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CARLEMAN ESTIMATES FOR ELLIPTIC OPERATORS WITH COMPLEX COEFFICIENTS PART I: BOUNDARY VALUE PROBLEMS

MOURAD BELLASSOUED AND JÉRÔME LE ROUSSEAU

ABSTRACT. We consider elliptic operators with complex coefficients and we derive microlocal and local Carleman estimates near a boundary, under sub-ellipticity and strong Lopatinskii conditions. Carleman estimates are weighted *a priori* estimates for the solutions of the associated elliptic boundary problem. The weight is of exponential form, $\exp(\tau\varphi)$, where τ is meant to be taken as large as desired. Such estimates have numerous applications in unique continuation, inverse problems, and control theory. Based on inequalities for interior and boundary differential quadratic forms, the proof relies on the microlocal factorization of the symbol of the conjugated operator in connection with the sign of the imaginary part of its roots. We further consider weight functions of the form $\varphi = \exp(\gamma\psi)$, with γ meant to be taken as large as desired, and we derive Carleman estimates where the dependency upon the two large parameters, τ and γ, is made explicit. Applications on unique continuation properties are given.

KEYWORDS: Carleman estimate; elliptic operators; boundary problem; Lopatinskii conditions.

AMS 2000 SUBJECT CLASSIFICATION: 35B45; 35J30; 35J40.

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1. Introduction and main result

Let Ω be a bounded and connected domain in $\mathbb{R}^n$ with a $\mathscr{C}^\infty$-boundary $\partial\Omega$. Points in Ω are denoted by $x = (x_1, \dots, x_n)$ and we write $D_j = -i\partial/\partial x_j$ where $i = \sqrt{-1}$. Let us consider a linear partial differential operator of order $m = 2\mu$, $\mu \geq 1$:

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha, \tag{1.1}$$

where the coefficients $a_\alpha(x)$ are bounded measurable complex-valued functions defined in $\overline{\Omega}$. The higher-order coefficients $a_\alpha(x)$ with $|\alpha| = m$ are required to be $\mathscr{C}^\infty$ in $\overline{\Omega}$. In what follows, we assume that the operator P is elliptic.

Moreover, we consider a system of linear boundary operators of order less than m

$$B^k = \sum_{|\alpha| \leq \beta_k} b_\alpha^k(x) D^\alpha, \quad k = 1, \dots, \mu = m/2, \tag{1.2}$$

where the coefficients $b_\alpha^k(x)$ are $\mathscr{C}^\infty$ complex-valued functions defined in some neighborhood of $\partial\Omega$.
The aim of the present article is to derive a Carleman estimate for the following elliptic boundary value problem

$$\begin{cases} Pu(x) = f(x), & x \in \Omega, \\ B^k u(x) = g^k(x), & x \in \partial\Omega, \quad k = 1, \dots, \mu. \end{cases}$$

Carleman estimates are weighted *a priori* inequalities for the solutions of a partial differential equation (PDE), where the weight is of exponential type. For the partial differential operator P *away from the boundary* it takes the form:

$$\|e^{\tau\varphi}w\|_{L^2} \leq C\,\|e^{\tau\varphi}Pw\|_{L^2}\,, \qquad w \in \mathscr{C}_c^\infty(\Omega),\ \tau \geq \tau_0. \tag{1.3}$$

The exponential weight involves a parameter τ that can be taken as large as desired. Additional terms in the l.h.s., involving derivatives of u, can be obtained depending on the order of P and on the *joint* properties of P and φ. For instance for a second-order operator P such an estimate can take the form

$$\tau^3\,\|e^{\tau\varphi}u\|_{L^2}^2 + \tau\,\|e^{\tau\varphi}\nabla_x u\|_{L^2}^2 \leq C\,\|e^{\tau\varphi}Pu\|_{L^2}^2\,, \qquad \tau \geq \tau_0,\ u \in \mathscr{C}_c^\infty(\Omega). \tag{1.4}$$

This type of estimate was used for the first time by T. Carleman [8] to achieve uniqueness properties for the Cauchy problem of an elliptic operator. Later, A.-P. Calderón and L. Hörmander further developed Carleman's method [7, 16]. To this day, Carleman estimates remain an essential method to prove unique continuation properties; see for instance [42] for an overview. On such questions more recent advances have been concerned with differential operators with singular potentials, starting with the contribution of D. Jerison and C. Kenig [25]. The reader is also referred to [40, 27, 28]. In more recent years, the field of applications of Carleman estimates has gone beyond the original domain; they are also used in the study of:

- Inverse problems, where Carleman estimates are used to obtain stability estimates for the unknown sought quantity (e.g. coefficient, source term) with respect to norms on measurements performed on the solution of the PDE, see e.g. [6, 23, 29, 21]; Carleman estimates are also fundamental in the construction of complex geometrical optic solutions that lead to the resolution of inverse problems such as the Calderón problem with partial data [26, 9].
- Control theory for PDEs; through unique continuation properties, Carleman estimates are used for the exact controllability of hyperbolic equations [2]. They also yield the null controllability of linear parabolic equations [34] and the null controllability of classes of semi-linear parabolic equations [14, 1, 13].

Here, we seek an estimate similar to (1.3) in the neighborhood of a point of the boundary $\partial\Omega$. The estimate we shall obtain will exhibit additional terms that account for the boundary conditions given by the operators B^k, $k = 1, \ldots, \mu$. This question was addressed by D. Tataru for general operators with real coefficients [41] and applied to the unique continuation problem near the boundary. Here, we shall focus on the case of general elliptic operators, yet allowing for complex coefficients. In such case there is no general theory for the derivation of Carleman estimates at the boundary. In [41] because of the generality of the types of operators treated, norms in the Carleman estimates are not optimal in the case of elliptic operators with real coefficients. Here we obtain norms that precisely coincide with those one could anticipate from the known estimates away from the boundary and from particular cases of operators for which such an estimate has been derived at the boundary, e.g. for the Laplace operator [34, 22].

The key conditions for the derivation of the Carleman estimate are compatibility properties between the elliptic operator P, the weight function φ, and the boundary operators B^k, $k = 1, \ldots, \mu$. Those are the sub-ellipticity and the strong Lopatinskii conditions. The former involves P and φ and is known to be necessary and sufficient for the estimate to hold away from the boundary in the case of an elliptic operator. The latter involves P, φ, and the B^k. The Lopatinskii condition is used in [41]. In the present article, by proper (tangential) microlocalizations at the boundary we show the precise action of this condition. These microlocalizations are important as the Lopatinskii condition is function of the sign of the imaginary parts of the roots of[1] $p_\varphi(x, \xi', \tau, \xi_n) = p(x, \xi + i\tau\varphi'(x))$ viewed as a polynomial in ξ_n. Of course the configuration

[1]Here to simplify we consider the case $\Omega = \{x_n > 0\}$. Then ξ_n corresponds the (co)normal direction at the boundary $\partial\Omega = \{x_n = 0\}$. In the main text we shall use change of variables to reach this configuration locally.

of the roots changes as the other parameters (x', ξ', τ) are modified. Roots can for instance cross the real axis. Each configuration needs to be addressed separately through a microlocalization procedure. For the Laplace operator at the boundary this was exploited to obtain a Carleman estimate in [35] for the purpose of proving a stabilization result for the wave equation.

As in [41] the method of the present article is based on the study of interior and boundary differential quadratic forms, an approach that originates in the work of [17] for estimates away from boundaries and in [38, 39, 37] for the treatment of boundaries. In connection with the microlocalizations described above we give a microlocal treatment of those differential quadratic forms. Positivity arguments rely on the Gårding inequality for homogeneous polynomials in connection with the position of the roots of the polynomial $p_\varphi(x, \xi', \tau, \xi_n)$. In fact the roots are split into three groups: roots with positive imaginary part, roots with negative imaginary part, and real roots. Accordingly, gathering the associated monomials we write p_φ as a product of three factors:

$$p_\varphi = p_\varphi^+ p_\varphi^- p_\varphi^0.$$

The regularity of each factor is important to carry pseudo-differential calculus and applying Gårding type inequalities. Roots can however cross and only their continuity is certain. Yet, using the Rouché theorem, the three factors can be shown smooth in proper microlocal regions.

