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# A MODELING METHOD OF SISO DISCRETE-EVENT SYSTEMS IN MAX-ALGEBRA

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## Abstract

This paper deals with the modeling of the time behavior of discrete-event systems in (max, +) algebra. The systems we consider are linearly modeled in this algebra. The proposed method is inspired by conventional linear system theory : from an ARMA form of the model and the system impulse response, the model parameters are computed by using a basic result of Residuation Theory in order to minimize an error criterion.

## 1 Introduction

In automatic control we need a mathematical model to characterize system properties (commandability, stability, ...) or to design the control law of a system. Therefore, it is important to obtain a model of the system which should, as far as possible, be both realistic and simple.

In conventional linear system theory many works deal with this problem. These studies have allowed to develop quite simple and relatively effective methods in order to identify conventional linear systems. The purpose of this paper is to propose a modeling method of discrete-event system (DES). This method uses the max-plus algebraic notation in order to do analogies with the conventional linear system theory. To have a linear representation of a system in (max, +) algebra (for shortness we will note max-algebra), we consider DES in which only synchronization phenomena

appear. Moreover, they are deterministic and we restrict ourselves to single-input single-output (SISO) systems.

In the second section we briefly recall the max-algebraic operations and the main representations of DES. A complete introduction to max-algebra can be found in [2], [3]. The third section is concerned with the identification method in max-algebra which is mainly based on the Residuation Theory [1], [2]. In the fourth section, we propose a practical modeling procedure.

## 2 Notations and Descriptions of DES in Max-Algebra

We consider the semiring  $(\mathcal{R} \cup \{-\infty, +\infty\}, \oplus, \otimes)$  where the laws  $\oplus, \otimes$  are defined by

$$a \oplus b = \max(a, b) \quad , \quad a \otimes b = a + b$$

The element  $-\infty$  is neutral for the law  $\oplus$  and absorbing for  $\otimes$ , the element 0 noted  $e$  is neutral for  $\otimes$ . We can note that  $\oplus$  is idempotent, i.e.,  $a \oplus a = a$ .

Firstly we recall the usual recurrent linear equations in max-algebra

$$\begin{cases} \underline{x}(k+1) = A \otimes \underline{x}(k) \oplus B \otimes u(k+1) \\ y(k) = C \otimes \underline{x}(k) \end{cases} \quad (1)$$

$\underline{x} = (x_1 \cdots x_n)^t$  is the  $n$  dimensional model state.  $u, x_i$  and  $y$  are called control,  $i^{\text{th}}$  model state and model output respectively. The model described by Eq. (1) can also be represented by a timed event graph where  $u, x_i$  and  $y$  are

transitions ;  $u(j)$ ,  $x_i(j)$  and  $y(j)$  represent the dates when  $u$ ,  $x_i$  and  $y$  are respectively fired for the  $j^{\text{th}}$  time. In the following, the sign  $\otimes$  will be omitted. The use of the  $\gamma$ -transform [2], [4], where  $\gamma$  operates as the  $z^{-1}$  operator of conventional system theory, leads to the following representation of Eq. (1)

$$\begin{cases} \underline{X}(\gamma) = A\gamma\underline{X}(\gamma) \oplus BU(\gamma) \\ Y(\gamma) = C\underline{X}(\gamma) \end{cases} \quad (2)$$

where  $U(\gamma)$ ,  $\underline{X}(\gamma)$  and  $Y(\gamma)$  are the  $\gamma$ -transform of  $u$ ,  $\underline{x}$  and  $y$  respectively. A basic theorem [2], [4] shows that the least solution of Eq. (2) is given by

$$Y(\gamma) = C(A\gamma)^* B U(\gamma) \quad \text{with } (A\gamma)^* = \bigoplus_{i=0}^{+\infty} (A\gamma)^i$$

and thus the input-output behavior of Eq. (2) can also be described by the transfer function

