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A Feedback Control in Max-Algebra

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Abstract

For timed event graphs, linear models were obtained using Max-Algebra. This paper presents a method to control such systems. After describing the optimal solution of a model tracking problem, we propose a feedback control structure in order to take into account a possible modeling error. We present its construction, its main properties and an algorithm for its practical implementation. An illustrative example is provided.

1 Introduction

This paper deals with the control of Discrete Event Systems (DES) which can be modeled by Timed Event Graphs (TEG). A theory using a new Algebra often called Max-Algebra has been developed for this particular class of Petri nets, and offers strong analogies with the conventional linear system theory. In particular, concepts of state representation, transfer relation, correlation and autocorrelation have been introduced ([1], [5], [7]).

An optimal solution has been defined in order to control the system when its model is exact, more precisely the system input is fired at the latest date such that the firing dates of the system (model) output occur at the latest dates before the desired ones. To keep good accuracy in spite of a difference often present in practice between the system and its model, we propose to improve the previous control solution by using the concept of modified control. By observation of the system output behavior, we modify the optimal model control in order to be as close as possible to the optimal system control. In theory we need to have all the system outputs behavior to compute this control law. In practice we will estimate the future (unavailable) system outputs behavior.

The paper is organized as follows: firstly, we briefly present in sections 2, 3 and 4 some elements of Max-Algebra, Residuation theory, state representation and transfer relation respectively. We recall optimal model control in section 5,

and we present the modified control method and its practical application in sections 6 and 7 respectively. We conclude by a short example in section 8.

2 Max-Algebra

Definition 1: A dioid is a set endowed with two internal operations denoted \oplus (addition) and \otimes (multiplication), both associative and both having neutral elements denoted ε and e respectively, such that: \oplus is commutative and idempotent ($a \oplus a = a$), \otimes is distributive with respect to \oplus , and ε is absorbing for \otimes ($a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$).

If \otimes is commutative, the dioid is said to be commutative.

Definition 2: A dioid is an ordered set with the natural order relation:

$$a \succeq b \Leftrightarrow a = a \oplus b$$

($a \oplus b$ is the least upper bound of a and b).

Definition 3: A dioid is complete if sums of infinite numbers of terms are always defined, and if multiplication distributes over infinite sums too. The sum of all elements of a dioid is denoted T (for $\square \text{top} \square$).

In a complete dioid, the greatest lower bound always exists for an arbitrary (possible infinite) set S . For example if

$$S = \{a, b\} \text{ this bound denoted } a \wedge b \text{ is equal to } \bigoplus_{\substack{x \preceq a \\ x \preceq b}} x.$$

Example: The set $\mathbb{Z} \cup \{-\infty, +\infty\}$ with \max as \oplus and usual addition as \otimes is a complete commutative dioid denoted $\bar{\mathbb{Z}}_{\max}$ with $\varepsilon = -\infty$, $T = +\infty$ and $e = 0$.

3 Residuation theory

Definition 4: A mapping $f : D \rightarrow C$, where D and C are ordered sets, is residuated if for all $y \in C$, the set $\{x \in D \mid f(x) \preceq y\}$ admits a least upper bound. Let us denote this (unique) greatest $\square \text{subsolution} \square f^\#(y)$. By definition we

have $f^\# [f(y)] \preceq y$ (see [1, chapter 4], [2], [7] for a detailed presentation of this theory).

Example: The mapping $f: D \xrightarrow{x} C \otimes x$, where D and C are complete dioids, is residuated and the residuated mapping $f^\#: C \rightarrow D$ will be denoted $y \rightarrow a \setminus y$. Let us present some properties of this mapping used in the following (see [1, chapter 4] for proofs):

- $y \rightarrow a \setminus y$ is isotone (1)
 - (h is isotone if $\forall a, b \quad a \preceq b \Rightarrow h(a) \preceq h(b)$)
 - $(ab) \setminus c = b \setminus (a \setminus c)$ (2)
 - If $C = D$ then $\forall a \in D, a \otimes (a \setminus a) = a$ (3)
 - If D is a complete dioid, then $D^m, m \in \mathbb{N}$, is a complete dioid too, and $f: D \xrightarrow{x} D^m \otimes x$,
- $a = (a_1 \dots a_m)^t \in D^m$ is residuated. In particular,
- $$\forall y \in D^m, a \setminus y = \bigwedge_{i=1}^m (a_i \setminus y_i) \text{ is a } \square \text{ scalar } (\in D). \quad (4)$$