The Carleman estimate we prove is of the form:

$$\|e^{\tau\varphi}u\|^2 + |e^{\tau\varphi}T(u)|^2 \leq C\big(\|e^{\tau\varphi}P(x,D)u\|^2 + \sum_{k=1}^{\mu} |e^{\tau\varphi}B^k(x,D)u_{|\partial\Omega}|^2\big),$$

for u supported near a point at the boundary, where $T(u)$ stands for the trace of $(u, D_\nu u, \dots, D_\nu^{m-1}u)$, the successive normal derivatives of u, at $\partial\Omega$. In this form, the estimate is incorrect as norms needs to be made precise. For a correct statement please refer to Theorem 1.6 below.

For Carleman estimates, one is often inclined to choose a weight function of the form $\varphi = \exp(\gamma\psi)$, with the parameter $\gamma > 0$ chosen large. Several authors have derived Carleman estimates for some operators in which the dependency upon the second parameters γ is kept explicit. See for instance [14]. Such results can be very useful to address systems of PDEs, in particular for the purpose of solving inverse problems. On such questions see for instance [10, 12, 24, 5].

Compatibility conditions need to be introduced between the operator P and the weight ψ. Those are the so-called strong pseudo-convexity conditions introduced by L. Hörmander [17, 20]. With the weight function φ of the form $\varphi = \exp(\gamma\psi)$, the parameter γ can be viewed as a convexification parameter. As shown in Proposition 28.3.3 in [20] the strong pseudo-convexity of the function ψ with respect to P implies the sub-ellipticity condition for φ mentioned above[2] for γ chosen sufficiently large. Away from the boundary, for a second-order estimate the resulting Carleman estimate can take the form (compare with (1.4)):

$$(\gamma\tau)^3\big\|\varphi^{3/2}e^{\tau\varphi}u\big\|_{L^2}^2 + \gamma\tau\big\|\varphi^{1/2}e^{\tau\varphi}\nabla_x u\big\|_{L^2}^2 \lesssim \|e^{\tau\varphi}Pu\|_{L^2}^2, \qquad \tau \geq \tau_0,\ \gamma \geq \gamma_0,\ u \in \mathscr{C}_c^\infty(\Omega). \tag{1.5}$$

We aim to extend such estimate in the neighborhood of the boundary. We then assume that the strong Lopatinskii condition holds for the operators P, B^k and the weight ψ. The work [30] provides a general framework for the analysis and the derivation of Carleman estimates with two large parameters away from boundaries. For that purpose it introduces a pseudo-differential calculus of the Weyl-Hörmander type that resembles the semi-classical calculus and takes into account the two large parameters τ and γ as well as the weight function $\varphi = \exp(\gamma\psi)$. Here, the analysis of [30] is adapted to the case of an estimate at the boundary. Estimates with the two large parameters τ and γ are derived in the case of general elliptic operators.

[2]The terminology for the strong pseudo-convexity condition and the sub-ellipticity condition are often confused by authors. Here we make a clear distinction of the two notions.

If we strengthen strong pseudo-convexity condition of ψ and P, assuming the so-called simple characteristic property, sharper estimates can be obtained [30]. We also derive such estimates at the boundary.

With the different Carleman estimate that we obtain here she shall be able to achieve unique continuation properties at a boundary across some hypersurface for some classes of elliptic operators and some products of such operators.

Perspectives. The treatment of transmission problems for elliptic operators is a natural extension of the present work. If elliptic operators are given on both sides of an interface and transmission conditions are given by interface operators, the potential derivation of a Carleman estimate is a natural question. It was studied for second-order elliptic operators for the purpose of stabilization of the associated wave equations [3] and the controllability of the associated heat equation [33, 32]. The treatment of general elliptic transmission problems is the subject of an ongoing joint work by the two authors of the present article [4].

Here, we consider Carleman estimates with the loss of a half derivative. It would be interesting to carry out a similar analysis for estimates with a larger loss of derivatives. Such estimates can be very important in some classes of inverse problems; see for instance [26, 9]. An important example of operator exhibiting a loss of a full derivative could be the bi-Laplace operator with clamped boundary conditions for which estimates cannot be deduced from estimates for the Laplace operator.

1.1. **Setting.** We shall now give more precision on the setting we consider. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote by $\xi = (\xi_1, \dots, \xi_n)$ the corresponding Fourier variables. Moreover, for every $\xi \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$ we define $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. We denote by

$$p(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha$$

the principal symbol of the operator P given in (1.1) and, for $k = 1, \dots \mu$, we denote by

$$b^k(x, \xi) = \sum_{|\alpha|=\beta_k} b_\alpha(x) \xi^\alpha$$

the principal symbol of the boundary operator B^k in (1.2).

Here, we assume that the operator P is elliptic, viz.,

$$p(x, \xi) \neq 0, \quad \forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Let $\nu = \nu(x)$ denote the unit outward conormal vector to $\partial\Omega$ at x. We assume that the system $B = (B^1, \dots, B^\mu)$ of boundary differential operators is normal at $x \in \partial\Omega$, that is,

$$0 \leq \beta_1 < \beta_2 < \cdots < \beta_\mu < m,$$

and, for all $k = 1, \dots, \mu$, that

$$b^k(x, \nu(x)) \neq 0, \quad \forall x \in \partial\Omega.$$

Moreover since P is elliptic we have that $\partial\Omega$ is not characteristic with respect the operator $P(x, D)$:

$$p(x, \nu(x)) \neq 0, \quad \forall x \in \partial\Omega.$$

We now review the definition of important properties that will be used in what follows: the sub-ellipticity and the strong Lopatinskii conditions.

1.2. **Sub-ellipticity condition.** For any two functions $f(x,\xi)$ and $g(x,\xi)$ in $\mathscr{C}^\infty(\Omega\times\mathbb{R}^n)$ we denote their Poisson bracket in phase-space by

$$\{f,g\}=\sum_{j=0}^{n}\big(\frac{\partial f}{\partial\xi_j}\frac{\partial g}{\partial x_j}-\frac{\partial f}{\partial\xi_j}\frac{\partial g}{\partial x_j}\big).$$

It is to be connected with the commutator of two (pseudo-)differential operators. In fact, if f and g are polynomials in ξ, then the principal symbol of the commutator $[f(x,D),g(x,D)]$ is precisely $-i\{f,g\}(x,\xi)$.

The sub-ellipticity condition connecting the symbol p and a weight function φ is the following (See [17, Chapter 8] and [20, Sections 28.2–3]).

Definition 1.1. *Let $\varphi(x)$ be a smooth functions on $\overline{\Omega}$ and let U be an open subset of Ω. The pair $\{P,\varphi\}$ satisfies the sub-ellipticity condition on $\overline{U}$ if $\varphi'(x):=d\varphi(x)\neq 0$ at every point in $\overline{U}$ and if*

$$p(x,\xi+i\tau\varphi'(x))=0\quad\Rightarrow\quad\frac{1}{2i}\left\{\overline{p}(x,\xi-i\tau\varphi'(x)),p(x,\xi+i\tau\varphi'(x))\right\}>0,$$

for all $x\in\overline{U}$ and all non-zero $\xi\in\mathbb{R}^n$, $\tau>0$.

For an elliptic operator p the sub-ellipticity condition is *necessary and sufficient* for a Carleman estimate of the form of (1.3) to hold away from the boundary [20, Section 28.2]. For a simple exposition of the derivation of Carleman estimates for second-order elliptic operators under the sub-ellipticity condition we refer to [31].

Note also that the sub-ellipticity condition is invariant under changes of coordinates. This is an important fact here as we shall work in local coordinates in what follows.

Remark 1.2. *Note that here, as the operator P is elliptic, we have $p(x,\xi)\neq 0$ for each $\xi\in\mathbb{R}^n$, $\xi\neq 0$. The sub-ellipticity condition thus holds naturally at $\tau=0$.*

Remark 1.3. *Setting $p_\varphi(x,\xi,\tau)=p(x,\xi+i\tau\varphi'(x))$ and writing $p_\varphi=a+ib$ with a and b real, we have*

$$\frac{1}{2i}\left\{\overline{p}(x,\xi-i\tau\varphi'(x)),p(x,\xi+i\tau\varphi'(x))\right\}=\frac{1}{2i}\left\{\overline{p_\varphi},p_\varphi\right\}(x,\xi,\tau)=\{a,b\}(x,\xi,\tau).$$

Below, we shall use the sub-ellipticity condition in the form

$$p(x,\xi+i\tau\varphi'(x))=0\quad\Rightarrow\quad\{a,b\}(x,\xi,\tau)>0,$$

for all $x\in\overline{U}$ and all non-zero $\xi\in\mathbb{R}^n$, $\tau>0$.

