lambda, \mathcal{Z}^{k_2}W_\Lambda\big)$, $k_1+k_2\leq k$, so as a combination of expressions deduced from (3.7).

$$\begin{aligned}
&\operatorname{Op}_h\big(a_0(\xi)\big)\Big[\big(\operatorname{Op}_h\big(a_1(\xi)\big)\mathcal{Z}^{k_1}w_\Lambda\big)\big(\operatorname{Op}_h\big(a_2(\xi)\big)\mathcal{Z}^{k_2}w_\Lambda\big)\Big],\\
&\operatorname{Op}_h\big(a_0(\xi)\big)\Big[\big(\operatorname{Op}_h\big(a_1(\xi)\big)\mathcal{Z}^{k_1}w_\Lambda\big)\big(\operatorname{Op}_h\big(a_2(\xi)\big)\mathcal{Z}^{k_2}\overline{w}_\Lambda\big)\Big],\\
&\operatorname{Op}_h\big(a_0(\xi)\big)\Big[\big(\operatorname{Op}_h\big(a_1(\xi)\big)\mathcal{Z}^{k_1}\overline{w}_\Lambda\big)\big(\operatorname{Op}_h\big(a_2(\xi)\big)\mathcal{Z}^{k_2}\overline{w}_\Lambda\big)\Big],
\end{aligned}\tag{4.35}$$

where a_0, a_1, a_2 are homogeneous of non negative order. We have just seen that $\mathcal{Z}^{k'}w_\Lambda$ is in $\big(h^{-\frac{1}{2}}\widetilde{\mathcal{B}}^{1,a}_\infty[K]\big)\cap\big(\widetilde{\mathcal{B}}^{0,a}_2[K]\big)$ with norm in that space bounded from above by $C\mathcal{F}_{k'}(v)$. It follows from Proposition 2.12 and the fact that a is large enough relatively to b that the first (resp. second, resp. third) expression (4.35) belongs to $h^{-\frac{1}{2}}\widetilde{\mathcal{B}}^{1,b}_2[K_2]$ (resp. $h^{-\frac{1}{2}}\widetilde{\mathcal{B}}^{1,b}_2[K_0]$, resp. $h^{-\frac{1}{2}}\widetilde{\mathcal{B}}^{1,a}_2[K_{-2}]$) where K_2 (resp. K_0, resp. K_{-2}) is a compact subset of $T^*(\mathbb{R}\setminus\{0\})$ contained in a small neighborhood of $2\cdot\Lambda$ (resp. $0\cdot\Lambda$, resp. $-2\cdot\Lambda$), and that the norm of these functions in those spaces is $O\big(\mathcal{F}_{k_1}(v)\mathcal{F}_{k_2}(v)\big)$. Consequently $\mathcal{Z}^k_j\Theta^*_{-j}\Delta^h_j Q_0(W_\Lambda)$ is microlocally supported far away from Λ. When we apply a $\operatorname{Op}_{h_j}(\gamma_\Lambda)$ cut-off as in (4.34), we gain a $O(h_j^\infty)=O(h^\infty)$ factor. We conclude that (4.34) is bounded from above $h^N\sum_{k_1+k_2\leq k}\mathcal{F}_{k_1}(v)\mathcal{F}_{k_2}(v)$ so that (4.34) is controlled by the $h^{1/8}$–term in (4.32).

To finish the proof of Proposition 4.6, we are left with showing

Lemma 4.9. *Assume that* (4.32) *holds. Then estimate* (4.27) *holds as well.*

Proof. The definition of $\mathcal{F}_k(v)$ and the fact that $w_j=\Theta^*_{-j}\Delta^h_j v$ implies that

$$\big\|\operatorname{Op}_{h_j}(\gamma_\Lambda)\mathcal{Z}^k_j w_j\big\|_{L^2}\leq C2^{\frac{j}{4}-j+a}\mathcal{F}_k(v).\tag{4.36}$$

Since $\big(2x\xi+|\xi|^{\frac{1}{2}}\big)=(\xi-\mathrm{d}\,\omega(x))g(x,\xi)$ for some elliptic symbol g, on a neighborhood of the support of γ_Λ, we deduce from (4.32), (4.36) and symbolic calculus that

$$\big\|\operatorname{Op}_{h_j}(\xi-\mathrm{d}\,\omega(x))\operatorname{Op}_{h_j}(\gamma_\Lambda)\mathcal{Z}^k_j w_j\big\|_{L^2}\leq h_j^{\frac{1}{2}}h^{\frac{1}{2}-0}M2^{-j+b}\tag{4.37}$$

where

$$M=C\left[\sum_{k_1+k_2+k_3\leq k}\prod_{\ell=1}^3\mathcal{F}_{k_\ell}(v)^{\frac{1}{3}}\mathcal{E}_{k_\ell}(v)^{\frac{2}{3}}+h^{\frac{1}{8}}P_k(v)+\mathcal{F}_{k+1}(v)\right]$$

for some P_k in $\mathcal{T}_k$. We may rewrite (4.36) and (4.37) as

$$\begin{aligned}
&\Big\|e^{-i\omega(x)/h_j}\operatorname{Op}_{h_j}(\gamma_\Lambda)\mathcal{Z}^k_j w_j\Big\|_{L^2}\leq C2^{\frac{j}{4}-j+a}\mathcal{F}_k(v),\\
&\Big\|\big(h_jD_x\big)\Big(e^{-i\omega(x)/h_j}\operatorname{Op}_{h_j}(\gamma_\Lambda)\mathcal{Z}^k_j w_j\Big)\Big\|_{L^2}\leq Ch_j^{\frac{1}{2}}h^{\frac{1}{2}-0}M2^{-j+b}.
\end{aligned}$$

Using that $\|f\|_{L^\infty} = O\Big(\|f\|_{L^2}^{\frac{1}{2}} \|D_x f\|_{L^2}^{\frac{1}{2}}\Big)$, we get

$$\begin{aligned}\big\|\mathrm{Op}_{h_j}(\gamma_\Lambda)\mathcal{Z}_j^k w_j\big\|_{L^\infty} &\le Ch^{-0} 2^{\frac{j}{4} - j + \frac{a+b}{2}} (\mathcal{F}_k(v)M)^{\frac{1}{2}} \\ &\le Ch^{-0} 2^{-j+b} M.\end{aligned}$$

This implies (4.27). $\square$

Proof of Proposition 4.1: We combine estimates (4.23) and (4.27). We obtain

$$\begin{aligned}\big\|\mathcal{Z}_j^k w_j\big\|_{L^\infty} \le Ch^{-0}\Big[&\sum_{k_1+k_2+k_3\le k}\prod_{\ell=1}^{3}\mathcal{F}_{k_\ell}(v)^{\frac{1}{3}}\mathcal{E}_{k_\ell}(v)^{\frac{2}{3}} \\ &+ h^{\frac{1}{8}} P_k(v) + \mathcal{F}_{k+1}(v)\Big] 2^{-j+b}\end{aligned} \tag{4.38}$$

for any j in $J(h,C)$ and some P_k in $\mathcal{T}_k$, assuming an a priori bound $\mathcal{E}_{k-1}(v) \le h^{-1/16}$.

We assume that (4.8) holds and that (4.9) has been proved up to order $k-1$. Consequently, by *(ii)* of Lemma 3.3, we know that

$$\begin{aligned}&\sup_j \Big(2^{j+b}\big\|\Delta_j^h \operatorname{Op}_h\big((1-\varphi_0)\big(h^{2\beta}\xi\big)\big)\mathcal{Z}^k v\big\|_{L^\infty}\Big) \le C_k \varepsilon h^{\frac{7}{8}}, \\ &\qquad\big\|\mathrm{Op}_h\big(\varphi_0\big(h^{-2(1-\sigma)}\xi\big)\mathcal{Z}^k v\big\|_{L^\infty} \le C_k \varepsilon h^{\frac{7}{16}-\sigma},\end{aligned}$$

where C_K depends only on $A_0, \ldots, A_k$.

On the other hand, (4.38) gives a control of $\big\|\Delta_j^h \mathcal{Z}^k v\big\|_{L^\infty}$ for $j \in J(h,C)$. Going back to the definition (4.4) of $\mathcal{E}_k(v)$ we obtain

$$\begin{aligned}\mathcal{E}_k(v) \le Ch^{-0}\Big[&\sum_{k_1+k_2+k_3\le k}\prod_{\ell=1}^{3}\mathcal{F}_{k_\ell}(v)^{\frac{1}{3}}\mathcal{E}_{k_\ell}(v)^{\frac{2}{3}}\Big] \\ &+ h^{\frac{1}{16}}\big[P_k(v) + C_k\varepsilon\big] + h^{-0}\mathcal{F}_{k+1}(v).\end{aligned} \tag{4.39}$$

Let us deduce from that that (4.9) holds at rank k. By the assumption (4.8) and the fact that by (4.1) $\delta'_k > \delta_{k+1}$, we may bound $h^{-0}\mathcal{F}_{k+1}(v)$ by $\varepsilon A'_k h^{-\delta'_k}$ for some $A'_k > 0$ depending only on A_{k+1}. The same is true for $h^{\frac{1}{16}} C_k \varepsilon$, with A'_k depending only on $A_0, \ldots, A_{k+1}$. Consider the $h^{\frac{1}{16}} P_k(v)$ contribution. By definition of the class $\mathcal{T}_k$ and (4.7), this term has modulus bounded from above by quantities of the form

$$h^{\frac{1}{16}}\mathcal{E}_{k_1}(v)\cdots\mathcal{E}_{k_{\ell'}}(v)\mathcal{F}_{k_{\ell'+1}}(v)\cdots\mathcal{F}_{k_{\ell+1}}(v) \tag{4.40}$$

where $\ell' \le \ell \le 4$, $k_1 + \cdots + k_\ell \le k$. Assume first that one of the k_j, $1 \le j \le \ell'$, is equal to k, so that the other ones equal 0. We obtain, according to assumption (4.8) a bound in $Ch^{\frac{1}{16}-(\ell'-1)\delta'_0-(\ell-\ell'+1)\delta_1} \times \mathcal{E}_k(v)$, with a constant C depending only on $\widetilde{A}_0, A_1$.

By (4.1) and (4.2) this is smaller than $C\mathcal{E}_k(v)h^{\frac{1}{32}}$ with a constant C depending only on $\widetilde{A}_0, A_1$. On the other hand, if all k_j, $0 \leq j \leq \ell'$, are strictly smaller than k, we may apply the induction hypothesis to estimate $\mathcal{E}_{k_j}(v)$ and (4.8) to control $\mathcal{F}_{k_{j+1}}(v)$. We obtain for (4.40) a bound in $C\varepsilon h^{\frac{1}{16}-\delta'_k}$, according to the first inequality (4.1), where the constant depends only on $\widetilde{A}_0, A_0, \dots, A_k$.

Let us study now the first term in the right hand side of (4.39). When $k_1 < k$, $k_2 < k$, $k_3 < k$, we write

$$
\begin{aligned}
h^{-0}\prod_{\ell=1}^{3}\mathcal{F}_{k_\ell}(v)^{\frac{1}{3}}\mathcal{E}_{k_\ell}(v)^{\frac{2}{3}} \leq \frac{h^{-0}}{3}\Big(&\mathcal{F}_{k_1}(v)\mathcal{E}_{k_2}(v)\mathcal{E}_{k_3}(v)\\
&+\mathcal{E}_{k_1}(v)\mathcal{F}_{k_2}(v)\mathcal{E}_{k_3}(v)+\mathcal{E}_{k_1}(v)\mathcal{E}_{k_2}(v)\mathcal{F}_{k_3}(v)\Big).
\end{aligned}
\tag{4.41}
$$

By (4.8), the fact that (4.9) is assumed to hold for $k_\ell < k$ and the first inequality (4.1), we get that (4.41) is $O\big(\varepsilon h^{-\delta'_k}\big)$ with a constant depending only on $\widetilde{A}_0$, $A_1, \dots, A_{k+1}$. Finally, we are left with studying the first term in the right hand side of (4.39) when one the k_j is equal to k, i.e.

$$
h^{-0}\mathcal{E}_k(v)^{\frac{2}{3}}\mathcal{F}_k(v)^{\frac{1}{3}}\mathcal{F}_0(v)^{\frac{2}{3}}\mathcal{E}_0(v)^{\frac{4}{3}} \leq \frac{2}{3}\delta\mathcal{E}_k(v)+\frac{1}{3}\delta^{-2}h^{-0}\mathcal{F}_k(v)\mathcal{F}_0(v)^2\mathcal{E}_0(v)^4
$$

for any $\delta > 0$ (where in the right-hand side, h^{-0} denotes $h^{-3\theta}$ if in the left hand side h^{-0} stands for $h^{-\theta}$ with $\theta > 0$ small). The last term in the above inequality is $O\big(\varepsilon h^{-\delta'_k}\big)$ according to assumption (4.8) and the second inequality (4.1), with a constant depending only on $\widetilde{A}_0, A_1, \dots, A_{k+1}$. Summing up, we have obtained

$$
\mathcal{E}_k(v) \leq \Big[\frac{2}{3}\delta + Ch^{\frac{1}{32}}\Big]\mathcal{E}_k(v) + \varepsilon A'_k h^{-\delta'_k}
$$

from which (4.9) at rank k follows if h and δ are taken small enough. $\square$

5 Decomposition of the solution in oscillating terms

The goal of this subsection is to give a description of the component w in the decomposition (3.19) of v in terms of oscillating contributions. More precisely, we expect w to be a sum of a main term, oscillating along the phase ω (i.e. a term which is a lagrangian distribution along Λ), of $O(\sqrt{h})$ terms, coming from the quadratic part of the nonlinearity, that will oscillate along the phases $\pm 2\omega$ (so, which are associated to the lagrangians $\pm 2\Lambda$), of $O(h)$ terms, coming from the cubic part of the nonlinearity, oscillating along the phases $\pm 3\omega$, $\pm\omega$, and a remainder. Moreover, we shall need, in preparation for next subsection, to get an explicit expression for contributions oscillating on $\pm 2\Lambda$.

We consider a solution v of (3.6) satisfying for h in some interval $]h', h_0]$ the a priori estimate (4.9) for $k' \leq \frac{s}{2} + N_1$ for some fixed $N_1 \ll s$. In particular, for $k \leq \frac{s}{2} + N_1$,

$$
\big\|\Delta_j^h \mathcal{Z}^k v\big\|_{L^\infty} \leq \varepsilon A'_k h^{-\delta'_k} 2^{-j+b}
\tag{5.1}
$$

for $j \in J(h,C)$. In this section, we shall denote by K compact subsets of $T^*(\mathbb{R}\setminus\{0\})$ contained in a small neighborhood of one of the lagrangians $\ell\cdot\Lambda$, $\ell\neq 0$, by L compact subsets of $T^*(\mathbb{R}\setminus\{0\})$ and by F closed subsets of $T^*\mathbb{R}$ whose second projection is compact in $\mathbb{R}\setminus\{0\}$.

We first obtain a rough decomposition of v.

Lemma 5.1. *One may write $v = v_L + w_\Lambda + w_{\Lambda^c} + v_H$, where v_L, v_H are defined in* (3.19) *and, for some compact subset K of $T^*(\mathbb{R}\setminus\{0\})$, lying in a small enough neighborhood of Λ and intersecting Λ, some closed set F as above, $\mathcal{Z}^k w_\Lambda$, $0\leq k\leq \frac{s}{2}+N_1$ is an $O(\varepsilon)$ element of $h^{-\delta'_{k+1}}L^\infty\tilde{I}^{0,b-2}_\Lambda[K]$ and $\mathcal{Z}^k w_{\Lambda^c}$, $0\leq k\leq\frac{s}{2}+N_1$ is an $O(\varepsilon)$ element of $h^{\frac{1}{2}-\delta'_{k+1}}\tilde{\mathcal{B}}^{0,b-2}_\infty[F] + h^{1-\delta'_{k+1}}\tilde{\mathcal{B}}^{-1,b-2}_\infty[F]$. Moreover, $\mathrm{Op}_h(x\xi)\mathcal{Z}^k w_{\Lambda^c}$ is in $h^{\frac{1}{2}-\delta'_{k+1}}\tilde{\mathcal{B}}^{1,b-2}_\infty[F] + h^{1-\delta'_{k+1}}\tilde{\mathcal{B}}^{0,b-2}_\infty[F]$.*

Proof. We have written in (3.19) $v = v_L + w + v_H$ and using notations (4.10), we may decompose $w = \sum_{j\in J(h,C)}\Theta_j^* w_j$. Recall definition (4.11) of symbol γ_Λ and set

$$w_{j,\Lambda} = \mathrm{Op}_{h_j}(\gamma_\Lambda)w_j, \quad j\in J(h,C)$$

so that $\mathcal{Z}^k w_\Lambda = \sum_{j\in J(h,C)}\Theta_j^*\big(\mathcal{Z}_j^k w_{j,\Lambda}\big)$. Since, when commuting $\mathcal{Z}_j^k$ to $\mathrm{Op}_{h_j}(\gamma_\Lambda)$, we get expressions of the form $\mathrm{Op}_{h_j}(\tilde{\gamma}^{k'}_\Lambda)\mathcal{Z}_j^{k'}$ with $k'\leq k$ and $\tilde{\gamma}^{k'}_\Lambda$ a new symbol, we deduce from estimates (4.9) that $\mathcal{Z}^k w_\Lambda$ belongs to $h^{-\delta'_k}\tilde{\mathcal{B}}^{0,b}_\infty[K]$ for a compact set K satisfying the conditions of the statement if γ_Λ is supported in a small enough neighborhood of Λ. Moreover, $\mathcal{Z}^k w_\Lambda$ is $O(\varepsilon)$ in the preceding space. If we use symbolic calculus, estimates (4.15), (4.16) and the assumed a priori estimates (4.9) together with (4.1), we get

$$\big\|\mathrm{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big)\mathcal{Z}_j^k w_{j,\Lambda}\big\|_{L^\infty} \leq C\varepsilon 2^{-j_+(b-2)}h^{-\delta'_{k+1}}\big[h^{\frac{1}{2}} + h_j\big] \tag{5.2}$$

i.e. $\big(\mathcal{Z}_j^k w_{j,\Lambda}\big)_j$ belongs to $h^{-\delta'_{k+1}}L^\infty I^{0,b-2}_\Lambda$ and is of size $O(\varepsilon)$ in that space. This gives the statement concerning w_Λ of the Lemma.

Set $w_{j,\Lambda^c} = \mathrm{Op}_{h_j}(\gamma^c_\Lambda)w_j$ so that $w_{\Lambda^c} = \sum_{j\in J(h,C)}\Theta_j^* w_{j,\Lambda^c}$. We use (4.22) with $p=\infty$. We estimate the first term in the right hand side of this inequality using (4.15), (4.16), the bounds (4.9) together with inequalities (4.1). We get

$$\big\|\mathrm{Op}_{h_j}(\gamma^c_\Lambda)\mathcal{Z}_j^k w_j\big\|_{L^\infty} \leq C\varepsilon 2^{-j_+(b-2)}h^{-\delta'_{k+1}}\big(h^{\frac{1}{2}} + h_j\big) + C_N\varepsilon h_j^N h^{-\delta'_k}2^{-j_+ b}, \tag{5.3}$$

where the last term has been estimated from (5.1).

If N is large enough, since $h_j\leq Ch^\sigma$ we get that $\mathcal{Z}^k w_{\Lambda^c}$ is in

$$h^{\frac{1}{2}-\delta'_{k+1}}\tilde{\mathcal{B}}^{0,b-2}_\infty[F] + h^{1-\delta'_{k+1}}\tilde{\mathcal{B}}^{-1,b-2}_\infty[F]$$

and is $O(\varepsilon)$ in that space (where F is a closed set as described before the statement of Lemma 5.1).

To study $\mathrm{Op}_h(x\xi)w_{\Lambda^c}$, we write

$$\mathrm{Op}_h(x\xi)w_{\Lambda^c} = \sum_{j\in J(h,C)} 2^{\frac{j}{2}}\Theta_j^*\,\mathrm{Op}_{h_j}(x\xi)w_{j,\Lambda^c}.$$

By symbolic calculus, we may write $\mathrm{Op}_{h_j}(x\xi)w_{j,\Lambda^c}$ from

$$\mathrm{Op}_{h_j}\big(\tilde{\gamma}_{\Lambda^c}\big)w_j, \quad \mathrm{Op}_{h_j}\big(\gamma_{\Lambda^c}\big)\big(\mathrm{Op}_{h_j}(x\xi)w_j\big),$$

where $\tilde{\gamma}_{\Lambda^c}$ is a cut-off with support contained in the one of γ_{Λ^c}. The L^∞-norm of the action of $\mathcal{Z}_j^k$ on the first of these expressions is bounded like (5.3). The second expression may be written from

$$\mathrm{Op}_{h_j}\big(\gamma_{\Lambda^c}\big)\,\mathrm{Op}_{h_j}\big(2x\xi+|\xi|^{\frac{1}{2}}\big)w_j, \quad \mathrm{Op}_{h_j}\big(\gamma_{\Lambda^c}\big)\,\mathrm{Op}_{h_j}\big(\,|\xi|^{\frac{1}{2}}\,\big)w_j.$$

The L^∞-norm of the action of $\mathcal{Z}_j$ on the last term is bounded using (5.3) by the right hand side of this inequality. For the first term, we use again (4.15), (4.16) as in the proof of (5.2) to get a similar upper bound. This concludes the proof. $\square$

The decomposition $w = w_\Lambda + w_{\Lambda^c}$, in terms of a contribution w_Λ localized close to Λ and another one w_{Λ^c} supported outside a neighborhood of Λ is not precise enough for our purposes. We need to refine it, writing w_{Λ^c} as a sum of terms oscillating on the lagrangians $\pm 2\Lambda$, of size of order $\sqrt{h}$, and of a remainder that is $O(h)$. Moreover, we need also to check that w_Λ is in $h^{-2\delta'_{k+1}}L^\infty\tilde{J}_\Lambda^{0,b'}[K]$. This is the goal of next proposition, that will be proved plugging the decomposition of Lemma 5.1 in the equation (3.23) satisfied by w, written under the form

$$\mathrm{Op}_h\big(2x\xi+|\xi|^{\frac{1}{2}}\big)w = -\sqrt{h}Q_0(W) + h\Big[\frac{i}{2}w - iZw - C_0(W)\Big] - h^{\frac{5}{4}}R(V). \tag{5.4}$$

Proposition 5.2. *Let $b' < b-5$ and $N_0 < N_1$ such that $(N_1 - N_0 - 1)\sigma \geq 1$. We may write the first decomposition of w*

$$w = w_\Lambda + \sqrt{h}\big(w_{2\Lambda} + w_{-2\Lambda}\big) + hg \tag{5.5}$$

where, for any $k \leq \frac{s}{2} + N_0$, $\mathcal{Z}^k w_{\pm 2\Lambda}$ is a $O(\varepsilon)$ element of $h^{-2\delta'_{k+1}}L^\infty\tilde{I}_{\pm 2\Lambda}^{2,b'+\frac{3}{2}}[K_{\pm 2}]$, $\mathcal{Z}^k w_\Lambda$ is an $O(\varepsilon)$ element of $h^{-\delta'_{k+1}}L^\infty\tilde{J}_\Lambda^{0,b'}[K]$, $\mathcal{Z}^k g$ is a $O(\varepsilon)$ element of $h^{-3\delta'_{k+1+N_1-N_0}}\tilde{\mathcal{B}}_\infty^{0,b'}[F]$ and $\mathcal{Z}^k\,\mathrm{Op}_h(x\xi)g$ is an $O(\varepsilon)$ element of $h^{-3\delta'_{k+1+N_1-N_0}}\tilde{\mathcal{B}}_\infty^{1,b'}[F]$, for some compact subsets $K_{\pm 2}$ of $T^*(\mathbb{R}\setminus\{0\})\setminus 0$ contained in small neighborhoods of $\pm 2\Lambda$, some closed subset F of $T^*\mathbb{R}$ whose second projection is compact in $\mathbb{R}\setminus\{0\}$. Moreover, $w_{\pm 2\Lambda}$ are given by

$$\begin{aligned} w_{2\Lambda} &= -i(1-\chi)(xh^{-\beta})\frac{1+\sqrt{2}}{4}\,|\mathrm{d}\,\omega(x)|\,w_\Lambda^2,\\ w_{-2\Lambda} &= -i(1-\chi)(xh^{-\beta})\frac{1-\sqrt{2}}{4}\,|\mathrm{d}\,\omega(x)|\,\overline{w}_\Lambda^2 \end{aligned} \tag{5.6}$$

where $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ close to zero has small enough support.

In order to prove the proposition, we shall compute the main contribution to $Q_0(W)$ obtained when plugging inside (3.7) the decomposition $w = w_\Lambda + w_{\Lambda^c}$ obtained in Lemma 5.1. We make at the same time a similar (and more precise) computation when one knows that an expansion of the form (5.5) holds.

Lemma 5.3. *i) Assume that* $w = w_\Lambda + w_{\Lambda^c}$*, where for all* $k \leq \frac{s}{2} + N_1$, $\mathcal{Z}^k w_\Lambda$ *(resp.* $\mathcal{Z}^k w_{\Lambda^c}$*) is an* $O(\varepsilon)$ *element of* $h^{-\delta'_{k+1}} L^\infty \tilde{I}_\Lambda^{0,b-2}[K]$ *(resp. of* $h^{\frac{1}{2}-\delta'_{k+1}} \tilde{\mathcal{B}}_\infty^{0,b-2}[F] + h^{1-\delta'_{k+1}} \tilde{\mathcal{B}}_\infty^{-1,b-2}[F]$ *such that* $\mathcal{Z}^k \operatorname{Op}_h(x\xi) w_{\Lambda^c}$ *is in* $h^{\frac{1}{2}-\delta'_{k+1}} \tilde{\mathcal{B}}_\infty^{1,b-2}[F] + h^{1-\delta'_{k+1}} \tilde{\mathcal{B}}_\infty^{0,b-2}[F]$*). Denote by* b' *any number* $b' < b - 5$*. Then, there are functions* $\widetilde{w}_{\pm 2\Lambda}$ *such that for* $k \leq \frac{s}{2} + N_1$

$$\mathcal{Z}^k \widetilde{w}_{\pm 2\Lambda} \text{ is } O(\varepsilon) \text{ in } h^{-2\delta'_{k+1}} L^\infty \tilde{I}_{\pm 2\Lambda}^{3,b'+\frac{3}{2}}[K_{\pm 2}]$$

so that

$$\begin{aligned} \sum_{j \in J(h,C)} \Delta_j^h Q_0(W) &= \widetilde{w}_{2\Lambda} + \widetilde{w}_{-2\Lambda} + \sqrt{h} \tilde{g}_2, \\ \sum_{j \in J(h,C)} \Delta_j^h \sqrt{h} C_0(W) &= \sqrt{h} \tilde{g}_3 \end{aligned} \tag{5.7}$$

where for any $k \leq \frac{s}{2} + N_1$,

$$\mathcal{Z}^k \tilde{g}_2, \mathcal{Z}^k \operatorname{Op}_h(x\xi) \tilde{g}_2 \in h^{-2\delta'_{k+1}} \tilde{\mathcal{B}}_\infty^{1,b'+\frac{1}{2}}[F], \quad \mathcal{Z}^k \tilde{g}_3, \mathcal{Z}^k \operatorname{Op}_h(x\xi) \tilde{g}_3 \in h^{-3\delta'_{k+1}-0} \tilde{\mathcal{B}}_\infty^{1,b'+\frac{1}{2}}[F]$$

for some new closed subset F *of* $T^*\mathbb{R}$*. Moreover one may write*

$$\begin{aligned} \widetilde{w}_{2\Lambda} &= -i(1-\chi)(xh^{-\beta}) \, |\mathrm{d}\,\omega|^{\frac{3}{2}} \, \frac{\sqrt{2}}{4} w_\Lambda^2, \\ \widetilde{w}_{-2\Lambda} &= -i(1-\chi)(xh^{-\beta}) \, |\mathrm{d}\,\omega|^{\frac{3}{2}} \, \frac{\sqrt{2}}{4} \overline{w}_\Lambda^2, \end{aligned} \tag{5.8}$$

for some $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ *close to zero, with small enough support.*

ii) Assume one is given a decomposition of w *of the form* (5.5) *and denote by* b' *any number* $b' < b - 8$*. Then there are elements* $\widetilde{w}_{\pm 2\Lambda}$ *such that for* $0 \leq k \leq \frac{s}{2} + N_0$, $\mathcal{Z}^k \widetilde{w}_{\pm 2\Lambda}$ *is* $O(\varepsilon)$ *in* $h^{-2\delta'_{k+1}} L^\infty \tilde{J}_{\pm 2\Lambda}^{3,b'+3/2}\big[K_{\pm 2}\big]$*, and for* $\ell \in \{\pm 1, \pm 3\}$ *elements* $\widetilde{w}_{\ell\Lambda}$*, such that, for any* $0 \leq k \leq \frac{s}{2} + N_0$, $\mathcal{Z}^k \widetilde{w}_{\ell\Lambda}$ *is* $O(\varepsilon)$ *in* $h^{-3\delta'_{k+1}} L^\infty \tilde{I}_{\ell\Lambda}^{3,b'+3/2}\big[K_\ell\big]$ *so that*

$$\sum_{j \in J(h,C)} \Delta_j^h \big[Q_0(W) + \sqrt{h} C_0(W)\big] = \widetilde{w}_{2\Lambda} + \widetilde{w}_{-2\Lambda} + \sqrt{h}\big(\widetilde{w}_{3\Lambda} + \widetilde{w}_\Lambda + \widetilde{w}_{-\Lambda} + \widetilde{w}_{-3\Lambda}\big) + h\tilde{g}, \tag{5.9}$$

where $\mathcal{Z}^k \tilde{g}$ *is in* $h^{-4\delta'_{k+P}} \tilde{\mathcal{B}}^{1,b'+\frac{1}{2}}[F]$ *and* $\mathcal{Z}^k \operatorname{Op}_h(x\xi) \tilde{g}$ *is in* $h^{-4\delta'_{k+P}} \tilde{\mathcal{B}}_\infty^{1,b'}[F]$ *with* $P = N_1 - N_0 + 1$.