4 State representation and transfer relation

A TEG is a class of Petri nets in which each place has exactly one upstream and one downstream transition. Let us consider the dynamic behavior of a TEG: in the \square dater \square representation, this one is described by linear equations under the standard form:

$$\begin{cases} \underline{x}(k+1) = A \otimes \underline{x}(k) \oplus B \otimes \underline{u}(k) \\ \underline{y}(k) = C \otimes \underline{x}(k) \end{cases}$$

where $\underline{u}, \underline{y}$ and \underline{x} are the dater vectors of source, sink and internal transitions respectively. $x_i(k)$ denotes the date when the transition x_i incurs its firing numbered k (all calculations are to be understood in the dioid \bar{Z}_{\max}).

In the next we only consider S.I.S.O. case, nevertheless all the presented results can easily be extended to M.I.M.O. case.

Using the analogue of the z-transform for daters, based on the 'backward shift' operator γ in the event domain (formally $\gamma u(k) = u(k-1)$), an input-output representation is obtained in the dioid $\bar{Z}_{\max}[[\gamma]]$ of formal power series in γ with positive and negative exponents and coefficients in \bar{Z}_{\max} ([1, chapter 5], [8]).

Let us denote: $Y = \bigoplus_{k \in \mathbb{Z}} y(k) \gamma^k, \underline{X} = \bigoplus_{k \in \mathbb{Z}} \underline{x}(k) \gamma^k$ and

$U = \bigoplus_{k \in \mathbb{Z}} u(k) \gamma^k$ the power series associated to the

daters $\{y(k)\}_{k \in \mathbb{Z}}, \{\underline{x}(k)\}_{k \in \mathbb{Z}}$, and $\{u(k)\}_{k \in \mathbb{Z}}$

respectively, the previous linear equations can be written in this dioid by:

$$\begin{cases} \underline{X} = A \otimes \underline{X} \oplus B \otimes U \\ Y = C \otimes \underline{X} \end{cases} \quad (5)$$

which also can be expressed as the following input-output relation $Y = M \otimes U$, where $M = CA^*B$ is the model

$$(A^* = \bigoplus_{n \in \mathbb{N}} A^n).$$

5 Optimal model control

Given a sequence $\{z(k)\}_{k \in \mathbb{Z}}$ of desired outputs (also called reference input) in \bar{Z}_{\max} , the optimal control of the previous model M is the *latest* controls sequence $\{u(k)\}_{k \in \mathbb{Z}}$ such that the model outputs sequence $\{y(k)\}_{k \in \mathbb{Z}}$ is matched as closely as possible before the desired outputs sequence. This can be formally written in $\bar{Z}_{\max}[[\gamma]]$ by:

$$\begin{aligned} &\text{given } Z, \text{ find the maximum } U \text{ such that} \\ &Y = M \otimes U \preceq Z. \end{aligned}$$

The solution to solve this optimal tracking problem exists and is given by an open loop control structure. It is a direct application of Residuation theory and noted

$$\tilde{U} = M \setminus Z = \bigoplus_{\{U \mid M \otimes U \preceq Z\}} U$$

(consider the transfer mapping $X \rightarrow M \otimes X$).

Remark: the sequence $\{z(k)\}_{k \in \mathbb{Z}}$ is such that $z(k) = +\infty, \forall k \geq k_f \in \mathbb{N}$. This condition means that the desired outputs sequence $\{z(k)\}_{k \in \mathbb{Z}}$ is given up to the finite event number k_f .

It can be proved (see [1, chapter 5 §6] and [6]) that the optimal model control \tilde{U} is the greatest solution of the following system (remindful of the co-system in conventional optimal control theory):

$$\begin{cases} \underline{E} = (A \setminus \underline{E}) \wedge (C \setminus Z) \\ U = B \setminus \underline{E} \end{cases} \quad (6)$$

This is equivalent to say that the vector \underline{E} (called co-state) must be selected as the greatest solution of (6). This solution is $\underline{E} = (CA^*) \setminus Z$ (see [1, Lemma 5.33] for proof). $e_i(k)$ indicates the latest date when the internal transition x_i should do its firing numbered k in order not to delay future outputs beyond the deadline provided by Z .