In connection with the symbol interpretation of the Poisson bracket given above, we see that the sub-ellipticity condition guarantees some positivity for the operator $i[a(x,D,\tau),b(x,D,\tau)]$ on the characteristic set of $p(x,D+i\tau\varphi')=a(x,D,\tau)+ib(x,D,\tau)$. A proper combination of $a(x,D,\tau)^*a(x,D,\tau)+b(x,D,\tau)^*b(x,D,\tau)$ and $i[a(x,D,\tau),b(x,D,\tau)]$ thus leads to a positive operator. This is the heart of the proof of Carleman estimates.

1.3. **Strong Lopatinskii condition.** Elliptic boundary value problems are well-posed only if boundary conditions are chosen appropriately. By well-posedness one usually means that the solution exists and is unique in some space, and it depends continuously on data (boundary conditions and source terms) and parameters. A sufficient condition to obtain well-posedness is the so-called Lopatinskii condition that is of algebraic nature. Here, we shall treat conditions of this type adapted to the elliptic operators we consider after conjugation by the Carleman weight function.

For $x\in\partial\Omega$ we denote by $N_x^*(\partial\Omega)$ the conormal space at x given by

$$N_x^*(\partial\Omega)=\left\{N\in T_x^*(\overline{\Omega});\,N(Z)=0,\,\forall Z\in T_x(\partial\Omega)\right\}.$$

The conormal bundle of $\partial\Omega$ is given by

$$N^*(\partial\Omega) = \left\{(x, N) \in T^*(\overline{\Omega});\, x \in \partial\Omega,\, N \in N^*_x(\partial\Omega)\right\}.$$

By a boundary quadruple $\omega = (x, Y, N, \tau)$ we shall mean $x \in \partial\Omega$, $Y \in T^*_x(\partial\Omega)$, $N \in N^*_x(\partial\Omega) \setminus \{0\}$ pointing inside Ω and $\tau \geq 0$. We also require $(\tau, Y) \neq (0, 0)$. For a boundary quadruple ω and $\lambda \in \mathbb{C}$, we set

$$\tilde{p}_\varphi(\omega, \lambda) := p\left(x, Y + \lambda N + i\tau d\varphi(x)\right). \tag{1.6}$$

For a fixed boundary quadruple $\omega_0 = (x_0, Y_0, N_0, \tau_0)$, we denote by σ_j the roots of $\tilde{p}_\varphi(\omega_0, \lambda)$ with multiplicity μ_j, viewed as a polynomial of degree m in λ, with leading-order coefficient c_0. We can then factorize this polynomial as follows:

$$\tilde{p}_\varphi(\omega_0, \lambda) = c_0 \tilde{p}^+_\varphi(\omega_0, \lambda)\tilde{p}^-_\varphi(\omega_0, \lambda)\tilde{p}^0_\varphi(\omega_0, \lambda),$$

with

$$\tilde{p}^\pm_\varphi(\omega_0, \lambda) = \prod_{\pm \operatorname{Im}\sigma_j > 0} (\lambda - \sigma_j)^{\mu_j}, \quad \tilde{p}^0_\varphi(\omega_0, \lambda) = \prod_{\operatorname{Im}\sigma_j = 0} (\lambda - \sigma_j)^{\mu_j}.$$

We define the polynomial $\kappa_\varphi(\omega_0, \lambda)$ by

$$\kappa_\varphi(\omega_0, \lambda) = \tilde{p}^+_\varphi(\omega_0, \lambda)\tilde{p}^0_\varphi(\omega_0, \lambda). \tag{1.7}$$

Similarly, for $B = \left\{B^k\right\}_{k=1,\dots,\mu}$ the set of boundary operators and $b^k(x, \xi)$ their principal symbols, for a boundary quadruple $\omega = (x, Y, N, \tau)$ we set

$$\tilde{b}^k_\varphi(\omega, \lambda) = b^k(x, Y + \lambda N + i\tau d\varphi(x)). \tag{1.8}$$

Definition 1.4. (1) *We say that $\{P, B^k, \varphi,\ k = 1, \dots, \mu\}$ satisfies the strong Lopatinskii condition at a boundary quadruple $\omega_0 = (x_0, Y_0, N_0, \tau_0)$ with N_0 pointing inside Ω, $\tau_0 \geq 0$, and $(\tau, Y_0) \neq (0, 0)$, if the set of polynomials $\{\tilde{b}^k_\varphi(\omega_0, \lambda)\}_{1\leq k\leq\mu}$ is complete modulo $\kappa_\varphi(\omega_0, \lambda)$ as polynomials in λ: for all $f(\lambda)$ polynomials there exist $q(\lambda)$ polynomial and $c_k \in \mathbb{C}$, $1 \leq k \leq \mu$, such that*

$$f(\lambda) = \sum_{k=1}^{\mu} c_k \tilde{b}^k_\varphi(\omega_0, \lambda) + q(\lambda)\kappa_\varphi(\omega_0, \lambda), \qquad \lambda \in \mathbb{R}.$$

(2) *We say that $\{P, B^k, \varphi,\ k = 1, \dots, \mu\}$ satisfies the strong Lopatinskii condition at $x_0 \in \partial\Omega$ if the previous property holds for all boundary quadruples $\omega = (x_0, Y, N, \tau)$ with $Y \in T^*_{x_0}(\partial\Omega)$, $N \in N^*_{x_0}(\partial\Omega)$ with N pointing inside Ω, $\tau \geq 0$, and $(\tau, Y) \neq 0$.*

Remark 1.5. (1) *Observe that the strong Lopatinskii condition only depends on $d\varphi$ rather than φ. It is thus a geometrical condition that concerns the level sets of φ (as here $d\varphi(x) \neq 0$ – see Definition 1.1) in connexion with the differential operators P and B^k, $k = 1, \dots, \mu$.*

(2) *Observe that for a polynomial $f(\lambda)$, the Euclidean division yields the existence of two polynomials $g(\lambda)$ and $q(\lambda)$, with $d^\circ g < d^\circ \kappa_\varphi(\omega_0, \tau_0, \lambda)$, such that*

$$f(\lambda) = g(\lambda) + q(\lambda)\kappa_\varphi(\omega_0, \lambda), \qquad \lambda \in \mathbb{R}.$$

In the statement of the strong Lopatinskii condition we may thus restrict ourselves to polynomials of degree less than that of $\kappa_\varphi(\omega_0, \lambda)$. Considering the definition of $\kappa_\varphi(\omega_0, \lambda)$ in (1.7) that depends on the roots of the polynomial of $\tilde{p}_\varphi(\omega_0, \lambda)$, in what follows we shall restrict ourselves to polynomials $f(\lambda)$ of degree less than or equal to $m - 1$.

(3) *Note that the strong Lopatinskii condition implies* $d^\circ\kappa_\varphi \leq m-1$. *Hence* $d^\circ\tilde{p}_\varphi^- > 0$. *In fact, otherwise, the vector space of the polynomial functions of degree less than or equal to* $m-1$, *of dimension* m, *is generated by a family of* $\mu = m/2$ *polynomials; a contradiction.*

Invariance by change of coordinates. We finish the presentation of the strong Lopatinskii condition by observing that this definition is of geometrical nature, independent of the choice of coordinates. This fact is important as we shall make use of local coordinates at the boundary $\partial\Omega$ of the open set Ω in what follows.

In fact, for a point $x \in \partial\Omega$ we consider an open neighborhood $X \in \mathbb{R}^n$ of x and two coordinate systems (X_1, ψ_1) and (X_2, ψ_2), that is $\psi_1 : X \to X_1$ and $\psi_2 : X \to X_2$ are diffeomorphisms and X_1, X_2 are open sets in $\mathbb{R}^n$. We set $x_1 = \psi_1(x)$ and $x_2 = \psi_2(x)$.

We then introduce the diffeomorphism $\kappa : X_1 \to X_2$ given by $\kappa = \psi_2 \circ \psi_1^{-1}$ and we have $\kappa(x_1) = x_2$. Let Y_1, N_1 (resp. Y_2, N_2) be the local versions of Y and N in the two coordinate systems. Similarly let $p^{(1)}$ and $b_k^{(1)}$, $k = 1, \dots, \mu$, (resp. $p^{(2)}$ and $b_k^{(2)}$) be the local versions of the principal symbols of the differential operators P and B^k. We also define $\varphi_1 = \varphi \circ \psi_1$ and $\varphi_2 = \varphi \circ \psi_2$ the local versions of the weight function in the coordinate patches.

