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A Feedback Control in Max-Algebra

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Abstract

For timed event graphs, linear models were obtained using Max-Algebra. This paper presents a method to control such systems. After describing the optimal solution of a model tracking problem, we propose a feedback control structure in order to take into account a possible modeling error. We present its construction, its main properties and an algorithm for its practical implementation. An illustrative example is provided.

1 Introduction

This paper deals with the control of Discrete Event Systems (DES) which can be modeled by Timed Event Graphs (TEG). A theory using a new Algebra often called Max-Algebra has been developed for this particular class of Petri nets, and offers strong analogies with the conventional linear system theory. In particular, concepts of state representation, transfer relation, correlation and autocorrelation have been introduced ([1], [5], [7]).

An optimal solution has been defined in order to control the system when its model is exact, more precisely the system input is fired at the latest date such that the firing dates of the system (model) output occur at the latest dates before the desired ones. To keep good accuracy in spite of a difference often present in practice between the system and its model, we propose to improve the previous control solution by using the concept of modified control. By observation of the system output behavior, we modify the optimal model control in order to be as close as possible to the optimal system control. In theory we need to have all the system outputs behavior to compute this control law. In practice we will estimate the future (unavailable) system outputs behavior.

The paper is organized as follows: firstly, we briefly present in sections 2, 3 and 4 some elements of Max-Algebra, Residuation theory, state representation and transfer relation respectively. We recall optimal model control in section 5, and we present the modified control method and its practical application in sections 6 and 7 respectively. We conclude by a short example in section 8.

2 Max-Algebra

Definition 1: A dioid is a set endowed with two internal operations denoted \( \oplus \) (addition) and \( \otimes \) (multiplication), both associative and both having neutral elements denoted \( \varepsilon \) and \( e \) respectively, such that: \( \oplus \) is commutative and idempotent (\( a \oplus a = a \)), \( \otimes \) is distributive with respect to \( \oplus \), and \( \varepsilon \) is absorbing for \( \otimes \) (\( a \otimes \varepsilon = \varepsilon = \varepsilon \otimes a = \varepsilon \)).

If \( \otimes \) is commutative, the dioid is said to be commutative.

Definition 2: A dioid is an ordered set with the natural order relation:
\( a \succeq b \iff a = a \oplus b \) (\( a \oplus b \) is the least upper bound of \( a \) and \( b \)).

Definition 3: A dioid is complete if sums of infinite numbers of terms are always defined, and if multiplication distributes over infinite sums too. The sum of all elements of a dioid is denoted \( T \) (for \( \text{top} \)).

In a complete dioid, the greatest lower bound always exists for an arbitrary (possible infinite) set \( S \). For example if \( S = \{a, b\} \) this bound denoted \( a \land b \) is equal to \( \begin{bmatrix} x & a \\ x & b \end{bmatrix} \).

Example: The set \( Z \cup \{-\infty, +\infty\} \) with \( \text{max} \) as \( \oplus \) and usual addition as \( \otimes \) is a complete commutative dioid denoted \( \mathbb{Z}_{\text{max}} \) with \( \varepsilon = -\infty \), \( T = +\infty \) and \( e = 0 \).

3 Residuation theory

Definition 4: A mapping \( f : D \to C \) , where \( D \) and \( C \) are ordered sets, is residuated if for all \( y \in C \), the set \( \{ x \in D \mid f(x) \leq y \} \) admits a least upper bound. Let us denote this (unique) greatest \( \square \) subsolution \( f^\#(y) \). By definition we
have \( f^* \left[ f(y) \right] \leq y \) (see [1, chapter 4], [2], [7] for a detailed presentation of this theory).