$$h(\gamma) = C(A\gamma)^* B$$

which can also be expressed as a polynomial expression of the form

$$h(\gamma) = p(\gamma) \oplus \gamma^v q(\gamma) (s\gamma^r)^* \quad (3)$$

where  $p(\gamma) = \bigoplus_{i=0}^{v-1} p_i \gamma^i$ ,  $q(\gamma) = \bigoplus_{j=0}^{r-1} q_j \gamma^j$  and  $s$  is a monomial [2]. Let us remind that  $h(\gamma)$  can be considered as the  $\gamma$ -transform of the impulse response of the model. Therefore, the  $h(\gamma)$  expression can express that the pattern represented by  $q(\gamma)$  is indefinitely reproduced because the multiplication by  $s\gamma^r$  symbolizes a  $r$  unit abscissa shift and a  $s$  unit ordinate shift. This periodic behavior begins after a transient behavior which may be represented by  $p(\gamma)$ . We must note that such a realization is not necessarily minimal [2, §6.5.4], however it allows a simple interpretation of  $h(\gamma)$ . For example, the transfer function

$$h(\gamma) = e \oplus 1\gamma \oplus 3\gamma^2 \oplus \gamma^3 (6 \oplus 8\gamma \oplus 9\gamma^2) (7\gamma^3)^*$$

leads to the impulse response illustrated in Fig. 1.

The purpose is to identify parameters of the transfer function  $h(\gamma)$  described by Eq. (3), i.e., parameters of polynomials  $p(\gamma)$  and  $q(\gamma)$  and monomial  $s$ , by using the system impulse response. The  $(s\gamma^r)^*$  term of the transfer function  $h(\gamma)$  implies an infinite repetition of the  $q(\gamma)$  pattern which means an infinite number of parameters to

compute. Hence as in conventional system theory, we transform the transfer function  $h(\gamma)$  into an ARMA form to obtain a finite number of parameters. A solution is given in [2, § 9.2.2] to obtain this ARMA form, however we prefer a simpler method which is :

Assuming that  $h(\gamma)$  is defined by Eq. (3), then  $Y(\gamma)$  and  $U(\gamma)$  satisfy the ARMA equation

$$Y(\gamma) \oplus s\gamma^r p(\gamma) U(\gamma) = [p(\gamma) \oplus \gamma^v q(\gamma)] U(\gamma) \oplus s\gamma^r Y(\gamma) \quad (4)$$

Proof is easily obtained by expressing Eq. (3) as

$$Y(\gamma) = [p(\gamma) \oplus \gamma^v q(\gamma) (e \oplus s\gamma^r \oplus (s\gamma^r)^2 \oplus \dots)] U(\gamma)$$

which can be written as

$$\underbrace{Y(\gamma)}_{a_1} = \underbrace{[p(\gamma) \oplus \gamma^v q(\gamma)] U(\gamma)}_{a_2} \oplus \underbrace{[\gamma^v q(\gamma) s\gamma^r (s\gamma^r)^*] U(\gamma)}_{a_3}$$

On the other hand the multiplication of Eq. (3) by  $s\gamma^r$  leads to

$$s\gamma^r Y(\gamma) = \underbrace{[s\gamma^r p(\gamma)] U(\gamma)}_{a_4} \oplus \underbrace{[s\gamma^r \gamma^v q(\gamma) (s\gamma^r)^*] U(\gamma)}_{a_5 \oplus a_6}$$

Because  $a_3 = a_6$ , hence we have

$$a_1 \oplus a_5 = a_2 \oplus a_3 \oplus a_5 = a_2 \oplus a_5 \oplus a_6 = a_2 \oplus a_4$$

which leads to the result.

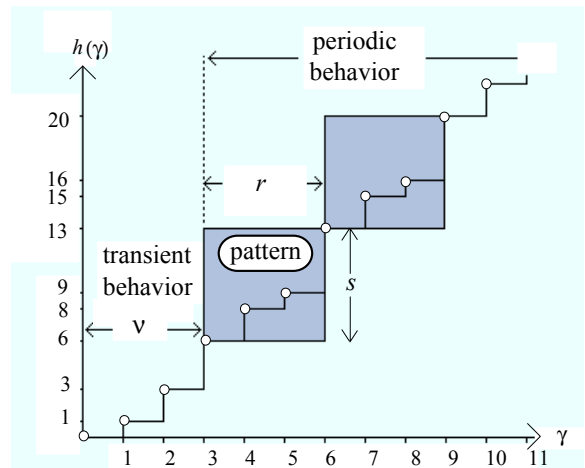


Figure 1 : Impulse response corresponding to  $h(\gamma)$ .  
( $v = 3, r = 3, s = 7$ )