With standard differential geometry arguments we have the following relations:

$$Y_1 = {}^t\kappa'(x_1)Y_2, \quad N_1 = {}^t\kappa'(x_1)N_2, \quad d\varphi_1(x_1) = {}^t\kappa'(x_1)d\varphi_2(x_2),$$
$$p^{(1)}(x,\xi) = p^{(2)}(\kappa(x), {}^t\kappa'(x)^{-1}\xi), \quad b_k^{(1)}(x,\xi) = b_k^{(2)}(\kappa(x), {}^t\kappa'(x)^{-1}\xi).$$

If we set $f_j(\lambda) = p^{(j)}(x_j, Y_j + i\tau d\varphi_j(x_j) + \lambda N_j)$, $j = 1, 2$, we find

$$\begin{aligned} f_1(\lambda) &= p^{(1)}(x_1, Y_1 + \lambda N_1 + i\tau d\varphi_1(x_1)) = p^{(2)}\big(\kappa(x_1), {}^t\kappa'(x_1)^{-1}(Y_1 + \lambda N_1 + i\tau d\varphi_1(x_1))\big) \\ &= p^{(2)}(x_2, Y_2 + \lambda N_2 + i\tau d\varphi_2(x_2)) = f_2(\lambda), \end{aligned}$$

which simply means that the polynomial function $\tilde{p}_\varphi$ defined in (1.6) does not depend on the coordinate system chosen. The same holds for the polynomial function $\tilde{b}_\varphi^k$ defined in (1.8), which allows one to conclude that the strong Lopatinskii condition of Definition 1.4 can be stated (and checked) in any coordinate system.

1.4. **Sobolev norms with a parameter.** The L^2 inner-products on Ω and $\partial\Omega$ will be denoted by $(.,.)$ and $(.,.)_\partial$ respectively.

Let $\tau \geq 0$ and $s \geq 0$. We introduce the Sobolev spaces $H_\tau^s(\Omega)$ and $H_\tau^s(\partial\Omega)$ defined by the following norms respectively:

$$\|u\|_{s,\tau}^2 = \tau^{2s}\|u\|_{L^2(\Omega)}^2 + \|u\|_{H^s(\Omega)}^2 \quad \text{and} \quad |u|_{s,\tau}^2 = \tau^{2s}|u|_{L^2(\partial\Omega)}^2 + |u|_{H^s(\partial\Omega)}^2,$$

where we denote the usual Sobolev norms on Ω and $\partial\Omega$ by $\|.\|_{H^s(\Omega)}$ and $|.|_{H^s(\partial\Omega)}$. Observe that for $\sigma \in [0, s]$ we have

$$\tau^{s-\sigma}\|u\|_{H^\sigma(\Omega)} \lesssim \|u\|_{s,\tau}, \qquad \tau^{s-\sigma}|u|_{H^\sigma(\partial\Omega)} \lesssim |u|_{s,\tau}.$$

For $m \in \mathbb{N}$ and $s \in \mathbb{R}$ we introduce the following boundary space

$$H_\tau^{m,s}(\partial\Omega) = \prod_{j=0}^m H_\tau^{m-j+s}(\partial\Omega),$$

equipped with the norm

$$|\mathbf{u}|_{m,s,\tau}^2 = \sum_{j=0}^m |u_j|_{m-j+s,\tau}^2, \quad \mathbf{u} = (u_0, \dots, u_m). \tag{1.9}$$

If $u \in \mathscr{C}^\infty(\overline{\Omega})$ we denote $T^m(u) = (T_0(u), \dots, T_m(u))$ where $T_j(u) = (\frac{1}{i}\partial_\nu)^j u$ is the trace of u of order j and we define

$$|T^m(u)|^2_{m,s,\tau} = \sum_{j=0}^{m} |T_j(u)|^2_{m-j+s,\tau}\,.$$

In what follows we shall write $T(u)$ in place of $T^m(u)$ for concision. We shall also write norms of the form $|e^{\tau\varphi}T(u)|^2_{m,s,\tau}$ actually meaning

$$|e^{\tau\varphi}T^m(u)|^2_{m,s,\tau} = \sum_{j=0}^{m} |e^{\tau\varphi}T_j(u)|^2_{m-j+s,\tau}\,.$$

1.5. **Statement of the main result.** We can now state the local Carleman estimate that we prove in the neighborhood of a point of the boundary, with the sub-ellipticity and strong lopatinskii conditions.

Theorem 1.6. *Let $x_0 \in \partial\Omega$ and let $\varphi \in \mathscr{C}^\infty(\Omega)$ be such that the pair $\{P, \varphi\}$ has the sub-ellipticity property of Definition 1.1 in a neighborhood of x_0 in $\overline{\Omega}$. Moreover, assume that $\{P, \varphi, B^k,\ k = 1, \dots, \mu\}$ satisfies the strong Lopatinskii condition at x_0. Then there exist a neighborhood W of x_0 in $\mathbb{R}^n$ and two constants C and $\tau_* > 0$ such that*

(1.10)

$$\tau^{-1}\,\|e^{\tau\varphi}u\|^2_{m,\tau} + |e^{\tau\varphi}T(u)|^2_{m-1,1/2,\tau} \le C\big(\,\|e^{\tau\varphi}P(x,D)u\|^2_{L^2} + \sum_{k=1}^{\mu} |e^{\tau\varphi}B^k(x,D)u_{|\partial\Omega}|^2_{m-1/2-\beta_k,\tau}\big),$$

for all $u = w_{|\Omega}$ with $w \in \mathscr{C}^\infty_c(W)$ and $\tau \ge \tau_*$.

First, this results will be established microlocally: at a boundary point x_0 we shall assume that the strong Lopatinskii condition holds for some Y_0 and N_0 in the cotangent space at x_0 and $\tau_0 \ge 0$ (as introduced in Section 1.3) and we shall prove that a Carleman estimate of the form above holds in a conic neighborhood of (x_0, Y_0, N_0, τ_0) in phase-space; localization in phase-space will be done by means of cut-off functions and associated pseudo-differential operators. We refer the reader to Section 4.4. Second, we will deduce Theorem 1.6 from such microlocal estimates.

Estimates of the form of (1.10) are local. Yet, they can be patched together to form global estimates. We do not cover such details here. Patching of local estimates away from the boundary can be found in [17, Lemma 8.3.1]; for estimates near the boundary one can for instance consult [31].

In Section 6 we shall prove Carleman estimates with a weight function of the form $\varphi(x) = \exp(\gamma\psi(x))$ as is usually done in practice with the parameter γ chosen as large as desired. We shall provide the precise dependency of the Carleman estimate with respect to this second large parameter.

1.6. **Local reduction of the problem.** Let $x_0 \in \partial\Omega$. There exists a neighborhood V of x_0 and a local system of coordinates $x = (x_1, \dots, x_n)$ where $V \cap \Omega \subset \{x_n > 0\}$ and $x' = (x_1, \dots, x_{n-1})$ parametrizes the boundary $V \cap \partial\Omega \subset \{x_n = 0\}$. We denote by $\mathbb{R}^n_+$ the half space $\{x_n > 0\}$ and $V_+ = V \cap \mathbb{R}^n_+$. For our purpose here, without any loss of generality, we may assume that V_+ is bounded. We shall write $\partial\Omega \cap V$ to denote $\{x \in V;\ x_n = 0\}$ in the local system of coordinates.

In such local coordinates, in V_+, the differential operator[3] P of order m with complex coefficients takes the form

$$P = P(x, D) = \sum_{j=1}^{m} P_j(x, D')D_n^j, \qquad D_n = \frac{1}{i}\partial_n,$$

where $P_j(x, D')$, $j = 1, \dots, m$, are tangential differential operators with complex coefficients of order $m - j$. We have $P_m = P_m(x) \neq 0$. Upon dividing by $P_m(x)$ we may assume that $P_m(x) = 1$.

[3]By abuse of notation, in the new local coordinates, we keep the notation P and B^k, $k = 1, \dots, \mu$, for the operators introduced in Section 1.

Similarly the boundary operators take the form

$$B^k = B^k(x,D) = \sum_{j=0}^{\beta_k} B_j^k(x,D')D_n^j, \quad 1 \le k \le \mu,$$

where $B_j^k(x,D')$, $j = 0,\dots,\beta_k$ are tangential differential operators of order $(\beta_k - j)$.