**Example:** The mapping \( f : D \rightarrow C \), where \( D \) and \( C \) are complete dioids, is residuated and the residuated mapping \( f^* : C \rightarrow D \) will be denoted \( y \rightarrow a \land y \). Let us present some properties of this mapping used in the following (see [1, chapter 4] for proofs):

- \( y \rightarrow a \land y \) is isotone
  
  \[ (h \text{ is isotone if } \forall a, b \quad a \leq b \implies h(a) \leq h(b)) \]  

- \( (ab) \land c = b \land (a \land c) \)
  
  \[ \text{(1)} \]

- If \( C = D \) then \( \forall a \in D, a \land (a \lor a) = a \)
  
  \[ \text{(2)} \]

- If \( D \) is a complete dioid, then \( D^m, m \in \mathbb{N} \), is a complete dioid too, and \( f : D \rightarrow D^m \), \( a = (a_1, \ldots, a_m) \) \( \in D^m \) is residuated. In particular,
  
  \[ \forall y \in D^m, a \land y = \bigwedge_{i=1}^{m} (a_i \land y_i) \]  

  is a \( \bigwedge \) scalar \((\in D)\).
  
  \[ \text{(3)} \]

4 State representation and transfer relation

A TEG is a class of Petri nets in which each place has exactly one upstream and one downstream transition. Let us consider the dynamic behavior of a TEG: in the \( \bigwedge \) dater \( \bigwedge \) representation, this one is described by linear equations under the standard form:

\[
\begin{align*}
\dot{x}(k+1) &= A \otimes x(k) \oplus B \otimes u(k) \\
y(k) &= C \otimes x(k)
\end{align*}
\]

where \( u, y \) and \( x \) are the dater vectors of source, sink and internal transitions respectively. \( x_i(k) \) denotes the date when the transition \( x_i \) incurs its firing numbered \( k \) (all calculations are to be understood in the dioid \( \mathbb{Z}_{\text{max}} \)).

In the next we only consider S.I.S.O. case, nevertheless all the presented results can easily be extended to M.I.M.O. case.

Using the analogue of the \( z \)-transform for daters, based on the 'backward shift' operator \( \gamma \) in the event domain (formally \( \gamma u(k) = u(k-1) \)), an input-output representation is obtained in the dioid \( \mathbb{Z}_{\text{max}}[[y]] \) of formal power series in \( y \) with positive and negative exponents and coefficients in \( \mathbb{Z}_{\text{max}} \) ([1, chapter 5], [8]).

Let us denote: \( Y = \bigoplus_{k \in \mathbb{Z}} y(k)\gamma^k \), \( X = \bigoplus_{k \in \mathbb{Z}} x(k)\gamma^k \) and \( U = \bigoplus_{k \in \mathbb{Z}} u(k)\gamma^k \) the power series associated to the daters \( \{ y(k) \}_{k \in \mathbb{Z}}, \{ z(k) \}_{k \in \mathbb{Z}} \), and \( \{ u(k) \}_{k \in \mathbb{Z}} \) respectively, the previous linear equations can be written in this dioid by:

\[
\begin{align*}
X &= A \otimes X \oplus B \otimes U \\
Y &= C \otimes X
\end{align*}
\]

which also can be expressed as the following input-output relation \( Y = M \otimes U \), where \( M = CA^*B \) is the model \( \left( A^* = \bigoplus_{a \in \mathbb{N}} A^a \right) \).

5 Optimal model control

Given a sequence \( \{ z(k) \}_{k \in \mathbb{Z}} \) of desired outputs (also called reference input) in \( \mathbb{Z}_{\text{max}} \), the optimal control of the previous model \( M \) is the latest controls sequence \( \{ u(k) \}_{k \in \mathbb{Z}} \) such that the model outputs sequence \( \{ y(k) \}_{k \in \mathbb{Z}} \) is matched as closely as possible before the desired outputs sequence. This can be formally written in \( \mathbb{Z}_{\text{max}}[[Y]] \) by:

\[
given Z, \text{find the maximum } U \text{ such that } Y = M \otimes U \leq Z.
\]

The solution to solve this optimal tracking problem exists and is given by an open loop control structure. It is a direct application of Residuation theory and noted

\[
\begin{align*}
\hat{U} &= M \setminus Z = \bigoplus_{U \setminus M \otimes U \leq Z} U \\
&\quad \text{(consider the transfer mapping } X \rightarrow M \otimes X \text{).}
\end{align*}
\]

**Remark:** the sequence \( \{ z(k) \}_{k \in \mathbb{Z}} \) is such that \( z(k) = +\infty, \forall k \geq k_f, \in \mathbb{N} \). This condition means that the desired outputs sequence \( \{ z(k) \}_{k \in \mathbb{Z}} \) is given up to the finite event number \( k_f \).