Moreover, $\widetilde{w}_\Lambda$ *is given in terms of* w_Λ *and of the functions in* (5.6) *by*

$$\begin{aligned} \widetilde{w}_\Lambda = {} & \frac{i}{2}(1-\chi)(xh^{-\beta}) \, |\mathrm{d}\,\omega|^{\frac{3}{2}} \, \overline{w}_\Lambda \big(w_{2\Lambda} - \overline{w}_{-2\Lambda}\big) \\ & + \frac{1}{4}(1-\chi)\big(xh^{-\beta}\big) |\,\mathrm{d}\,\omega(x)|^{\frac{5}{2}} \big|w_\Lambda\big|^2 w_\Lambda, \end{aligned} \tag{5.10}$$

for some $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ close to zero, with small enough support, and $\widetilde{w}_{3\Lambda}$, $\widetilde{w}_{-\Lambda}$, $\widetilde{w}_{-3\Lambda}$ have similar expressions

$$
\begin{aligned}
\widetilde{w}_{3\Lambda} &= \frac{i}{2}(1-\chi)(xh^{-\beta})\,|\mathrm{d}\,\omega|^{\frac{3}{2}}\, w_\Lambda\big(\mu_3' w_{2\Lambda} + \mu_3''\overline{w}_{-2\Lambda}\big)\\
&\quad + (1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega(x)|^{\frac{5}{2}}\mu_3''' w_\Lambda^3,\\
\widetilde{w}_{-\Lambda} &= \frac{i}{2}(1-\chi)(xh^{-\beta})\,|\mathrm{d}\,\omega|^{\frac{3}{2}}\, w_\Lambda\big(\mu_{-1}' w_{-2\Lambda} + \mu_{-1}''\overline{w}_{2\Lambda}\big)\\
&\quad + (1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega(x)|^{\frac{5}{2}}\mu_{-1}'''|w_\Lambda|^2\overline{w}_\Lambda,\\
\widetilde{w}_{-3\Lambda} &= \frac{i}{2}(1-\chi)(xh^{-\beta})\,|\mathrm{d}\,\omega|^{\frac{3}{2}}\, \overline{w}_\Lambda\big(\mu_{-3}' w_{-2\Lambda} + \mu_{-3}''\overline{w}_{2\Lambda}\big)\\
&\quad + (1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega(x)|^{\frac{5}{2}}\mu_{-3}'''\overline{w}_\Lambda^3,
\end{aligned}
\tag{5.11}
$$

for some real coefficients μ_ℓ', μ_ℓ'', μ_ℓ''', $\ell \in \{-3,-1,3\}$.

Before starting the proof, we make the following remark that will be used several times below.

Remark 5.4. Let κ be a smooth function on $\mathbb{R}^*$, such that for some real numbers ℓ, ℓ' and for any integer k, $\partial_x^k \kappa = O\big(|x|^{-\ell-k}\langle x\rangle^{-\ell'}\big)$. Let χ be in $C_0^\infty(\mathbb{R})$, equal to one close to zero and let r be an element of $\tilde{\mathcal{B}}_\infty^{\mu,\gamma}[K]$ for some compact subset of $T^*(\mathbb{R}\setminus\{0\})$. Then $(1-\chi)(xh^{-\beta})\kappa(x)r$ belongs to $\tilde{\mathcal{B}}_\infty^{\mu+(\ell+\ell'),\gamma+\ell'/2}[K]$.

Proof. We decompose $r = \sum_{j\in J(h,C)} \Theta_j^* r_j$ where $(r_j)_j$ is a bounded family in $\mathcal{B}_\infty^{\mu,\gamma}[K]$. Then

$$
(1-\chi)(xh^{-\beta})\kappa r = \sum_{j\in J(h,C)} \Theta_j^* \tilde{r}_j
$$

with $\tilde{r}_j = (1-\chi)(x2^{-j/2}h^{-\beta})\kappa(2^{-j/2}x)r_j$. Since r_j is microlocally supported in K we may, modulo a $O(h_j^\infty) = O(h^\infty)$ remainder, replace r_j by $\theta(x)r_j$ for some $\theta \in C_0^\infty(\mathbb{R})$, equal to one on a large enough compact subset of $\mathbb{R}^*$. Since

$$
x \to (1-\chi)(x2^{-j/2}h^{-\beta})2^{-\frac{j}{2}(\ell+\ell')+j+\frac{\ell'}{2}}\kappa\big(2^{-\frac{j}{2}}x\big)\theta(x)
$$

is in $C_0^\infty(\mathbb{R}^*)$ and has derivatives uniformly estimated in j, h, we see that $(\tilde{r}_j)_j$ is microlocally supported on K and satisfies uniform bounds in $\mathcal{B}_\infty^{\mu+(\ell+\ell'),\gamma+\frac{\ell'}{2}}[K]$. □

Remark 5.5. Let χ_1, χ_2 be two $C_0^\infty(\mathbb{R})$ functions equal to one close to zero and r be in $\tilde{\mathcal{B}}_\infty^{\mu,\gamma}[K]$ for some compact set K of $T^*(\mathbb{R}\setminus\{0\})$. Then, if $\operatorname{Supp}\chi_1$ and $\operatorname{Supp}\chi_2$ are small enough, $(1-\chi_1)(xh^{-\beta})r$ and $(1-\chi_2)(xh^{-\beta})r$ coincide modulo $O(h^\infty)$ (so that they are identified).

Proof. We write again

$$
\Big[(1-\chi_1)(xh^{-\beta}) - (1-\chi_2)(xh^{-\beta})\Big]r = \sum_{j\in J(h,C)} \Theta_j^*\Big[(\chi_2-\chi_1)(x2^{-j/2}h^{-\beta})r_j\Big].
$$

As above, modulo $O(h^\infty)$, we may insert some cut-off θ against r_j. We may then notice that $(\chi_2-\chi_1)(x2^{-j/2}h^{-\beta})\theta(x)\equiv 0$ if $\operatorname{Supp}\chi_\ell$ is small enough, as $2^{-j/2}h^{-\beta}>c$ for some $c>0$ since j is in $J(h,C)$. $\square$

To prove lemma 5.3, it will be necessary to compute explicitly the action of some multilinear operators on functions of the type $w=w_\Lambda+w_{\Lambda^c}$.

Let us fix some notation. If p_1,p_2 are in $\mathbb{Z}^*$, K_{p_1}, K_{p_2} are compact subsets contained in small neighborhoods of $p_1\cdot\Lambda$, $p_2\cdot\Lambda$ and if $w^\ell_{p_\ell\cdot\Lambda}$ is an element of $L^\infty\tilde{I}^{\mu_\ell,\gamma_\ell}_{p_\ell\cdot\Lambda}\big[K_{p_\ell}\big]$, Proposition 2.12 shows that the product $w^1_{p_1\cdot\Lambda}\cdot w^2_{p_2\cdot\Lambda}$ belongs to $L^\infty\tilde{I}^{\mu_1+\mu_2,\gamma_1+\gamma_2}_{(p_1+p_2)\cdot\Lambda}\big[K_{p_1+p_2}\big]$ for some compact subset $K_{p_1+p_2}$ of $T^*(\mathbb{R}\setminus\{0\})$ contained in a small neighborhood of $(p_1+p_2)\cdot\Lambda$, if K_{p_1} and K_{p_2} were contained in small enough neighborhoods of $p_1\cdot\Lambda$, $p_2\cdot\Lambda$ respectively. In the sequel, to avoid heavy notations, we shall eventually denote by K_{p_ℓ} different compact subsets of $T^*(\mathbb{R}\setminus\{0\})$ contained in a small enough neighborhood of $p_\ell\cdot\Lambda$. All of them will be constructed from a compact subset K of $T^*(\mathbb{R}\setminus\{0\})$ contained in a small enough neighborhood of Λ. To simplify some notations, $p\Lambda$ will sometimes stand for $p\cdot\Lambda$. We shall also denote by L some compact subset of $T^*(\mathbb{R}\setminus\{0\})$ which may vary from line to line.

Lemma 5.6. *Let $b_\ell\colon\mathbb{R}^*\to\mathbb{C}$, $\ell=1,2,3$ be smooth functions positively homogeneous of degree d_ℓ and $a_0,a_1\colon\mathbb{R}^*\to\mathbb{C}$ be smooth, positively homogeneous of degree m_0,m_1.*

Let p_ℓ be in $\mathbb{Z}^*$, $|p_\ell|\le 3$, $\ell=1,2,3$. If $p_1+p_2=0$ (resp. $p_1+p_2+p_3=0$), assume moreover that a_1 (resp. a_0) is an homogeneous polynomial of order $m_1\in\mathbb{N}^*$ (resp. $m_0\in\mathbb{N}^*$). Let b' be a large enough positive number. Let χ be in $C_0^\infty(\mathbb{R})$, $\chi\equiv 1$ close to zero, with small enough support.

i) Assume given for $\ell=1,2,3$ functions $w^\ell_{p_\ell\Lambda}$ such that for some N and any $k\le\frac{s}{2}+N$, $\mathcal{Z}^k w^\ell_{p_\ell\Lambda}$ is in $h^{-\delta'_{k+1}}L^\infty\tilde{I}^{0,b'}_{p_\ell\cdot\Lambda}\big[K_{p_\ell}\big]$, for compact subsets K_{p_ℓ} satisfying the above conditions. Denote $\mu_2=2(m_1+d_1+d_2)$, $\mu_3=2(m_0+m_1+d_1+d_2+d_3)$. Then,

$$\operatorname{Op}_h(a_1)\Big[\big(\operatorname{Op}_h(b_1)w^1_{p_1\Lambda}\big)\big(\operatorname{Op}_h(b_2)w^2_{p_2\Lambda}\big)\Big] \tag{5.12}$$

may be written as the sum of

$$(1-\chi)(xh^{-\beta})a_1\big((p_1+p_2)\,\mathrm{d}\,\omega\big)b_1(p_1\,\mathrm{d}\,\omega)b_2(p_2\,\mathrm{d}\,\omega)w^1_{p_1\Lambda}w^2_{p_2\Lambda}, \tag{5.13}$$

which is an element of $h^{-2\delta'_1}L^\infty\tilde{I}^{\mu_2,2b'}_{(p_1+p_2)\cdot\Lambda}\big[K_{p_1+p_2}\big]$ such that the action of $\mathcal{Z}^k$ on it gives an element of $h^{-2\delta'_{k+1}}L^\infty\tilde{I}^{\mu_2,2b'}_{(p_1+p_2)\cdot\Lambda}\big[K_{p_1+p_2}\big]$ for $k\le\frac{s}{2}+N$, and of a remainder R such that, for those k's, $\mathcal{Z}^kR$ is in

$$h^{\frac12-2\delta'_{k+1}}\tilde{\mathcal{B}}^{\mu_2-1,2b'-\frac12}_\infty[L]+h^{1-2\delta'_{k+1}}\tilde{\mathcal{B}}^{\mu_2-2,2b'-1}[L]. \tag{5.14}$$

In the same way, a cubic term

$$\operatorname{Op}_h(a_0)\Big[\operatorname{Op}_h(a_1)\Big\{\big(\operatorname{Op}_h(b_1)w^1_{p_1\Lambda}\big)\big(\operatorname{Op}_h(b_2)w^2_{p_2\Lambda}\big)\Big\}\big(\operatorname{Op}_h(b_3)w^3_{p_3\Lambda}\big)\Big] \tag{5.15}$$

may be written as the sum of

$$(1-\chi)(xh^{-\beta})a_0\big((p_1+p_2+p_3)\,\mathrm{d}\,\omega\big)a_1\big((p_1+p_2)\,\mathrm{d}\,\omega\big)\prod_{\ell=1}^{3}b_\ell(p_\ell\,\mathrm{d}\,\omega)w^\ell_{p_\ell\Lambda}, \tag{5.16}$$

which is a function such that the action of $\mathcal{Z}^k$ *on it,* $k\le \frac{s}{2}+N$, *gives an element of*

$$h^{-3\delta'_{k+1}}L^\infty\tilde{I}^{\mu_3,3b'}_{(p_1+p_2+p_3)\cdot\Lambda}\big[K_{p_1+p_2+p_3}\big]$$

and of a remainder R *such that* $\mathcal{Z}^kR$ *is in*

$$\sum_{j=1}^{3}h^{\frac{j}{2}-3\delta'_{k+1}}\tilde{\mathcal{B}}^{\mu_3-j,3b'-j/2}_\infty[L]. \tag{5.17}$$

ii) Assume that we are given a function

$$w^\ell = w^\ell_{p_\ell\cdot\Lambda}+\sqrt{h}\Big[w^\ell_{2p_\ell\cdot\Lambda}+w^\ell_{-2p_\ell\cdot\Lambda}\Big] \tag{5.18}$$

where for $k\le\frac{s}{2}+N$,

$$\begin{aligned}&\mathcal{Z}^kw^\ell_{\pm2p_\ell\cdot\Lambda}\ \text{ is in } h^{-2\delta'_{k+1}}L^\infty\tilde{I}^{0,b'}_{\pm2p_\ell\cdot\Lambda}\big[K_{\pm2p_\ell}\big],\\ &\mathcal{Z}^kw^\ell_{p_\ell\cdot\Lambda}\quad\ \text{ is in } h^{-\delta'_{k+1}}L^\infty\tilde{J}^{0,b'}_{p_\ell\cdot\Lambda}\big[K_{p_\ell}\big].\end{aligned}$$

Assume also that $p_1\pm2p_2\neq0$, $p_2\pm2p_1\neq0$. *Then* (5.12) *may be written as the sum of a quadratic term, given by* (5.13), *which is such that the action of* $\mathcal{Z}^k$ *on it gives an element of* $h^{-2\delta'_{k+1}}L^\infty\tilde{J}^{\mu_2,2b'}_{(p_1+p_2)\cdot\Lambda}\big[K_{p_1+p_2}\big]$, *of a cubic term, which may be written as the product of* $\sqrt{h}$ *and*

$$\begin{aligned}(1-\chi)(xh^{-\beta})\Big[&a_1\big((p_1+2p_2)\,\mathrm{d}\,\omega\big)b_1(p_1\,\mathrm{d}\,\omega)b_2(2p_2\,\mathrm{d}\,\omega)w^1_{p_1\Lambda}w^2_{2p_2\Lambda}\\ &+a_1\big((p_1-2p_2)\,\mathrm{d}\,\omega\big)b_1(p_1\,\mathrm{d}\,\omega)b_2(-2p_2\,\mathrm{d}\,\omega)w^1_{p_1\Lambda}w^2_{-2p_2\Lambda}\\ &+a_1\big((2p_1+p_2)\,\mathrm{d}\,\omega\big)b_1(2p_1\,\mathrm{d}\,\omega)b_2(p_2\,\mathrm{d}\,\omega)w^1_{2p_1\Lambda}w^2_{p_2\Lambda}\\ &+a_1\big((-2p_1+p_2)\,\mathrm{d}\,\omega\big)b_1(-2p_1\,\mathrm{d}\,\omega)b_2(p_2\,\mathrm{d}\,\omega)w^1_{-2p_1\Lambda}w^2_{p_2\Lambda}\Big]\end{aligned} \tag{5.19}$$

and of a remainder term R. *Moreover the action of* $\mathcal{Z}^k$ *on* (5.19), $0\le k\le\frac{s}{2}+N$, *gives an element of*

$$\sum_{+,-}h^{-3\delta'_{k+1}}L^\infty\tilde{I}^{\mu_2,2b'}_{(p_1\pm2p_2)\cdot\Lambda}\big[K_{p_1\pm2p_2}\big]+\sum_{+,-}h^{-3\delta'_{k+1}}L^\infty\tilde{I}^{\mu_2,2b'}_{(p_2\pm2p_1)\cdot\Lambda}\big[K_{p_2\pm2p_1}\big]$$

and the action of $\mathcal{Z}^k$ *on* R *gives an element of*

$$h^{1-4\delta'_{k+1}}\tilde{\mathcal{B}}^{\mu_2-1,2b'-\frac{1}{2}}_\infty[L]. \tag{5.20}$$

Proof. *i)* We use Proposition 2.11. By (2.17) applied with a symbol $a \equiv b_\ell(\xi)$, $\ell = 1, 2$, we may write

$$\mathrm{Op}_h(b_\ell) w^\ell_{p_\ell \Lambda} = (1-\chi)\big(xh^{-\beta}\big) b_\ell(p_\ell \,\mathrm{d}\,\omega) w^\ell_{p_\ell\Lambda} + \tilde{r}^\ell \tag{5.21}$$

where the action of $\mathcal{Z}^k$ on $\tilde{r}^\ell$ (resp. on the left hand side, resp. on the first term in the right hand side) of (5.21) is in $h^{\frac{1}{2}-\delta'_{k+1}}\tilde{\mathcal{B}}^{2d_\ell-1,b'-\frac{1}{2}}_\infty[L]$ (resp. in $h^{-\delta'_{k+1}} L^\infty \tilde{I}^{2d_\ell,b'}_{p_\ell\Lambda}[K_{p_\ell}]$).

By Proposition 2.12,

$$Z^k\Big[\big(\mathrm{Op}_h(b_1) w^1_{p_1\Lambda}\big)\big(\mathrm{Op}_h(b_2) w^2_{p_2\Lambda}\big) - (1-\chi)\big(xh^{-\beta}\big)^2 b_1(p_1 \,\mathrm{d}\,\omega) b_2(p_2\,\mathrm{d}\,\omega) w^1_{p_1\Lambda} w^2_{p_2\Lambda}\Big] \tag{5.22}$$

is in

$$h^{\frac{1}{2}-2\delta'_{k+1}}\tilde{\mathcal{B}}^{2(d_1+d_2)-1,2b'-\frac{1}{2}}_\infty[L] + h^{1-2\delta'_{k+1}}\tilde{\mathcal{B}}^{2(d_1+d_2)-2,2b'-1}_\infty[L]$$

and the second term in (5.22) belongs to

$$h^{-2\delta'_{k+1}} L^\infty \tilde{I}^{2(d_1+d_2),2b'}_{(p_1+p_2)\cdot\Lambda}\big[K_{p_1+p_2}\big].$$

We make act $\mathrm{Op}_h(a_1)$ on the bracket in (5.22). By Proposition 2.11, this gives a remainder R satisfying the conclusions of the statement. Moreover, the action of $\mathrm{Op}_h(a_1)$ on

$$(1-\chi)\big(xh^{-\beta}\big)^2 b_1(p_1\,\mathrm{d}\,\omega) b_2(p_2\,\mathrm{d}\,\omega) w^1_{p_1\Lambda} w^2_{p_2\Lambda}$$

may be written as (5.13) modulo similar remainders. Notice that the second remark after the statement of Lemma 5.3 allows one to replace any power $(1-\chi)^p\big(xh^{-\beta}\big)$ by $(1-\chi)\big(xh^{-\beta}\big)$ if $\operatorname{Supp}\chi$ is small enough.

One studies the cubic expressions (5.15) in the same way.

ii) We start from the stronger assumption (5.18). By (2.17) and the lines following that formula we may write $\mathrm{Op}_h(b_\ell) w^\ell$ as

$$\begin{aligned}
&(1-\chi)\big(xh^{-\beta}\big) b_\ell(p_\ell\,\mathrm{d}\,\omega) w^\ell_{p_\ell\Lambda}\\
&\quad + \sqrt{h}(1-\chi)\big(xh^{-\beta}\big)\Big[b_\ell(2p_\ell\,\mathrm{d}\,\omega) w^\ell_{2p_\ell\Lambda} + b_\ell(-2p_\ell\,\mathrm{d}\,\omega) w^\ell_{-2p_\ell\Lambda}\Big]\\
&\quad + \tilde{r}^\ell
\end{aligned} \tag{5.23}$$

where the action of $\mathcal{Z}^k$ on $\tilde{r}^\ell$ is in $h^{1-2\delta'_{k+1}}\tilde{\mathcal{B}}^{2d_\ell-1,b'-\frac{1}{2}}_\infty[L]$.

Moreover $\mathcal{Z}^k\big[(1-\chi)\big(xh^{-\beta}\big) b_\ell(p_\ell\,\mathrm{d}\,\omega) w^\ell_{p_\ell\Lambda}\big]$ (resp. $\mathcal{Z}^k\big[(1-\chi)\big(xh^{-\beta}\big) b_\ell(\pm 2p_\ell\,\mathrm{d}\,\omega) w^\ell_{\pm 2p_\ell\Lambda}\big]$) belongs to

$$h^{-\delta'_{k+1}} L^\infty \tilde{J}^{2d_\ell,b'}_{p_\ell\cdot\Lambda}\big[K_{p_\ell}\big] \quad \text{(resp. } h^{-2\delta'_{k+1}} L^\infty \tilde{I}^{2d_\ell,b'}_{\pm 2p_\ell\cdot\Lambda}\big[K_{\pm 2p_\ell}\big]\text{)}.$$

Applying Proposition 2.12, we obtain that $\big(\mathrm{Op}_h(b_1) w^1\big)\big(\mathrm{Op}_h(b_2) w^2\big)$ may be written as the sum of quadratic terms

$$(1-\chi)\big(xh^{-\beta}\big)^2 b_1(p_1\,\mathrm{d}\,\omega) b_2(p_2\,\mathrm{d}\,\omega)\big(w^1_{p_1\Lambda}\big)\big(w^2_{p_2\Lambda}\big), \tag{5.24}$$

of cubic terms

$$\sqrt{h}(1-\chi)\big(xh^{-\beta}\big)^2\Big[b_1(p_1\,\mathrm{d}\,\omega)b_2(2p_2\,\mathrm{d}\,\omega)\big(w^1_{p_1\Lambda}\big)\big(w^2_{2p_2\Lambda}\big)$$
$$(5.25)\qquad + b_1(p_1\,\mathrm{d}\,\omega)b_2(-2p_2\,\mathrm{d}\,\omega)\big(w^1_{p_1\Lambda}\big)\big(w^2_{-2p_2\Lambda}\big)$$
$$+ b_1(2p_1\,\mathrm{d}\,\omega)b_2(p_2\,\mathrm{d}\,\omega)\big(w^1_{2p_1\Lambda}\big)\big(w^2_{p_2\Lambda}\big)$$
$$+ b_1(-2p_1\,\mathrm{d}\,\omega)b_2(p_2\,\mathrm{d}\,\omega)\big(w^1_{-2p_1\Lambda}\big)\big(w^2_{p_2\Lambda}\big)\Big]$$

and of a remainder $\tilde{R}^\ell$ such that $\mathcal{Z}^k\tilde{R}^\ell$ is in $h^{1-4\delta'_{k+1}}\tilde{\mathcal{B}}_\infty^{2(d_1+d_2)-1,2b'-\frac{1}{2}}[L]$.

To study (5.12), we make act $\mathrm{Op}_h(a_0)$ on (5.24), (5.25) and on the remainder. We know from Proposition 2.12 that $(1-\chi)\big(xh^{-\beta}\big)^2 w^1_{p_1\Lambda}w^2_{p_2\Lambda}$ is in $h^{-2\delta'_{k+1}}L^\infty\tilde{J}^{0,2b'}_{(p_1+p_2)\cdot\Lambda}\big[K_{p_1+p_2}\big]$ and that $(1-\chi)\big(xh^{-\beta}\big)^2 w^1_{p_1\Lambda}w^2_{\pm2p_2\Lambda}$ (resp. $(1-\chi)\big(xh^{-\beta}\big)^2 w^1_{\pm2p_1\Lambda}w^2_{p_2\Lambda}$) belongs to

$$h^{-3\delta'_{k+1}}L^\infty\tilde{I}^{0,2b'}_{(p_1\pm2p_2)\cdot\Lambda}\big[K_{p_1\pm2p_2}\big]\quad(\text{resp. } h^{-3\delta'_{k+1}}L^\infty\tilde{I}^{0,2b'}_{(\pm2p_1+p_2)\cdot\Lambda}\big[K_{\pm2p_1+p_2}\big]).$$

To study the action of $\mathrm{Op}_h(a_0)$ on (5.24), (5.25), we may use (2.18), noticing that, since $\mathrm{d}\,\omega$ is homogeneous of degree -2,

$$a_1(\xi)b_1(p_1\,\mathrm{d}\,\omega(x))b_2(p_2\,\mathrm{d}\,\omega(x)),$$
$$a_1(\xi)b_1(p_1\,\mathrm{d}\,\omega(x))b_2(\pm2p_2\,\mathrm{d}\,\omega(x)),$$
$$a_1(\xi)b_1(\pm2p_1\,\mathrm{d}\,\omega(x))b_2(p_2\,\mathrm{d}\,\omega(x))$$

satisfy the assumptions (2.15) with (ℓ,ℓ',d,d') replaced by $(-2(d_1+d_2),0,m_1,0)$. We conclude that (5.12) is given by the sum of (5.13), (5.19) and remainders R such that the action of $\mathcal{Z}^k$ on R gives elements of $h^{1-4\delta'_{k+1}}\tilde{\mathcal{B}}_\infty^{\mu_2-1,2b'-\frac{1}{2}}[L]$. Moreover, (5.13) and (5.19) belong to the spaces indicated in the statement of the lemma. This concludes the proof. $\square$

Proof of Lemma 5.3. i) Let us prove the first equality (5.7). Recall that we denote $W=(w,\bar{w})$. In the same way, set $W_\Lambda=(w_\Lambda,\bar{w}_\Lambda)$, $W_{\Lambda^c}=(w_{\Lambda^c},\bar{w}_{\Lambda^c})$. If B_0 denotes the polar form of Q_0, we have

$$Q_0(W)=Q_0\big(W_\Lambda\big)+2B_0\big(W_\Lambda,W_{\Lambda^c}\big)+Q_0\big(W_{\Lambda^c}\big).$$

For j in $J(h,C)$, we set

$$\tilde{g}_{2,j}=h^{-\frac{1}{2}}\Theta^*_{-j}\Delta^h_j\big[2B_0\big(W_\Lambda,W_{\Lambda^c}\big)+Q_0\big(W_{\Lambda^c}\big)\big].$$

Let us show that $\tilde{g}_2=\sum_{j\in J(h,C)}\Theta^*_j\tilde{g}_{2,j}$ satisfies the conclusions of the lemma. By assumption $\mathcal{Z}^k w_\Lambda$ is in $h^{-\delta'_{k+1}}\tilde{\mathcal{B}}_\infty^{0,b-2}[K]$ and $\mathcal{Z}^k w_{\Lambda^c}$ is in $h^{\frac{1}{2}-\delta'_{k+1}}\tilde{\mathcal{B}}_\infty^{0,b-2}[F]+h^{1-\delta'_{k+1}}\tilde{\mathcal{B}}_\infty^{-1,b-2}[F]$. We plug these informations inside (3.12). We get

$$\big\|\Delta^h_j\mathcal{Z}^kB_0\big(W_\Lambda,W_{\Lambda^c}\big)\big\|_{L^\infty}\le C2^j\sum_{\substack{\max(j_1,j_2)\ge j-C\\ j_1,j_2\in J(h,C)}}2^{\frac{1}{2}\min(j_1,j_2)}\Big[h^{\frac{1}{2}-2\delta'_{k+1}}+h^{1-2\delta'_{k+1}}2^{-\frac{j_2}{2}}\Big]$$
$$\times 2^{-(j_{1+}+j_{2+})(b-2)}.$$

Summing using the fact that the number of negative j_ℓ's in $J(h,C)$ is $O(|\log(h)|)$, we obtain a bound in $2^{j-j_+(b-2)}h^{\frac{1}{2}-2\delta'_{k+1}}$ which is the bound characterizing elements of $h^{\frac{1}{2}-2\delta'_{k+1}}\mathcal{B}^{2,b-2}_{\infty}[F]$ (where F is a closed set of the form $C^{-1}\leq|\xi|\leq C$). In the same way

$$\begin{aligned}\big\|\Delta_j^h\mathcal{Z}^kB_0\big(W_{\Lambda^c},W_{\Lambda^c}\big)\big\|_{L^\infty}\leq C2^j\sum_{\substack{\max(j_1,j_2)\geq j-C\\ j_1,j_2\in J(h,C)}}2^{\frac{1}{2}\min(j_1,j_2)}&\Big[h^{\frac{1}{2}-\delta'_{k+1}}+h^{1-\delta'_{k+1}}2^{-\frac{j_1}{2}}\Big]\\ &\times\Big[h^{\frac{1}{2}-\delta'_{k+1}}+h^{1-\delta'_{k+1}}2^{-\frac{j_2}{2}}\Big]\\ &\times 2^{-(j_{1+}+j_{2+})(b-2)}.\end{aligned}$$

Summing we get a bound in $C\Big[2^jh^{1-2\delta'_{k+1}}+2^{\frac{j}{2}}h^{2-0-2\delta'_{k+1}}\Big]2^{-j_+(b-2)}$. This characterizes an element of $h^{1-2\delta'_{k+1}}\mathcal{B}^{2,b-2}_{\infty}[F]+h^{2-0-2\delta'_{k+1}}\mathcal{B}^{1,b-2}_{\infty}[F]$. Summarizing, we get finally that $\tilde{g}_2$ is in $h^{-2\delta'_{k+1}}\tilde{\mathcal{B}}^{1,b-\frac{5}{2}}_{\infty}[F]$ which is the wanted conclusion since we assume $b'<b-3$.