Calling (ξ',ξ_n) the Fourier variables corresponding to (x',x_n) we have, for the principal symbol of P,

$$p(x,\xi) = \sum_{j=0}^{m} p_j(x,\xi')\xi_n^j,$$

which is a polynomial homogeneous of degree m in the n variables (ξ',ξ_n).

We introduce $p_\varphi(x,\xi,\tau) := p\left(x,\xi + i\tau\varphi'(x)\right)$. Setting $\varrho' = (x,\xi',\tau)$ and $\varrho = (\varrho',\xi_n)$, for simplicity we shall write $p_\varphi(\varrho)$ in place of $p_\varphi(x,\xi,\tau)$ and often $p_\varphi(\varrho',\xi_n)$ to emphasize that the symbol is polynomial in ξ_n.

For a fixed point $\varrho_0' = (x_0,\xi_0',\tau_0) \in$ with $x_0 \in \partial\Omega$ and $(\xi_0',\tau_0) \in \mathbb{S}_+^{n-1} = \{(\xi',\tau) \in \mathbb{R}^n,\ \tau \ge 0,\ |(\xi',\tau)| = 1\}$, the closed upper semi-sphere in $\mathbb{R}^n$, we denote the roots of $p_\varphi(\varrho_0',\xi_n)$ by $\alpha_1,\dots,\alpha_N$, with respective multiplicities $\mu_1,\dots,\mu_N$ satisfying $\mu_1 + \dots + \mu_N = m$. By Lemma A.2, there exists a conic open neighborhood $\mathscr{U}$ of ϱ_0' such that

$$p_\varphi(\varrho',\xi_n) = p_\varphi^+(\varrho',\xi_n)p_\varphi^-(\varrho',\xi_n)p_\varphi^0(\varrho',\xi_n), \quad \varrho' \in \mathscr{U},\ \xi_n \in \mathbb{R},$$

with $p_\varphi^\pm$ and p_φ^0, polynomials in ξ_n of constant degrees in $\mathscr{U}$, smooth and homogeneous; in $\mathscr{U}$ the imaginary parts of the roots of $p_\varphi^+(\varrho',\xi_n)$ (resp. $p_\varphi^-(\varrho',\xi_n)$) are all positive (resp. negative) and we have

$$p_\varphi^\pm(\varrho_0',\xi_n) = \prod_{\pm\operatorname{Im}\alpha_j > 0} (\xi_n - \alpha_j)^{\mu_j}, \quad p_\varphi^0(\varrho_0',\xi_n) = \prod_{\operatorname{Im}\alpha_j = 0} (\xi_n - \alpha_j)^{\mu_j}.$$

The polynomial p_φ is thus decomposed into three factors in a neighborhood of ϱ_0'. For $p_\varphi^\pm$ the sign of the imaginary part of their roots remain constant equal to $\pm$ respectively; for p_φ^0 this sign may change and the roots are precisely real at $\varrho' = \varrho_0'$.

We then define the polynomial $\kappa_\varphi(\varrho',\xi_n)$ by

$$\kappa_\varphi(\varrho',\xi_n) = p_\varphi^+(\varrho',\xi_n)p_\varphi^0(\varrho',\xi_n). \tag{1.11}$$

For $B = \left\{B^k\right\}_{k=1,\dots,\mu}$ the set of boundary operators and $b^k(x,\xi)$ their principal symbols, we set $b_\varphi^k(x,\xi,\tau) = b^k(x,\xi + i\tau\varphi')$. As above we write $b_\varphi^k(\varrho',\xi_n)$ where $\varrho' = (x,\xi',\tau)$ to emphasize that the symbol is polynomial in ξ_n. We have

$$b_\varphi^k(\varrho',\xi_n) = \sum_{j=0}^{\beta_k} b_{\varphi,j}^k(\varrho')\xi_n^j,$$

with $b_{\varphi,j}^k(\varrho')$ homogeneous of degree $\beta_k - j$ in (ξ',τ).

The strong Lopatinskii of condition Definition 1.4 is invariant under change of variables as seen at the end of Section 1.3. A conormal vector N is given by $(0,\dots,0,N_n)$ in the present coordinate system. For the statement of the strong Lopatinskii condition we can choose $N = (0,\dots,0,1)$ without any loss of generality since N is asked to point inside Ω. In the local coordinate system (x',x_n) in V, a boundary quadruple $\omega = (x,Y,N,\tau)$, with $Y = (\xi',0)$ can thus be identified with $\varrho' = (x,\xi',\tau)$. The strong Lopatinskii condition at $\varrho_0' = (x_0,\xi_0',\tau_0)$, with $\tau_0 \ge 0$ and $(\tau_0,\xi_0') \ne (0,0)$, thus reads as follows:

The set $\left\{b_\varphi^k(\varrho_0',\xi_n)\right\}_{k=1,\dots,\mu}$ is complete modulo $\kappa_\varphi(\varrho_0',\xi_n)$ as polynomials in ξ_n.

We shall now prove that this property remains true for ϱ' in a small conic neighborhood of ϱ'_0.

We set $m^- = d^\circ\big(p^-_\varphi(\varrho',.)\big)$ that is *independent* of $\varrho' \in \mathscr{U}$, with the open conic neighborhood $\mathscr{U}$ as introduced above, and we let $\kappa_\varphi(\varrho',\xi_n)$ be the polynomial function given in (1.11). It takes the form

$$\kappa_\varphi(\varrho',\xi_n) = \sum_{j=0}^{m-m^-} \kappa_{\varphi,j}(\varrho')\xi_n^j, \quad \varrho' \in \mathscr{U},\ \xi_n \in \mathbb{R},$$

where $\kappa_{\varphi,j}$ is homogeneous of degree $(m-m^- - j)$ w.r.t. (ξ',τ).

Remark 1.7. *If we denote by $M^-(\varrho')$ the number of roots (counted with their multiplicities) with negative imaginary parts of $p_\varphi(\varrho',\xi_n)$, polynomial in ξ_n with $\varrho' \in \mathscr{U}$, we have $m^- = M^-(\varrho'_0)$. We may however have $m^- \neq M^-(\varrho')$ for some $\varrho' \in \mathscr{U}$. Note that in such case we have $m^- \leq M^-(\varrho')$ from the construction of the neighborhood $\mathscr{U}$ given in Lemma A.2.*

We set $m' = m^- + \mu$ and for $k = 1,\dots,m'$, we shall introduce a family of polynomial functions of degree less than or equal to $m-1$ denoted by $e^k_\varphi(\varrho',\xi_n)$, all taking the form

$$e^k_\varphi(\varrho',\xi_n) = \sum_{j=0}^{m-1} e^k_{\varphi,j}(\varrho')\xi_n^j, \quad \varrho' \in \mathscr{U}, \tag{1.12}$$

with $e^k_{\varphi,j}$ homogeneous w.r.t. (ξ',τ). This family of polynomials is composed of two different sets:

(1) For $k = 1,\dots,\mu$, we set $e^k_\varphi = b^k_\varphi$, yielding

$$e^k_{\varphi,j} = \begin{cases} b^k_j & \text{if } j \leq \beta_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $e^k_{\varphi,j}(\varrho')$ is homogeneous of degree $\beta_k - j$ w.r.t. (ξ',τ).

(2) For $k = \mu+1,\dots,m'$, we set $e^k_\varphi(\varrho',\xi_n) = \kappa_\varphi(\varrho',\xi_n)\xi_n^{k-(\mu+1)}$, yielding

$$e^k_{\varphi,j} = \begin{cases} \kappa_{\varphi,j-k+\mu+1} & \text{if } k-\mu-1 \leq j \leq m-m'+k-1, \\ 0 & \text{otherwise.} \end{cases}$$

Setting $\beta_k = m - m^- + k - (\mu+1)$ we have that $e^k_{\varphi,j}$ is homogeneous of degree $\beta_k - j$ w.r.t. (ξ',τ).