6 Modified control

Let us note G the transfer relation of the system, and apply the previous open loop control $\tilde{U} = M \setminus Z$ to the system G (see fig. 1).

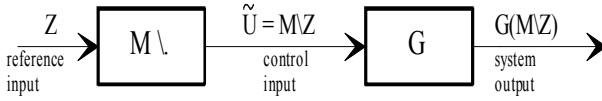


fig 1: Optimal model control applied to the system.

Such a control cannot take into account mismatch between the system and its model. A closed loop control structure inspired by the Internal Model Control usually used in the conventional control theory is proposed in [3, 4]. In the continuity of this approach, we propose another closed loop control structure consisting in applying an open loop control which is modified by using system output behavior. Indeed, the model is initially supposed exact ($G = M$) hence we apply the optimal control $\tilde{U} = M \setminus Z$. Then we modify this control by using system outputs in order to be as close as possible to the optimal system control.

6.1. Construction

Let us note $Y_G = G \otimes \tilde{U}$, then the greatest modification $\Delta\tilde{U}$ such that $Y_G \otimes \Delta\tilde{U} \preceq Z$ is given by Residuation theory and is equal to $\Delta\tilde{U} = Y_G \setminus Z$ (consider the transfer mapping: $X \rightarrow Y_G \otimes X$).

Hence we have:

$$\begin{aligned} Y_G \otimes \Delta\tilde{U} &= (G \otimes \tilde{U}) \otimes \Delta\tilde{U} \\ &= G \otimes (\tilde{U} \otimes \Delta\tilde{U}) \preceq Z \quad \text{by associativity of } \otimes \end{aligned}$$

which yields the modified control:

$$U_m = \tilde{U} \otimes \Delta\tilde{U} = (M \setminus Z) \otimes (Y_G \setminus Z)$$

Remark: $Y_G \setminus Z (= \Delta\tilde{U})$ can be interpreted as a correlation measure of Y_G and Z .

More precisely, in the S.I.S.O. case, Y_G and Z are scalars ($\in \bar{Z}_{\max}[[\gamma]]$) and $Y_G \setminus Z$ is called the correlation of Y_G with Z (see [1, chapter 6 §6] and [9] for a detailed presentation).

In the M.I.M.O. case, Y_G and Z are vectors ($\in \bar{Z}_{\max}^n[[\gamma]]$)

and by (4), we have $Y_G \setminus Z = \bigwedge_{i=1}^n (y_{G_i} \setminus z_i)$.

6.2. Properties

Property 1: $\Delta\tilde{U} = \tilde{U} \setminus U_{opt}$ where $U_{opt} = G \setminus Z$ is the optimal system control.

Proof: By this statement we mean that $\Delta\tilde{U}$ is equal to the correlation measure between the optimal system control ($U_{opt} = G \setminus Z$) and the optimal model control ($\tilde{U} = M \setminus Z$).

$$\begin{aligned} \Delta\tilde{U} &= Y_G \setminus Z = (G \otimes \tilde{U}) \setminus Z \\ &= \tilde{U} \setminus (G \setminus Z) \quad \text{by (2)} \\ &= \tilde{U} \setminus U_{opt} \end{aligned}$$

In the next property we state that in case of perfect modeling, the modified control U_m is equal to the optimal system (model) control.

Property 2: If $G = M$ then $U_m = U_{opt}$.

Proof: Since $G = M$, $\tilde{U} = U_{opt}$.

$$\begin{aligned} U_m &= \tilde{U} \otimes \Delta\tilde{U} \\ &= \tilde{U} \otimes (\tilde{U} \setminus U_{opt}) \quad \text{by property 1} \\ &= U_{opt} \otimes (U_{opt} \setminus U_{opt}) \\ &= U_{opt} \quad \text{by (3)} \end{aligned}$$

Property 3: U_m is the (unique) greatest control of the form $\tilde{U} \otimes \alpha$ less than or equal to the optimal system control $U_{opt} = G \setminus Z$.