To estimate $\mathrm{Op}_h(x\xi)\tilde{g}_2$, we have to perform similar estimates replacing $B_0\big(W_\Lambda,W_{\Lambda^c}\big)$ (resp. $B_0\big(W_{\Lambda^c},W_{\Lambda^c}\big)$) by $(xhD_x)B_0\big(W_\Lambda,W_{\Lambda^c}\big)$ (resp. $(xhD_x)B_0\big(W_{\Lambda^c},W_{\Lambda^c}\big)$). If $S(\xi)$ is a positively homogeneous function of order $\lambda>0$, smooth outside zero,

$$[\mathrm{Op}_h(x\xi),\mathrm{Op}_h(S(\xi))]=i\lambda h\,\mathrm{Op}_h(S).$$

Consequently, the expression (3.7) of Q_0 and Leibniz rule show that $\mathrm{Op}_h(x\xi)Q_0(V)$ may be expressed from $B_0\big(\mathrm{Op}_h(x\xi)V,V\big)$ and from $h\tilde{B}_0(V,V)$, where $\tilde{B}_0$ is a bilinear form satisfying the same estimates (3.12) as B_0 (Actually, $\tilde{B}_0$ is either a multiple of the polar form of the quadratic form in the first line of the right hand side of (3.7), or a multiple of the polar forms of the sum of the second and third lines). The last property stated in Lemma 5.1 implies that

$$\mathcal{Z}^k\,\mathrm{Op}_h(x\xi)w_{\Lambda^c}\in h^{\frac{1}{2}-\delta'_{k+1}}\tilde{\mathcal{B}}^{0,b-\frac{5}{2}}_{\infty}[F]+h^{1-\delta'_{k+1}}\tilde{\mathcal{B}}^{-1,b-\frac{5}{2}}_{\infty}[F].$$

Moreover, still because of this lemma, $\mathcal{Z}^kw_\Lambda$ is in $h^{-\delta'_{k+1}}\tilde{\mathcal{B}}^{0,b-2}_{\infty}[K]$ for a compact subset K of $T^*(\mathbb{R}\setminus\{0\})$. It follows from (2.13) and the fact that $x\xi$ restricted to such a compact set is in the class of symbols $S(1)$, that $\mathcal{Z}^k\,\mathrm{Op}_h(x\xi)w_\Lambda$ is in $h^{-\delta'_{k+1}}\tilde{\mathcal{B}}^{1,b-2}_{\infty}[K]\subset h^{-\delta'_{k+1}}\tilde{\mathcal{B}}^{0,b-\frac{5}{2}}_{\infty}[F]$. This shows that to estimate $\mathrm{Op}_h(x\xi)B_0(W_\Lambda,W_{\Lambda^c})$, $\mathrm{Op}_h(x\xi)B_0(W_{\Lambda^c},W_{\Lambda^c})$, it suffices to use the bounds obtained above for $B_0(W_\Lambda,W_{\Lambda^c})$, $B_0(W_{\Lambda^c},W_{\Lambda^c})$ replacing b by $b-\frac{1}{2}$. We conclude that $\mathrm{Op}_h(x\xi)\tilde{g}_2$ is in $h^{-2\delta'_{k+1}}\tilde{\mathcal{B}}^{1,b-3}_{\infty}[F]\subset h^{-2\delta'_{k+1}}\tilde{\mathcal{B}}^{1,b'+\frac{1}{2}}_{\infty}[F]$.

We compute next $Q_0(W_\Lambda)$ from (3.7). Let us examine first the contributions that are bilinear in $(w_\Lambda, \bar{w}_\Lambda)$ i.e.

$$
\begin{aligned}
(5.26)\qquad & -\frac{i}{4}\operatorname{Op}_h(|\xi|^{\frac{1}{2}})\Big[\big(\operatorname{Op}_h(\xi|\xi|^{-\frac{1}{2}})w_\Lambda\big)\big(\operatorname{Op}_h(\xi|\xi|^{-\frac{1}{2}})\bar{w}_\Lambda\big)\Big]\\
& -\frac{i}{4}\operatorname{Op}_h(|\xi|^{\frac{1}{2}})\Big[\big(\operatorname{Op}_h(|\xi|^{\frac{1}{2}})w_\Lambda\big)\big(\operatorname{Op}_h(|\xi|^{\frac{1}{2}})\bar{w}_\Lambda\big)\Big]\\
& +\frac{i}{4}\operatorname{Op}_h(|\xi|)\Big[w_\Lambda\operatorname{Op}_h(|\xi|^{\frac{1}{2}})\bar{w}_\Lambda-\bar{w}_\Lambda\operatorname{Op}_h(|\xi|^{\frac{1}{2}})w_\Lambda\Big]\\
& -\frac{i}{4}\operatorname{Op}_h(\xi)\Big[w_\Lambda\operatorname{Op}_h(\xi|\xi|^{-\frac{1}{2}})\bar{w}_\Lambda-\bar{w}_\Lambda\operatorname{Op}_h(\xi|\xi|^{-\frac{1}{2}})w_\Lambda\Big].
\end{aligned}
$$

We use that (5.12) may be computed from (5.13), up to a remainder given by (5.14) with $\mu_2 = 3$, $b' = b-2$, that contributes to $\sqrt{h}\tilde{g}_2$ in (5.7) (since $b' < b - \frac{5}{2}$ and b is large enough). Notice that the main contribution, computed from (5.13) vanishes. For the terms inside the first two brackets in (5.26), this follows from a two by two cancellation between the two contributions in each bracket. For the last term in (5.26), we remark that the symbol ξ of the outside operator $\operatorname{Op}_h(\xi)$ is an homogeneous polynomial, which allows us to make use of expansion (5.13) with $a_1 \equiv \xi$, $p_1 + p_2 = 0$, and implies as well the vanishing of that term.

We are left with studying the quadratic terms in w_Λ and the quadratic terms in $\bar{w}_\Lambda$ in (3.7). We may apply to both of them $i)$ of Lemma 5.6 with $(p_1, p_2) = (1,1)$ or $(p_1,p_2) = (-1,-1)$. We get the contribution to $Q_0(W_\Lambda)$ given by the sum of the two expressions (5.8).

To study $C_0(W)$, we use that the assumptions imply that $\mathcal{Z}^k w$ is in $h^{-\delta'_{k+1}}\tilde{\mathcal{B}}^{0,b-2}_\infty[F]$ (This follows from the fact that $h\tilde{\mathcal{B}}^{-1,b-2}_\infty[F] \subset h^\sigma\tilde{\mathcal{B}}^{0,b-2}_\infty[F]$, as a consequence of the inequality $h_j = O(h^\sigma)$). To bound $\big\|\Delta^h_j\mathcal{Z}^k C_0(W)\big\|_{L^\infty}$, we apply (3.15) with $p=\infty$, $d = b-\alpha-2-0$, $V_1 = V_2 = V_3 = W$. Our assumptions on w and d imply that

$$
\big\|\mathcal{Z}^k\langle hD_x\rangle^{\alpha+d}w\big\|_{L^\infty} = O\big(h^{-\delta'_{k+1}-0}\big)
$$

as is seen from the expansion $w = \sum_{j\in J(h,C)}\Theta^*_j w_j$ and the bounds on the w_j's. It follows from (3.15) that

$$
\big\|\Delta^h_j\mathcal{Z}^k C_0(W)\big\|_{L^\infty} = O\Big(2^{\frac{j}{2}-j_+(b-\alpha-2-0)}h^{-3\delta'_{k+1}-0}\Big). \tag{5.27}
$$

The conclusion $\mathcal{Z}^k g_3 \in h^{-3\delta'_{k+1}-0}\tilde{\mathcal{B}}^{1,b'+\frac{1}{2}}_\infty[F]$ follows if we assume $b' < b - \frac{9}{2}$ (since α is any number strictly larger than 2).

To obtain that $\mathcal{Z}^k\operatorname{Op}_h(x\xi)\tilde{g}_3$ is in $h^{-3\delta'_{k+1}-0}\tilde{\mathcal{B}}^{1,b'+\frac{1}{2}}_\infty[F]$ we make act $\operatorname{Op}_h(x\xi)$ on $C_0(W)$ and we argue as in the study of quadratic terms, distributing xhD_x on the different factors using Leibniz rule. We have seen that $\mathcal{Z}^k\operatorname{Op}_h(x\xi)W$ is $h^{-\delta'_{k+1}}\tilde{\mathcal{B}}^{0,b-\frac{5}{2}}_\infty[F]$. It follows as above that we get for $\big\|\Delta^h_j\mathcal{Z}^k\operatorname{Op}_h(x\xi)C_0(W)\big\|_{L^\infty}$ the same estimate as (5.27), with b replaced by $b-1/2$. This gives the wanted bound as $b' < b-5$. This concludes the proof of $i)$ of the lemma.

$ii)$ Let us show first that we may replace in the quadratic (resp. cubic) part of the left hand side of (5.9) w by $w_p = w_\Lambda + \sqrt{h}\big(w_{2\Lambda} + w_{-2\Lambda}\big)$ (resp. by w_Λ) up to a contribution to the

$h\tilde{g}$ term in the right hand side. By assumption, $\mathcal{Z}^k w_p$ is in $h^{-\delta'_{k+1}}\tilde{\mathcal{B}}^{0,b'}_\infty[F]$ for some $b' < b - \frac{9}{2}$. Set $W_p = (w_p, \bar{w}_p)$, $G = (g, \bar{g})$ and let us show that the contributions of $\mathcal{Z}^k B_0(W_p, G)$ and $h\mathcal{Z}^k B_0(G, G)$ are in $h^{-4\delta'_{k+P}}\tilde{\mathcal{B}}^{2,b'}_\infty[F]$. We use (3.12) and the assumption that $\mathcal{Z}^k g$ is in $h^{-3\delta'_{k+P}}\tilde{\mathcal{B}}^{0,b'}_\infty[F]$ to bound using (3.12)

$$\big\|\Delta^h_j \mathcal{Z}^k B_0\big(W_p, G\big)\big\|_{L^\infty} \leq C 2^j \sum_{\substack{\max(j_1,j_2)\geq j-C \\ j_1,j_2\in J(h,C)}} 2^{\frac{1}{2}\min(j_1,j_2)} h^{-\delta'_{k+1}-3\delta'_{k+P}} 2^{-(j_{1+}+j_{2+})b'}.$$

We get an estimate in $O\big(h^{-4\delta'_{k+P}} 2^{j-j_+ b'}\big)$ which shows the wanted conclusion. One argues in the same way for $hB_0(G, G)$.

Considering the cubic term, we write $w = w_\Lambda + \sqrt{h}g'$, where $g' = (w_{2\Lambda} + w_{-2\Lambda}) + \sqrt{h}g$ satisfies $\mathcal{Z}^k g' \in h^{-2\delta'_{k+1}}\tilde{\mathcal{B}}^{0,b'}_\infty[F]$ and where $\mathcal{Z}^k w_\Lambda$ is in $h^{-\delta'_{k+1}}\tilde{\mathcal{B}}^{0,b'}_\infty[F]$. If we set $G' = (g', \bar{g}')$, we have to study $\big\|\Delta^h_j \mathcal{Z}^k\big[C_0(W_\Lambda + \sqrt{h}G') - C_0(W_\Lambda)\big]\big\|_{L^\infty}$ i.e. $\sqrt{h}\big\|\Delta^h_j \mathcal{Z}^k T_0(W_\Lambda, W_\Lambda, G')\big\|_{L^\infty}$, $h\big\|\Delta^h_j \mathcal{Z}^k T_0(W_\Lambda, G', G')\big\|_{L^\infty}$ and $h^{\frac{3}{2}}\big\|\Delta^h_j \mathcal{Z}^k T_0(G', G', G')\big\|_{L^\infty}$. If $d = b' - \alpha - 0$, we bound

$$\big\|\mathcal{Z}^k \langle hD_x\rangle^{\alpha+d} W_\Lambda\big\|_{L^\infty} = O\big(h^{-0-\delta'_{k+1}}\big), \quad \big\|\mathcal{Z}^k \langle hD_x\rangle^{\alpha+d} G'\big\|_{L^\infty} = O\big(h^{-2\delta'_{k+1}-0}\big).$$

Plugging these estimates in (3.15), we get for the quantities under study a bound in terms of $2^{\frac{j}{2}-j_+(b'-\alpha-0)}h^{-4\delta'_{k+1}+\frac{1}{2}}$. Let us study as well

$$\mathrm{Op}_h(x\xi)B_0(W_\Lambda, G), \quad \mathrm{Op}_h(x\xi)B_0(G, G), \quad \mathrm{Op}_h(x\xi)\big[C_0\big(W_\Lambda + \sqrt{h}G'\big) - C_0\big(W_\Lambda\big)\big].$$

As in the proof of i), we may express these quantities from

$$\begin{gathered} B_0\big(\mathrm{Op}_h(x\xi)W_p, G\big), \quad B_0\big(W_p, \mathrm{Op}_h(x\xi)G\big), \\ \sqrt{h}T_0\big(\mathrm{Op}_h(x\xi)W_\Lambda, W_\Lambda, G'\big), \quad \sqrt{h}T_0\big(W_\Lambda, W_\Lambda, \mathrm{Op}_h(x\xi)G'\big), \\ hT_0\big(W_\Lambda, \mathrm{Op}_h(x\xi)G', G'\big), \quad hT_0\big(W_\Lambda, \mathrm{Op}_h(x\xi)G', G'\big), \quad h^{\frac{3}{2}}T_0\big(\mathrm{Op}_h(x\xi)G', G', G'\big), \end{gathered}$$

and from quadratic and cubic quantities of the form of those already estimated. By assumption, the g-term in (5.5) satisfies $\mathcal{Z}^k \mathrm{Op}_h(x\xi)g \in h^{-3\delta'_{k+P}}\tilde{\mathcal{B}}^{1,b'}_\infty[F] \subset h^{-3\delta'_{k+P}}\tilde{\mathcal{B}}^{0,b'-\frac{1}{2}}_\infty[F]$. Moreover, by definition of that quantity, $\mathcal{Z}^k \mathrm{Op}_h(x\xi)w_p$ is in $h^{-\delta'_{k+1}}\tilde{\mathcal{B}}^{0,b'}_\infty[F]$. The above estimate of B_0, with b' replaced by $b' - \frac{1}{2}$, shows that

$$\big\|\Delta^h_j \mathcal{Z}^k B_0\big(\mathrm{Op}_h(x\xi)W_p, G\big)\big\|_{L^\infty} + \big\|\Delta^h_j \mathcal{Z}^k B_0\big(W_p, \mathrm{Op}_h(x\xi)G\big)\big\|_{L^\infty} = O\big(h^{-4\delta'_{k+P}} 2^{j-j_+(b'-\frac{1}{2})}\big).$$

We obtain a $h^{-4\delta'_{k+P}}\tilde{\mathcal{B}}^{2,b'-\frac{1}{2}}_\infty[F] \subset h^{-4\delta'_{k+P}}\tilde{\mathcal{B}}^{1,b'-\frac{3}{2}}_\infty[F]$ contribution to $\mathcal{Z}^k \mathrm{Op}_h(x\xi)\tilde{g}$ in the action of $\mathrm{Op}_h(x\xi)$ on (5.9). The definition of g' implies that $\mathcal{Z}^k \mathrm{Op}_h(x\xi)g'$ is in $h^{-2\delta'_{k+1}}\tilde{\mathcal{B}}^{0,b'-\frac{1}{2}}_\infty[F]$. Bounding, with $d = b' - \frac{1}{2} - \alpha - 0$

$$\begin{aligned} \big\|\mathcal{Z}^k \langle hD_x\rangle^{\alpha+d}\mathrm{Op}_h(x\xi)W_\Lambda\big\|_{L^\infty} &= O\big(h^{-0-\delta'_{k+1}}\big), \\ \big\|\mathcal{Z}^k \langle hD_x\rangle^{\alpha+d}\mathrm{Op}_h(x\xi)G'\big\|_{L^\infty} &= O\big(h^{-0-2\delta'_{k+1}}\big), \end{aligned}$$

we get again from (2.15) that

$$\left\|\Delta_j^h \mathcal{Z}^k \operatorname{Op}_h(x\xi)\left[C_0\left(W_\Lambda+\sqrt{h}G'\right)-C_0\left(W_\Lambda\right)\right]\right\|_{L^\infty}=O\left(2^{\frac{j}{2}-j_+(b'-\frac{1}{2}-\alpha-0)}h^{-4\delta'_{k+1}+\frac{1}{2}}\right).$$

If we replace above b' by $b'+3$ (since $\alpha=2+0$), which corresponds to decreasing by 3 units the assumption made on $b'-b$ in Proposition 5.2 (i.e. imposing $b'<b-8$), we obtain finally that the contributions of $Q_0(W)-Q_0(W_p)$ and $C_0(W)-C_0(W_\Lambda)$ to (5.9) may be incorporated into the $h\tilde{g}$ term of the right hand side. We are reduced to the study of

$$Q_0\left(W_\Lambda+\sqrt{h}(W_{2\Lambda}+W_{-2\Lambda})\right),\quad C_0(W_\Lambda).$$

To treat the first expression, we use *ii*) of Lemma 5.6, which allows us to compute expressions (3.7) using (3.11). The remainders satisfy bounds of the form (5.20) with $\mu_2=3$, so may be incorporated to the $h\tilde{g}$ term in (5.9). We have already seen in the proof of *i*) that the $O(1)$ term in (5.9) is given by (5.8).

The $O(\sqrt{h})$ term is computed from (5.19) applied to the different contributions to Q_0 given by (3.7). We need to compute explicitly only the Λ-oscillating term i.e. the contributions to (5.19) corresponding to $p_1\pm 2p_2=1$ and $p_2\pm 2p_1=1$ ($p_1,p_2\in\{-1,1\}$). From the expression (3.7) of Q_0 and (5.19), we get a contribution

$$(1-\chi)\left(xh^{-\beta}\right)\frac{i}{2}\,|\mathrm{d}\,\omega|^{\frac{3}{2}}\,\overline{w}_\Lambda\left(w_{2\Lambda}-\overline{w}_{-2\Lambda}\right). \tag{5.28}$$

In the same way, using (5.16), we compute the Λ-oscillating cubic term coming from the expression (3.8) of $C_0(W_\Lambda)$. We obtain a contribution $\frac{1}{4}(1-\chi)\left(xh^{-\beta}\right)|\,\mathrm{d}\,\omega(x)|^{\frac{5}{2}}|w_\Lambda|^2 w_\Lambda$. Summing up, we get

$$\begin{aligned} Q_0(w,\bar{w})+\sqrt{h}C_0(w,\bar{w}) &= \widetilde{w}_{2\Lambda}+\widetilde{w}_{-2\Lambda}\\ &\quad+\sqrt{h}\left(\widetilde{w}_{3\Lambda}+\widetilde{w}_{\Lambda}+\widetilde{w}_{-\Lambda}+\widetilde{w}_{-3\Lambda}\right)\\ &\quad+h\tilde{g}_1 \end{aligned}$$

where according to (5.17) and (5.20),

$$\mathcal{Z}^k\tilde{g}_1\in h^{-4\delta'_{k+1}}\tilde{\mathcal{B}}_\infty^{2,b'+1}[F]\subset h^{-4\delta'_{k+1}}\tilde{\mathcal{B}}_\infty^{1,b'+\frac{1}{2}}[F]$$

(if b' is large enough), and where $\widetilde{w}_\Lambda$ is given by (5.10). The contributions $\widetilde{w}_{3\Lambda}$, $\widetilde{w}_{-\Lambda}$, $\widetilde{w}_{-3\Lambda}$ have expressions (5.11), for which we do not need to compute explicitly the coefficients μ'_ℓ, μ''_ℓ. This concludes the proof of the lemma. □

The next step of the proof of Proposition 5.2 will be to deduce from equation (3.10), and from the description provided by Lemma 5.3 of the right hand side of this equation, an expansion of $\operatorname{Op}_h(\gamma_\Lambda^c)w$, exploiting that $2x\xi+|\xi|^{\frac{1}{2}}$ is an elliptic symbol on the support of γ_Λ^c. In a first step, we establish some a priori bounds for the components w_j of w cut-off outside a neighborhood of Λ.

Lemma 5.7. *Assume that (4.9) holds for $k \leq \frac{s}{2} + N_1$ for some integer N_1 satisfying $(N_1 - N_0)\sigma \geq 1$. Then for any symbol a in $S(1)$, microlocally supported outside a neighborhood of Λ, the following estimate holds for $k \leq \frac{s}{2} + N_0$, and any j in $J(h, C)$,*

$$\big\|\mathrm{Op}_{h_j}(a)\mathcal{Z}_j^k w_j\big\|_{L^\infty} \leq C_k \varepsilon h^{\frac{1}{2}-\delta'_{k+N_1-N_0}}\big(2^{j/2} + h^{1/2}\big)2^{-j+b'} \tag{5.29}$$

for any $b' < b - \alpha < b - 2$.

Proof. Let us construct for any $1 \leq \ell \leq N_1 - N_0$, any $k \leq \frac{s}{2} + N_1 - \ell$, a family of symbols in $S(1)$, $(b^\ell_{\ell'})_{0\leq \ell' \leq \ell}$, vanishing close to Λ and a sequence $(r_j^{\ell,k})_{j\in J(h,C)}$ with

$$\big\|r_j^{\ell,k}\big\|_{L^\infty} \leq C\varepsilon h^{\frac{1}{2}-\delta'_{k+\ell}}\big(2^{j/2} + h^{1/2}\big)2^{-j+b'} \tag{5.30}$$

such that

$$\mathcal{Z}_j^k \,\mathrm{Op}_{h_j}(a) w_j = h_j^\ell \left[\sum_{\ell'=0}^{\ell} \mathcal{Z}_j^{k+\ell'}\,\mathrm{Op}_{h_j}\big(b^\ell_{\ell'}\big) w_j^{(\ell,\ell')}\right] + r_j^{\ell,k} \tag{5.31}$$

where $w_j^{(\ell,\ell')}$ denotes a function defined like $w_j = \Theta^*_{-j}\Delta_j^h w$ but with Δ_j^h replaced by another cut-off of the same type. Then (5.31) with $\ell = N_1 - N_0$ implies (5.29) since $h_j^{N_1-N_0} \leq h^{\sigma(N_1-N_0)} \leq h$ and since $w_j^{(\ell,\ell')}$ satisfies the same L^∞ estimates (4.9) as w_j.

We remark that to prove (5.31), we just need to treat the case $\ell = 1$ and iterate the formula. Finally, to obtain (5.31) with $\ell = 1$, we use that, by the symbolic calculus of appendix, and since a vanishes close to Λ, we may find a symbol q in $S\big(\langle x\rangle^{-1}\big) \subset S(1)$, vanishing close to Λ, a symbol ρ in $S(1)$ such that

$$\mathrm{Op}_{h_j}(a) = \mathrm{Op}_{h_j}(q)\,\mathrm{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big) + h_j^N\,\mathrm{Op}_{h_j}(\rho)$$

for an arbitrary integer N. If N is large enough, the fact that (4.9) holds implies that $\mathcal{Z}_j^k\big(h_j^N\,\mathrm{Op}_{h_j}(\rho)w_j\big)$ satisfies estimate (5.30) with $\ell = 1$ for all $j \in J(h, C)$. We are thus reduced to showing that

$$\mathcal{Z}_j^k\,\mathrm{Op}_{h_j}(q)\,\mathrm{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big)w_j \tag{5.32}$$

may be written as the right hand side of (5.31) with $\ell = 1$. We use (4.17). On the one hand, we get a contribution to (5.32) of the form

$$\mathcal{Z}_j^k\Big[h_j\,\mathrm{Op}_{h_j}(q)\Big(\frac{i}{2}\widetilde{w}_j - iZw_j\Big)\Big]$$

that forms part of the sum in (5.31) with $\ell = 1$. On the other hand, the nonlinear terms in (4.17) bring an expression

$$\mathcal{Z}_j^k\,\mathrm{Op}_{h_j}(q)\Big[-\sqrt{h}2^{-\frac{j}{2}}\Theta^*_{-j}\Delta_j^h Q_0(W) - h2^{-\frac{j}{2}}\Theta^*_{-j}\Delta_j^h C_0(W) - 2^{-\frac{j}{2}}h^{\frac{5}{4}}\Theta^*_{-j}\Delta_j^h R(V)\Big].$$

Let us check that these terms satisfy estimates (5.30) with $\ell = 1$. For the quadratic terms, this follows from (3.12) (with $p = p_{k_1} = p_{k_2} = \infty$) and from the assumption $\delta'_{k_1} + \delta'_{k_2} \leq \delta'_{k+1}$ if $k_1 + k_2 \leq k$, that follows from (4.1). For the cubic term (resp. the remainder) we use (3.15) (resp. (3.17)) and the estimates of $\left\|\mathcal{Z}^k \langle hD_x\rangle^{\alpha+d} V\right\|_{L^\infty}$ deduced from (4.9) with $d = b-\alpha-0$, $b' < b-\alpha-0$. This concludes the proof. $\square$

We shall use the preceding lemma to give an asymptotic expansion of $\mathrm{Op}_{h_j}(\gamma^c_\Lambda)w_j$, assuming that we know a priori that $Q_0(w,\bar w)$ admits the expansion given by equality (5.7) in Lemma 5.3, or that $Q_0(w,\bar w) + \sqrt{h}C_0(w,\bar w)$ obeys the equality (5.9) of the same lemma.

Lemma 5.8. *i) Assume $b' < b-5$ and that $Q_0(W)$ satisfies (5.7). Then there are functions $w_{\pm 2\Lambda} = \sum_{j\in J(h,C)} \Theta^*_j w_{\pm 2\Lambda,j}$ such that for any $k \leq \frac{s}{2} + N_0$, $\mathcal{Z}^k w_{\pm 2\Lambda}$ is an $O(\varepsilon)$ element of $h^{-2\delta'_{k+1}} L^\infty \tilde I^{2,b'+\frac{3}{2}}_{\pm 2\Lambda}\left[K_{\pm 2}\right]$ and a function $g = \sum_{j\in J(h,C)} \Theta^*_j g_j$ such that $\mathcal{Z}^k g$ is an $O(\varepsilon)$ element of $h^{-3\delta'_{k+1+N_1-N_0}} \tilde{\mathcal{B}}^{0,b'}_\infty[F]$ for some closed subset F of $T^*\mathbb{R}$ whose second projection is compact in $\mathbb{R}^*$ and $\mathcal{Z}^k\,\mathrm{Op}_h(x\xi)g$ is an $O(\varepsilon)$ element of $h^{-3\delta'_{k+1+N_1-N_0}} \tilde{\mathcal{B}}^{1,b'}_\infty[F]$, so that*

$$
\begin{aligned}
w_{2\Lambda} &= -i(1-\chi)\big(xh^{-\beta}\big)\frac{1+\sqrt{2}}{4}\,|\mathrm{d}\,\omega(x)|\,w_\Lambda^2\\
w_{-2\Lambda} &= -i(1-\chi)\big(xh^{-\beta}\big)\frac{1-\sqrt{2}}{4}\,|\mathrm{d}\,\omega(x)|\,\bar w_\Lambda^2
\end{aligned}
\tag{5.33}
$$

and for any j in $J(h,C)$

$$
\mathrm{Op}_{h_j}(\gamma^c_\Lambda)w_j = \sqrt{h}\big(w_{2\Lambda,j} + w_{-2\Lambda,j}\big) + hg_j.
\tag{5.34}
$$

ii) Assume that $b' < b-8$ and $Q_0(w,\bar w) + \sqrt{h}C_0(w,\bar w)$ obeys (5.9). Then there are functions $w_{\pm 2\Lambda}$ such that for any $k \leq \frac{s}{2} + N_0$, $\mathcal{Z}^k w_{\pm 2\Lambda}$ is an $O(\varepsilon)$ element of $h^{-2\delta'_{k+1}} L^\infty \tilde J^{2,b'+\frac{3}{2}}_{\pm 2\Lambda}\left[K_{\pm 2}\right]$, there are functions $w_{\ell\Lambda} = \sum_{j\in J(h,C)} \Theta^*_j w_{\ell\Lambda,j}$ for $\ell = -3,-1,3$, such that for $k \leq \frac{s}{2} + N_0$, $\mathcal{Z}^k w_{\ell\Lambda}$ is an $O(\varepsilon)$ element in $h^{-3\delta'_{k+1}} L^\infty \tilde I^{2,b'+\frac{3}{2}}_{\ell\Lambda}\left[K_\ell\right]$, a function $g = \sum_{j\in J(h,C)} \Theta^*_j g_j$ such that $\mathcal{Z}^k g$ is $O(\varepsilon)$ in $h^{-4\delta'_{k+1+N_1-N_0}} \tilde{\mathcal{B}}^{0,b'}_\infty[L]$ and $\mathcal{Z}^k\,\mathrm{Op}_h(x\xi)g$ is an $O(\varepsilon)$ element of $h^{-4\delta'_{k+1+N_1-N_0}} \tilde{\mathcal{B}}^{0,b'-\frac{1}{2}}_\infty[F]$ so that

$$
\begin{aligned}
\mathrm{Op}_{h_j}(\gamma^c_\Lambda)w_j = \sqrt{h}&\big(w_{2\Lambda,j} + w_{-2\Lambda,j}\big)\\
&+ h\big(w_{3\Lambda,j} + w_{-\Lambda,j} + w_{-3\Lambda,j}\big)\\
&+ h^{1+\sigma}g_j.
\end{aligned}
\tag{5.35}
$$

Moreover, $w_{\pm 2\Lambda}$ is still given by (5.33) and

$$
\begin{aligned}
w_{3\Lambda} &= (1-\chi)\big(xh^{-\beta}\big)\lambda_3\,|\mathrm{d}\,\omega(x)|^2\,w_\Lambda^3\\
w_{-\Lambda} &= (1-\chi)\big(xh^{-\beta}\big)\lambda_{-1}\,|\mathrm{d}\,\omega(x)|^2\,|w_\Lambda|^2\,\bar w_\Lambda\\
w_{-3\Lambda} &= (1-\chi)\big(xh^{-\beta}\big)\lambda_{-3}\,|\mathrm{d}\,\omega(x)|^2\,\bar w_\Lambda^3
\end{aligned}
\tag{5.36}
$$

for some real constants λ_3, λ_{-1}, λ_{-3}.