The strong Lopatinskii condition of Definition 1.4 (using Remark 1.5) means precisely that the family $(e^k_\varphi(\varrho',\xi_n))_{1\leq k\leq m'}$ generates the space of polynomials of degree less than or equal to $m-1$ in ξ_n for $\varrho' = \varrho'_0$, implying $m' \geq m$ and that the matrix

$$\mathcal{M}(\varrho'_0) = (e^k_{\varphi,j}(\varrho'_0))_{\substack{0\leq j\leq m-1\\ 1\leq k\leq m'}}$$

is of rank m. Then there exists a $m\times m$ sub-matrix $\mathcal{M}_0(\varrho'_0)$ such that $\det \mathcal{M}_0(\varrho'_0) \neq 0$. As the coefficients of $\mathcal{M}(\varrho')$ are continuous and homogeneous of degree $\beta_k - j$ we then have $\det \mathcal{M}_0(\varrho') \neq 0$ for ϱ' in a small conic neighborhood $\mathscr{V} \subset \mathscr{U}$ of ϱ'_0. Note that the homogeneity of the coefficients is important for $\mathscr{V}$ to be chosen conic since $\det \mathcal{M}_0(\varrho')$ is itself homogeneous w.r.t. (ξ',τ). The rank of $\mathcal{M}(\varrho')$ thus remains equal to m in $\mathscr{V}$, meaning that the strong Lopatinskii condition is valid in the whole $\mathscr{V}$.

We have thus reached the following result.

Proposition 1.8. *Let the strong Lopatinskii condition hold at $\varrho'_0 = (x_0,\xi'_0,\tau_0)$. Then we have $m' = m^- + \mu \geq m$. Moreover there exists a conic neighborhood $\mathscr{V}$ of ϱ'_0 such that the strong Lopatinskii condition remains true at every point ϱ' of $\mathscr{V}$.*

This result can be commented in view of the proof of the Carleman estimate we give below. In fact, with the factorization $p_\varphi = p_\varphi^- \kappa_\varphi$, the following states roughly the proof strategy we shall adopt:

(1) The factor p_φ^- associated with roots with negative imaginary part yields a perfect elliptic estimate at the boundary.
(2) The factor κ_φ yields an estimate at the boundary that involves trace terms. These terms will be estimated via the actions of the boundary operators B_φ^k by means of to the strong Lopatinskii condition.

The inequality $\mu \geq m - m^-$ thus indicates that we shall have at hand a sufficiently large number of boundary operators to control the terms originating from the estimate with the factor κ_φ that is of degree $m - m^-$.

As here $\mu = m/2$ note also that we have $m^- \geq m/2$.

1.7. **Some examples.** Here give simple examples of operators for which the present analysis applies.

A natural example is P second-order elliptic with real coefficients. We can find local coordinates at the boundary such that $V_+ = \{x_n > 0\}$ and $P = D_{x_n} + r(x, D')$ where $r(x, D') = r(x', x_n, D')$ is a x_n-family of elliptic operators with $r(x, \xi') \geq C|\xi'|^2$. For any smooth ψ the pair $\{P, \varphi\}$, with $\varphi = \exp(\gamma\psi)$, satisfies the sub-ellipticity condition of Definition 1.1 if $\psi' \neq 0$ and γ is chosen sufficiently large (see e.g. [31]). First for simplicity we consider $\varphi = \varphi(x_n)$. If $\partial_{x_n}\varphi > 0$, the strong Lopatinskii condition is for example satisfied in the following cases:

(1) $Bu = u$, Dirichlet condition;
(2) $Bu = D_{x_n}u + a(x)u$, Robin conditions.
(3) $Bu = D_{x_n}u + iaD_{x_1}u$ with $a^2 < r$.

These results remain true if we consider $\varphi = \varphi(x', x_n)$ with $|\partial_{x'}\varphi| \ll |\partial_{x_n}\varphi|$ allowing for small variations of φ in the tangential direction. With Theorem 1.6 we thus recover known results for second-order operators [34, 35].

For a simple example of higher-order operators we consider $P = D_{x_1}^4 + D_{x_2}^4$ in $V_+ = \{x_2 > 0\}$. Here also for any smooth ψ the pair $\{P, \varphi\}$, with $\varphi = \exp(\gamma\psi)$, satisfies the sub-ellipticity condition of Definition 1.1 if $\psi' \neq 0$ and γ is chosen sufficiently large (see e.g. [30]). Here also, considering $\varphi = \varphi(x_2)$, if $\partial_{x_2}\varphi > 0$, the strong Lopatinskii condition is for example satisfied in the following cases:

(1) $B^1u = u$, $B^2u = D_{x_2}u$;
(2) $B^1u = u$, $B^2u = \Delta u$;
(3) $B^1u = u$, $B^2u = D_{x_2}\Delta u$.

This list of examples for $P = D_{x_1}^4 + D_{x_2}^4$ is by far not exhaustive. Here also, including small variations of φ in the tangential direction preserves these properties.

Details on these examples are given in Appendix A.1.

1.8. **Notation.** If $V \subset \overline{\mathbb{R}}^n_+$ we denote the semi-classical unit half cosphere bundle over V by

$$\mathbb{S}^*_{\mathsf{T},\tau}(V) = \{(x, \xi', \tau);\ x \in V,\ \xi' \in \mathbb{R}^{n-1},\ \tau \in \mathbb{R}_+,\ |\xi'|^2 + \tau^2 = 1\}.$$

The canonical inner product in $\mathbb{C}^m$ is denoted by $(\mathbf{z}, \mathbf{z}')_{\mathbb{C}^m} = \sum_{j=0}^{m-1} z_j \overline{z'}_j$, for $\mathbf{z} = (z_0, \ldots, z_{m-1}), \mathbf{z}' = (z'_0, \ldots, z'_{m-1}) \in \mathbb{C}^m$. The associated norm will be denoted $|\mathbf{z}|^2_{\mathbb{C}^m} = \sum_{j=0}^{m-1} |z_j|^2$.

We shall use some spaces of smooth functions in the closed half space. We set

$$\mathscr{S}(\overline{\mathbb{R}}^n_+) = \{u_{|\overline{\mathbb{R}}^n_+};\ u \in \mathscr{S}(\mathbb{R}^n)\}.$$

For two $u, v \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$ we set

$$(u, v)_+ = (u, v)_{L^2(\mathbb{R}^n_+)} \qquad \left(u_{|x_n=0^+}, v_{|x_n=0^+}\right)_\partial = \left(u_{|x_n=0^+}, v_{|x_n=0^+}\right)_{L^2(\mathbb{R}^{n-1})}.$$

In this article, when the constant C is used, it refers to a constant that is independent of the large parameter τ. Its value may however change from one line to another. If we want to keep track of the value of a constant we shall use another letter.

In what follows, for concision, we shall sometimes use the notation $\lesssim$ for $\leq C$, with a constant $C > 0$. We shall write $a \asymp b$ to denote $a \lesssim b \lesssim a$.

1.9. **Outline.** We start by a review of pseudo-differential calculus with a large parameter in Section 2, including regularity results on appropriate Sobolev spaces. Section 3 is an exposition of results concerning interior and boundary differential quadratic forms. In particular we write a (microlocal) Gårding inequality at the boundary for operators that are differential in the direction normal to the boundary and homogeneous. We also write a generalized Green formula.

Section 4 is devoted to the proof of the Carleman estimate of Theorem 1.6. First a microlocal Carleman estimate is proven (Theorem 4.4). The proof exploits the factorization $p_\varphi = p_\varphi^+ p_\varphi^- p_\varphi^0$. To ease the reading of the proof we have separated the action of each factor and corresponding condition to form partial estimates. The factor p_φ^- yields a perfect elliptic estimate (Section 4.1). The factor $\kappa_\varphi = p_\varphi^+ p_\varphi^0$ yields an estimate controlling the traces of the unknown function at the boundary with the operators B^k, through the strong Lopatinskii condition (see Section 4.2). The sub-ellipticity condition is exploited in Section 4.3 and, based on the generalized Green formula of Proposition 3.15, the derivation leads to a control of the norm of the unknown function in Ω yet with remainder terms involving the traces of the function at $\partial\Omega$. Collecting the different arguments we obtain the microlocal Carleman estimate in Section 4.4. Then in Section 4.5 we show how the patching of such estimates yields the result of Theorem 1.6.

In Section 5 we present the pseudo-differential calculus with two large parameters and how the analysis of differential quadratic forms can be revisited.

In Section 6, with the weight function $\varphi = \exp(\gamma\psi)$, to prove the Carleman estimate with two large parameters, τ and γ, by means of the Gårding inequality at the boundary we need some positivity results on the symbol of some homogeneous differential operator. This follows from the strong pseudo-convexity condition on ψ and P. In Section 6.5 the approach of Section 4 is then adapted to prove a microlocal Carleman estimate with two large parameters. This estimate is finally improved if the strong pseudo-convexity condition is replaced by the simple characteristic property.