Proof: The greatest solution of the inequality $\tilde{U} \otimes \alpha \preceq U_{opt}$ is $\alpha = \tilde{U} \setminus U_{opt}$

$$= \Delta\tilde{U} \quad \text{by property 1}$$

The uniqueness is straightforward as the greatest solution of an inequation.

Property 4: control U_m is the solution of the following recurrent equation (due to the feedback control structure used):

$$\begin{cases} U_1 = \tilde{U} \\ U_{n+1} = U_n \otimes [(G \otimes U_n) \setminus Z] \quad n \geq 1 \end{cases}$$

and this sequence converges on U_m for $n = 2$.

Proof:

$$\begin{aligned}
U_2 &= \tilde{U} \otimes (Y_G \setminus Z) = \tilde{U} \otimes (\tilde{U} \setminus U_{opt}) = U_m \\
U_3 &= U_2 \otimes [(G \otimes U_2) \setminus Z] \\
&= U_2 \otimes (U_2 \setminus U_{opt}) \quad \text{by property 1} \\
&= [\tilde{U} \otimes (\tilde{U} \setminus U_{opt})] \otimes [(\tilde{U} \otimes (\tilde{U} \setminus U_{opt})) \setminus U_{opt}] \\
&= [\tilde{U} \otimes (\tilde{U} \setminus U_{opt})] \otimes [(\tilde{U} \setminus U_{opt}) \setminus (\tilde{U} \setminus U_{opt})] \quad \text{by (2)} \\
&= \tilde{U} \otimes \left\{ (\tilde{U} \setminus U_{opt}) \otimes [(\tilde{U} \setminus U_{opt}) \setminus (\tilde{U} \setminus U_{opt})] \right\} \\
&\quad \text{by associativity of } \otimes \\
&= \tilde{U} \otimes (\tilde{U} \setminus U_{opt}) \quad \text{by (3)} \\
&= U_m
\end{aligned}$$

Let us note that if U_m was applied to the system input (instead of $\tilde{U} = M \setminus Z$) then the corresponding modified control would be again U_m .

Property 5: $\Delta\tilde{U}$ is the greatest solution of the inequation $\underline{X}_G \otimes \alpha \preceq \underline{E}_G$, where \underline{X}_G and \underline{E}_G are the state vector and the co-state vector of system G respectively.

Hence considering the system outputs or its internal states is equivalent to search the optimal correction.

Proof:

Let us note A_G, B_G and C_G the matrices associated to the system state representation, we have $G = C_G A_G^* B_G$, $\underline{X}_G = A_G^* B_G U$ and $\underline{E}_G = (C_G A_G^*) \setminus Z$.

The greatest solution of $\underline{X}_G \otimes \alpha \preceq \underline{E}_G$ is equal to

$$\begin{aligned}
\underline{X}_G \setminus \underline{E}_G &= (A_G^* B_G \tilde{U}) \setminus [(C_G A_G^*) \setminus Z] \\
&= (C_G A_G^* A_G^* B_G \tilde{U}) \setminus Z \quad \text{by (2)} \\
&= (C_G A_G^* B_G \tilde{U}) \setminus Z \quad \text{since } (A_G^* \otimes A_G^* = A_G^*) \\
&= (G \otimes \tilde{U}) \setminus Z \\
&= \Delta\tilde{U}
\end{aligned}$$

7 Application

The computation of U_m implies to have $G \otimes \tilde{U}$, i.e., all the system outputs sequence Y_G , which is not realistic. In practice, we compute control U_c (an estimation of U_m) at the instant t by using the observed system output sequence

$\{y_G(k) \mid y_G(k) \leq t; k \in \mathbb{Z}\}$ completed by an estimation of the future system outputs (see [10]). Before presenting the computing algorithm of this control U_c , let us introduce the following notations:

Let $\{v(k)\}_{k \in \mathbb{Z}}$ be a dater, $V = \bigoplus_{k \in \mathbb{Z}} v(k) \gamma^k$ its associated

formal series and $K \in \mathbb{Z}$, we define: $V_{(\rightarrow K)} = \bigoplus_{k \leq K} v(k) \gamma^k$

and $V_{(K \rightarrow)} = \bigoplus_{k > K} v(k) \gamma^k$

Algorithm:

step 1:

$$U_1 = M \setminus Z = \tilde{U}$$

(the optimal model control \tilde{U} is initially applied: the model is supposed to be exact, i.e., $G = M$).

step $n+1$: ($n \geq 1$)

1. Let K_n an event sample and t_n the date of the firing numbered K_n of $G \otimes U_n$.

We compute:

$$\hat{G}U_n = (GU_n)_{(\rightarrow K_n)} \oplus (\tilde{G}U_n)_{(K_n \rightarrow)}$$

where $(GU_n)_{(\rightarrow K_n)}$ is the observed system output until t_n

and $(\tilde{G}U_n)_{(K_n \rightarrow)}$ is an estimation of $(GU_n)_{(K_n \rightarrow)}$.

2. Then the modified control is:

$$\hat{U}_{n+1} = U_n \otimes (\hat{G}U_n \setminus Z)$$

3. Let $K'_n = \text{Sup} \{k \in \mathbb{Z} \mid u_n(k) \leq t_n\}$.

As the set of controls $\{u_n(k), k \leq K'_n\}$ has been applied to the system input until instant t_n , this leads to consider the control:

$$U_{n+1} = (U_n)_{(\rightarrow K'_n)} \oplus (\hat{U}_{n+1})_{(K'_n \rightarrow)}$$

This control will be applied to the system input until t_{n+1} which is the time of the next control computation (U_{n+2}). In other words, we have:

$$U_{c(\rightarrow K'_{n+1})} = U_{n+1(\rightarrow K'_{n+1})}.$$

8 Example

We consider the manufacturing system described by TEG of fig. 2 modeling a machine producing 4 parts by 4 time units. The corresponding model is described by TEG of fig. 3. Let us note that the system and its model are different except for the production rate.

The reference input is defined by :

$$\begin{aligned}
 z(k) &= \varepsilon \text{ (} = -\infty \text{)} & \text{for } k < 0; \\
 z(k) &= z(k-1) + 1 & \text{for } k = 1, 2, 3, \quad \text{with } z(0) = 0; \\
 z(k) &= z(k-1) + 2 & \text{for } k = 4, \dots, 8; \\
 z(k) &= z(k-1) + 1 & \text{for } k = 9, 10; \\
 z(k) &= T \text{ (} = +\infty \text{)} & \text{for } k > 10 \\
 & & \text{(end of the desired production).}
 \end{aligned}$$

Figure 4 represents the trajectories of the modified control U_m , the optimal system control U_{opt} and control U_c computed by the previous algorithm. Figure 5 represents the corresponding outputs trajectories.

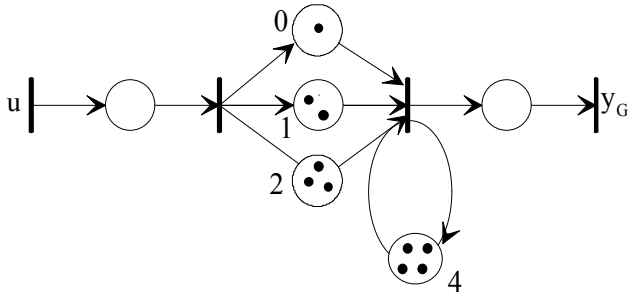


fig. 2. TEG of system G

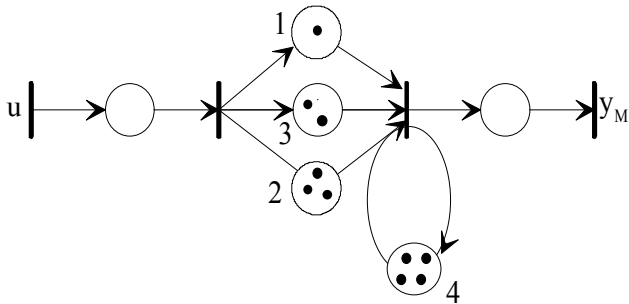


fig. 3. TEG of model M

In this particular example U_m is equal to U_{opt} (see fig. 4). Initially (step 1 of the algorithm), the optimal model control is applied until instant $t_1 = 1$, the date of the firing numbered K_1 of $G \otimes U_1$. $K_1 = 3$ (see fig. 5). Then, at each observed output (i.e., $K_{n+1} = K_n + 1$) a new control trajectory U_n is computed. We see that beyond time t_1 , the trajectories associated to the computed control U_c and the modified control U_m are identical.