It can be proved (see [1, chapter 5 §6] and [6]) that the optimal model control \( \hat{U} \) is the greatest solution of the following system (remindful of the co-system in conventional optimal control theory):

\[
\begin{align*}
\hat{E} &= (A \setminus E) \land (C \setminus Z) \\
\hat{U} &= B \setminus \hat{E}
\end{align*}
\]

(6)
This is equivalent to say that the vector $E$ (called co-state) must be selected as the greatest solution of (6). This solution is $E = (CA^*) \setminus Z$ (see [1, Lemma 5.33] for proof). $e_j(k)$ indicates the latest date when the internal transition $x_j$ should do its firing numbered $k$ in order not to delay future outputs beyond the deadline provided by $Z$.

6 Modified control

Let us note $G$ the transfer relation of the system, and apply the previous open loop control $\tilde{U} = M \setminus Z$ to the system $G$ (see fig. 1).

Such a control cannot take into account mismatch between the system and its model. A closed loop control structure inspired by the Internal Model Control usually used in the conventional control theory is proposed in [3, 4]. In the continuity of this approach, we propose another closed loop control structure consisting in applying an open loop control which is modified by using system output behavior. Indeed, the model is initially supposed exact ($G = M$) hence we apply the optimal control $\tilde{U} = M \setminus Z$. Then we modify this control by using system outputs in order to be as close as possible to the optimal control $U_{\text{opt}}$.

6.1. Construction

Let us note $Y_G = G \otimes \tilde{U}$, then the greatest modification $\Delta \tilde{U}$ such that $Y_G \otimes \Delta \tilde{U} \leq Z$ is given by Residuation theory and is equal to $\Delta \tilde{U} = Y_G \setminus Z$ (consider the transfer mapping: $X \rightarrow Y_G \otimes X$).

Hence we have:

$$Y_G \otimes \Delta \tilde{U} = (G \otimes \tilde{U}) \otimes \Delta \tilde{U} = G \otimes (\tilde{U} \otimes \Delta \tilde{U}) \leq Z$$

by associativity of $\otimes$ which yields the modified control:

$$U_m = \tilde{U} \otimes \Delta \tilde{U} = (M \setminus Z) \otimes (Y_G \setminus Z)$$

Remark: $Y_G \setminus Z$ is $\Delta \tilde{U}$ can be interpreted as a correlation measure of $Y_G$ and $Z$.

More precisely, in the S.I.S.O. case, $Y_G$ and $Z$ are scalars ($\in Z_{\text{max}}(Y)$) and $Y_G \setminus Z$ is called the correlation of $Y_G$ with $Z$ (see [1, chapter 6 §6] and [9] for a detailed presentation).

In the M.I.M.O. case, $Y_G$ and $Z$ are vectors ($\in Z^2_{\text{max}}(Y)$) and by (4), we have $Y_G \setminus Z = \bigwedge_{i=1}^{n} \langle y_{G_i} \setminus z_i \rangle$.

6.2. Properties

Property 1: $\Delta \tilde{U} = \tilde{U} \setminus U_{\text{opt}}$ where $U_{\text{opt}} = G \setminus Z$ is the optimal system control.

Proof: By this statement we mean that $\Delta \tilde{U}$ is equal to the correlation measure between the optimal system control $(U_{\text{opt}} = G \setminus Z)$ and the optimal model control $(\tilde{U} = M \setminus Z)$.

$$\Delta \tilde{U} = Y_G \setminus Z = (G \otimes \tilde{U}) \setminus (G \otimes Z) = \tilde{U} \setminus U_{\text{opt}}$$

In the next property we state that in case of perfect modeling, the modified control $U_m$ is equal to the optimal system (model) control.