Section 7 is devoted to the application of the Carleman estimates of the previous sections to obtain unique continuation properties near a boundary across a hypersurface. Strong pseudo-convexity is assumed for the hypersurface and the strong Lopatinskii condition is assumed at the boundary. Similar results are obtained in the case of the product of two operators. For one of them the above assumptions are made, for the second one the simple characteristic property is further assumed.

In Appendix A we have collected some intermediate technical results.

2. Pseudo-differential operators with a large parameter

Parameter-dependent pseudo-differential operators have proven to be important tools for the derivation of Carleman estimates. The general aim is to obtain a pseudo-differential calculus with a large parameter, and then to derive estimates with constants that are independent of the parameter. Often such a pseudo-differential calculus is referred to as a semi-classical calculus.

2.1. **Classes of symbols.** We first introduce symbols that depend on a parameter.

Definition 2.1. *Let $a(\varrho) \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $\varrho = (x, \xi, \tau)$, with τ as a parameter in $[\tau_{\min}, +\infty)$, $\tau_{\min} > 0$, and $m \in \mathbb{R}$, be such that for all multi-indices $\alpha, \beta \in \mathbb{N}^n$ we have*

$$\left|\partial_x^\alpha \partial_\xi^\beta a(\varrho)\right| \leq C_{\alpha,\beta} \lambda^{m-|\beta|}, \quad x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n,\ \tau \in [\tau_{\min}, +\infty), \tag{2.1}$$

where $\lambda = |(\xi,\tau)| = \big(|\xi|^2+\tau^2\big)^{\frac12}$. *Thus differentiation with respect to* ξ *improves the decay in* ξ *and* τ *simultaneously. We write* $a \in S^m_\tau(\mathbb{R}^n\times\mathbb{R}^n)$ *or simply* S^m_τ. *For* $a\in S^m_\tau$ *we denote by* $\sigma(a)$ *its principal part, that is, its equivalence class in* S^m_τ/S^{m-1}_τ.

We also introduce tangential symbols. Let $a(\varrho')\in\mathscr{C}^\infty(\overline{\mathbb{R}}^n_+\times\mathbb{R}^{n-1})$, $\varrho'=(x,\xi',\tau)$, *with* τ *as a parameter in* $[\tau_{\min},+\infty)$, $\tau_{\min}>0$, *and* $m\in\mathbb{R}$, *be such that for all multi-indices* $\alpha\in\mathbb{N}^n$, $\beta\in\mathbb{N}^{n-1}$ *we have*

$$\left|\partial_x^\alpha\partial_{\xi'}^\beta a(\varrho')\right| \le C_{\alpha,\beta}\lambda_{\mathsf{T}}^{m-|\beta|},\quad x\in\overline{\mathbb{R}}^n_+,\ \xi'\in\mathbb{R}^{n-1},\ \tau\in[\tau_{\min},+\infty),$$

where $\lambda_{\mathsf{T}}=|(\xi',\tau)|=\big(|\xi'|^2+\tau^2\big)^{\frac12}$. *We write* $a\in S^m_{\mathsf{T},\tau}(\overline{\mathbb{R}}^n_+\times\mathbb{R}^{n-1})$ *or simply* $S^m_{\mathsf{T},\tau}$. *For* $a\in S^m_{\mathsf{T},\tau}$ *we denote by* $\sigma(a)$ *its principal part, that is, its equivalence class in* $S^m_{\mathsf{T},\tau}/S^{m-1}_{\mathsf{T},\tau}$.

We also introduce symbol classes that behave polynomially in the ξ_n *variable. Let* $a(\varrho)\in\mathscr{C}^\infty(\overline{\mathbb{R}}^n_+\times\mathbb{R}^n)$, *with* τ *as a parameter in* $[\tau_{\min},+\infty)$, $\tau_{\min}>0$, *and* $m\in\mathbb{N}$ *and* $r\in\mathbb{R}$, *be such that*

$$a(\varrho)=\sum_{j=0}^m a_j(\varrho')\xi_n^j,\quad a_j\in S^{m-j+r}_{\mathsf{T},\tau},\ \ \varrho=(\varrho',\xi_n),\ \varrho'=(x,\xi',\tau),$$

with $x\in\overline{\mathbb{R}}^n_+$, $\xi\in\mathbb{R}^n$, $\tau\ge\tau_{\min}$, *and* $\xi_n\in\mathbb{R}$. *We write* $a(\varrho)\in S^{m,r}_\tau(\overline{\mathbb{R}}^n_+\times\mathbb{R}^n)$ *or simply* $S^{m,r}_\tau$.

Note that we have $S^{m,r}_\tau\subset S^{m+m',r-m'}_\tau$, if $m,m'\in\mathbb{N}$ and $r\in\mathbb{R}$. We shall call the principal symbol of a the symbol

$$\sigma(a)(\varrho)=\sum_{j=0}^m\sigma(a_j)(\varrho')\xi_n^j,$$

which is a representative of the class of a in $S^{m,r}_\tau/S^{m,r-1}_\tau$.

Note that $S^{m,r}_\tau\not\subset S^{m+r}_\tau$. For example consider $a(x,\xi,\tau)=|(\xi',\tau)|\xi_n$ for $|(\xi',\tau)|\ge1$. We have $a\in S^{2,0}_\tau\cap S^{1,1}_\tau$ and yet $a\notin S^2_\tau$. In fact observe that differentiating with respect to ξ' yields

$$|\partial_{\xi'}^\alpha a(x,\xi,\tau)\le C_\alpha|(\xi',\tau)|^{1-|\alpha|}|\xi_n|.$$

An estimate of the form of (2.1) is however not achieved for $|\alpha|\ge2$. A microlocalization is required to repair this flaw and to use the two different symbol classes in a pseudo-differential calculus (See [19, Theorem 18.1.35]).

Finally, we define the corresponding spaces of poly-homogeneous symbols. Such symbols are often referred to as classical symbols; they are characterized by an asymptotic expansion where each term is positively homogeneous with respect to (ξ,τ) (resp. (ξ',τ)):

Definition 2.2. *We shall say* $a\in S^m_{\tau,\mathrm{cl}}(\mathbb{R}^n\times\mathbb{R}^n)$ *or simply* $S^m_{\tau,\mathrm{cl}}$ *(resp.* $S^m_{\mathsf{T},\tau,\mathrm{cl}}(\overline{\mathbb{R}}^n_+\times\mathbb{R}^{n-1})$ *or simply* $S^m_{\mathsf{T},\tau,\mathrm{cl}}$*) if there exists* $a_j\in S^{m-j}_\tau$ *(resp.* $S^{m-j}_{\mathsf{T},\tau}$*), homogeneous of degree* $m-j$ *in* (ξ,τ) *for* $|(\xi,\tau)|\ge r_0$, *(resp.* (ξ',τ) *for* $|(\xi',\tau)|\ge r_0$*), with* $r_0\ge0$, *such that*

$$a\sim\sum_{j\ge0}a^{(j)},\quad \textit{in the sense that}\quad a-\sum_{j=0}^N a^{(j)}\in S^{m-N-1}_\tau\ (\textit{resp. } S^{m-N-1}_{\mathsf{T},\tau}). \tag{2.2}$$

A representative of the principal part is then given by the first term in the expansion.

Finally for $m\in\mathbb{N}$ *and* $r\in\mathbb{R}$, *we shall say that* $a(\varrho)\in S^{m,r}_{\tau,\mathrm{cl}}(\overline{\mathbb{R}}^n_+\times\mathbb{R}^n)$ *or simply* $S^{m,r}_{\tau,\mathrm{cl}}$, *if*

$$a(\varrho)=\sum_{j=0}^m a_j(\varrho')\xi_n^j,\quad \textit{with } a_j\in S^{m-j+r}_{\mathsf{T},\tau,\mathrm{cl}},\ \ \varrho=(\varrho',\xi_n).$$

The principal part is given by $\sum_{j=0}^m\sigma(a_j)(\varrho')\xi_n^j$ *and is homogeneous of degree* m *in* (ξ,τ).