Proof. *i)* By Corollary A.3 of the appendix, we may find symbols a in $S(\langle x\rangle^{-1})$, c in $S(1)$, supported in a domain $C_0^{-1} \leq |\xi| \leq C_0$ and outside a neighborhood of Λ such that $\gamma_\Lambda^c = a\#(2x\xi + |\xi|^{\frac{1}{2}}) + h_j^N c$. Moreover, we may write

$$a = (2x\xi + |\xi|^{\frac{1}{2}})^{-1}\gamma_\Lambda^c + h_j a_1 \tag{5.37}$$

for some symbol a_1 in $S(\langle x\rangle^{-1})$. We get

$$\operatorname{Op}_{h_j}(\gamma_\Lambda^c)w_j = \operatorname{Op}_{h_j}(a)\operatorname{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big)w_j + h_j^N \operatorname{Op}_{h_j}(c)w_j. \tag{5.38}$$

Since $h_j \leq h^\sigma$, taking N large enough, we see that assumption (4.9) implies that the last term in (5.38) may be written as $h^{\frac{3}{2}}g_j$ with $\left\|\mathcal{Z}_j^k g_j\right\|_{L^\infty} = O(h^{-\delta_k'}2^{-j_+b'})$ for $k \leq \frac{s}{2} + N_0$. By construction, g_j is microlocally supported in a closed set of the form $C_0^{-1} \leq |\xi| \leq C_0$.

Moreover, since by Lemma 5.1 $\mathcal{Z}^k w_\Lambda$ is in $h^{-\delta_{k+1}'}\tilde{\mathcal{B}}_\infty^{0,b-2}[K]$, we get that $\mathcal{Z}^k\operatorname{Op}_h(x\xi)w_\Lambda$ belongs to $h^{-\delta_{k+1}'}\tilde{\mathcal{B}}_\infty^{1,b-2}[K]$. Since $\mathcal{Z}^k\operatorname{Op}_h(x\xi)w_{\Lambda^c}$ is by the same lemma in $h^{\frac{1}{2}-\delta_{k+1}'}\tilde{\mathcal{B}}_\infty^{1,b-2}[F] + h^{1-\delta_{k+1}'}\tilde{\mathcal{B}}_\infty^{0,b-2}[F]$, we get that $\mathcal{Z}^k\operatorname{Op}_h(x\xi)w$ belongs to $h^{-\delta_{k+1}'}\tilde{\mathcal{B}}_\infty^{1,b-2}[F] + h^{1-\delta_{k+1}'}\tilde{\mathcal{B}}_\infty^{0,b-2}[F]$. Since $h2^{-\frac{j}{2}} = h_j = O(h^\sigma) = O(1)$, this implies for $\left\|\mathcal{Z}_j^k\operatorname{Op}_{h_j}(x\xi)w_j\right\|_{L^\infty}$ a bound in

$$2^{-\frac{j}{2}}h^{-\delta_{k+1}'}\left[2^{\frac{j}{2}-j_+(b-2)} + h2^{-j_+(b-2)}\right] \leq Ch^{-\delta_{k+1}'-j_+(b-2)}$$

This implies that $\left\|\mathcal{Z}_j^k\operatorname{Op}_{h_j}(x\xi)g_j\right\|_{L^\infty}$ is $O\big(h^{-\delta_{k+1}'}2^{-j_+b'}\big)$ so that $\sum_{j\in J(h,C)}\Theta_j^* g_j$ brings a contribution to the g function in the statement of the lemma.

We use expression (4.17) to study the first term in the right hand side of (5.38). The contribution of

$$\mathcal{Z}_j^k\Big[h_j\operatorname{Op}_{h_j}(a)\Big(\frac{i}{2}\widetilde{w}_j - iZw_j\Big)\Big] \tag{5.39}$$

has according to (5.29) a bound of the form

$$C_k\varepsilon h^{\frac{1}{2}-\delta_{k+1+N_1-N_0}'}\big(h + h_j h^{\frac{1}{2}}\big)2^{-j_+b'} \leq C_k\varepsilon h^{1+\sigma-\delta_{k+1+N_1-N_0}'}2^{-j_+b'} \tag{5.40}$$

using $h_j = O(h^\sigma)$. This will give a contribution to g_j in (5.34) since the action of $\operatorname{Op}_{h_j}(x\xi)$ on (5.39) admits similar bounds as a is in $S(\langle x\rangle^{-1})$.

Let us examine the contribution of

$$\begin{aligned}\mathcal{Z}_j^k\Big[&-\sqrt{h}2^{-j/2}\operatorname{Op}_{h_j}(a)\Theta_{-j}^*\Delta_j^h Q_0(W) - h2^{-j/2}\operatorname{Op}_{h_j}(a)\Theta_{-j}^*\Delta_j^h C_0(W)\\ &- 2^{-\frac{j}{2}}h^{\frac{5}{4}}\operatorname{Op}_{h_j}(a)\Theta_{-j}^*\Delta_j^h R(V)\Big]\end{aligned} \tag{5.41}$$

to (5.38). We use expressions (5.7). The contributions of $\tilde{g}_2$, $\tilde{g}_3$ to (5.41) induce in (5.34) an expression contributing to hg_j. Actually, they give terms whose L^∞-norm is $O\big(h^{-2\delta'_{k+1}+1}2^{-j_+ b'}\big)$. Moreover, the action of $\mathrm{Op}_{h_j}(x\xi)$ on these terms admit similar bounds, again because (5.41) contains a $\mathrm{Op}_{h_j}(a)$ operator in factor, with a in $S(\langle x\rangle^{-1})$. In the same way, the L^∞-norm of the last term in (5.41) (and of the action of $\mathrm{Op}_{h_j}(x\xi)$ on it) may be estimated using (3.17) with $d = b' - \alpha - 0$ and the fact that $\big\|\mathcal{Z}^k\langle hD_x\rangle^{\alpha+d}V\big\|_{L^\infty}$ is bounded using (4.9) (The loss in $2^{j_+(\alpha+0)} = O(h^{-2\beta(\alpha+0)})$ coming from the right hand side of (3.17) is absorbed by part of the $h^{1/4}$-extra factor in the last term in (5.41)). This brings another contribution to hg_j. Consequently the only contribution to (5.41) that we are left with is

$$-\mathcal{Z}_j^k\Big[2^{-j/2}\sqrt{h}\,\mathrm{Op}_{h_j}(a)\Theta^*_{-j}\Delta_j^h\big(\widetilde{w}_{2\Lambda}+\widetilde{w}_{-2\Lambda}\big)\Big]. \tag{5.42}$$

We shall study this expression in part $ii)$ of the proof below.

$ii)$ We assume that $Q_0+\sqrt{h}C_0$ obeys (5.9) and write again (5.38), expressing the right hand side from (4.17). The contribution (5.39) brings, according to (5.40), part of the term $h^{1+\sigma}g$ in (5.35). The same holds for the remainder term in (4.17). We are thus reduced to the study of the quadratic and cubic terms in (5.41). By (5.9), we have an expression for $Q_0(W)+\sqrt{h}C_0(W)$. The term $\tilde{g}$ in that expansion will bring part of the g_j term in (5.35). Consequently, we are reduced to studying

$$\begin{aligned}&\mathcal{Z}_j^k\Big[\sqrt{h}2^{-j/2}\,\mathrm{Op}_{h_j}(a)\Theta^*_{-j}\Delta_j^h\big(\widetilde{w}_{2\Lambda}+\widetilde{w}_{-2\Lambda}\big)\Big],\\ &\mathcal{Z}_j^k\Big[h2^{-j/2}\,\mathrm{Op}_{h_j}(a)\Theta^*_{-j}\Delta_j^h\big(\widetilde{w}_{3\Lambda}+\widetilde{w}_{\Lambda}+\widetilde{w}_{-\Lambda}+\widetilde{w}_{-3\Lambda}\big)\Big].\end{aligned} \tag{5.43}$$

We notice first that $\widetilde{w}_\Lambda$ is microlocally supported close to Λ while $\mathrm{Op}_{h_j}(a)$ cut-offs outside a neighborhood of that set. Consequently, the $\widetilde{w}_\Lambda$ term in the second formula (5.43) gives rise to a remainder. For $\ell\in\{-3,-2,-1,2,3\}$ and j in $J(h,C)$, set

$$w^{(1)}_{\ell\Lambda,j} = -2^{-j/2}\,\mathrm{Op}_{h_j}(a)\Theta^*_{-j}\Delta_j^h\widetilde{w}_{\ell\Lambda}. \tag{5.44}$$

Expressions (5.41), (5.42), (5.43) show that (5.35) holds with $w_{\ell\Lambda,j}$ replaced by $w^{(1)}_{\ell\Lambda,j}$. We have to show that, up to a modification of g_j in (5.35), $w^{(1)}_{\ell\Lambda,j}$ may be replaced by a function $w_{\ell\Lambda,j}$ such that $\sum_{j\in J(h,C)}\Theta_j^* w_{\ell\Lambda,j} = w_{\ell\Lambda}$ satisfies the conclusions of the lemma.

We write $\widetilde{w}_{\ell\Lambda} = \sum_{j'\in J(h,C)}\Theta^*_{j'}\widetilde{w}_{\ell\Lambda,j'}$ and set

$$\widetilde{w}^{(1)}_{\ell\Lambda,j} := \Theta^*_{-j}\Delta_j^h\widetilde{w}_{\ell\Lambda} = \sum_{j'\in J(h,C)}\Theta^*_{-j+j'}\,\mathrm{Op}_{h_{j'}}\big(\varphi(2^{-j+j'}\xi)\big)\widetilde{w}_{\ell\Lambda,j'}.$$

Since $\widetilde{w}_{\ell\Lambda,j'} = \mathrm{Op}_{h_{j'}}\big(\widetilde{\varphi}(\xi)\big)\widetilde{w}_{\ell\Lambda,j'}$ for some $\widetilde{\varphi}$ in $C_0^\infty(\mathbb{R}^*)$, we may limit the sum above to those j' satisfying $|j-j'|\le M$ for some M. This shows that $\big(\mathcal{Z}_j^k\widetilde{w}^{(1)}_{\ell\Lambda,j}\big)_j$ is a bounded family in

$$h^{-2\delta'_{k+1}}L^\infty J^{3,b'+\frac{3}{2}}_{\pm2\Lambda}\big[K_{\pm2}\big]\subset h^{-2\delta'_{k+1}}L^\infty J^{2,b'+1}_{\pm2\Lambda}\big[K_{\pm2}\big]$$

when $\ell = \pm 2$ and in $h^{-3\delta'_{k+1}} L^\infty I^{3,b'+\frac{3}{2}}_{\ell\Lambda}\big[K_\ell\big]$ if $\ell \in \{-3,-1,3\}$, according to the assumptions made on $\widetilde{w}_{\ell\Lambda}$. In the expression (5.44) of

$$w^{(1)}_{\ell\Lambda,j} = -2^{-j/2}\,\mathrm{Op}_{h_j}(a)\widetilde{w}^{(1)}_{\ell\Lambda,j},$$

we insert the decomposition (5.37) of a. Since γ^c_Λ may be assumed to be equal to one close to $\ell\Lambda$, $\ell \neq 1$, if the support of γ_Λ is close enough to Λ, we may write

$$a|_{\ell\Lambda} = \big(|\ell|^{\frac{1}{2}} - \ell\big)^{-1}\,|\mathrm{d}\,\omega(x)|^{-\frac{1}{2}} + h_j a_1|_{\ell\Lambda}$$

for $\ell \in \{-3,-2,-1,2,3\}$. Consequently, (5.44) may be written as the sum of

$$w_{\ell\Lambda,j} = -2^{-\frac{j}{2}}\big(|\ell|^{\frac{1}{2}} - \ell\big)^{-1}\,|\mathrm{d}\,\omega(x)|^{-\frac{1}{2}}\,\Theta^*_{-j}\Delta^h_j\widetilde{w}_{\ell\Lambda} \tag{5.45}$$

and of

$$-2^{-\frac{j}{2}}\,\mathrm{Op}_{h_j}\big(c^\ell + h_j d^\ell\big)\widetilde{w}^{(1)}_{\ell\Lambda,j} \tag{5.46}$$

where c^ℓ, d^ℓ are symbols, with c^ℓ vanishing on $\ell\cdot\Lambda$.

Let us show first that (5.46) multiplied by $\sqrt{h}$ when $\ell = \pm 2$ and by h when ℓ belongs to $\{-3,-1,3\}$ provides a contribution to $h^{1+\sigma}g_j$ in (5.35). Since $\big(\mathcal{Z}^k_j\widetilde{w}^{(1)}_{\pm 2\Lambda,j}\big)_j$ is a $O(\varepsilon)$ family in $h^{-2\delta'_{k+1}}L^\infty J^{2,b'+1}_{\pm 2\Lambda}\big[K_{\pm 2}\big]$ and $c^{\pm 2}$ vanishes on $\pm 2\Lambda$, we see that the L^∞-norm of the action of $\mathcal{Z}^k_j$ on (5.46) with $\ell = \pm 2$ is bounded from above by

$$C\varepsilon 2^{-\frac{j}{2}} h^{-2\delta'_{k+1}} 2^{j - j_+(b'+1)} h_j \le C\varepsilon h^{1-2\delta'_{k+1}} 2^{-j_+(b'+1)}.$$

Consequently, when $\ell = \pm 2$, if we make act $\mathcal{Z}^k_j$ on (5.46) multiplied by $\sqrt{h}$, we obtain an element of $h^{\frac{3}{2}-2\delta'_{k+1}}\mathcal{B}^{0,b'}_\infty[L]$ i.e. a contribution to $h^{1+\sigma}g_j$ in (5.35). In the same way, when $\ell \in \{-3,-1,3\}$, using that $\big(\mathcal{Z}^k_j\widetilde{w}^{(1)}_{\ell\Lambda,j}\big)_j$ is in $h^{-3\delta'_{k+1}}L^\infty I^{2,b'+1}_{\ell\Lambda}\big[K_\ell\big]$, we may estimate the L^∞-norm of (5.46) on which acts $\mathcal{Z}^k_j$ by

$$C\varepsilon 2^{-\frac{j}{2}} h^{-3\delta'_{k+1}} 2^{j - j_+(b'+1)}\big[h^{\frac{1}{2}} + h_j\big] \le C\varepsilon h^{\frac{1}{2}-3\delta'_{k+1}} 2^{-j_+(b'+\frac{1}{2})}.$$

Again, after multiplication by h, this gives a contribution to $h^{1+\sigma}g_j$ in (5.35). Notice that the fact that $\mathrm{Op}_h(x\xi)g = \sum_{j\in J(h,C)} 2^{\frac{j}{2}}\Theta^*_j\,\mathrm{Op}_{h_j}(x\xi)g_j$ satisfies the same estimates as g, with b' replaced by $b' - \frac{1}{2}$, follows from the above bounds since $\widetilde{w}^{(1)}_{\ell\Lambda,j}$ is microlocally supported in a compact set of $T^*(\mathbb{R}\setminus\{0\})$.

We have thus shown that $\mathrm{Op}_{h_j}(\gamma^c_\Lambda)w_j$ is given by the right hand side of (5.35), with $w_{\ell\Lambda,j}$ given by (5.45). In particular since $|\mathrm{d}\,\omega|$ is positively homogeneous of degree -2, we get

$$w_{\ell\Lambda} = \sum_{j\in J(h,C)}\Theta^*_j w_{\ell\Lambda,j} = -\big(|\ell|^{\frac{1}{2}} - \ell\big)^{-1}\,|\mathrm{d}\,\omega(x)|^{-\frac{1}{2}}\,\widetilde{w}_{\ell\Lambda}.$$

Combining this with (5.8), (5.11), we obtain (5.33) and (5.36). Moreover, expressions (5.45) and the properties of $\widetilde{w}_{\ell\Lambda}$ obtained in *ii*) if Lemma 5.3 show that $\mathcal{Z}^k w_{\pm 2\Lambda}$ is in the space $h^{-2\delta'_{k+1}} L^\infty \tilde{J}^{2,b'+\frac{3}{2}}_{\pm 2\Lambda}\big[K_{\pm 2}\big]$ and that $w_{\ell\Lambda}$, $\ell \in \{-3,-1,3\}$ belongs to $L^\infty \tilde{I}^{2,b'+\frac{3}{2}}_{\ell\Lambda}\big[K_\ell\big]$, and are $O(\varepsilon)$ in these spaces. □

Proof of Proposition 5.2. Let b' satisfying the assumption of the proposition.

By Lemma 5.1, $\mathcal{Z}^k w_\Lambda$ is an $O(\varepsilon)$ element of $h^{-\delta'_{k+1}} L^\infty \tilde{I}^{0,b-2}_\Lambda[K]$ and $\mathcal{Z}^k w_{\Lambda^c}$ is an $O(\varepsilon)$ element of $h^{\frac{1}{2}-\delta'_{k+1}} \tilde{\mathcal{B}}^{0,b-2}_\infty[F] + h^{1-\delta'_{k+1}} \tilde{\mathcal{B}}^{-1,b-2}_\infty[F]$, for $0 \le k \le \frac{s}{2} + N_1$. We may therefore apply *i*) of Lemma 5.3 which shows that (5.7) holds. This allows us to use *i*) of Lemma 5.8. In that way, we obtain functions $w_{\pm 2\Lambda}$, in the spaces indicated in the statement of that lemma, such that (5.6) holds. Writing

$$w = w_\Lambda + \sum_{j \in J(h,C)} \Theta_j^* \operatorname{Op}_{h_j}(\gamma_\Lambda^c) w_j$$

and using (5.34), we obtain equality (5.5).

We still have to check that $\mathcal{Z}^k w_\Lambda$ is an $O(\varepsilon)$ element in $h^{-2\delta'_{k+1}} L^\infty \tilde{J}^{0,b'}_\Lambda[K]$ since Lemma 5.1 was only ensuring that this function is $O(\varepsilon)$ in the space $h^{-\delta'_{k+1}} L^\infty \tilde{I}^{0,b'}_\Lambda[K]$. To do so, we must show that

$$\big\| \mathcal{Z}_j^k \operatorname{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big) w_{\Lambda,j} \big\|_{L^\infty} \le C\varepsilon h^{-2\delta'_{k+1}} h_j 2^{-j+b'}. \tag{5.47}$$

We notice that, by symbolic calculus and assumption (4.9)

$$\big\| \mathcal{Z}_j^k \big[\operatorname{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big), \operatorname{Op}_{h_j}(\gamma_\Lambda) \big] w_j \big\|_{L^\infty}$$

satisfies the wanted bound, since the commutators between the vector fields and $\operatorname{Op}_{h_j}(e)$, for a symbol e, are of the form $\operatorname{Op}_{h_j}(\tilde{e})$ for another symbol $\tilde{e}$. We may therefore study

$$\big\| \mathcal{Z}_j^k \operatorname{Op}_{h_j}(\gamma_\Lambda) \operatorname{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big) w_j \big\|_{L^\infty}.$$

Using the commutation relation

$$\big[tD_t + xD_x, \operatorname{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big)\big] = i \operatorname{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big)$$

we see that the above quantity may be estimated from

$$\big\| \operatorname{Op}_{h_j}(\widetilde{\gamma}_\Lambda) \operatorname{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big) \mathcal{Z}_j^{k'} w_j \big\|_{L^\infty}$$

for $k' \le k$ and $\widetilde{\gamma}_\Lambda$ a symbol with $\operatorname{Supp} \widetilde{\gamma}_\Lambda \subset \operatorname{Supp} \gamma_\Lambda$. We use now (4.15), which provides the wanted bound of type (5.47), up to a similar estimate for

$$2^{-\frac{j}{2}} \sqrt{h} \big\| \operatorname{Op}_{h_j}(\widetilde{\gamma}_\Lambda) \mathcal{Z}_j^{k'} \Theta_{-j}^* \Delta_j^h Q_0(W) \big\|_{L^\infty}.$$

To study this quantity, we need to exploit the structure of $Q_0(W)$ given by (5.7). The remainder in the first equation (5.7) gives a contribution bounded by the right hand side of (5.47). On the other hand,

$$\left\|\mathrm{Op}_{h_j}\left(\widetilde{\gamma}_\Lambda\right)\mathcal{Z}_j^{k'}\Theta_{-j}^*\Delta_j^h\left(\widetilde{w}_{2\Lambda}+\widetilde{w}_{-2\Lambda}\right)\right\|_{L^\infty}$$

is $O\big(\varepsilon h^\infty\big)$ since $\widetilde{\gamma}_\Lambda$ cuts-off on a neighborhood of Λ while $\widetilde{w}_{\pm 2\Lambda}$ are supported close to $\pm 2\Lambda$, so outside such a neighborhood. This concludes the proof of (5.47), whence the proposition. □

Let us deduce from expansion (5.5) of w a second refined decomposition of w in oscillating factors.

Corollary 5.9. *Under the assumptions of Proposition 5.2 with moreover $b' < b - 8$, we may write*

$$w = w_\Lambda + \sqrt{h}\big(w_{2\Lambda} + w_{-2\Lambda}\big) + h\big(w_{3\Lambda} + w_{-\Lambda} + w_{-3\Lambda}\big) + h^{1+\sigma}g \tag{5.48}$$

where $w_{\pm 2\Lambda}$, $w_{\pm 3\Lambda}$, $w_{-\Lambda}$ satisfy the conclusions of ii) of Lemma 5.8 and are given in terms of w_Λ by (5.33), (5.36), and where for $k \le \frac{s}{2} + N_0$, $\mathcal{Z}^k g$ is $O(\varepsilon)$ in $h^{-4\delta'_{k+1+N_1-N_0}}\tilde{\mathcal{B}}_\infty^{0,b'}[F]$ and $\mathcal{Z}^k\,\mathrm{Op}_h(x\xi)g$ is $O(\varepsilon)$ in $h^{-4\delta'_{k+1+N_1-N_0}}\tilde{\mathcal{B}}_\infty^{0,b'-\frac{1}{2}}[F]$.

Proof. By Proposition 5.2, w may be written as (5.5). Consequently, the assumptions of *ii)* of Lemma 5.3 hold and this lemma implies a decomposition (5.9) for

$$\sum_{j\in J(h,C)} \Delta_j^h\big[Q_0(W) + \sqrt{h}C_0(W)\big].$$

This shows that the assumptions of *ii)* of Lemma 5.8 hold. According to this lemma, $\mathrm{Op}_{h_j}\left(\gamma_\Lambda^c\right)w_j$ is given by (5.35). We define $w_{\ell\Lambda} = \sum_{j\in J(h,C)}\Theta_j^* w_{\ell\Lambda,j}$ and get (5.48), remembering that we defined $w_\Lambda = \sum_{j\in J(h,C)}\Theta_j^*\,\mathrm{Op}_{h_j}(\gamma_\Lambda)w_j$ if $w = \sum_{j\in J(h,C)}\Theta_j^* w_j$. The expansion in terms of $w_\Lambda, \overline{w}_\Lambda$ follow from (5.33), (5.36). □

We have seen in Proposition 5.2, that $\mathcal{Z}^k w_\Lambda$ is an $O(\varepsilon)$ element in $h^{-\delta'_{k+1}}L^\infty\tilde{J}_\Lambda^{0,b'}[K]$. We need a more precise description of this quantity.

Proposition 5.10. *Let $w_\Lambda = \sum_{j\in J(h,C)}\Theta_j^* w_{\Lambda,j}$ be the function introduced in Proposition 5.2. There are elements $f = \sum_{j\in J(h,C)}\Theta_j^* f_j$ where $\mathcal{Z}^k f$ is $O(\varepsilon)$ in $h^{-3\delta'_{k+2}}L^\infty\tilde{I}_\Lambda^{0,b'}[K]$ for $k \le \frac{s}{2} + N_0$ and $r = \sum_{j\in J(h,C)}\Theta_j^* r_j$, with $\mathcal{Z}^k r$ of size $O(\varepsilon)$ in $h^{-4\delta'_{k+N_1-N_0+1}}\tilde{\mathcal{B}}_\infty^{0,b'}[F]$ such that*

$$\mathrm{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big)w_{\Lambda,j} = h_j\big[f_j + h^{\frac{1}{4}}r_j\big]. \tag{5.49}$$

Proof. We use the definition of $w_{j,\Lambda} = \operatorname{Op}_{h_j}(\gamma_\Lambda) w_j$ and (4.17) to write

$$\begin{aligned}
\operatorname{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big) w_{j,\Lambda} = {} & \big[\operatorname{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big), \operatorname{Op}_{h_j}(\gamma_\Lambda)\big] w_j \\
& - \sqrt{h} 2^{-\frac{j}{2}} \operatorname{Op}_{h_j}(\gamma_\Lambda) \Theta^*_{-j} \Delta^h_j Q_0(W) \\
& + h 2^{-\frac{j}{2}} \operatorname{Op}_{h_j}(\gamma_\Lambda) \Big[\frac{i}{2} \widetilde{w}_j - iZw_j - \Theta^*_{-j} \Delta^h_j C_0(W)\Big] \\
& - 2^{-\frac{j}{2}} h^{\frac{5}{4}} \operatorname{Op}_{h_j}(\gamma_\Lambda) \Theta^*_{-j} \Delta^h_j R(V).
\end{aligned} \tag{5.50}$$

The commutator term may be written $h_j \operatorname{Op}_{h_j}(e) w_j$ for some symbol e in $S(1)$, with support contained in $\operatorname{Supp} \gamma_\Lambda$ (up to a $O\big(h_j^\infty\big) = O(h^\infty)$ remainder). Consequently $\operatorname{Op}_{h_j}(e) w_j$ will satisfy the same type of properties as $w_{\Lambda,j}$ i.e. by Lemma 5.1, $\big(\mathcal{Z}^k_j \operatorname{Op}_{h_j}(e) w_j\big)_j$ will be a $O(\varepsilon)$ family in $h^{-\delta'_{k+1}} L^\infty I^{0,b'}_\Lambda[K]$ so that the first term in the right hand side of (5.50) contributes to $h_j f_j$ in (5.49).

To study the quadratic and cubic terms in (5.50), we use expression (5.9) for $Q_0(W) + \sqrt{h} C_0(W)$. The remainder $\tilde{g}$ in (5.9) will bring a contribution to $h_j h^{\frac{1}{2}} r_j$ in (5.49). The contribution

$$\sqrt{h} 2^{-\frac{j}{2}} \operatorname{Op}_{h_j}(\gamma_\Lambda) \Theta^*_{-j} \Delta^h_j \Big(\widetilde{w}_{2\Lambda} + \widetilde{w}_{-2\Lambda} + \sqrt{h}\big(\widetilde{w}_{3\Lambda} + \widetilde{w}_{-\Lambda} + \widetilde{w}_{-3\Lambda}\big)\Big)$$

and its $\mathcal{Z}$-derivatives are $O(\varepsilon h^\infty)$ since γ_Λ cuts-off close to Λ, while the terms on which it acts are supported close to $\ell \cdot \Lambda$, $|\ell| \leq 3$, $\ell \neq 1$. Consequently, the only remaining term coming from (5.9) is

$$-h 2^{-\frac{j}{2}} \operatorname{Op}_{h_j}(\gamma_\Lambda) \Theta^*_{-j} \Delta^h_j \widetilde{w}_\Lambda.$$

Since

$$\mathcal{Z}^k \widetilde{w}_\Lambda = \sum_{j' \in J(h,C)} \mathcal{Z}^k \Theta^*_{j'} \widetilde{w}_{\Lambda,j'} \quad \text{is } O(\varepsilon) \text{ in } h^{-3\delta'_{k+1}} L^\infty \tilde{I}^{0,b'}_\Lambda[K]$$

we get a contribution to $h_j f_j$ in (5.49).

The remainder term $2^{-\frac{j}{2}} h^{\frac{5}{4}} \operatorname{Op}_{h_j}(\gamma_\Lambda) \Theta^*_{-j} \Delta^h_j R(V)$ will contribute to the last term in (5.36) as it has been seen in the estimate of the last term in (5.41).