Property 2: If $G = M$ then $U_m = U_{\text{opt}}$.

Proof: Since $G = M$, $\tilde{U} = U_{\text{opt}}$.

$$U_m = \tilde{U} \otimes \Delta \tilde{U} = \tilde{U} \otimes (U_{\text{opt}} \setminus U_{\text{opt}}) = U_{\text{opt}}$$

by property 1

Property 3: $U_m$ is the (unique) greatest control of the form $\tilde{U} \otimes \alpha$ less than or equal to the optimal system control $U_{\text{opt}} = G \setminus Z$.

Proof: The greatest solution of the inequality $\tilde{U} \otimes \alpha \leq U_{\text{opt}}$ is $\alpha = \tilde{U} \setminus U_{\text{opt}} = \Delta \tilde{U}$ by property 1.

The uniqueness is straightforward as the greatest solution of an inequation.

Property 4: control $U_m$ is the solution of the following recurrent equation (due to the feedback control structure used):

$$U_1 = \tilde{U}$$

$$U_{n+1} = U_n \otimes (G \otimes U_n) \setminus Z \quad n \geq 1$$
and this sequence converges on \( U_m \) for \( n = 2 \).

**Proof:**

\[
U_2 = \bar{U} \otimes (Y_G \setminus Z) = \bar{U} \otimes (\bar{U} \setminus U_{opt}) = U_m
\]

\[
U_3 = U_2 \otimes [(G \otimes U_2) \setminus Z]
\]

\[
= U_2 \otimes (U_2 \setminus U_{opt}) \quad \text{by property 1}
\]

\[
= [\bar{U} \otimes (\bar{U} \setminus U_{opt})] \otimes [(\bar{U} \otimes (\bar{U} \setminus U_{opt}) \setminus U_{opt})]
\]

\[
= \bar{U} \otimes \left\{ (\bar{U} \setminus U_{opt}) \otimes (\bar{U} \setminus U_{opt}) \right\} \quad \text{by (2)}
\]

\[
\varepsilon = \bar{U} \otimes (\bar{U} \setminus U_{opt}) \quad \text{by (3)}
\]

\[
= U_m
\]

Let us note that if \( U_m \) was applied to the system input (instead of \( \bar{U} = M \setminus Z \)) then the corresponding modified control would be again \( U_m \).

**Property 5:** \( \Delta \bar{U} \) is the greatest solution of the inequation \( X_G \otimes \varepsilon \leq E_G \), where \( X_G \) and \( E_G \) are the state vector and the co-state vector of system \( G \) respectively.

Hence considering the system outputs or its internal states is equivalent to search the optimal correction.

**Proof:**

Let us note \( A_G, B_G \) and \( C_G \) the matrices associated to the system state representation, we have \( G = C_G A_G B_G \), \( X_G = A_G B_G U \) and \( E_G = (C_G A_G) \setminus Z \).

The greatest solution of \( X_G \otimes \varepsilon \leq E_G \) is equal to

\[
X_G \setminus E_G = \left( (A_G \star B_G) U, (C_G A_G) \setminus Z \right)
\]

\[
= \left( C_G A_G \star B_G U \right) / Z \quad \text{by (2)}
\]

\[
= (C_G A_G) \setminus Z \quad \text{since } (A_G \star A_G) = A_G
\]

\[
= (G \otimes U) \setminus Z
\]

\[
= \Delta \bar{U}
\]

7 Application

The computation of \( U_m \) implies to have \( G \otimes \bar{U} \), i.e., all the system outputs sequence \( Y_G \), which is not realistic. In practice, we compute control \( U_\varepsilon \) (an estimation of \( U_m \)) at the instant \( t \) by using the observed system output sequence \( \{y_G(k) \mid y_G(k) \leq t; k \in Z \} \) completed by an estimation of the future system outputs (see [10]). Before presenting the computing algorithm of this control \( U_\varepsilon \), let us introduce the following notations:

Let \( \{v(k)\}_{k \in Z} \) be a dater, \( V = \bigoplus_{k \in Z} v(k)\gamma^k \) its associated formal series and \( K \in Z \), we define: \( V_{(\leq K)} = \bigoplus_{k \leq K} v(k)\gamma^k \)

and \( V_{(\geq K)} = \bigoplus_{k > K} v(k)\gamma^k \)

**Algorithm:**

**step 1:**

\( U_1 = M \setminus Z = \bar{U} \)

(the optimal model control \( \bar{U} \) is initially applied: the model is supposed to be exact, i.e., \( G = M \)).