### 3 Identification Method

In this section we develop an identification method of an ARMA model in max-algebra which will be used to compute the parameters of the transfer function  $h(\gamma)$ . The proposed approach offers an analogy with conventional discrete time system theory. Let us remind that a linear discrete time system can be expressed as the ARMA equation

$$y(k) + \sum_{j=1}^n a_j y(k-j) = \sum_{i=0}^m b_i u(k-i)$$

To estimate the parameters of this ARMA equation, i.e.,  $a_1, \dots, a_n, b_0, \dots, b_m$ , some identification methods consider the following prediction error

$$\mathbf{E}(k) = y_G(k) + \sum_{j=1}^n \hat{a}_j y_G(k-j) - \sum_{i=0}^m \hat{b}_i u(k-i)$$

where  $y_G$  is the measured system output and  $\hat{a}_1, \dots, \hat{a}_n, \hat{b}_0, \dots, \hat{b}_m$  are the estimated parameters. Such a prediction error allows to define a criterion which is minimized for the set of parameters searched. For example a quadratic criterion is used in the well known least square method [5, chap. 7].

#### 3.1 Identification Method of Max-Algebra ARMA Model

Let us suppose that model can be described in max-algebra by the ARMA equation

$$Y(\gamma) = [b_0 \oplus \dots \oplus b_m \gamma^m] U(\gamma) \oplus [a_1 \gamma \oplus \dots \oplus a_n \gamma^n] Y(\gamma) \quad (5)$$

The identification method we propose is based on the system impulse response, i.e.,  $u(k) = \begin{cases} 0 & \forall k \geq 0 \\ -\infty & \text{otherwise} \end{cases}$  Similarly to

the conventional system theory, we define the prediction error

$$\mathbf{E}(k) = y_G(k) - (\underline{\phi}_k^t \otimes \hat{\underline{\theta}})$$

where  $y_G$  is the measured system output,

$$\underline{\phi}_k^t = [u(k) \cdots u(k-m) y_G(k-1) \cdots y_G(k-n)]$$

is the regression vector at event  $k$  and

$\hat{\underline{\theta}} = [\hat{b}_0 \cdots \hat{b}_m \hat{a}_1 \cdots \hat{a}_n]^t$  is the estimated parameters vector.

By using the data of  $u(k)$  and  $y_G(k)$  with  $k = 0$  to  $N$  and  $N > n + m$ , one can obtain the matrix expression

$$\underline{\mathbf{E}} = \underline{Y}_G - (M_G \otimes \hat{\underline{\theta}})$$

where  $\underline{\mathbf{E}} = [\mathbf{E}(n) \cdots \mathbf{E}(N)]^t$  is the prediction error vector,  $\underline{Y}_G = [y_G(n) \cdots y_G(N)]^t$  is the measured system output vector and  $M_G$  is the matrix  $[\underline{\phi}_n \cdots \underline{\phi}_N]^t$ . Vector  $\underline{\mathbf{E}}$  makes possible to define a criterion  $J$  as

$$J(\hat{\underline{\theta}}) = \sum_{k=n}^N |\mathbf{E}(k)|$$

To determine the estimated parameters vector  $\hat{\underline{\theta}}$  which minimizes this criterion, we consider a basic result of Residuation Theory [1], [2] which states that

$$\underline{\theta}^{\square} = (-M_G)^{\vee} \underline{Y}_G \quad (6)$$

is the greatest subsolution of  $\underline{Y}_G = M_G \otimes \hat{\underline{\theta}}$  where  $\otimes$  refers to the multiplication of two matrices in which the min-operation is used rather than the max-operation [2]. Thus, by using this greatest subsolution, each element of the prediction error vector  $\underline{\mathbf{E}}$  is always positive and  $J(\hat{\underline{\theta}})$  is minimized.

#### 3.2 Identification Method of Transfer Function $h(\gamma)$

The identification method proposed in the previous section needs to have a particular linear ARMA form (see Eq. (5)). Because this condition is not verified by the ARMA equation (4) (owing to the  $s \gamma^r p(\gamma) U(\gamma)$  term), we propose to identify transient part and periodic part separately. This procedure leads us to consider the two following equations rather than Eq. (4).