Finally, we are left with studying

$$h 2^{-\frac{j}{2}} \operatorname{Op}_{h_j}(\gamma_\Lambda) \Big[\frac{i}{2} \widetilde{w}_j - iZw_j\Big] \tag{5.51}$$

We use *iii)* of Lemma 4.4 to bound the action of $\mathcal{Z}^k_j \operatorname{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big)$ on (5.51). We obtain an L^∞ bound of the form

$$C h_j 2^{-j+b'} \Big[h^{\frac{1}{2}} \sum_{k_1+k_2 \leq k+1} \mathcal{E}_{k_1}(v) \mathcal{E}_{k_2}(v) + h^{\frac{1}{2}} \sum_{k_1+k_2+k_3 \leq k+1} \mathcal{E}_{k_1}(v) \mathcal{E}_{k_2}(v) \mathcal{E}_{k_3}(v) + h_j \mathcal{E}_{k+2}(v)\Big].$$

Using (4.1) we bound this by $Ch_j\big(h^{\frac{1}{2}}+h_j\big)\mathcal{E}_{k+2}(v)$ so, according to (4.9), by

$$Ch_jh^{-\delta'_{k+2}}\big(h^{\frac{1}{2}}+h_j\big)2^{-j+b'}.$$

We obtain in that way a contribution to h_jf_j in (5.49). $\square$

In the following section, we shall need estimates not only for w, but also for

$$w^{(\ell)}=\mathrm{Op}_h\left(\langle\xi\rangle^{\ell}\right)w,\quad w_\Lambda^{(\ell)}=\mathrm{Op}_h\left(\langle\xi\rangle^{\ell}\right)w_\Lambda,$$

where ℓ is an integer $0\le\ell\le\frac{s}{2}+N_0+b$. Let us deduce from Corollary 5.9 an expansion for $w^{(\ell)}$ in terms of $w_\Lambda^{(\ell)}$, $\ell=0,\dots,\frac{s}{2}+N_0+b'$.

Corollary 5.11. *Under the assumptions of Proposition 5.2, for $\ell=0,\dots,\frac{s}{2}+N_0+b'$ we may write*

$$w^{(\ell)}=w_\Lambda^{(\ell)}+\sqrt{h}\big(w_{2\Lambda}^{(\ell)}+w_{-2\Lambda}^{(\ell)}\big)+h\big(w_{3\Lambda}^{(\ell)}+w_{-\Lambda}^{(\ell)}+w_{-3\Lambda}^{(\ell)}\big)+h^{1+\sigma}g^{(\ell)}\tag{5.52}$$

where for any $k\le\min\big(\frac{s}{2}+N_0,\frac{s}{2}+N_0+b'-\ell\big)$, $\mathcal{Z}^kg^{(\ell)}$ is in $h^{-4\delta'_{k+1+N_1-N_0}}\tilde{\mathcal{B}}_\infty^{0,b'-\ell}[F]$, $\mathcal{Z}^k\,\mathrm{Op}_h(x\xi)g^{(\ell)}$ is in $h^{-4\delta'_{k+1+N_1-N_0}}\tilde{\mathcal{B}}_\infty^{0,b'-\ell-\frac{1}{2}}[F]$ and of size $O(\varepsilon)$ in that space and

$$\begin{aligned}
w_{2\Lambda}^{(\ell)}&=-i(1-\chi)\big(xh^{-\beta}\big)\langle2\,\mathrm{d}\,\omega\rangle^{\ell}\langle\mathrm{d}\,\omega\rangle^{-2\ell}\,|\mathrm{d}\,\omega|\,\frac{1+\sqrt{2}}{4}\big(w_\Lambda^{(\ell)}\big)^2\\
w_{-2\Lambda}^{(\ell)}&=-i(1-\chi)\big(xh^{-\beta}\big)\langle2\,\mathrm{d}\,\omega\rangle^{\ell}\langle\mathrm{d}\,\omega\rangle^{-2\ell}\,|\mathrm{d}\,\omega|\,\frac{1-\sqrt{2}}{4}\big(\overline{w}_\Lambda^{(\ell)}\big)^2\\
w_{3\Lambda}^{(\ell)}&=\quad(1-\chi)\big(xh^{-\beta}\big)\langle3\,\mathrm{d}\,\omega\rangle^{\ell}\langle\mathrm{d}\,\omega\rangle^{-3\ell}\,|\mathrm{d}\,\omega|^2\,\lambda_3^{\ell}\big(w_\Lambda^{(\ell)}\big)^3\\
w_{-\Lambda}^{(\ell)}&=\quad(1-\chi)\big(xh^{-\beta}\big)\langle\mathrm{d}\,\omega\rangle^{-2\ell}\,|\mathrm{d}\,\omega|^2\,\lambda_{-1}^{\ell}\big|w_\Lambda^{(\ell)}\big|^2\overline{w}_\Lambda^{(\ell)}\\
w_{-3\Lambda}^{(\ell)}&=\quad(1-\chi)\big(xh^{-\beta}\big)\langle3\,\mathrm{d}\,\omega\rangle^{\ell}\langle\mathrm{d}\,\omega\rangle^{-3\ell}\,|\mathrm{d}\,\omega|^2\,\lambda_{-3}^{\ell}\big(\overline{w}_\Lambda^{(\ell)}\big)^3
\end{aligned}\tag{5.53}$$

where $\lambda_3^\ell,\lambda_{-1}^\ell,\lambda_{-3}^\ell$ are real constants, $\chi\in C_0^\infty(\mathbb{R})$, $\chi\equiv1$ close to zero, with small enough support.

Finally, in the decomposition

$$w_\Lambda^{(\ell)}=\sum_{j\in J(h,C)}\Theta_j^*w_{\Lambda,j}^{(\ell)}$$

of $w_\Lambda^{(\ell)}$ deduced from the one of w_Λ, we may write

$$\mathrm{Op}_{h_j}\big(2x\xi+|\xi|^{\frac{1}{2}}\big)w_{\Lambda,j}^{(\ell)}=h_j\big(f_j^{(\ell)}+h^{\frac{1}{4}}r_j^{(\ell)}\big)\tag{5.54}$$

where

$$\begin{aligned}
&\big(\mathcal{Z}_j^kf_j^{(\ell)}\big)_j\ \textit{is a } O(\varepsilon)\textit{ family in } h^{-3\delta'_{k+2}}L^\infty I_\Lambda^{0,b'-\ell}[K]\\
&\big(\mathcal{Z}_j^kr_j^{(\ell)}\big)_j\ \textit{is a } O(\varepsilon)\textit{ family in } h^{-4\delta'_{k+1+N_1-N_0}}\tilde{\mathcal{B}}_\infty^{0,b'-\ell}[L].
\end{aligned}$$

Proof. By definition, $w_\Lambda = \mathrm{Op}_h\big(\langle\xi\rangle^{-\ell}\big)w_\Lambda^{(\ell)}$ and $\mathcal{Z}^k w_\Lambda^{(\ell)}$ belongs to $h^{-\delta'_{k+1}}L^\infty\tilde{J}_\Lambda^{0,b'-\ell}[K]$. By Proposition 2.11, we may write

$$w_\Lambda = (1-\chi)\big(xh^{-\beta}\big)\langle \mathrm{d}\,\omega\rangle^{-\ell}w_\Lambda^{(\ell)} + hr_1$$

where $\mathcal{Z}^k r_1$ is in $h^{-\delta'_{k+1}}\tilde{\mathcal{B}}_\infty^{-1,b'}[K]$ and the action of $\mathcal{Z}^k$ on $(1-\chi)\big(xh^{-\beta}\big)\langle \mathrm{d}\,\omega\rangle^{-\ell}w_\Lambda^{(\ell)}$ is in $h^{-\delta'_{k+1}}L^\infty\tilde{J}_\Lambda^{0,b'}[K]$. We apply Proposition 2.12 to compute powers of w_Λ

$$\begin{aligned}
\big(w_\Lambda\big)^2 &= (1-\chi)\big(xh^{-\beta}\big)^2\langle \mathrm{d}\,\omega\rangle^{-2\ell}\big(w_\Lambda^{(\ell)}\big)^2 + hr_2\\
\big(\overline{w}_\Lambda\big)^2 &= (1-\chi)\big(xh^{-\beta}\big)^2\langle \mathrm{d}\,\omega\rangle^{-2\ell}\big(\overline{w}_\Lambda^{(\ell)}\big)^2 + hr_{-2}\\
\big(w_\Lambda\big)^3 &= (1-\chi)\big(xh^{-\beta}\big)^3\langle \mathrm{d}\,\omega\rangle^{-3\ell}\big(w_\Lambda^{(\ell)}\big)^3 + hr_3\\
\big|w_\Lambda\big|^2 w_\Lambda^2 &= (1-\chi)\big(xh^{-\beta}\big)^3\langle \mathrm{d}\,\omega\rangle^{-3\ell}\big|w_\Lambda^{(\ell)}\big|^2 w_\Lambda^{(\ell)} + hr_1\\
\big|w_\Lambda\big|^2 \overline{w}_\Lambda^2 &= (1-\chi)\big(xh^{-\beta}\big)^3\langle \mathrm{d}\,\omega\rangle^{-3\ell}\big|w_\Lambda^{(\ell)}\big|^2 \overline{w}_\Lambda^{(\ell)} + hr_{-1}\\
\big(\overline{w}_\Lambda\big)^3 &= (1-\chi)\big(xh^{-\beta}\big)^3\langle \mathrm{d}\,\omega\rangle^{-3\ell}\big(\overline{w}_\Lambda^{(\ell)}\big)^3 + hr_{-3}
\end{aligned}\tag{5.55}$$

where the action of $\mathcal{Z}^k$ on r_q gives an element of $h^{-2\delta'_{k+1}}\tilde{\mathcal{B}}_\infty^{-2,2b'-\frac{1}{2}}\big[K_q\big]$ if $q=\pm 2$ and of $h^{-3\delta'_{k+1}}\tilde{\mathcal{B}}_\infty^{-3,3b'-1}\big[K_q\big]$ if $q=3,1,-1,-3$.

On the other hand, consider the contributions $w_{q\Lambda}$, $|q|\leq 3$, $q\neq 1$, to the expansion (5.48) and define $w_{q\Lambda}^{(\ell),1} = \mathrm{Op}_h\big(\langle\xi\rangle^{\ell}\big)w_{q\Lambda}$ so that (5.48) may be written

$$w^{(\ell)} = w_\Lambda^{(\ell)} + \sqrt{h}\big(w_{2\Lambda}^{(\ell),1} + w_{-2\Lambda}^{(\ell),1}\big) + h\big(w_{3\Lambda}^{(\ell),1} + w_{-\Lambda}^{(\ell),1} + w_{-3\Lambda}^{(\ell),1}\big) + h^{1+\sigma}g^{(\ell)}\tag{5.56}$$

where $g^{(\ell)}$ satisfies the bounds of the remainder in (5.52). We apply again Proposition 2.11 to get an expansion of $w_{q\Lambda}^{(\ell),1}$. Since by *ii*) of Lemma 5.8,

$$\begin{aligned}
&\mathcal{Z}^k w_{\pm 2\Lambda} \text{ is in } h^{-2\delta'_{k+1}}L^\infty\tilde{J}_\Lambda^{2,b'+\frac{3}{2}}\big[K_{\pm 2}\big],\\
&\mathcal{Z}^k w_{q\Lambda}\ \ \text{ is in } h^{-3\delta'_{k+1}}L^\infty\tilde{I}_\Lambda^{2,b'+\frac{3}{2}}\big[K_q\big],\ \text{for } q=-3,-1,3,
\end{aligned}$$

we obtain that

$$\begin{aligned}
w_{\pm 2\Lambda}^{(\ell),1} &= (1-\chi)\big(xh^{-\beta}\big)\langle 2\,\mathrm{d}\,\omega\rangle^{\ell}w_{\pm 2\Lambda} + hr_{\pm 2}^{(\ell),1}\\
w_{q\Lambda}^{(\ell),1} &= (1-\chi)\big(xh^{-\beta}\big)\langle q\,\mathrm{d}\,\omega\rangle^{\ell}w_{q\Lambda} + h^{\frac{1}{2}}r_q^{(\ell),1},\quad q=-3,-1,3,
\end{aligned}\tag{5.57}$$

where $\mathcal{Z}^k r_{\pm 2}^{(\ell),1}$ is in $h^{-2\delta'_{k+1}}\tilde{\mathcal{B}}_\infty^{1,b'+\frac{3}{2}-\ell}\big[K_2\big]$ and for $q=-3,-1,3$,

$$\mathcal{Z}^k r_q^{(\ell),1} \text{ is in } h^{-3\delta'_{k+1}}\tilde{\mathcal{B}}_\infty^{1,b'+1-\ell}\big[K_q\big].$$

We deduce from (5.56) that

$$
\begin{aligned}
w^{(\ell)} &= w_\Lambda^{(\ell)} + \sqrt{h}\langle 2\,\mathrm{d}\,\omega\rangle^\ell (1-\chi)\big(xh^{-\beta}\big)\big(w_{2\Lambda}+w_{-2\Lambda}\big)\\
&\quad + h(1-\chi)\big(xh^{-\beta}\big)\big(\langle 3\,\mathrm{d}\,\omega\rangle^\ell w_{3\Lambda} + \langle \mathrm{d}\,\omega\rangle^\ell w_{-\Lambda} + \langle 3\,\mathrm{d}\,\omega\rangle^\ell w_{-3\Lambda}\big)\\
&\quad + h^{1+\sigma} g^{(\ell)}
\end{aligned}
\tag{5.58}
$$

with a new remainder $g^{(\ell)}$ as in (5.52). We use next (5.33), (5.36) to express $w_{q\Lambda}$ from w_Λ, $\overline{w}_\Lambda$ and (5.55) to compute the resulting quantities from $w_\Lambda^{(\ell)}$, $\overline{w}_\Lambda^{(\ell)}$. We get expressions (5.53), with $(1-\chi)$ replaced eventually by some of its powers. As already seen, these powers may be replaced by $(1-\chi)$, up to $O(h^\infty)$ remainders.

The remainders coming from the ones in (5.55) may be expressed as the product of $h^{\frac{3}{2}}$ (resp. h^2) with $(1-\chi)\big(xh^{-\beta}\big)\,|\mathrm{d}\,\omega|\,\langle 2\,\mathrm{d}\,\omega\rangle^\ell r_{\pm 2}$ (resp. $(1-\chi)\big(xh^{-\beta}\big)\,|\mathrm{d}\,\omega|^2\,\langle q\,\mathrm{d}\,\omega\rangle^\ell r_\ell$, $\ell = -3,-1,3$). By Proposition 2.11, and since $|\mathrm{d}\,\omega|\,\langle 2\,\mathrm{d}\,\omega\rangle^\ell$ (resp. $|\mathrm{d}\,\omega|^2\,\langle q\,\mathrm{d}\,\omega\rangle^\ell$) satisfies (2.15) with (ℓ,ℓ',d,d') replaced by $(-2\ell-2,2\ell,0,0)$ (resp. $(-2\ell-4,2\ell,0,0)$), we obtain that the action of $\mathcal{Z}^k$ on these functions gives elements belonging to

$$
h^{-2\delta'_{k+1}}\tilde{\mathcal{B}}_\infty^{0,2b'-\frac{1}{2}-\ell}[K_\ell] \subset h^{-2\delta'_{k+1}}\tilde{\mathcal{B}}_\infty^{0,b'-\ell}[K_\ell]
$$

(resp. $h^{-3\delta'_{k+1}}\tilde{\mathcal{B}}_\infty^{1,3b'-1-\ell}[K_\ell] \subset h^{-3\delta'_{k+1}}\tilde{\mathcal{B}}_\infty^{0,b'-\ell}[K_\ell]$) for $\ell\in\{-3,\dots,3\}$ so that we obtain again a contribution to g^ℓ. (Notice that the action of $\mathrm{Op}_h(x\xi)$ on these remainders give elements of the same spaces with b replaced by $b-1/2$, since they are microlocally supported in a compact subset of $T^*(\mathbb{R}\setminus\{0\})$). This concludes the proof of (5.52).

To prove (5.54), we first write, according to the definition of $w_\Lambda^{(\ell)}$ and (2.13), that $w_{\Lambda,j}^{(\ell)} = \mathrm{Op}_{h_j}\big(\langle 2^j\xi\rangle^\ell\big)w_{\Lambda,j}$. Making act $\mathrm{Op}_{h_j}\big(\langle 2^j\xi\rangle^\ell\big)$ on (5.49), we get for the left hand side of (5.54) an expression given by its right hand side, modulo a term

$$
\Big[\mathrm{Op}_{h_j}\big(2x\xi+|\xi|^{\frac{1}{2}}\big), \mathrm{Op}_{h_j}\big(\langle 2^j\xi\rangle^\ell\big)\Big]w_{\Lambda,j}.
$$

Since $\big(\mathcal{Z}^k w_{\Lambda,j}\big)_j$ is a bounded family in $h^{-3\delta'_{k+1}}L^\infty J_\Lambda^{0,b'}[K]$, we see using symbolic calculus, that this expression contributes to the $h_j f_j^{(\ell)}$ term in (5.54). $\square$

6 Ordinary differential equation for w_Λ

We consider a solution v of (3.6), satisfying for h in some interval $]h',1]$ the a priori estimate (4.8) for $k'\le k+1$, with $k\le s-a-1$. By Proposition 4.1, we know that v satisfies then (4.9), and by Corollary 5.9, that $v = v_L + w + v_H$, where w has an expansion (5.48). Our goal here is to deduce from that and from the equation satisfied by w, a uniform estimate for $\big\|\mathrm{Op}_h\big(\langle\xi\rangle^\ell\big)w(t,\cdot)\big\|_{L^\infty}$, and estimates for $\big\|\mathcal{Z}^k\,\mathrm{Op}_h\big(\langle\xi\rangle^\ell\big)w(t,\cdot)\big\|_{L^\infty}$ which are not uniform, but which are better than (4.9) (i.e. that involve exponents closer to zero than the $\delta'_{k'}$).

For $0 \leq \ell \leq \frac{s}{2} + N_0$ and $0 \leq k \leq \frac{s}{2} + N_0 - \ell$ we define

$$W^{k,(\ell)} = \big(Z^{k'} w^{(\ell)}\big)_{0 \leq k' \leq k} \tag{6.1}$$

where

$$w^{(\ell)} = \mathrm{Op}_h\big(\langle\xi\rangle^{\ell}\big) w,$$

as in the preceding section. The estimates we are looking for will follow from an ordinary differential equation satisfied by $W^{k,(\ell)}$.

Proposition 6.1. *Under the preceding assumptions, the function $w^{(\ell)}$ satisfies the equation*

$$\begin{aligned} D_t w^{(\ell)} = {} & \frac{1}{2}(1-\chi)\big(xh^{-\beta}\big)\, |\mathrm{d}\,\omega(x)|^{\frac{1}{2}}\, w^{(\ell)} \\ & - i\frac{\sqrt{h}}{8}(1-\chi)\big(xh^{-\beta}\big)\, |\mathrm{d}\,\omega|^{\frac{3}{2}}\, \langle \mathrm{d}\,\omega\rangle^{-2\ell} \langle 2\,\mathrm{d}\,\omega\rangle^{\ell}\Big[(1+\sqrt{2})\big(w^{(\ell)}\big)^2 - 3(1-\sqrt{2})\big(\overline{w}^{(\ell)}\big)^2\Big] \\ & + h\Big[\Phi_3^{(\ell)}\big(w^{(\ell)}\big)^3 + \Phi_1^{(\ell)}|w^{(\ell)}|^2 w^{(\ell)} + \Phi_{-1}^{(\ell)}|w^{(\ell)}|^2\overline{w}^{(\ell)} + \Phi_{-3}^{(\ell)}\big(\overline{w}^{(\ell)}\big)^3\Big] \\ & + h^{1+\kappa} r^{(\ell)}(t,x) \end{aligned} \tag{6.2}$$

where χ is in $C_0^\infty(\mathbb{R})$, equal to one close to zero, with small enough support, where κ is a small positive number, where $\Phi_j^{(\ell)}$, $-3 \leq j \leq 3$ are given by

$$\begin{aligned} \Phi_1^{(\ell)}(x) &= (1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega|^{\frac{5}{2}}\langle \mathrm{d}\,\omega\rangle^{-2\ell}\left[\frac{\langle 2\,\mathrm{d}\,\omega\rangle^{2\ell}}{\langle \mathrm{d}\,\omega\rangle^{2\ell}}\frac{3(3-2\sqrt{2})}{16} + \frac{1}{2}\right] \\ \Phi_j^{(\ell)}(x) &= (1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega|^{\frac{5}{2}}\Gamma_j^{(\ell)}(\mathrm{d}\,\omega) \quad \ell \neq 1 \end{aligned} \tag{6.3}$$

for some real valued symbols of order -2ℓ, $\Gamma_j^{(\ell)}$, and where $\big\|(hD_x)^p \mathcal{Z}^k r^{(\ell)}(t,x)\big\|_{L^\infty}$ is $O(\varepsilon)$ for any integers k, p, ℓ with $k \leq \frac{s}{2} + N_0 - \ell$, $0 \leq p \leq b' - 1$. Moreover, $W^{k,(\ell)}$ defined by (6.1) satisfies a system of the form

$$\begin{aligned} D_t W^{k,(\ell)} = {} & \frac{1}{2}(1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega(x)|^{\frac{1}{2}} W^{k,(\ell)} \\ & + \sqrt{h} Q^{k,(\ell)}\big[x, h; W^{k,(\ell)}, \overline{W}^{k,(\ell)}\big] \\ & + h\mathcal{C}^{k,(\ell)}\big[x, h; W^{k,(\ell)}, \overline{W}^{k,(\ell)}\big] + h^{1+\kappa} R^{k,(\ell)}(t,x) \end{aligned} \tag{6.4}$$

- *where $\big\|(hD_x)^p Z^{k'} R^{k,(\ell)}(t,\cdot)\big\|_{L^\infty} = O(\varepsilon)$ for $0 \leq p \leq b'' \leq b' - 2/\beta$ and $k' \leq \frac{s}{2} + N_0 - \ell - k$;*

- *where $Q^{k,(\ell)}$ is a vector valued quadratic map in $(W^{k,(\ell)}, \overline{W}^{k,(\ell)})$ whose components are linear combination of functions of the form*

$$\begin{aligned} &\theta\big(xh^{-\beta}\big)\Phi(x)\big(Z^{k_1} w^{(\ell)}\big)\big(Z^{k_2} w^{(\ell)}\big) \\ &\theta\big(xh^{-\beta}\big)\Phi(x)\big(Z^{k_1} w^{(\ell)}\big)\big(Z^{k_2} \overline{w}^{(\ell)}\big) \\ &\theta\big(xh^{-\beta}\big)\Phi(x)\big(Z^{k_1} \overline{w}^{(\ell)}\big)\big(Z^{k_2} \overline{w}^{(\ell)}\big) \end{aligned} \tag{6.5}$$

for $k_1+k_2\leq k$, with smooth functions θ bounded as well as their derivatives, $\theta\equiv 0$ close to zero and Φ satisfying $\left|Z^k\Phi(x)\right|\leq C|x|^{2\ell-3}\langle x\rangle^{-2\ell}$ for any k;

- *where $\mathcal{C}^{k,(\ell)}$ is a vector valued cubic map, whose components are linear combination of quantities*

$$\begin{aligned}
&\theta\big(xh^{-\beta}\big)\Phi(x)\big(Z^{k_1}w^{(\ell)}\big)\big(Z^{k_2}w^{(\ell)}\big)\big(Z^{k_3}w^{(\ell)}\big)\\
&\theta\big(xh^{-\beta}\big)\Phi(x)\big(Z^{k_1}w^{(\ell)}\big)\big(Z^{k_2}w^{(\ell)}\big)\big(Z^{k_3}\overline{w}^{(\ell)}\big)\\
&\theta\big(xh^{-\beta}\big)\Phi(x)\big(Z^{k_1}w^{(\ell)}\big)\big(Z^{k_2}\overline{w}^{(\ell)}\big)\big(Z^{k_3}\overline{w}^{(\ell)}\big)\\
&\theta\big(xh^{-\beta}\big)\Phi(x)\big(Z^{k_1}\overline{w}^{(\ell)}\big)\big(Z^{k_2}\overline{w}^{(\ell)}\big)\big(Z^{k_3}\overline{w}^{(\ell)}\big)
\end{aligned}\tag{6.6}$$

for $k_1+k_2+k_3\leq k$, $\left|Z^k\Phi(x)\right|\leq C|x|^{4\ell-5}\langle x\rangle^{-4\ell}$ for any k.

We shall prove first (6.2), deducing it from (3.23) on which we make act $\mathrm{Op}_h\big(\langle\xi\rangle^\ell\big)$. Let us study first the action of this operator on the nonlinearity.

As in the preceding section, we shall call K or K_ℓ compact subsets of $T^*(\mathbb{R}\setminus\{0\})$ contained in a small neighborhood of $\ell\Lambda$ for $\ell\in\{\pm3,\pm2,\pm1\}$, by L compact subsets of $T^*(\mathbb{R}\setminus\{0\})$ and by F closed subsets of $T^*\mathbb{R}$ whose second projection is compact in $\mathbb{R}\setminus\{0\}$.

Lemma 6.2. *Under the assumptions of the proposition, we may write for $\ell\leq\frac{s}{2}+N_0$,*

$$\begin{aligned}
\sum_{j\in J(h,C)}&\Delta_j^h\Big[\mathrm{Op}_h\big(\langle\xi\rangle^\ell\big)\big[\sqrt{h}Q_0(W)+hC_0(W)\big]\Big]\\
&=-i\sqrt{h}(1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega|^{\frac{3}{2}}\frac{\sqrt{2}}{4}\langle 2\,\mathrm{d}\,\omega\rangle^\ell\langle\mathrm{d}\,\omega\rangle^{-2\ell}\Big[\big(w_\Lambda^{(\ell)}\big)^2+\big(\overline{w}_\Lambda^{(\ell)}\big)^2\Big]\\
&\quad+\frac{h}{2}(1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega|^{\frac{5}{2}}\langle\mathrm{d}\,\omega\rangle^{-3\ell}\langle 3\,\mathrm{d}\,\omega\rangle^\ell\Big[\lambda_3^{(\ell)}\big(w_\Lambda^{(\ell)}\big)^3+\lambda_{-3}^{(\ell)}\big(\overline{w}_\Lambda^{(\ell)}\big)^3\Big]\\
&\quad+\frac{h}{2}(1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega|^{\frac{5}{2}}\langle\mathrm{d}\,\omega\rangle^{-2\ell}\Big[\big|w_\Lambda^{(\ell)}\big|^2w_\Lambda^{(\ell)}+\lambda_{-1}^{(\ell)}\big|w_\Lambda^{(\ell)}\big|^2\overline{w}_\Lambda^{(\ell)}\Big]\\
&\quad+h^{\frac{3}{2}}r^{(\ell)}
\end{aligned}\tag{6.7}$$

where for $k\leq\frac{s}{2}+N_0-\ell$ $Z^kr^{(\ell)}$ belongs to $h^{-3\delta'_{k+1+\ell}}\tilde{\mathcal{B}}_\infty^{0,b'}[F]$ and $\lambda_{\pm3}^{(\ell)}$, $\lambda_{-1}^{(\ell)}$ are real constants.

Proof. We apply $\mathrm{Op}_h\big(\langle\xi\rangle^\ell\big)$ to (5.9) and write the resulting right hand side as in (6.7). By *ii*) if Lemma 5.3, we know that $\mathcal{Z}^{k+\ell}\tilde{w}_{\pm2\Lambda}$ is $O(\varepsilon)$ in $h^{-2\delta'_{k+\ell+1}}L^\infty\tilde{J}_{\pm2\Lambda}^{3,b'+\frac{3}{2}}\big[K_{\pm2}\big]$, so $Z^k\tilde{w}_{\pm2\Lambda}$ is $O(\varepsilon)$ in $h^{-2\delta'_{k+\ell+1}}L^\infty\tilde{J}_{\pm2\Lambda}^{3,b'+\frac{3}{2}+\ell}\big[K_{\pm2}\big]$.