**step \( n+1 \): \( n \geq 1 \)

1. Let \( K_n \) an event sample and \( t_n \) the date of the firing numbered \( K_n \) of \( G \otimes U_n \).

   We compute:

   \[
   \hat{G} U_n = (G U_n)_{(\leq K_n)} \oplus \left( \hat{G} U_{(\leq K_n)} \right)
   \]

   where \( (G U_n)_{(\leq K_n)} \) is the observed system output until \( t_n \) and \( \hat{G} U_{(\leq K_n)} \) is an estimation of \( (G U_n)_{(\leq K_n)} \).

2. Then the modified control is:

\[
\hat{U}_{n+1} = U_n \oplus \left( \hat{G} U_n \setminus Z \right)
\]

3. Let \( K_n' = \text{Sup} \{ k \in Z \mid u_n(k) \leq t_n \} \).

   As the set of controls \( \{ u_n(k), \quad k \leq K_n' \} \) has been applied to the system input until instant \( t_n \), this leads to consider the control:

\[
\hat{U}_{n+1} = (U_n)_{(\leq K_n')} \oplus \left( \hat{U}_{n+1} \right)_{(K_n' \rightarrow K_{n+1})}
\]

This control will be applied to the system input until \( t_{n+1} \) which is the time of the next control computation (\( U_{n+2} \)). In other words, we have:

\[
U_{c(\rightarrow K_{n+1})} = U_{n+1(\rightarrow K_{n+1})}
\]

8 Example

We consider the manufacturing system described by TEG of fig. 2 modeling a machine producing 4 parts by 4 time units. The corresponding model is described by TEG of fig. 3. Let us note that the system and its model are different except for the production rate.
The reference input is defined by:

\[ z(k) = \varepsilon ( = -\infty ) \quad \text{for} \quad k < 0; \]
\[ z(k) = z(k - 1) + 1 \quad \text{for} \quad k = 1, 2, 3, \quad \text{with} \quad z(0) = 0; \]
\[ z(k) = z(k - 1) + 2 \quad \text{for} \quad k = 4, \ldots, 8; \]
\[ z(k) = z(k - 1) + 1 \quad \text{for} \quad k = 9, 10; \]
\[ z(k) = T ( = +\infty ) \quad \text{for} \quad k > 10 \]

(end of the desired production).

Figure 4 represents the trajectories of the modified control \( U_m \), the optimal system control \( U_{opt} \) and control \( U_c \) computed by the previous algorithm. Figure 5 represents the corresponding outputs trajectories.

In this particular example \( U_m \) is equal to \( U_{opt} \) (see fig. 4). Initially (step 1 of the algorithm), the optimal model control is applied until instant \( t_1 = 1 \), the date of the firing numbered \( K_I \) of \( G \otimes U_1 \), \( K_I = 3 \) (see fig. 5).

Then, at each observed output (i.e., \( K_n + 1 = K_n + 1 \)) a new control trajectory \( U_{n+1} \) is computed.

We see that beyond time \( t_1 \), the trajectories associated to the computed control \( U_c \) and the modified control \( U_m \) are identical.
Conclusion

The proposed feedback control is applied to DES modeled in Max-Algebra. This control allows taking into account mismatch between the system and its model. Its principle is to modify the optimal model control to be as close as possible to the optimal system control (see property 3). Without modeling error this control remains equal to the optimal model control. These optimisation problems where the \textit{inversion} notion is underlying, essentially use the Residuation theory. The use of the system output to compute the control law necessarily raises estimation problem of system outputs behavior which is not addressed in this paper.

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