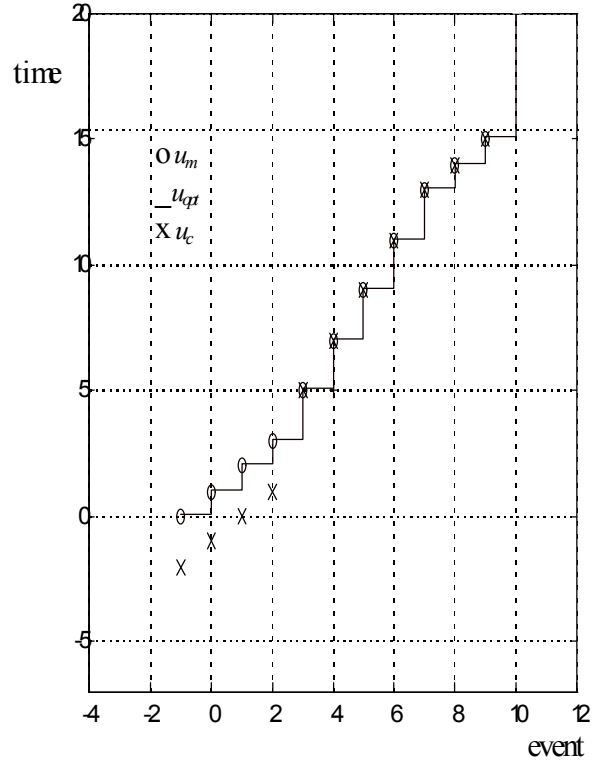


fig. 4. Controls behavior

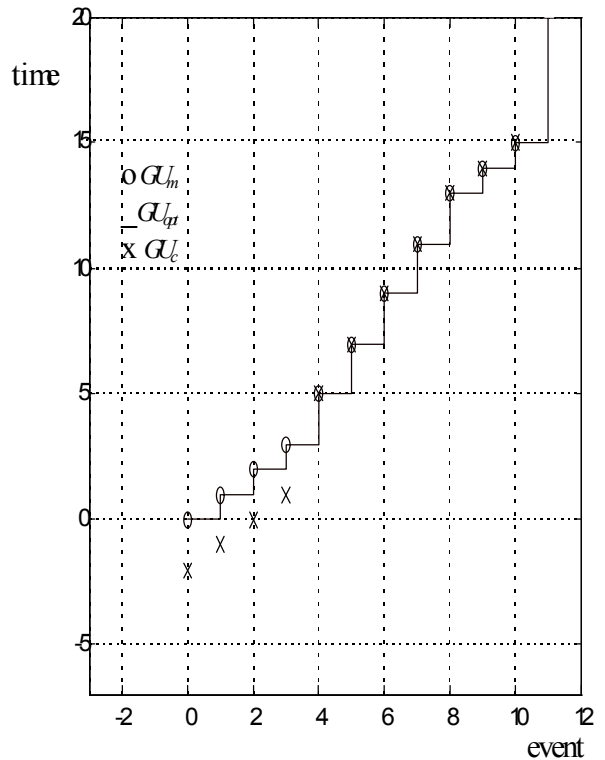


fig. 5. Outputs behavior

Conclusion

The proposed feedback control is applied to DES modeled in Max-Algebra.

This control allows taking into account mismatch between the system and its model. Its principle is to modify the optimal model control to be as close as possible to the optimal system control (see property 3). Without modeling error this control remains equal to the optimal model control. These optimisation problems where the \square -inversion \square notion is underlying, essentially use the Residuation theory.

The use of the system output to compute the control law necessarily raises estimation problem of system outputs behavior which is not addressed in this paper.

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