In the same way $Z^k\tilde{w}_{q\Lambda}$ is $O(\varepsilon)$ in $h^{-3\delta'_{k+\ell+1}}L^\infty\tilde{I}_{q\Lambda}^{3,b'+\frac{3}{2}+\ell}\big[K_q\big]$, for $q\in\{\pm1,\pm3\}$. Consequently, Proposition 2.11 shows that

$$\begin{aligned}
\mathrm{Op}_h\big(\langle\xi\rangle^\ell\big)\tilde{w}_{\pm2\Lambda}&=(1-\chi)\big(xh^{-\beta}\big)\langle 2\,\mathrm{d}\,\omega\rangle^\ell\tilde{w}_{\pm2\Lambda}+h\tilde{r}_{\pm2}^{(\ell)},\\
\mathrm{Op}_h\big(\langle\xi\rangle^\ell\big)\tilde{w}_{q\Lambda}&=(1-\chi)\big(xh^{-\beta}\big)\langle q\,\mathrm{d}\,\omega\rangle^\ell\tilde{w}_{q\Lambda}+h^{\frac{1}{2}}\tilde{r}_q^{(\ell)},
\end{aligned}$$

where $Z^k\tilde{r}^{(\ell)}_{\pm 2}$ (resp. $Z^k\tilde{r}^{(\ell)}_q$) is $O(\varepsilon)$ in $h^{-2\delta'_{k+\ell+1}}\tilde{\mathcal{B}}^{2,b'+\frac{3}{2}}_\infty\big[K_{\pm 2}\big]$ (resp. $h^{-3\delta_{k+1+\ell}}\tilde{\mathcal{B}}^{2,b'+1}_\infty\big[K_q\big]$). We combine this with the expressions (5.8), (5.10), (5.11) of $\tilde{w}_{q\Lambda}$ in terms of w_Λ, $w_{\pm 2\Lambda}$, and with the formulas (5.33) expressing $w_{\pm 2\Lambda}$ in terms of w_Λ. If moreover we compute the powers of w_Λ, $\overline{w}_\Lambda$ from $w^{(\ell)}_\Lambda$, $\overline{w}^{(\ell)}_\Lambda$ using (5.55), we get

$$\begin{aligned}\mathrm{Op}_h\big(\langle\xi\rangle^\ell\big)\tilde{w}_{2\Lambda} &= -i(1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega|^{\frac{3}{2}}\frac{\sqrt{2}}{4}\langle 2\,\mathrm{d}\,\omega\rangle^\ell\langle\mathrm{d}\,\omega\rangle^{-2\ell}\big(w^{(\ell)}_\Lambda\big)^2 + hr^{(\ell)}_2,\\ \mathrm{Op}_h\big(\langle\xi\rangle^\ell\big)\tilde{w}_{-2\Lambda} &= -i(1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega|^{\frac{3}{2}}\frac{\sqrt{2}}{4}\langle 2\,\mathrm{d}\,\omega\rangle^\ell\langle\mathrm{d}\,\omega\rangle^{-2\ell}\big(\overline{w}^{(\ell)}_\Lambda\big)^2 + hr^{(\ell)}_{-2},\end{aligned}$$

with remainders $r^{(\ell)}_{\pm 2}$ satisfying again, because of Proposition 2.11, that $Z^k r^{(\ell)}_{\pm 2}$ is $O(\varepsilon)$ in $h^{-2\delta'_{k+1+\ell}}\tilde{\mathcal{B}}^{0,b'}_\infty\big[K_{\pm 2}\big]$. In the same way

$$\mathrm{Op}_h\big(\langle\xi\rangle^\ell\big)\tilde{w}_{q\Lambda} = (1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega|^{\frac{5}{2}}\lambda^{(\ell)}_q\langle q\,\mathrm{d}\,\omega\rangle^\ell\langle\mathrm{d}\,\omega\rangle^{-3\ell}P_q\big(w^{(\ell)}_\Lambda,\overline{w}^{(\ell)}_\Lambda\big) + h^{\frac{1}{2}}r^{(\ell)}_q,$$

where $P_q\big(w^{(\ell)}_\Lambda,\overline{w}^{(\ell)}_\Lambda\big)$ is equal to $\big(w^{(\ell)}_\Lambda\big)^3$ (resp. $\big|w^{(\ell)}_\Lambda\big|^2 w^{(\ell)}_\Lambda$, resp. $\big|w^{(\ell)}_\Lambda\big|^2\overline{w}^{(\ell)}_\Lambda$, resp. $\big(\overline{w}^{(\ell)}_\Lambda\big)^3$) if $q=3$ (resp. $q=1$, resp. $q=-1$, resp. $q=-3$), where $\lambda^{(\ell)}_q$ are real constants with $\lambda^{(\ell)}_1 = \frac{1}{2}$, and where $Z^k r^{(\ell)}_q$ is $O(\varepsilon)$ in $h^{-3\delta'_{k+1+\ell}}\tilde{\mathcal{B}}^{0,b'}_\infty\big[K_q\big]$. (We used again remark 5.5 to replace different powers of $(1-\chi)\big(xh^{-\beta}\big)$ by 1.)

This concludes the proof of the lemma. $\square$

Let us study next the action of $\mathrm{Op}_h\big(\langle\xi\rangle^\ell\big)$ on the linear term $\mathrm{Op}_h\big(x\xi + |\xi|^{\frac{1}{2}}\big)w$ of (3.23), writing $w = \mathrm{Op}_h\big(\langle\xi\rangle^{-\ell}\big)w^{(\ell)}$.

Lemma 6.3. *One may write, for $\ell \le \frac{s}{2} + N_0$*

$$\begin{aligned}&\mathrm{Op}_h\big(\langle\xi\rangle^\ell\big)\,\mathrm{Op}_h\big(x\xi + |\xi|^{\frac{1}{2}}\big)\,\mathrm{Op}_h\big(\langle\xi\rangle^{-\ell}\big)w^{(\ell)} + i\ell h\,\mathrm{Op}_h\Big(\frac{\xi^2}{\langle\xi\rangle^2}\Big)w^{(\ell)}\\ &\quad= (1-\chi)\big(xh^{-\beta}\big)\Big[\Big(\frac{1}{2}|\,\mathrm{d}\,\omega|^{\frac{1}{2}} + \frac{i}{2}h\Big)w^{(\ell)}_\Lambda\\ &\qquad - i\frac{\sqrt{h}}{4}|\,\mathrm{d}\,\omega|^{\frac{3}{2}}\langle 2\,\mathrm{d}\,\omega\rangle^\ell\langle\mathrm{d}\,\omega\rangle^{-2\ell}\Big[\big(w^{(\ell)}_\Lambda\big)^2 - \big(\overline{w}^{(\ell)}_\Lambda\big)^2\Big]\\ &\qquad + h|\,\mathrm{d}\,\omega|^{\frac{5}{2}}\langle\mathrm{d}\,\omega\rangle^{-3\ell}\Big[\langle 3\,\mathrm{d}\,\omega\rangle^\ell\Big(\mu^{(\ell)}_3\big(w^{(\ell)}_\Lambda\big)^3 + \mu^{(\ell)}_{-3}\big(\overline{w}^{(\ell)}_\Lambda\big)^3\Big)\\ &\qquad\qquad + \langle\mathrm{d}\,\omega\rangle^\ell\mu^{(\ell)}_{-1}\big|w^{(\ell)}_\Lambda\big|^2\overline{w}^{(\ell)}_\Lambda\Big]\Big]\\ &\quad + h^{1+\sigma}r^{(\ell)}\end{aligned}\tag{6.8}$$

for some real constants $\mu^{(\ell)}_3$, $\mu^{(\ell)}_{-1}$, $\mu^{(\ell)}_{-3}$ and where for $k \le \frac{s}{2} + N_0 - \ell$, $Z^k r^{(\ell)}$ belongs to $h^{-4\delta'_{k+\ell+1+N_1-N_0}}\tilde{\mathcal{B}}^{0,b'-\frac{1}{2}}_\infty[F]$.

The proof of the above lemma will use

Lemma 6.4. *We may write for $\ell \leq \frac{s}{2} + N_0$*

$$\operatorname{Op}_h\big(x\xi + |\xi|^{\frac{1}{2}}\big)w_\Lambda^{(\ell)} = \frac{1}{2}(1-\chi)\big(xh^{-\beta}\big)\Big[|\operatorname{d}\omega|^{\frac{1}{2}}w_\Lambda^{(\ell)} + ihw_\Lambda^{(\ell)}\Big] + h^{1+\sigma}r_1^{(\ell)} \tag{6.9}$$

where for $k \leq \frac{s}{2} + N_0 - \ell$, $Z^k r_1^{(\ell)}$ is in $h^{-4\delta'_{k+2+\ell}}\tilde{\mathcal{B}}_\infty^{0,b'}[L]$.

Proof. We write remembering (2.13)

$$\operatorname{Op}_h\big(x\xi + |\xi|^{\frac{1}{2}}\big)w_\Lambda^{(\ell)} = \sum_{j\in J(h,C)} 2^{\frac{j}{2}}\Theta_j^* \operatorname{Op}_{h_j}\big(x\xi + |\xi|^{\frac{1}{2}}\big)w_{\Lambda,j}^{(\ell)}. \tag{6.10}$$

Let us show that we may write

$$\begin{aligned}\operatorname{Op}_{h_j}\big(x\xi + |\xi|^{\frac{1}{2}}\big) &= \frac{1}{2}|\operatorname{d}\omega(x)|^{\frac{1}{2}} + \operatorname{Op}_{h_j}(e_1)\Big(\operatorname{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big)\Big)^2\\ &\quad + ih_j\operatorname{Op}_{h_j}(e_2)\operatorname{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big)\\ &\quad - ih_j\operatorname{Op}_{h_j}(e_1)\operatorname{Op}_{h_j}\big(|\xi|^{\frac{1}{2}}\big) + h_j^2\operatorname{Op}_{h_j}(e_3)\end{aligned} \tag{6.11}$$

where e_j are symbols in $S(1,K)$, for some large enough compact subset K of $T^*(\mathbb{R}\setminus\{0\})$, satisfying

$$e_1|_\Lambda = -\frac{1}{2}|\operatorname{d}\omega(x)|^{-\frac{1}{2}}. \tag{6.12}$$

Denote $a(x,\xi) = x\xi + |\xi|^{\frac{1}{2}}$ and take

$$e_1(x,\xi) = \frac{a(x,\xi) - a(x,\operatorname{d}\omega)}{\big(2x\xi + |\xi|^{\frac{1}{2}}\big)^2}.$$

A direct computation shows that the numerator vanishes at second order on Λ, so that the quotient is smooth, and that its restriction to Λ is given by (6.12). If we set $e(x,\xi) = 2x\xi + |\xi|^{\frac{1}{2}}$, we obtain by symbolic calculus

$$e\#e = e^2 - ih_j\partial_\xi e\partial_x e + h_j^2\tilde{e}$$

for some symbol $\tilde{e}$, so that by an immediate computation

$$e^2 = e\#e + 2ih_je - ih_j\,|\xi|^{\frac{1}{2}} - h_j^2\tilde{e}. \tag{6.13}$$

On the other hand, by symbolic calculus $e_1e^2 = e_1\#e^2 + h_je'_2\#e + h_j^2\tilde{e}'$ for some symbols $e'_2, \tilde{e}'$ in $S(1,K)$, so that taking (6.13) into account

$$e_1e^2 = e_1\#e\#e - ih_je_1\#\,|\xi|^{\frac{1}{2}} + ih_je_2\#e + h_j^2e_3$$

for new symbols e_2, e_3. Since $e_1 e^2 = a(x,\xi) - a(x, \mathrm{d}\,\omega) = x\xi + |\xi|^{\frac{1}{2}} - \frac{1}{2}\,|\mathrm{d}\,\omega(x)|^{\frac{1}{2}}$, we obtain (6.11) by quantification.

Let us use (6.11) to show that (6.9) holds. Actually, the contribution of the first term in the right hand side of (6.11) to (6.9) gives the $|\mathrm{d}\,\omega|^{\frac{1}{2}}$ term in the right hand side of (6.9) (Again, we may insert a cut-off $(1-\chi)\big(xh^{-\beta}\big)$ as $w_{\Lambda,j}$ is microlocally supported on a compact subset of $T^*(\mathbb{R}\setminus\{0\})$ and j stays in $J(h,C)$, if we accept some $O(h^\infty)$ remainder). The contribution of the last but one term in (6.11) to (6.10) may be written

$$-ih \sum_{j\in J(h,C)} \Theta_j^* \operatorname{Op}_{h_j}(e_1) \operatorname{Op}_{h_j}\big(\,|\xi|^{\frac{1}{2}}\,\big) w_{\Lambda,j}^{(\ell)} = -ih \operatorname{Op}_h(e_1) \operatorname{Op}_h\big(\,|\xi|^{\frac{1}{2}}\,\big) w_{\Lambda}^{(\ell)}.$$

By Proposition 2.11 and (6.12), this is equal to

$$\frac{i}{2} h(1-\chi)\big(xh^{-\beta}\big) w_\Lambda^{(\ell)} + h^2 r_{1,1}^{(\ell)}$$

where $Z^k r_{1,1}^{(\ell)}$ belongs to $h^{-\delta'_{k+1+\ell}} \tilde{\mathcal{B}}_\infty^{-1,b'}[K]$. Actually noticing that $e_1(x,\xi) = |x| e_1\big(\frac{x}{|x|}, |x|^2\xi\big)$, and that $(\partial_x^\alpha \partial_\eta^\beta e_1)(\pm 1, \eta) = O\big(|\eta|^{-1-|\beta|}\big)$, $|\eta| \to 0$ and $|\eta| \to +\infty$, one checks that $e_1(x,\xi)$ satisfies (2.15) with $(\ell,\ell',d,d') = (-1,0,-1,0)$ so that $e_1(x,\xi)|\xi|^{\frac{1}{2}}$ obeys these estimates for $(\ell,\ell',d,d') = (-1,0,-1/2,0)$. Since $Z^k w_\Lambda^{(\ell)}$ is in $h^{-\delta'_{k+1+\ell}} L^\infty \tilde{J}_\Lambda^{0,b'}[K]$, the above statement holds. Since for $j \in J(h,C)$, $2^j \geq h^{2(1-\sigma)}$, the remainder may be rewritten as the product of $h^{1+\sigma}$ with an element whose $\mathcal{Z}^k$-derivatives are in $h^{-\delta'_{k+1+\ell}} \tilde{\mathcal{B}}_\infty^{0,b'}[K]$ i.e. contributes to $h^{1+\sigma} r_1^{(\ell)}$ in (6.9).

We are reduced to showing that the contributions of the second, third and last terms in the right hand side of (6.11) provide remainders. This is evident for the last term as $2^{\frac{j}{2}} h_j^2 = h h_j = O(h^{1+\sigma})$. Using (5.54), we may write the sum of the two remaining terms

$$h_j \operatorname{Op}_{h_j}(e_1) \operatorname{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big)\Big(f_j^{(\ell)} + h^{\frac{1}{4}} r_j^{(\ell)}\Big) + i h_j^2 \operatorname{Op}_{h_j}(e_2)\Big(f_j^{(\ell)} + h^{\frac{1}{4}} r_j^{(\ell)}\Big). \tag{6.14}$$

Since $\big(Z^k f_j^{(\ell)}\big)_j$ is $O(\varepsilon)$ in $h^{-3\delta'_{k+2+\ell}} L^\infty I_\Lambda^{0,b'}[K]$ and $\big(Z^k r_j^{(\ell)}\big)_j$ is $O(\varepsilon)$ in $h^{-4\delta'_{k+1+\ell}} \tilde{\mathcal{B}}_\infty^{0,b'}[L]$, the last term as well as the $r_j^{(\ell)}$ contribution to the first one induce in (6.10) a contribution that may be included in the $h^{1+\sigma} r_1^{(\ell)}$ remainder term of (6.9). On the other hand, the fact that $\big(Z^k f_j^{(\ell)}\big)_j$ is $O(\varepsilon)$ in $h^{-3\delta'_{k+2+\ell}} L^\infty I_\Lambda^{0,b'}[K]$ implies that $Z^k \operatorname{Op}_{h_j}\big(2x\xi + |\xi|^{\frac{1}{2}}\big) f_j^{(\ell)}$ belongs to a ε-neighborhood of zero in $h^{-3\delta'_{k+2+\ell}}\big(h^{\frac{1}{2}} + h_j\big) \tilde{\mathcal{B}}_\infty^{0,b'}[L]$. Consequently, the first term in (6.14) induces also in (6.10) a contribution forming part to the $h^{1+\sigma} r_1^{(\ell)}$ term in (6.9). This concludes the proof of the lemma. $\square$

Proof of Lemma 6.3. We notice first that

$$\operatorname{Op}_h\big(\langle\xi\rangle^\ell\big) \operatorname{Op}_h\big(x\xi + |\xi|^{\frac{1}{2}}\big) \operatorname{Op}_h\big(\langle\xi\rangle^{-\ell}\big) + i\ell h \operatorname{Op}_h\Big(\frac{\xi^2}{\langle\xi\rangle^2}\Big) = \operatorname{Op}_h\big(x\xi + |\xi|^{\frac{1}{2}}\big). \tag{6.15}$$

We make act this operator on the expression of $w^{(\ell)}$ from $w_\Lambda^{(\ell)}$ given in (5.52). The action of $\mathrm{Op}_h\big(x\xi + |\xi|^{\frac12}\big)$ on $w_\Lambda^{(\ell)}$ has been computed in Lemma 6.4. Let us study

$$\mathrm{Op}_h\big(x\xi + |\xi|^{\frac12}\big)\Big(\sqrt{h}\big(w_{2\Lambda}^{(\ell)} + w_{-2\Lambda}^{(\ell)}\big)\Big).$$

One may express $w_{\pm 2\Lambda}^{(\ell)}$ from $w_\Lambda^{(\ell)}$, $\overline{w}_\Lambda^{(\ell)}$ by (5.53). Since, according to Proposition 5.2, $Z^k w_\Lambda^{(\ell)}$ is in $h^{-\delta'_{k+1+\ell}} L^\infty \tilde{J}_\Lambda^{0,b'}[K]$, it follows from Proposition 2.12 that $Z^k\big(w_\Lambda^{(\ell)}\big)^2$ (resp. $Z^k\big(\overline{w}_\Lambda^{(\ell)}\big)^2$) belongs to $h^{-2\delta'_{k+1+\ell}} L^\infty \tilde{J}_{2\Lambda}^{0,2b'}[K_2]$ (resp. $h^{-2\delta'_{k+1+\ell}} L^\infty \tilde{J}_{-2\Lambda}^{0,2b'}[K_{-2}]$).

We apply next Proposition 2.11, with a replaced by $(x\xi + |\xi|^{\frac12})\langle 2\,\mathrm{d}\,\omega\rangle^\ell \langle \mathrm{d}\,\omega\rangle^{-2\ell}|\,\mathrm{d}\,\omega|$. Since, because of the fact that $w_{\pm 2\Lambda}^{(\ell)}$ is microlocally supported close to $\pm 2\Lambda$, we may assume that $x\xi + |\xi|^{\frac12}$ is cut-off close to this manifold, we see that the above symbol satisfies (2.15) with $(\ell, \ell', d, d') = (-2 + 2\ell, -2\ell, 1/2, 0)$ or $(-1 + 2\ell, -2\ell, 1, 0)$. It follows from (2.17) that

$$\begin{aligned}
&\sqrt{h}\,\mathrm{Op}_h\big(x\xi + |\xi|^{\frac12}\big)\big(w_{2\Lambda}^{(\ell)} + w_{-2\Lambda}^{(\ell)}\big)\\
&\qquad = -\frac{i}{4}\sqrt{h}(1-\chi)\big(xh^{-\beta}\big)\langle 2\,\mathrm{d}\,\omega\rangle^\ell\langle \mathrm{d}\,\omega\rangle^{-2\ell}|\,\mathrm{d}\,\omega|^{\frac32}\Big[\big(w_\Lambda^{(\ell)}\big)^2 - \big(\overline{w}_\Lambda^{(\ell)}\big)^2\Big]\\
&\qquad\quad + h^{\frac32} r^{(\ell)}
\end{aligned} \tag{6.16}$$

where $Z^k r^{(\ell)}$ is in $h^{-2\delta'_{k+1+\ell}} \tilde{\mathcal{B}}_\infty^{2,2b'}[L] \subset h^{-2\delta'_{k+1+\ell}} \tilde{\mathcal{B}}_\infty^{0,b'}[L]$.

In the same way

$$\begin{aligned}
&h\,\mathrm{Op}_h\big(x\xi + |\xi|^{\frac12}\big)\big(w_{3\Lambda}^{(\ell)} + w_{-\Lambda}^{(\ell)} + w_{-3\Lambda}^{(\ell)}\big)\\
&\quad = h|\,\mathrm{d}\,\omega|^{\frac52}\langle \mathrm{d}\,\omega\rangle^{-3\ell}\Big[\langle 3\,\mathrm{d}\,\omega\rangle^\ell\Big(\mu_3^{(\ell)}\big(w_\Lambda^{(\ell)}\big)^3 + \mu_{-3}^{(\ell)}\big(\overline{w}_\Lambda^{(\ell)}\big)^3\Big) + \langle \mathrm{d}\,\omega\rangle^\ell \mu_{-1}^{(\ell)}|w_\Lambda^{(\ell)}|\overline{w}_\Lambda^{(\ell)}\Big]\\
&\quad\quad + h^2 r^{(\ell)}
\end{aligned} \tag{6.17}$$

with $Z^k r^{(\ell)}$ in $h^{-3\delta'_{k+1+\ell}} \tilde{\mathcal{B}}_\infty^{4,3b'}[L] \subset h^{-3\delta'_{k+1+\ell}} \tilde{\mathcal{B}}_\infty^{0,b'}[L]$ and some real constants $\mu_{\pm 3}^{(\ell)}$, $\mu_{-1}^{(\ell)}$.

Finally, since the action of $\mathrm{Op}_h\big(x\xi + |\xi|^{\frac12}\big)$ on the remainder $g^{(\ell)}$ of (5.52) gives a function $r^{(\ell)}$ such that $Z^k r^{(\ell)}$ is in $h^{-4\delta'_{k+N_1-N_0+1+\ell}} \tilde{\mathcal{B}}_\infty^{0,b'-\frac12}[F]$ we conclude, summing (6.9), (6.16), (6.17) that (6.15) is given by formula (6.8). $\square$

We may now prove Proposition 6.1.

Proof of Proposition 6.1. Let us compute

$$D_t w^{(\ell)} = D_t\,\mathrm{Op}_h\big(\langle\xi\rangle^\ell\big) w = i\ell h\,\mathrm{Op}_h\Big(\frac{\xi^2}{\langle\xi\rangle^2}\Big) w^{(\ell)} + \mathrm{Op}_h\big(\langle\xi\rangle^\ell\big) D_t w.$$

According to (3.23) this is the sum of $-\frac{i}{2}hw^{(\ell)}$, of (6.7), of (6.8) and of a remainder $h^{\frac{5}{4}}R^{(\ell)}(V)$ where $\mathcal{Z}^k R^{(\ell)}(V)$ is in $\mathcal{R}^b_\infty$. (Notice that by definition of w, we may always insert on the left hand side of (3.23) a cut-off $\sum_{j\in J(h,C)}\Delta^h_j$ for some large enough C, so that the sum of quadratic and cubic contributions is really given by (6.7)). Remembering the expression (5.52) of $w^{(\ell)}$ in terms of $w^{(\ell)}_\Lambda$, we may write

$$-\frac{i}{2}hw^{(\ell)} = -\frac{i}{2}h(1-\chi)\big(xh^{-\beta}\big)w^{(\ell)}_\Lambda + h^{\frac{3}{2}}r^{(\ell)}$$

with $Z^k r^{(\ell)}$ in $h^{-4\delta'_{k+N_1-N_0+1+\ell}}\tilde{\mathcal{B}}^{0,b'}_\infty[F]$ (We used again that the microlocal support properties of $w^{(\ell)}_\Lambda$ allow to multiply it by some cut-off $(1-\chi)\big(xh^{-\beta}\big)$ up to a $O(h^\infty)$-remainder). We obtain

(6.18)

$$\begin{aligned}
D_t w^{(\ell)} &= \frac{1}{2}(1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega|^{\frac{1}{2}}w^{(\ell)}_\Lambda \\
&\quad - i\frac{\sqrt{h}}{4}(1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega|^{\frac{3}{2}}\langle 2\,\mathrm{d}\,\omega\rangle^{\ell}\langle\mathrm{d}\,\omega\rangle^{-2\ell}\Big[(1+\sqrt{2})\big(w^{(\ell)}_\Lambda\big)^2 + (\sqrt{2}-1)\big(\overline{w}^{(\ell)}_\Lambda\big)^2\Big] \\
&\quad + \frac{h}{2}(1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega|^{\frac{5}{2}}\langle\mathrm{d}\,\omega\rangle^{-3\ell}\Big[\langle 3\,\mathrm{d}\,\omega\rangle^{\ell}\Big(\mu^{(\ell)}_3\big(w^{(\ell)}_\Lambda\big)^3 + \mu^{(\ell)}_{-3}\big(\overline{w}^{(\ell)}_\Lambda\big)^3\Big) \\
&\qquad\qquad + \langle\mathrm{d}\,\omega\rangle^{\ell}\big|w^{(\ell)}_\Lambda\big|^2 w^{(\ell)}_\Lambda + \langle\mathrm{d}\,\omega\rangle^{\ell}\mu^{(\ell)}_{-1}|w^{(\ell)}_\Lambda|\overline{w}^{(\ell)}_\Lambda\Big] \\
&\quad + h^{1+\sigma}R(V)
\end{aligned}$$

where $Z^k R(V)$ is $O(\varepsilon)$ in $h^{-4\delta'_{k+N_1-N_0+1+\ell}}\mathcal{R}^{b'-\frac{1}{2}}_\infty$, and where $\mu^{(\ell)}_q$ are some new real constants.

We express next $w^{(\ell)}_\Lambda$ from $w^{(\ell)}$ inverting relation (5.52) i.e. writing, taking (5.53) into account,

(6.19)

$$\begin{aligned}
w^{(\ell)}_\Lambda &= w^{(\ell)} + i\frac{\sqrt{h}}{4}(1-\chi)\big(xh^{-\beta}\big)\langle 2\,\mathrm{d}\,\omega\rangle^{\ell}\langle\mathrm{d}\,\omega\rangle^{-2\ell}|\,\mathrm{d}\,\omega|\Big[(1+\sqrt{2})\big(w^{(\ell)}\big)^2 + (1-\sqrt{2})\big(\overline{w}^{(\ell)}\big)^2\Big] \\
&\quad + h|\,\mathrm{d}\,\omega|^2(1-\chi)\big(xh^{-\beta}\big)\Big[\Gamma^{(\ell)}_3(\mathrm{d}\,\omega)\big(w^{(\ell)}\big)^3 + \Gamma^{(\ell)}_1(\mathrm{d}\,\omega)\big|w^{(\ell)}\big|^2 w^{(\ell)} \\
&\qquad\qquad + \Gamma^{(\ell)}_{-1}(\mathrm{d}\,\omega)\big|w^{(\ell)}\big|^2\overline{w}^{(\ell)} + \Gamma^{(\ell)}_{-3}(\mathrm{d}\,\omega)\big(\overline{w}^{(\ell)}\big)^3\Big] \\
&\quad + h^{1+\sigma}g^{(\ell)},
\end{aligned}$$

where $Z^k g^{(\ell)}$ is in $h^{-4\delta'_{k+N_1-N_0+1+\ell}}\mathcal{R}^{b'}_\infty$, and where $\Gamma^{(\ell)}_q(\zeta)$ is a symbol of order -2ℓ, with

$$\Gamma^{(\ell)}_1(\zeta) = \langle 2\zeta\rangle^{2\ell}\langle\zeta\rangle^{-4\ell}\left(\frac{3-2\sqrt{2}}{8}\right).$$

We plug this expansion in (6.18) to get (6.2). The remainder satisfies the conditions of the statement of the proposition if we assume that $4\delta'_{k+N_1-N_0+1+\ell} < \frac{\sigma}{2}$, so that we may take $\kappa = \sigma/2$.

We prove now (6.4) by induction from (6.2).

To deduce (6.4) at order k from the similar equality at order $k-1$, we notice first that the action of Z on the quadratic (resp. cubic, resp. remainder) terms of (6.4) at order $k-1$ gives contributions to $Q^{k,(\ell)}$ (resp. $\mathcal{C}^{k,(\ell)}$, resp. $\mathcal{R}^{k,(\ell)}$). Moreover,

$$\begin{aligned}\Big[Z, D_t - \frac{1}{2}(1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega(x)|^{\frac{1}{2}}\Big]&\\ &= -\Big(D_t - \frac{1}{2}(1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega(x)|^{\frac{1}{2}}\Big)\\ &\quad + \frac{1+\beta}{2}\big(xh^{-\beta}\big)\chi'\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega(x)|^{\frac{1}{2}}.\end{aligned}$$

The product of the last term with $W^{k-1,(\ell)}$ may be computed from expressions of the form $h^{-\beta}\Gamma\big(xh^{-\beta}\big)Z^{k'}w^{(\ell)}$ where $k' \leq k-1$ and Γ is in $C_0^\infty(\mathbb{R}^*)$. We just have to check that such terms contribute to the remainder in (6.4). Because of the expression (5.52) of $w^{(\ell)}$ in terms of $w_\Lambda^{(\ell)}$, we see that we need to check that for $p \leq b''$

$$\Big\|(hD_x)^p\,\Big[\Gamma\big(xh^{-\beta}\big)Z^{k'}w_\Lambda^{(\ell)}\Big]\Big\|_{L^\infty} = O\big(h^{1+\kappa+\beta}\big). \tag{6.20}$$

(The contribution coming from the remainder in (5.52) satisfies the wanted bound as we assume after (4.2) that $\beta < \sigma/2$.) We remember that $w_\Lambda^{(\ell)} = \sum_{j\in J(h,C)}\Theta_j^* w_{\Lambda,j}^{(\ell)}$, where $w_{\Lambda,j}^{(\ell)}$ is microlocally supported for x in a compact subset of $\mathbb{R}^*$, so that

$$\Gamma\big(xh^{-\beta}\big)Z^{k'}w_\Lambda^{(\ell)} = \sum_{j\in J(h,C)}\Gamma\big(xh^{-\beta}\big)Z^{k'}\Big(\Gamma_1\big(2^{\frac{j}{2}}x\big)\Theta_j^* w_{\Lambda,j}^{(\ell)}\Big)$$

for some Γ_1 in $C_0^\infty(\mathbb{R})$. This shows that the sum is limited to those j for which $2^j \sim h^{-2\beta}$.

Since $Z^{k'}w_\Lambda^{(\ell)}$ is in $h^{-\delta'_{k'+\ell+1}}L^\infty\tilde{J}_\Lambda^{0,b'}[K]$ according to Proposition 5.2,

$$\big\|\Theta_j^* Z^{k'}w_{\Lambda,j}^{(\ell)}\big\|_{L^\infty} = O\big(h^{-\delta'_{k'+\ell+1}}2^{-j+b'}\big).$$

Using that $2^j \sim h^{-2\beta}$, we bound $2^{-j+b'} \leq 2^{-j+b''}h^{\delta'_{k'+\ell+1}+\beta+\kappa+2}$ since the assumption on b'' relatively to b' implies that $2\beta(b'-b'') > \delta'_{k+\ell+1}+\kappa+\beta+2$, as $\delta'_{k+\ell+1}$, κ, β are small enough. Consequently, we get (6.20) for all integers $p \leq b''$.