When we consider an impulse response without periodic behavior, i.e.,  $q(\gamma) = -\infty$ , Eq. (3) is reduced to the simple MA equation

$$Y(\gamma) = p(\gamma) U(\gamma) \quad (7)$$

Similarly, an impulse response without transient behavior, i.e.,  $p(\gamma) = -\infty$ , leads by using Eq. (4) to

$$Y(\gamma) = \gamma^v q(\gamma) U(\gamma) \oplus s \gamma^r Y(\gamma) \quad (8)$$

Let us remind that  $v-1$  and  $r-1$  are the polynomial orders of  $p(\gamma)$  and  $q(\gamma)$  respectively. Therefore we obtain an estimation of the  $p(\gamma)$  parameters by applying the result of section 3.1 to Eq. (7)

$$\hat{\underline{\theta}} = \left[ -\underline{\varphi}_0 \cdots -\underline{\varphi}_{v-1} \right] \odot \underline{Y}_G \quad (9)$$

where  $\hat{\underline{\theta}} = [\hat{p}_0 \cdots \hat{p}_{v-1}]^t$

is the estimated vector of the  $p(\gamma)$  parameters and

$$\underline{\varphi}_k = [u(k) \cdots u(k-v+1)]^t$$

$$\underline{Y}_G = [y_G(0) \cdots y_G(v-1)]^t$$

Let us note that we naturally need the first data of the measured system output to correctly identify the transient part.

Similarly, we obtain an estimation of the parameters of  $q(\gamma)$  and  $s$  by applying the result of section 3.1 to Eq. (8)

$$\hat{\underline{\theta}} = \left[ -\underline{\varphi}_v \cdots -\underline{\varphi}_N \right] \odot \underline{Y}_G \quad (10)$$

with  $N \geq v+r$

where  $\hat{\underline{\theta}} = [\hat{q}_0 \cdots \hat{q}_{r-1} \hat{s}]^t$

is the estimated vector of the parameters of  $q(\gamma)$  and  $s$  and

$$\underline{Y}_G = [y_G(v) \cdots y_G(N)]^t$$

$$\underline{\varphi}_k = [u(k-v) \cdots u(k-v-r+1) y_G(k-r)]^t$$

According to the assumption which leads to Eq. (8)  $p(\gamma) = -\infty$ , we consider that  $y_G(0) = \cdots = y_G(v-1) = -\infty$  to compute  $\underline{\varphi}_k$ .

#### 4 Practical Modeling of $h(\gamma)$

As in conventional linear system theory [5, chap. 16], the identification of the transfer function  $h(\gamma)$  by using Eqs. (9) and (10) imposes to know the number of coefficients of polynomials  $p(\gamma)$  and  $q(\gamma)$ , i.e.,  $v$  and  $r$  respectively.

In practice the knowledge of these values is an important problem of the modeling. Indeed the model structure should as far as possible be both complex enough to correctly qualify the system and simple enough to reduce the number of parameters to be identified.

To obtain this compromise we need a preliminary analysis to have an estimation of  $v$  noted  $l_p$ , it can be coarser but it must be greater than the exact transient length. Then we propose a modeling method based on the analysis of the criterion J (see

section 3.1) as a function of  $l_p$  and  $l_q$  (defined as the estimation of the pattern length  $r$ ).

In order to estimate  $r$  :

- we initiate  $l_q$  to 1 ;

- we compute  $q_i$  ( $i=0, \dots, l_q-1$ ) and  $s$  by using

Eq. (10) ;

we repeat the latter calculation with  $l_q = l_q + 1$  until criterion J is minimal.

To estimate  $v$  we fix  $r$  to its previous estimation. Then we decrease  $l_p$  and we calculate  $q_i$  ( $i=0, \dots, r-1$ ) and  $s$  by using Eq. (10) until criterion J increases.

This critical value of  $l_p$  corresponds to the estimation of transient part length ( $v$ ).

## 5 Conclusion

We have proposed an identification method of DES in max-algebra. This method makes possible the identification of SISO systems. It offers an analogy with the conventional linear system theory : it is based both on the analysis of the system impulse response and on a linear ARMA form of the model. To obtain this ARMA form we separate the transient behavior and the periodic one. Such property makes possible the use of a basic result of Residuation Theory which leads to a simple method of parameters model estimation. However we cannot guarantee the minimality of the model due to the considered expression of the transfer function, we keep working on this problem. On the other hand it will be interesting to characterize the action of disturbances on the method.

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