This concludes the proof of the proposition. $\square$

Proposition 6.5. *Let T_0 be a large enough positive number, κ a small positive constant, $C_0 > 0$.*

Let ℓ be an integer, with $\ell \leq \frac{s}{2}+N_0$. Assume given a function $(t,x) \to r^{(\ell)}(t,x)$ from a domain $[T_0, T[\times\mathbb{R}$ to $\mathbb{C}$, satisfying for $p \leq b''$,

$$\sup_x\Big|(hD_x)^p r^{(\ell)}(t,x)\Big| \leq C_0\varepsilon$$

for any $t \in [T_0, T[$. Assume given a solution $w^{(\ell)}\colon [T_0,T[\times\mathbb{R} \to \mathbb{C}$ of equation (6.2), such that $\left|w^{(\ell)}(T_0,x)\right| \leq C_0\varepsilon$ for any x.

Then there are $\varepsilon_0 > 0$, $C_1 > 0$, depending only on C_0, such that for any $\varepsilon \in]0,\varepsilon_0[$,

$$\sup_{[T_0,T[} \left\|w^{(\ell)}(t,\cdot)\right\|_{L^\infty} \leq C_1\varepsilon. \tag{6.21}$$

Moreover, if we assume that $r^{(\ell)}$ is defined and satisfies the above assumption on $[T_0,+\infty[\times\mathbb{R}$, then $w^{(\ell)}$ is defined on $[T_0,+\infty[\times\mathbb{R}$ and there are a continuous bounded function $\alpha\colon \mathbb{R}\to\mathbb{C}$, vanishing like $|x|^{2b''}$ when x goes to zero, $(t,x)\to\rho(t,x)$ a bounded function on $[T_0,+\infty[\times\mathbb{R}$ with values in $\mathbb{C}$ and $\kappa > 0$ such that

$$w(t,x) = \varepsilon\alpha(x)\exp\left[\frac{i}{4|x|}\int_{T_0}^t (1-\chi)\big(\tau^\beta x\big)\,d\tau + \frac{i}{64}\varepsilon^2\frac{|\alpha(x)|^2}{|x|^5}\log t\right] + \varepsilon t^{-\kappa}\rho(t,x) \tag{6.22}$$

where $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ close to zero.

Remark. We may write (6.22) on a more explicit fashion. Assume that $b'' > \kappa/(2\beta)$. The contribution of the first term in expansion (6.22) localized for $|x| < Ct^{-\kappa/(2b'')}$ may be incorporated to the remainder, because of the vanishing of α at order $2b''$ at $x = 0$. On the other hand, if $|x| > Ct^{-\kappa/(2b'')}$, our assumption on b'' implies that $|x|^{-1/\beta} < C^{-1/\beta}t$, so that, if C is large enough,

$$\int_{T_0}^t \chi(\tau^\beta x)\,d\tau = \int_{T_0}^{+\infty}\chi(\tau^\beta x)\,d\tau.$$

If we define

$$\underline{\alpha}(x) = \alpha(x)\exp\left[-\frac{i}{4|x|}T_0 - \frac{i}{4|x|}\int_{T_0}^{+\infty}\chi\big(\tau^\beta x\big)\,d\tau\right]$$

we obtain

$$w(t,x) = \varepsilon\underline{\alpha}(x)\exp\left[\frac{it}{4|x|} + \frac{i}{64}\varepsilon^2\frac{|\underline{\alpha}(x)|^2}{|x|^5}\log t\right] + \varepsilon t^{-\kappa}\underline{\rho}(t,x) \tag{6.23}$$

for a new bounded remainder $\underline{\rho}$.

Proof. We shall establish the proposition performing a normal form transform on equation (6.2). Denote by $\mathcal{G}$ the space of continuous bounded functions on $[T_0,T[\times\mathbb{R}$. Let χ_0 be in $C_0^\infty(\mathbb{R})$, $\chi_0 \equiv 1$ close to zero, $\operatorname{Supp}\chi_0 \subset \{x\,;\,\chi(x)\equiv 1\}$ and set

(6.24)

$$\begin{aligned}
f^{(\ell)} &= w^{(\ell)}\\
&\quad + i\frac{\sqrt{h}}{4}(1-\chi_0)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega|\langle 2\,\mathrm{d}\,\omega\rangle^\ell\langle\mathrm{d}\,\omega\rangle^{-2\ell}\Big[(1+\sqrt{2})\big(w^{(\ell)}\big)^2 + (1-\sqrt{2})\big(\overline{w}^{(\ell)}\big)^2\Big]\\
&\quad + h(1-\chi_0)\big(xh^{-\beta}\big)^2|\,\mathrm{d}\,\omega|^2\Big[M_3^{(\ell)}(\mathrm{d}\,\omega)\big(w^{(\ell)}\big)^3 + M_{-3}^{(\ell)}(\mathrm{d}\,\omega)\big(\overline{w}^{(\ell)}\big)^3\\
&\qquad + M_{-1}^{(\ell)}(\mathrm{d}\,\omega)\big|w^{(\ell)}\big|^2\overline{w}^{(\ell)}\Big]
\end{aligned}$$

where $M_p^{(\ell)}(\zeta)$ are symbols of order -2ℓ in ζ to be chosen, $p=-3,-1,3$.

We consider the polynomial map $\Phi\colon \begin{pmatrix} w^{(\ell)} \\ \overline{w}^{(\ell)} \end{pmatrix} \to \begin{pmatrix} f^{(\ell)} \\ \overline{f}^{(\ell)} \end{pmatrix}$ defined on $\mathcal{G}$. For $h=t^{-1}$ small enough (i.e. $t \geq T_0$ large enough), this is a local diffeomorphism at zero in $\mathcal{G}$. The inverse Φ^{-1} sends $\begin{pmatrix} f^{(\ell)} \\ \overline{f}^{(\ell)} \end{pmatrix}$ to $\begin{pmatrix} w^{(\ell)} \\ \overline{w}^{(\ell)} \end{pmatrix}$, where $w^{(\ell)}$ may be expressed explicitly as

(6.25)

$$\begin{aligned}
w^{(\ell)} &= f^{(\ell)} \\
&\quad - i\frac{\sqrt{h}}{4}(1-\chi_0)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega|\langle 2\,\mathrm{d}\,\omega\rangle^{\ell}\langle \mathrm{d}\,\omega\rangle^{-2\ell}\Big[(1+\sqrt{2})\big(f^{(\ell)}\big)^2+(1-\sqrt{2})\big(\overline{f}^{(\ell)}\big)^2\Big] \\
&\quad + h|\,\mathrm{d}\,\omega|^2(1-\chi_0)^2\big(xh^{-\beta}\big)\Big[\widetilde{M}_3^{\ell}(\mathrm{d}\,\omega)\big(f^{(\ell)}\big)^3 \\
&\qquad\qquad + \frac{3-2\sqrt{2}}{8}\langle 2\,\mathrm{d}\,\omega\rangle^{2\ell}\langle \mathrm{d}\,\omega\rangle^{-4\ell}\big|f^{(\ell)}\big|^2 f^{(\ell)} \\
&\qquad\qquad + \widetilde{M}_{-1}^{\ell}(\mathrm{d}\,\omega)\big|f^{(\ell)}\big|^2\overline{f}^{(\ell)} \\
&\qquad\qquad + \widetilde{M}_{-3}^{\ell}(\mathrm{d}\,\omega)\big(\overline{f}^{(\ell)}\big)^3\Big] \\
&\quad + h^{1+\kappa}R_4\big(x,h;f^{(\ell)},\overline{f}^{(\ell)}\big)
\end{aligned}$$

where κ is some positive constant, where $R_4\big(x,h;f^{(\ell)},\overline{f}^{(\ell)}\big)$ is some analytic function of $\big(f^{(\ell)},\overline{f}^{(\ell)}\big)$, vanishing at order four at zero, with bounds uniform in (x,h), and $\widetilde{M}_p^{(\ell)}(\zeta) = -M_p^{(\ell)}(\zeta)+\Gamma_p^{\ell}(\zeta)$, $p=-3,-1,1$ for symbols Γ_p^{ℓ} of order -2ℓ, independent of M_p^{ℓ}.

We compute $D_t f^{(\ell)}$ from (6.24), expressing in the right hand side $D_t w^{(\ell)}$, $D_t\overline{w}^{(\ell)}$ using (6.2). We get

(6.26)

$$\begin{aligned}
D_t f^{(\ell)} &= \frac{1}{2}(1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega(x)|^{\frac{1}{2}}w^{(\ell)} \\
&\quad + i\frac{\sqrt{h}}{8}(1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega(x)|^{\frac{3}{2}}\langle 2\,\mathrm{d}\,\omega\rangle^{\ell}\langle \mathrm{d}\,\omega\rangle^{-2\ell}\Big[(1+\sqrt{2})\big(w^{(\ell)}\big)^2+(1-\sqrt{2})\big(\overline{w}^{(\ell)}\big)^2\Big] \\
&\quad + h(1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega(x)|^{\frac{5}{2}}\Big[\frac{1}{2}\langle \mathrm{d}\,\omega\rangle^{-2\ell}\big|w^{(\ell)}\big|^2 w^{(\ell)} \\
&\qquad + \Big(\frac{3}{2}M_3^{(\ell)}(\mathrm{d}\,\omega)+\widetilde{\Gamma}_3^{(\ell)}(\mathrm{d}\,\omega)\Big)\big(w^{(\ell)}\big)^3 + \Big(-\frac{1}{2}M_{-1}^{(\ell)}(\mathrm{d}\,\omega)+\widetilde{\Gamma}_{-1}^{(\ell)}(\mathrm{d}\,\omega)\Big)\big|w^{(\ell)}\big|^2\overline{w}^{(\ell)} \\
&\qquad + \Big(-\frac{3}{2}M_{-3}^{(\ell)}(\mathrm{d}\,\omega)+\widetilde{\Gamma}_{-3}^{(\ell)}(\mathrm{d}\,\omega)\Big)\big(\overline{w}^{(\ell)}\big)^3\Big] \\
&\quad + h^{1+\kappa}\Big(r^{(\ell)}(t,x)+R_2\big(x,h;w^{(\ell)},\overline{w}^{(\ell)}\big)\Big)
\end{aligned}$$

where $\widetilde{\Gamma}_p^{(\ell)}(\zeta)$ are symbols of order -2ℓ in ζ, that depend only on the coefficients of $\big(w^{(\ell)}\big)^2$, $\big(w^{(\ell)}\big)^3$, …in the right hand side of (6.2), where $r^{(\ell)}$ is the remainder in (6.2) and where $R_2\big(x,h;w^{(\ell)},\overline{w}^{(\ell)}\big)$ is some polynomial in $\big(w^{(\ell)},\overline{w}^{(\ell)}\big)$, vanishing at order 2 at zero, with uniform bounds in (x,h). We express $w^{(\ell)}$ in the right hand side of (6.26) using formula (6.25). The quadratic terms in the definition (6.24) of $f^{(\ell)}$ have been chosen in such a way that the quadratic contributions in the right hand side of the resulting expression for $D_t f^{(\ell)}$ vanish. We get

$$
\begin{aligned}
D_t f^{(\ell)} &= \frac{1}{2}(1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega(x)|^{\frac{1}{2}} f^{(\ell)}\\
&\quad + h(1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega(x)|^{\frac{5}{2}}\Big[\frac{1}{2}\langle \mathrm{d}\,\omega\rangle^{-2\ell}\big|f^{(\ell)}\big|^2 f^{(\ell)}\\
&\qquad + \Big(M_3^{(\ell)}(\mathrm{d}\,\omega)-\widetilde{\Gamma}'^{(\ell)}_3(\mathrm{d}\,\omega)\Big)\big(f^{(\ell)}\big)^3\\
&\qquad + \Big(-M_{-1}^{(\ell)}(\mathrm{d}\,\omega)-\widetilde{\Gamma}'^{(\ell)}_{-1}(\mathrm{d}\,\omega)\Big)\big|f^{(\ell)}\big|^2\overline{f}^{(\ell)}\\
&\qquad + \Big(-2M_{-3}^{(\ell)}(\mathrm{d}\,\omega)-\widetilde{\Gamma}'^{(\ell)}_{-3}(\mathrm{d}\,\omega)\Big)\big(\overline{f}^{(\ell)}\big)^3\Big]\\
&\quad + h^{1+\kappa}\Big(r^{(\ell)}(t,x)+R_2\big(x,h;f^{(\ell)},\overline{f}^{(\ell)}\big)\Big)
\end{aligned}
\tag{6.27}
$$

where $\widetilde{\Gamma}'^{(\ell)}_p(\zeta)$ are new symbols of order -2ℓ that do not depend on $M_p^{(\ell)}$, and where R_2 is a new analytic function of $(f^{(\ell)},\overline{f}^{(\ell)})$ vanishing at order 2 at zero, with uniform bounds in (x,h).

We choose now the free symbols $M_p^{(\ell)}$, $p=3,-1,-3$ introduced in the definition (6.24) of $f^{(\ell)}$ so that the coefficients of $\big(f^{(\ell)}\big)^3$, $\big|f^{(\ell)}\big|^2\overline{f}^{\ell}$ and $\big(\overline{f}^{(\ell)}\big)^3$ vanish. In that way, we are reduced to

$$
\begin{aligned}
D_t f^{(\ell)} &= \frac{1}{2}(1-\chi)\big(xh^{-\beta}\big)|\,\mathrm{d}\,\omega(x)|^{\frac{1}{2}}\Big[1+\frac{|\,\mathrm{d}\,\omega|^2}{t}\langle \mathrm{d}\,\omega\rangle^{-2\ell}\big|f^{(\ell)}\big|^2\Big]f^{(\ell)}\\
&\quad + t^{-1-\kappa}r^{(\ell)}(t,x)+t^{-1-\kappa}R_2\big(x,h;f^{(\ell)},\overline{f}^{(\ell)}\big)
\end{aligned}
\tag{6.28}
$$

where $\big\|(hD_x)^p Z^k r^{(\ell)}(t,x)\big\|_{L^\infty}$ is $O(\varepsilon)$ for any $p\le b''$, $k+\ell\le \frac{s}{2}+N_0$. It follows from (6.28) that $\Big|\partial_t\big|f^{(\ell)}\big|^2\Big|\le \big(C_0\varepsilon + C_0'\big|f^{(\ell)}\big|^2\big)t^{-1-\kappa}$ as long as $\big|f^{(\ell)}\big|$ stays smaller than 1. Since at time $t=T_0$, $\big|f^{(\ell)}\big|=O(\varepsilon)$, we obtain that $\big|f^{(\ell)}(t,x)\big|\le C_1'\varepsilon$ for some constant $C_1'>0$, as long as the solution exists. Using expression (6.25) for $w^{(\ell)}$ in terms of $f^{(\ell)}$, we get (6.21). If r_ℓ is defined for $t\in[T_0,+\infty[$, we get that $f^{(\ell)}$ and thus $w^{(\ell)}$ is defined on $[T_0,+\infty[\times\mathbb{R}$.

Let us prove the asymptotic expansion for w. If $\ell\le\frac{s}{2}+N_0$, we define

$$
\Theta_\ell(t,x)=\frac{1}{2}|\,\mathrm{d}\,\omega(x)|^{\frac{1}{2}}\int_{T_0}^t (1-\chi)\big(\tau^\beta x\big)\Big[1+\frac{|\,\mathrm{d}\,\omega(x)|^2}{\tau}\langle \mathrm{d}\,\omega\rangle^{-2\ell}\big|f^{(\ell)}(\tau,x)\big|^2\Big]\,d\tau.
$$

Then (6.28) and the uniform a priori bound just obtained for $f^{(\ell)}$ show that

$$\frac{d}{dt}\Big[f^{(\ell)}(t,x)\exp\Big[-i\Theta_\ell(t,x)\Big]\Big] = O\big(\varepsilon t^{-1-\kappa}\big)$$

uniformly for $x \in \mathbb{R}$. It follows that the uniform limit when t goes to $+\infty$ of

$$f^{(\ell)}(t,x)\exp\big[-i\Theta_\ell(t,x)\big]$$

exists and defines a continuous function $\varepsilon\alpha_\ell(x)$ on $\mathbb{R}$, which is $O(\varepsilon)$ in $L^\infty(\mathbb{R})$. Moreover

$$\big\|f^{(\ell)}(t,x) - \varepsilon\alpha_\ell(x)\exp\big(i\Theta_\ell(t,x)\big)\big\|_{L^\infty} = O\big(\varepsilon t^{-\kappa}\big), \quad t \to +\infty. \tag{6.29}$$

We write

$$\begin{aligned}
\Theta_\ell(t,x) = {} & \frac{1}{2}|\operatorname{d}\omega(x)|^{\frac{1}{2}}\int_{T_0}^t (1-\chi)\big(\tau^\beta x\big)\,d\tau \\
& + \frac{\varepsilon^2}{2}(1-\chi)\big(t^\beta x\big)|\operatorname{d}\omega(x)|^{\frac{5}{2}}\langle \operatorname{d}\omega(x)\rangle^{-2\ell}\,|\alpha_\ell(x)|^2\log t \\
& - \frac{\varepsilon^2}{2}(1-\chi)\big(T_0^\beta x\big)|\operatorname{d}\omega(x)|^{\frac{5}{2}}\langle \operatorname{d}\omega(x)\rangle^{-2\ell}\,|\alpha_\ell(x)|^2\log T_0 \\
& + \frac{\varepsilon^2}{2}\int_{T_0}^t \beta\tau^\beta x\chi'\big(\tau^\beta x\big)\,|\operatorname{d}\omega(x)|^{\frac{5}{2}}\,\langle \operatorname{d}\omega(x)\rangle^{-2\ell}\,|\alpha_\ell(x)|^2\,\frac{\log\tau}{\tau}\,d\tau \\
& + \frac{1}{2}\int_{T_0}^t (1-\chi)\big(\tau^\beta x\big)\,|\operatorname{d}\omega(x)|^{\frac{5}{2}}\,\langle \operatorname{d}\omega(x)\rangle^{-2\ell}\Big(\big|f^{(\ell)}\big|^2 - |\varepsilon\alpha_\ell(x)|^2\Big)\,\frac{d\tau}{\tau}.
\end{aligned} \tag{6.30}$$

We notice that $|\operatorname{d}\omega(x)|^{\frac{5}{2}}\,\langle\operatorname{d}\omega\rangle^{-2\ell}$ is $O(\langle x\rangle^{-5})$ if $\ell \geq 5/4$ and $\tau^\beta x\chi'(\tau^\beta x)\,|\operatorname{d}\omega(x)|^{\frac{5}{2}}\,\langle\operatorname{d}\omega(x)\rangle^{-2\ell}$ is $O\big(\tau^{-\beta}\langle x\rangle^{-6}\big)$ if $\ell \geq 3/2$.

These bounds and the estimate $\big\|\big|f^{(\ell)}\big|^2 - \varepsilon^2\,|\alpha_\ell|^2\big\|_{L^\infty} = O\big(\varepsilon^2 t^{-\kappa}\big)$ that follows from (6.29) imply that the last three terms in (6.30) may be written as $\varepsilon^2\Gamma_\ell(x) + \varepsilon^2 R_\ell(t,x)$ for some continuous function $\Gamma_\ell(x)$, which is $O\big(\langle x\rangle^{-5}\big)$ and some remainder $R(t,x)$ satisfying $|R(t,x)| = O\big(t^{-\kappa}\langle x\rangle^{-5}\big)$ (assuming $0 < \kappa < \beta$). Modifying the definition of that remainder, we get finally

$$\begin{aligned}
\Theta_\ell(t,x) = {} & \frac{1}{2}\,|\operatorname{d}\omega(x)|^{\frac{1}{2}}\int_{T_0}^t (1-\chi)(\tau^\beta x)\,d\tau \\
& + \frac{\varepsilon^2}{2}\,|\operatorname{d}\omega(x)|^{\frac{5}{2}}\,\langle\operatorname{d}\omega\rangle^{-2\ell}\,|\alpha_\ell(x)|^2\log t \\
& + \varepsilon^2\Gamma(x) + \varepsilon^2 R(t,x)
\end{aligned}$$

when $\ell > 3/2$. It follows from this and from (6.29) that

$$\begin{aligned}
f^{(\ell)}(t,x) = {} & \varepsilon\widetilde{\alpha}_\ell(x)\exp\left[\frac{i}{4|x|}\int_{T_0}^t (1-\chi)\big(\tau^\beta x\big)\,d\tau + \frac{i}{64}\varepsilon^2\frac{\big|\widetilde{\alpha}_\ell(x)\big|^2}{|x|^5}\langle\operatorname{d}\omega\rangle^{-2\ell}\log t\right] \\
& + \varepsilon t^{-\kappa}\rho(t,x)
\end{aligned} \tag{6.31}$$

where $\widetilde{\alpha}_\ell(x) = e^{i\varepsilon^2\Gamma(x)}\alpha_\ell(x)$ and where $\rho(t,x)$ is uniformly bounded. If we express $w^{(\ell)}$ from $f^{(\ell)}$ using (6.25), we conclude that the same expansion (6.31) holds for $w^{(\ell)}$ (with a different remainder). Let us compute $w(t,\cdot) = \mathrm{Op}_h\big(\langle\xi\rangle^{-\ell}\big)w^{(\ell)}$. The action of $\mathrm{Op}_h\big(\langle\xi\rangle^{-\ell}\big)$ on the remainder gives a term of the same type, if ℓ is large enough. On the other hand, by the expression (5.52), (5.53) of $w^{(\ell)}$ from $w_\Lambda^{(\ell)}$ (and the converse expression), we get that $\mathrm{Op}_h\big(\langle\xi\rangle^{-\ell}\big)w = \mathrm{Op}_h\big(\langle\xi\rangle^{-\ell}\big)w_\Lambda$ up to a remainder bounded in $L^\infty(dx)$ by $C\varepsilon t^{-\kappa}$. As w_Λ is in $h^{-\delta_1'}L^\infty\tilde{J}_\Lambda^{0,b'}[K]$, Proposition 2.11 applies and shows that $\mathrm{Op}_h\big(\langle\xi\rangle^{-\ell}\big)w_\Lambda$ may be written as $\langle \mathrm{d}\,\omega\rangle^{-\ell}w_\Lambda$ modulo a remainder in $h^{1-\delta_1'}\tilde{\mathcal{B}}_\infty^{-1,b'}[K] \subset h^{\sigma-\delta_1'}\tilde{\mathcal{B}}_\infty^{0,b'}[K]$, which is $O(\varepsilon t^{-\kappa})$ in L^∞ for small enough $\kappa > 0$ since by (4.2) $\delta_1' < \sigma/8$. Using again the expression of w_Λ from w deduced from (5.52), we deduce that

$$\Big\|w(t,\cdot) - \langle \mathrm{d}\,\omega\rangle^{-\ell}w^{(\ell)}(t,\cdot)\Big\|_{L^\infty} = O(\varepsilon t^{-\kappa}).$$

If we define $\alpha(x) = \langle \mathrm{d}\,\omega\rangle^{-\ell}\widetilde{\alpha}_\ell(x)$ with ℓ equal to b'', we obtain a function continuous and bounded on $\mathbb{R}$, vanishing like $|x|^{2b''}$ when x goes to zero and such that $w(t,x)$ is given by the asymptotic expansion (6.22). This concludes the proof. $\square$

We prove now a statement concerning the Z-derivatives of $w^{(\ell)}$. Let $(A_k'')_{k\geq 1}$ be a sequence of positive numbers satisfying $A_k'' \geq A_{k_1}'' + A_{k_2}'' + A_{k_3}''$ if $k_1+k_2+k_3 = k$, $k_j < k$, $j = 1,2,3$ and A_k'' large enough relatively to the constant C_1 in (6.21).

Proposition 6.6. *There is a constant $C_2 > 0$ such that, if we set $\widetilde{\delta}_k' = A_k''\varepsilon^2$, for any k,ℓ with $k+\ell \leq \frac{s}{2} + N_0 - 2$, the solution $w^{(\ell)}$ of (6.2) satisfies*

$$\big\|Z^k w^{(\ell)}(t,\cdot)\big\|_{L^\infty} \leq C_2\varepsilon t^{\widetilde{\delta}_k'}. \tag{6.32}$$

Remark. The gain in (6.32), in comparison with (4.9), is that the exponents $\widetilde{\delta}_k'$ depend only on the size ε of the Cauchy data and not on the exponents δ_k that are used in the L^2-estimates. In particular, taking ε small enough, we may arrange so that $\widetilde{\delta}_k' \ll \delta_k$.

Proof. We apply a normal forms method to remove the quadratic terms in (6.4). For (k,ℓ) satisfying $k+\ell \leq \frac{s}{2} + N_0$, we define a new quadratic map $\widetilde{Q}^{k,(\ell)}\Big[x,h;W^{k,(\ell)},\overline{W}^{k,(\ell)}\Big]$ in the following way: The components of this map are defined taking the same linear combinations as those used to define the components of $Q^{k,(\ell)}$ from (6.5) of the quantities

$$\begin{aligned}
&\frac{2\theta\big(xh^{-\beta}\big)\Phi(x)}{(1-\chi)\big(xh^{-\beta}\big)\,|\mathrm{d}\,\omega|^{\frac12}}\big(Z^{k_1}w^{(\ell)}\big)\big(Z^{k_2}w^{(\ell)}\big)\\
&\frac{-2\theta\big(xh^{-\beta}\big)\Phi(x)}{(1-\chi)\big(xh^{-\beta}\big)\,|\mathrm{d}\,\omega|^{\frac12}}\big(Z^{k_1}w^{(\ell)}\big)\big(Z^{k_2}\overline{w}^{(\ell)}\big)\\
&\frac{-2\theta\big(xh^{-\beta}\big)\Phi(x)}{3(1-\chi)\big(xh^{-\beta}\big)\,|\mathrm{d}\,\omega|^{\frac12}}\big(Z^{k_1}\overline{w}^{(\ell)}\big)\big(Z^{k_2}\overline{w}^{(\ell)}\big).
\end{aligned} \tag{6.33}$$

If we make act $D_t - \frac{1}{2}(1-\chi)\big(xh^{-\beta}\big)\,|\mathrm{d}\,\omega(x)|^{\frac{1}{2}}$ on each line of (6.33), we see that we obtain, using (6.4), the corresponding line of (6.5) and the following contributions

• Quantities of the form $\sqrt{h}\widetilde{\mathcal{C}}^{k,(\ell)}\Big[x,h;W^{k,(\ell)},\overline{W}^{k,(\ell)}\Big]$ for cubic forms $\widetilde{\mathcal{C}}^{k,(\ell)}$ which have the same structure (6.6) as $\mathcal{C}^{k,(\ell)}$.

• Quantities given by the product of h and of homogeneous expressions of order 4 in

$$\big(Z^{k_j}w^{(\ell)}, Z^{k_j}\overline{w}^{(\ell)}\big), \quad k_1+\cdots+k_4 \le k$$

with coefficients depending on x which are $O(h^{-7\beta})$. If we use that $Z^{k_j}w^{(\ell)}$ satisfies the a priori estimates (4.9), we see that these contributions may be written as $h^{\frac{1}{2}+\kappa}R^{k,(\ell)}$ for some $\kappa > 0$ and a bounded function $R^{k,(\ell)}$.

• Contributions coming from the remainder in (6.4) or from the action of D_t on the cut-offs in (6.33), that may be written also as $h^{\frac{1}{2}+\kappa}R^{k,(\ell)}$.

Consequently, if we set for $k+\ell \le \frac{s}{2}+N_0$,

$$\widetilde{W}^{k,(\ell)} = W^{k,(\ell)} - \sqrt{h}\widetilde{Q}^{k,(\ell)}\Big[x,h;W^{k,(\ell)},\overline{W}^{k,(\ell)}\Big],$$

we obtain that $\widetilde{W}^{k,(\ell)}$ satisfies bounds of the form (4.9) and solves the equation

$$\Big(D_t - \frac{1}{2}(1-\chi)\big(xh^{-\beta}\big)\,|\mathrm{d}\,\omega|^{\frac{1}{2}}\Big)\widetilde{W}^{k,(\ell)} = h\mathcal{C}^{k,(\ell)}\Big[x,h;W^{k,(\ell)},\overline{W}^{k,(\ell)}\Big] + h^{1+\kappa}\mathcal{R}^{k,(\ell)} \tag{6.34}$$

where $\mathcal{C}^{k,(\ell)}$ is a new cubic map given in terms of monomials of the form (6.6) and $\mathcal{R}^{k,(\ell)}$ a uniformly bounded remainder. Notice that, up to a modification of $\mathcal{R}^{k,(\ell)}$, we may replace $W^{k,(\ell)}$ by $\widetilde{W}^{k,(\ell)}$ in the argument of $\mathcal{C}^{k,(\ell)}$.

Assume by induction that for given k,ℓ with $k+\ell \le \frac{s}{2}+N_0$, $\ell \ge 2$, (6.32) has been established with k replaced by $k-1$. Then $W^{k-1,(\ell)}$ and $\widetilde{W}^{k-1,(\ell)}$ are under control, and we need to obtain (6.32) for the last component $Z^k w^{(\ell)}$ of $W^{k,(\ell)}$, or equivalently, for the last component $\widetilde{W}^{k,(\ell)}_k$ of $\widetilde{W}^{k,(\ell)}$. We sort the different contributions to

$$\mathcal{C}^{k,(\ell)}\Big[x,h;\widetilde{W}^{k,(\ell)},\overline{\widetilde{W}}^{k,(\ell)}\Big]. \tag{6.35}$$

On the one hand, we get terms given by expressions of the form

$$\begin{aligned}
&\theta\big(xh^{-\beta}\big)\Phi(x)\widetilde{W}^{k,(\ell)}_{k_1}\widetilde{W}^{k,(\ell)}_{k_2}\widetilde{W}^{k,(\ell)}_{k_3}\\
&\theta\big(xh^{-\beta}\big)\Phi(x)\widetilde{W}^{k,(\ell)}_{k_1}\widetilde{W}^{k,(\ell)}_{k_2}\overline{\widetilde{W}}^{k,(\ell)}_{k_3}\\
&\theta\big(xh^{-\beta}\big)\Phi(x)\widetilde{W}^{k,(\ell)}_{k_1}\overline{\widetilde{W}}^{k,(\ell)}_{k_2}\overline{\widetilde{W}}^{k,(\ell)}_{k_3}\\
&\theta\big(xh^{-\beta}\big)\Phi(x)\overline{\widetilde{W}}^{k,(\ell)}_{k_1}\overline{\widetilde{W}}^{k,(\ell)}_{k_2}\overline{\widetilde{W}}^{k,(\ell)}_{k_3}
\end{aligned} \tag{6.36}$$

where θ, Φ satisfy the same conditions as in (6.6), so are bounded since $\ell \geq 2$, and where two among k_1, k_2, k_3 are zero and the other one is equal to k. We call F the sum of contributions of that type, so that

$$|F(t,x)| \leq C\big\|\widetilde{W}_0^{k,(\ell)}(t,\cdot)\big\|_{L^\infty}^2 \big|\widetilde{W}_k^{k,(\ell)}(t,x)\big|.$$

Proposition 6.5 gives a uniform estimate for $\big\|w^{(\ell)}(t,\cdot)\big\|_{L^\infty}$, so also for

$$\widetilde{W}_0^{k,(\ell)} = w^{(\ell)} - \sqrt{h}\widetilde{Q}_0^{k,(\ell)}\Big[x,h;W^{k,(\ell)},\overline{W}^{k,(\ell)}\Big].$$

We conclude that for some constant $B > 0$, depending only on the constant C_1 in (6.21),

$$|F(t,x)| \leq B\varepsilon^2\big|\widetilde{W}_k^{k,(\ell)}(t,x)\big|. \tag{6.37}$$

On the other hand, (6.35) is also made of terms of the form (6.36) with

$$k_1 + k_2 + k_3 \leq k, \quad k_1, k_2, k_3 < k.$$

The assumption of induction, together with the inequality between the constants A_k made in the statement of the proposition, imply that the contribution G of these terms satisfies

$$|G(t,x)| \leq C\varepsilon^3 t^{\widetilde{\delta}'_k}. \tag{6.38}$$

We deduce from the equation for the last component $\widetilde{W}_k^{k,(\ell)}$ of $\widetilde{W}^{k,(\ell)}$ given by (6.34)

$$\begin{aligned}\big|\widetilde{W}_k^{k,(\ell)}(t,x)\big|^2 \leq \big|\widetilde{W}_k^{k,(\ell)}(T_0,x)\big|^2 &+ \int_{T_0}^t |F(\tau,x)|\big|\widetilde{W}_k^{k,(\ell)}(\tau,x)\big|\,\frac{d\tau}{\tau}\\ &+ \int_{T_0}^t \big|G(\tau,x)\big|\big|\widetilde{W}_k^{k,(\ell)}(\tau,x)\big|\,\frac{d\tau}{\tau}\\ &+ \int_{T_0}^t \big|R_k^{k,(\ell)}(\tau,x)\big|\big|\widetilde{W}_k^{k,(\ell)}(\tau,x)\big|\,\frac{d\tau}{\tau^{1+\kappa}}.\end{aligned}$$

Using (6.37), (6.38), and the fact that at $t = T_0$, $\widetilde{W}_k^{k,(\ell)}(T_0,\cdot)$ is $O(\varepsilon)$ we deduce that

$$\begin{aligned}\big|\widetilde{W}_k^{k,(\ell)}(t,x)\big| \leq C\varepsilon + B\varepsilon^2 &\int_{T_0}^t \big|\widetilde{W}_k^{k,(\ell)}(\tau,x)\big|\,\frac{d\tau}{\tau}\\ &+ C\varepsilon^2\int_{T_0}^t \tau^{\widetilde{\delta}'_k - 1}\,d\tau + C\varepsilon\int_{T_0}^t \frac{d\tau}{\tau^{1+\kappa}}.\end{aligned} \tag{6.39}$$

If we use Gronwall inequality, and assume that the constant A_k in the definition $\widetilde{\delta}'_k = A_k\varepsilon^2$ of $\widetilde{\delta}'_k$ is large enough relatively to B, we deduce from (6.39) that

$$\big|\widetilde{W}_k^{k,(\ell)}(t,x)\big| \leq C'\varepsilon t^{\widetilde{\delta}'_k}$$

when $k+\ell \leq \frac{s}{2}+N_0$, $\ell \geq 2$. By definition of $\widetilde{W}_k^{k,(\ell)}$, the same inequality holds for $Z^k w^{(\ell)}$. Since $w^{(\ell-2)} = \mathrm{Op}_h\left(\langle\xi\rangle^{-2}\right)w^{(\ell)}$, we conclude that $\big\|Z^k w^{(\ell)}(t,\cdot)\big\|_{L^\infty}$ is $O(\varepsilon t^{\widetilde{\delta}'_k})$ when $k+\ell \leq \frac{s}{2}+N_0-2$. This concludes the proof of the proposition. $\square$

To finish this section, we deduce from the results established so far the proof of Theorem 1.6. This will conclude the demonstration of our main theorem.

Proof of Theorem 1.6. We notice first that it is enough to prove the following apparently weaker statement: Assume that for some constants $B_2 > 0$, $\tilde{A}_0 > 0$, any $t \in [T_0, T[$, any $\epsilon \in]0,1]$, any $k \leq s_1$

$$
\begin{aligned}
&M_s^{(k_1)}(t) \leq B_2 \epsilon t^{\delta_k},\ N_\rho^{(s_0)}(t) \leq \sqrt{\epsilon} < 1,\\
&\|\eta(t,\cdot)\|_{C^\gamma} + \||D_x|^{1/2}\psi(t,\cdot)\|_{C^{\gamma-\frac{1}{2}}} \leq \tilde{A}_0 t^{-\frac{1}{2}+\delta_0'}.
\end{aligned} \tag{6.40}
$$

Then, (1.20) holds.

Actually, if the preceding implication is proved with $\rho \geq \gamma$, and if we assume only (1.19), then (6.40) holds true on some interval $[T_0, T']$, $T' > T_0$ taking $\tilde{A}_0$ large enough in function of T_0 (because the last condition in (6.40) follows then from the second one, taking T' close enough to T_0). We conclude that (1.20) holds on $[T_0, T']$, and taking $\epsilon < \epsilon_0'$ small enough so that $\epsilon B_\infty < \tilde{A}_0$, we see that, by continuity, (6.40) holds on some interval $[T_0, T'']$ with $T'' > T'$. By bootstrap, we conclude that (1.20) will then be true on the whole interval $[T_0, T]$.

Consequently, we have reduced ourselves to the proof of the fact that (6.40) implies (1.20).

Recall that we have fixed in (4.3) large enough numbers a, b. We introduced also at the beginning of Section 4 integers N_0, N_1 and we assumed in Proposition 5.2 that $(N_1 - N_0 - 1)\sigma \geq 1$. Let us fix $\gamma \in]\max(7/2, b), +\infty[\setminus \frac{1}{2}\mathbb{N}$, and assume that N_0 is taken large enough so that $N_0 \geq 2\gamma + \frac{13}{2}$. We define

$$
s_1 = \frac{s}{2} + N_1 + 1, \quad s_0 = \frac{s}{2} + N_0 - 3 - [\gamma] \tag{6.41}
$$

where s is an even integer taken large enough so that the following conditions hold

$$
s \geq s_1 \geq s_0 \geq \frac{1}{2}(s + 2\gamma) \tag{6.42}
$$

and that moreover

$$
s_1 \leq s - a - \frac{1}{2}. \tag{6.43}
$$

We set $\rho = s_0 + \gamma$. It follows from equation (5.2.157) of the companion paper [5] that if $\mathcal{C}_{s_0} N_\rho^{(s_0)} = C\big(N_\rho^{(s_0)}\big) N_\rho^{(s_0)}$ is small enough, we have for any $k \leq s_1$

$$
\big\|Z^k \eta(t)\big\|_{H^{s-k}} + \big\||D_x|^{\frac{1}{2}} Z^k \psi(t)\big\|_{H^{s-k-\frac{1}{2}}} \leq B_2 \varepsilon t^{\delta_k}
$$

for a new value of the constant B_2. The smallness condition above is satisfied for $\varepsilon < \varepsilon_0' \ll 1$ using the second estimate (6.40). Since we have set at the beginning of Section 3

$$
u(t,x) = |D_x|^{\frac{1}{2}} \psi + i\eta \text{ and } u(t,x) = \frac{1}{\sqrt{t}} v\left(t, \frac{x}{t}\right)
$$

it follows, denoting by the same notation Z the vector field in (t,x) and in $(t,\frac{x}{t})$-coordinates, that

$$\left\|(hD_x)^\ell Z^{k'}v(t,\cdot)\right\|_{L^2} \leq B_2\varepsilon t^{\delta_k}$$

for $k' \leq s_1$, $\ell \leq a$ since $s_1+a \leq s-\frac{1}{2}$. This, together with the definition (4.5) of $\mathcal{F}_k$, shows that the second condition in (4.8) holds with $k = s_1 - 1$. The first condition (4.8) holds because of the second estimate (6.40) and the fact that $\rho \geq \gamma > b$. Consequently, Proposition 4.1 implies that (4.9) holds for any $k \leq s_1 - 1 = \frac{s}{2} + N_1$, with constants $A'_{k'}$ depending only on B_2 in (6.40). The assumption (5.1) is thus satisfied, and since we assumed $(N_1 - N_0 - 1)\sigma \geq 1$, we may apply Proposition 5.2 and Corollary 5.9 which provides development (5.48). This development is the assumption that allows one to apply the results of Section 6: in particular inequality (6.32) will hold, with a constant C_2 depending only on the constant B_2 of (6.40) (and of universal quantities). If B_∞ is taken large enough relatively to B_2 and if B'_∞ is larger than the constant $A''_{s_0+[\gamma]+1}$ introduced in Proposition 6.6, we deduce from (6.32)

$$\left\|(h\partial_x)^\ell Z^k w(t,\cdot)\right\|_{L^\infty} \leq \frac{1}{2}B_\infty\varepsilon t^{B'_\infty\varepsilon^2}$$

for $k+\ell \leq s_0 + [\gamma] + 1$ (since $\frac{s}{2} + N_0 - 2 = s_0 + [\gamma] + 1$ by our choice (6.41) of s_0). Coming back to the expression of $u = |D_x|^{\frac{1}{2}}\psi + i\eta$ from $t^{-\frac{1}{2}}v$, and using that by definition $\rho = s_0 + \gamma$ this will give the bound

$$N_\rho^{(s_0)}(t) \leq \frac{1}{2}B_\infty\varepsilon t^{-\frac{1}{2}+B'_\infty\varepsilon^2} \tag{6.44}$$

if we prove that in the decomposition $v = v_L + w + v_H$, the contributions v_L and v_H satisfy also a bound of the form

$$\left\|Z^k v_H(t,\cdot)\right\|_{C^{\rho-k}} + \left\|Z^k v_L(t,\cdot)\right\|_{C^{\rho-k}} \leq \frac{1}{4}B_\infty\varepsilon t^{B'_\infty\varepsilon^2} \tag{6.45}$$

if $k \leq s_0$. Since our assumption (6.40) implies that (4.8) holds (with constants $A_{k'}$ depending only on B_2), for $k' \leq s_1$, we deduce from (4.5) and the definition (3.19) of v_L that

$$\left\|Z^k v_L(t,\cdot)\right\|_{L^2} \leq \varepsilon A_k h^{-\delta_k}, \quad k \leq s_1.$$

Since v_L is spectrally supported for $h|\xi| = O\big(h^{2(1-\sigma)}\big)$, we deduce from that by Sobolev injection that

$$\left\|Z^k v_L(t,\cdot)\right\|_{L^\infty} = O\big(\varepsilon h^{-\delta_k+\frac{1}{2}-\frac{\sigma}{2}}\big) \tag{6.46}$$

with constants depending only on B_2, which gives for v_L a better estimate than the one (6.45) we are looking for (since v_L is spectrally supported for small frequencies, estimating L^∞ or $C^{\rho-k}$ norms is equivalent).

Consider next the v_H-contribution. As (4.9) holds for $k = s_1 - 1$ with constants depending only on B_2, we may write for any $j \geq j_0(h,C)$, any k,ℓ with $k+\ell \leq s_1 - 1$,

$$\left\|\Delta_j^h(hD)^\ell Z^k v_H(t,\cdot)\right\|_{L^\infty} \leq C\varepsilon 2^{-j+b}h^{-\delta'_k}.$$

This holds in particular for $k+\ell \leq s_0+\gamma+1$ as $s_0+\gamma+1 \leq s_1-1$ by (6.41). Since v_H is spectrally supported for $|h\xi| \geq ch^{-\beta}$, we conclude that

$$\left\|Z^k v_H(t,\cdot)\right\|_{C^{\rho-k}} \leq C\varepsilon h^{b\beta-\delta'_k} \leq C\varepsilon h^{2-\delta'_k} \tag{6.47}$$

with a constant C depending only on B_2, as we assumed in (4.3) that $b\beta > 2$. This largely implies estimate (6.45) for v_H, and so concludes the proof of (6.44).

We thus have obtained the first inequality (1.20). We are left with showing the second estimate. This follows from (6.21) that holds for $\ell \leq \frac{s}{2}+N_0$, so for $\ell \leq s_0+\gamma+1$. This concludes the proof of Theorem 1.6. $\square$

Final remark on the proof of Theorem 1.4: In Section 1.3, we did not justify the asymptotic expansion (1.12) of $u(t,x) = \frac{1}{\sqrt{t}}v\big(t,\frac{x}{t}\big)$. This follows from (6.23), since we have seen in the proof above that in the decomposition $v = v_L + w + v_H$, v_L and v_H are $O(\varepsilon t^{-\kappa})$ for some $\kappa > 0$ (see (6.46) and (6.47)).

A Appendix: Semi-classical pseudo-differential operators

We recall here some definitions and results concerning semi-classical pseudo-differential operators in one dimension. We refer to the books of Dimassi-Sjöstrand [33] Martinez [51] and Zworski [69].

Let h be a parameter in $]0,1]$. An order function m is a function $m\colon (x,\xi) \mapsto m(x,\xi)$ from $T^*\mathbb{R}$ (identified with $\mathbb{R}\times\mathbb{R}$) to $\mathbb{R}_+$, smooth, such that there are constants $N_0 \in \mathbb{N}$, $C_0 > 0$ with

$$m(x,\xi) \leq C_0(1+|x-y|+|\xi-\eta|)^{N_0} m(y,\eta)$$

for any (x,ξ), (y,η) in $T^*\mathbb{R}$.

Definition A.1. *Let m be an order function on $T^*\mathbb{R}$. One denotes by $S(m)$ the set of functions $a\colon T^*\mathbb{R}\times]0,1] \to \mathbb{C}$, $(x,\xi,h) \mapsto a(x,\xi,h)$ such that for any (α,β) in $\mathbb{N}\times\mathbb{N}$, there is $C_{\alpha\beta} > 0$, and for any $(x,\xi) \in T^*\mathbb{R}$, any h in $]0,1]$*

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi,h)\right| \leq C_{\alpha\beta} m(x,\xi).$$

If $(u_h)_h$ is a family indexed by $h \in]0,1]$ of elements of $\mathcal{S}'(\mathbb{R})$, and $a \in S(m)$, we define a family of elements of $\mathcal{S}'(\mathbb{R})$ by

$$\mathrm{Op}_h(a)u_h = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi} a(x,h\xi,h)\widehat{u_h}(\xi)\,d\xi. \tag{A.1}$$

If $m \equiv 1$, $\mathrm{Op}_h(a)$ is a bounded family indexed by $h \in]0,1]$ of bounded operators in $L^2(\mathbb{R})$. If moreover $\xi \mapsto a(x,\xi,h)$ is supported in a compact subset independent of (x,h), the kernel of $\mathrm{Op}_h(a)$, is

$$K_h(x,y) = \frac{1}{h}k_h\left(x,\frac{x-y}{h}\right)$$

where $k_h(x,z) = (\mathcal{F}_\xi^{-1}a)(x,z,h)$ is a smooth function satisfying estimates $\left|\partial_x^\alpha \partial_z^\beta k_h(x,z)\right| \leq C_{\alpha\beta N}(1+|z|)^{-N}$ for any α, β, N so that $\mathrm{Op}_h(a)$ is uniformly bounded on any L^p-space, $p \in [1, \infty]$.

Let us recall the main result of symbolic calculus (Theorem 7.9, Proposition 7.7, formulas (7.16) and (7.3) in [33]).

Theorem A.2. *Let m_1, m_2 be two order functions, a_j an element of $S(m_j)$, $j = 1, 2$. There is an element $a_1 \# a_2$ of $S(m_1 m_2)$ such that $\mathrm{Op}_h(a_1 \# a_2) = \mathrm{Op}_h(a_1)\,\mathrm{Op}_h(a_2)$. Moreover, one has the expansion*

$$a_1 \# a_2 - \sum_{j=0}^{N} \frac{1}{j!}\left(\frac{h}{i}\right)^j (\partial_\xi^j a_1)(\partial_x^j a_2) \in h^{N+1} S(m_1 m_2). \tag{A.2}$$

Let m be an order function, a an element of $S(m)$. There is b in $S(m)$ such that $\mathrm{Op}_h(a)^* = \mathrm{Op}_h(b)$. Moreover, $b = \bar{a} + h b_1$ with b_1 in $S(m)$.

Corollary A.3. *Let m be an order function such that m^{-1} is also an order function. Let a be in $S(1)$, e be in $S(m)$ and assume that $e \geq cm$ for some $c > 0$ on a neighborhood of the support of a. Then for any $N \in \mathbb{N}$, there are $q \in S(m^{-1})$, $r \in S(1)$ such that $a = e \# q + h^N r$ (resp. $a = q \# e + h^N r$). Moreover, we may write $q = q_0 + h q_1$ where q_0, q_1 are in $S(m^{-1})$ and $q_0 = \frac{a}{e}$.*

Proof. We define $q_0 = \frac{a}{e}$, which is an element of $S(m^{-1})$ by assumption. Then Theorem A.2 shows that $a - e \# q_0$ (resp. $a - q_0 \# e$) may be written $h a_1 + h^N r_0$ with a_1 in $S(1)$, $\mathrm{Supp}\, a_1 \subset \mathrm{Supp}\, a$. We iterate the construction to get the result. $\square$

References

[1] T. Alazard, N. Burq, and C. Zuily. On the Cauchy problem for gravity water waves. arXiv:1212.0626.

[2] T. Alazard, N. Burq, and C. Zuily. On the water-wave equations with surface tension. *Duke Math. J.*, 158(3):413–499, 2011.

[3] T. Alazard, N. Burq, and C. Zuily. Strichartz estimates for water waves. *Ann. Sci. Éc. Norm. Supér. (4)*, 44(5):855–903, 2011.

[4] T. Alazard, N. Burq, and C. Zuily. The water-wave equations: from Zakharov to Euler. In *Studies in Phase Space Analysis with Applications to PDEs*, pages 1–20. Springer, 2013.

[5] T. Alazard and J.-M. Delort. Sobolev estimates for two dimensional gravity water waves. Preprint, 2013.

[6] T. Alazard and G. Métivier. Paralinearization of the Dirichlet to Neumann operator, and regularity of three-dimensional water waves. *Comm. Partial Differential Equations*, 34(10-12):1632–1704, 2009.

[7] S. Alinhac. Paracomposition et opérateurs paradifférentiels. *Comm. Partial Differential Equations*, 11(1):87–121, 1986.

[8] B. Alvarez-Samaniego and D. Lannes. Large time existence for 3D water-waves and asymptotics. *Invent. Math.*, 171(3):485–541, 2008.

[9] J. T. Beale. Large-time regularity of viscous surface waves. *Arch. Rational Mech. Anal.*, 84(4):307–352, 1983/84.

[10] T. B. Benjamin and P. J. Olver. Hamiltonian structure, symmetries and conservation laws for water waves. *J. Fluid Mech.*, 125:137–185, 1982.

[11] K. Beyer and M. Günther. On the Cauchy problem for a capillary drop. I. Irrotational motion. *Math. Methods Appl. Sci.*, 21(12):1149–1183, 1998.

[12] G. Bourdaud. Realizations of homogeneous Sobolev spaces. *Complex Var. Elliptic Equ.*, 56(10-11):857–874, 2011.

[13] B. Buffoni, M. D. Groves, S.-M. Sun, and E. Wahlén. Existence and conditional energetic stability of three-dimensional fully localised solitary gravity-capillary water waves. *J. Differential Equations*, 254(3):1006–1096, 2013.

[14] A. Castro, D. Córdoba, C. Fefferman, F. Gancedo, and J. Gómez-Serrano. Finite time singularities for the free boundary incompressible Euler equations. arXiv:1112.2170.

[15] A. Castro, D. Córdoba, C. L. Fefferman, F. Gancedo, and J. Gómez-Serrano. Splash singularity for water waves. *Proc. Natl. Acad. Sci. USA*, 109(3):733–738, 2012.

[16] A. Castro, D. Córdoba, C. L. Fefferman, F. Gancedo, and M. López-Fernández. Turning waves and breakdown for incompressible flows. *Proc. Natl. Acad. Sci. USA*, 108(12):4754–4759, 2011.

[17] A. L. Cauchy. Théorie de la propagation des ondes à la surface d'un fluide pesant d'une profondeur indéfinie. p.5-318. Mémoires présentés par divers savants à l'Académie royale des sciences de l'Institut de France et imprimés par son ordre. Sciences mathématiques et physiques. Tome I, imprimé par autorisation du Roi à l'Imprimerie royale; 1827. Disponible sur le site http://mathdoc.emath.fr/.

[18] R. M. Chen, J. L. Marzuola, D. Spirn, and J. D. Wright. On the regularity of the flow map for the gravity-capillary equations. *J. Funct. Anal.*, 264(3):752–782, 2013.

[19] H. Christianson, V. M. Hur, and G. Staffilani. Strichartz estimates for the water-wave problem with surface tension. *Comm. Partial Differential Equations*, 35(12):2195–2252, 2010.

[20] A. Córdoba, D. Córdoba, and F. Gancedo. Interface evolution: water waves in 2-D. *Adv. Math.*, 223(1):120–173, 2010.

[21] D. Coutand and S. Shkoller. On the finite-time splash and splat singularities for the 3-d free-surface Euler equations. arXiv:1201.4919.

[22] D. Coutand and S. Shkoller. Well-posedness of the free-surface incompressible Euler equations with or without surface tension. *J. Amer. Math. Soc.*, 20(3):829–930 (electronic), 2007.

[23] W. Craig. An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits. *Comm. Partial Differential Equations*, 10(8):787–1003, 1985.

[24] W. Craig. Birkhoff normal forms for water waves. In *Mathematical problems in the theory of water waves (Luminy, 1995)*, volume 200 of *Contemp. Math.*, pages 57–74. Amer. Math. Soc., Providence, RI, 1996.

[25] W. Craig. Non-existence of solitary water waves in three dimensions. *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.*, 360(1799):2127–2135, 2002. Recent developments in the mathematical theory of water waves (Oberwolfach, 2001).

[26] W. Craig, U. Schanz, and C. Sulem. The modulational regime of three-dimensional water waves and the Davey-Stewartson system. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 14(5):615–667, 1997.

[27] W. Craig and C. Sulem. Numerical simulation of gravity waves. *J. Comput. Phys.*, 108(1):73–83, 1993.

[28] W. Craig, C. Sulem, and P.-L. Sulem. Nonlinear modulation of gravity waves: a rigorous approach. *Nonlinearity*, 5(2):497–522, 1992.

[29] W. Craig and C. E. Wayne. Mathematical aspects of surface waves on water. *Uspekhi Mat. Nauk*, 62(3(375)):95–116, 2007.

[30] A. D. D. Craik. *Wave interactions and fluid flows*. Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge University Press, Cambridge, 1988.

[31] O. Darrigol. *Worlds of flow*. Oxford University Press, New York, 2005. A history of hydrodynamics from the Bernoullis to Prandtl.

[32] J.-M. Delort. Existence globale et comportement asymptotique pour l'équation de Klein-Gordon quasi linéaire à données petites en dimension 1. *Ann. Sci. École Norm. Sup. (4)*, 34(1), 2001. Erratum : Ann. Sci. École Norm. Sup. (4) 39 (2006), no. 2, 335–345.

[33] M. Dimassi and J. Sjöstrand. *Spectral asymptotics in the semi-classical limit*, volume 268 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1999.

[34] P. Germain, N. Masmoudi, and J. Shatah. Global existence for capillary water waves. arXiv:1210.1601.

[35] P. Germain, N. Masmoudi, and J. Shatah. Global solutions for the gravity water waves equation in dimension 3. *Ann. of Math. (2)*, 175(2):691–754, 2012.

[36] Y. Guo and I. Tice. Decay of viscous surface waves without surface tension. arXiv:1011.5179.

[37] N. Hayashi and P. I. Naumkin. Asymptotics of small solutions to nonlinear Schrödinger equations with cubic nonlinearities. *Int. J. Pure Appl. Math.*, 3(3):255–273, 2002.

[38] L. Hörmander. *Lectures on nonlinear hyperbolic differential equations*, volume 26 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 1997.

[39] V. M. Hur. No solitary waves exist on 2d deep water. arXiv:1209.1926.

[40] A. Ionescu and F. Pusateri. Global solutions for the gravity water waves system in 2d. arXiv:1303.5357.

[41] A. Ionescu and F. Pusateri. Nonlinear fractional Schrödinger equations in one dimension. arXiv:1209.4943.

[42] G. Iooss and P. Plotnikov. Multimodal standing gravity waves: a completely resonant system. *J. Math. Fluid Mech.*, 7(suppl. 1):S110–S126, 2005.

[43] G. Iooss and P. I. Plotnikov. Small divisor problem in the theory of three-dimensional water gravity waves. *Mem. Amer. Math. Soc.*, 200(940):viii+128, 2009.

[44] S. Klainerman. Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions. *Comm. Pure Appl. Math.*, 38(5):631–641, 1985.

[45] S. Klainerman. Uniform decay estimates and the Lorentz invariance of the classical wave equation. *Comm. Pure Appl. Math.*, 38(3):321–332, 1985.

[46] D. Lannes. Space time resonances [after Germain, Masmoudi, Shatah]. Séminaire BOURBAKI 64ème année, 2011-2012, no 1053.

[47] D. Lannes. Water waves: mathematical analysis and asymptotics. To appear.

[48] D. Lannes. Well-posedness of the water-waves equations. *J. Amer. Math. Soc.*, 18(3):605–654 (electronic), 2005.

[49] D. Lannes. A Stability Criterion for Two-Fluid Interfaces and Applications. *Arch. Ration. Mech. Anal.*, 208(2):481–567, 2013.

[50] H. Lindblad. Well-posedness for the motion of an incompressible liquid with free surface boundary. *Ann. of Math. (2)*, 162(1):109–194, 2005.

[51] A. Martinez. *An introduction to semiclassical and microlocal analysis.* Universitext. Springer-Verlag, New York, 2002.

[52] N. Masmoudi and F. Rousset. Uniform regularity and vanishing viscosity limit for the free surface Navier-Stokes equations. arXiv:1202.0657.

[53] M. Ming, F. Rousset, and N. Tzvetkov. Multi-solitons and related solutions for the water-waves system. arXiv:1304.5263.

[54] V. I. Nalimov. The Cauchy-Poisson problem. *Dinamika Splošn. Sredy*, (Vyp. 18 Dinamika Zidkost. so Svobod. Granicami):104–210, 254, 1974.

[55] G. Schneider and C. E. Wayne. The rigorous approximation of long-wavelength capillary-gravity waves. *Arch. Ration. Mech. Anal.*, 162(3):247–285, 2002.

[56] J. Shatah. Normal forms and quadratic nonlinear Klein-Gordon equations. *Comm. Pure Appl. Math.*, 38(5):685–696, 1985.

[57] J. Shatah and C. Zeng. Geometry and a priori estimates for free boundary problems of the Euler equation. *Comm. Pure Appl. Math.*, 61(5):698–744, 2008.

[58] J. Shatah and C. Zeng. A priori estimates for fluid interface problems. *Comm. Pure Appl. Math.*, 61(6):848–876, 2008.

[59] M. Shinbrot. The initial value problem for surface waves under gravity. I. The simplest case. *Indiana Univ. Math. J.*, 25(3):281–300, 1976.

[60] C. Sulem and P.-L. Sulem. *The nonlinear Schrödinger equation*, volume 139 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1999. Self-focusing and wave collapse.

[61] S.-M. Sun. Some analytical properties of capillary-gravity waves in two-fluid flows of infinite depth. *Proc. Roy. Soc. London Ser. A*, 453(1961):1153–1175, 1997.

[62] N. Totz and S. Wu. A rigorous justification of the modulation approximation to the 2D full water wave problem. *Comm. Math. Phys.*, 310(3):817–883, 2012.

[63] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. Math.*, 130(1):39–72, 1997.

[64] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *J. Amer. Math. Soc.*, 12(2):445–495, 1999.

[65] S. Wu. Almost global wellposedness of the 2-D full water wave problem. *Invent. Math.*, 177(1):45–135, 2009.

[66] S. Wu. Global wellposedness of the 3-D full water wave problem. *Invent. Math.*, 184(1):125–220, 2011.

[67] H. Yosihara. Gravity waves on the free surface of an incompressible perfect fluid of finite depth. *Publ. Res. Inst. Math. Sci.*, 18(1):49–96, 1982.

[68] V. E. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Journal of Applied Mechanics and Technical Physics*, 9(2):190–194, 1968.

[69] M. Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.