2.2. **Classes of semi-classical pseudo-differential operators.** For $a \in S^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$ (resp. $S^m_{\tau,\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}^n)$) we define the following pseudo-differential operator in $\mathbb{R}^n$:

$$a(x,D,\tau)u(x) = \mathrm{Op}(a)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x,\xi)} a(x,\xi,\tau)\hat{u}(\xi)\, d\xi, \qquad u \in \mathscr{S}(\mathbb{R}^n), \tag{2.3}$$

where $\hat{u}$ is the Fourier transform of u. In the sense of oscillatory integrals we have

$$a(x,D,\tau)u(x) = \mathrm{Op}(a)u(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y,\xi)} a(x,\xi,\tau)u(y)\, d\xi\, dy.$$

We write $\mathrm{Op}(a) \in \Psi^m_\tau(\mathbb{R}^n)$ or simply $\Psi^m_{\tau,\mathrm{cl}}$ (resp. $\Psi^m_\tau(\mathbb{R}^n)$ or simply $\Psi^m_{\tau,\mathrm{cl}}$). Here D denotes D_x. The principal symbol of $\mathrm{Op}(a)$ is $\sigma(\mathrm{Op}(a)) = \sigma(a) \in S^m_\tau / S^{m-1}_\tau$ (resp. $S^m_{\tau,\mathrm{cl}}/S^{m-1}_{\tau,\mathrm{cl}}$).

Tangential operators are defined similarly. For $a \in S^m_{\mathsf{T},\tau}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$ (resp. $S^m_{\mathsf{T},\tau,\mathrm{cl}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$) we set

$$a(x,D',\tau)u(x) = \mathrm{Op}(a)u(x) = (2\pi)^{-(n-1)} \iint_{\mathbb{R}^{2n-2}} e^{i(x'-y',\xi')} a(x,\xi',\tau)u(y',x_n)\, d\xi'\, dy', \tag{2.4}$$

for $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$, where $x \in \overline{\mathbb{R}}^n_+$. Here D' denotes $D_{x'}$. We write $A = \mathrm{Op}(a) \in \Psi^m_{\mathsf{T},\tau}(\overline{\mathbb{R}}^n_+)$ or simply $\Psi^m_{\mathsf{T},\tau}$ (resp. $\Psi^m_{\mathsf{T},\tau,\mathrm{cl}}(\overline{\mathbb{R}}^n_+)$ or simply $\Psi^m_{\mathsf{T},\tau,\mathrm{cl}}$). The principal symbol of $A = \mathrm{Op}(a)$ is $\sigma(A) = \sigma(a) \in S^m_{\mathsf{T},\tau}/S^{m-1}_{\mathsf{T},\tau}$ (resp. $S^m_{\mathsf{T},\tau,\mathrm{cl}}/S^{m-1}_{\mathsf{T},\tau,\mathrm{cl}}$).

Finally for $m \in \mathbb{N}$, $r \in \mathbb{R}$, and $a \in S^{m,r}_\tau$ (resp. $S^{m,r}_{\tau,\mathrm{cl}}$) with

$$a(\varrho) = \sum_{j=0}^m a_j(\varrho')\xi_n^j, \quad a_j \in S^{m-j+r}_{\mathsf{T},\tau} (\text{resp. } S^{m-j+r}_{\mathsf{T},\tau,\mathrm{cl}}), \quad \varrho = (\varrho', \xi_n),$$

we set

$$a(x,D,\tau) = \mathrm{Op}(a) = \sum_{j=0}^m a_j(x,D',\tau)D_n^j,$$

and we write $A = \mathrm{Op}(a) \in \Psi^{m,r}_\tau(\mathbb{R}^n_+)$ or simply $\Psi^{m,r}_\tau$ (resp. $\Psi^{m,r}_{\tau,\mathrm{cl}}(\mathbb{R}^n_+)$ or simply $\Psi^{m,r}_{\tau,\mathrm{cl}}$). The principal symbol of A is $\sigma(A)(\varrho) = \sigma(a)(\varrho) = \sum_{j=0}^m \sigma(a_j)(\varrho')\xi_n^j$ in $S^{m,r}_\tau / S^{m,r-1}_\tau$ (resp. $S^{m,r}_{\tau,\mathrm{cl}}/S^{m,r-1}_{\tau,\mathrm{cl}}$).

We provide some basic calculus rules in the case of tangential operators.

Proposition 2.3 (composition)**.** *Let $a \in S^m_{\mathsf{T},\tau}$ (resp. $S^m_{\mathsf{T},\tau,\mathrm{cl}}$)and $b \in S^{m'}_{\mathsf{T},\tau}$(resp. $S^{m'}_{\mathsf{T},\tau,\mathrm{cl}}$) be two tangential symbols. Then $\mathrm{Op}(a)\,\mathrm{Op}(b) = \mathrm{Op}(c) \in \Psi^{m+m'}_{\mathsf{T},\tau}$ (resp. $\Psi^{m+m'}_{\mathsf{T},\tau,\mathrm{cl}}$) with $c \in S^{m+m'}_{\mathsf{T},\tau}$ (resp. $S^{m+m'}_{\mathsf{T},\tau,\mathrm{cl}}$) defined by the (oscillatory) integral:*

$$\begin{aligned} c(\varrho') = (a \# b)(\varrho') &= (2\pi)^{-(n-1)} \iint e^{-i(y',\eta')} a(x,\xi'+\eta',\tau)\, b(x'+y',x_n,\xi',\tau)\, dy'\, d\eta' \\ &= \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_{\xi'} a(\varrho')\, \partial^\alpha_{x'} b(\varrho') + r_N, \end{aligned}$$

where $r_N \in S^{m+m'-N}_{\mathsf{T},\tau}$ (resp. $S^{m+m'-N}_{\mathsf{T},\tau,\mathrm{cl}}$) is given by

$$r_N = \frac{(-i)^N}{(2\pi)^{(n-1)}} \sum_{|\alpha|=N} \int_0^1 \frac{N(1-s)^{N-1}}{\alpha!} \iint e^{-i(y',\eta')} \partial^\alpha_{\xi'} a(x,\xi'+\eta',\tau) \partial^\alpha_{x'} b(x'+sy',x_n,\xi',\tau)\, dy' d\eta' ds.$$

Proposition 2.4 (formal adjoint)**.** *Let $a \in S^m_{\mathsf{T},\tau}$ (resp. $S^m_{\mathsf{T},\tau,\mathrm{cl}}$). There exists $a^* \in S^m_{\mathsf{T},\tau}$ (resp. $S^m_{\mathsf{T},\tau,\mathrm{cl}}$) such that*

$$(\mathrm{Op}(a)u, v)_+ = (u, \mathrm{Op}(a^*)v)_+, \qquad u, v \in \mathscr{S}(\overline{\mathbb{R}}^n_+).$$

and a^* is given be the following asymptotic expansion

$$a^*(\varrho') = (2\pi)^{-(n-1)} \iint e^{-i(y',\eta')}\overline{a}(x'+y', x_n, \xi'+\eta', \tau)\, dy'\, d\eta'$$
$$= \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!}\partial_{\xi'}^{\alpha}\partial_{x'}^{\alpha}\overline{a}(\varrho') + r_N, \quad r_N \in S^{m-N}_{\mathsf{T},\tau} (\textit{resp. } S^{m-N}_{\mathsf{T},\tau,\mathrm{cl}}),$$

where

$$r_N = \frac{(-i)^N}{(2\pi)^{(n-1)}} \sum_{|\alpha|=N} \int_0^1 \frac{N(1-s)^{N-1}}{\alpha!} \iint e^{-i(y',\eta')}\partial_{\xi'}^{\alpha}\partial_{x'}^{\alpha}\overline{a}(x'+sy', x_n, \xi'+\eta', \tau)\, dy' d\eta' ds.$$

We denote $\mathrm{Op}(a)^* = \mathrm{Op}(a^*)$. *We refer to* $\mathrm{Op}(a)^*$ *as to the formal adjoint of* $\mathrm{Op}(a)$.

A consequence of the previous calculus results is the following proposition.

Proposition 2.5. *Let $a(\varrho') \in S^m_{\mathsf{T},\tau,\mathrm{cl}}$ (resp. $S^m_{\mathsf{T},\tau,\mathrm{cl}}$) and $b(\varrho') \in S^{m'}_{\mathsf{T},\tau}$ (resp. $S^{m'}_{\mathsf{T},\tau,\mathrm{cl}}$), with $m, m' \in \mathbb{R}$. Define $h(\varrho') = D_{x'}(b\partial_{\xi'}\overline{a})(\varrho') \in S^{m+m'-1}_{\tau}$. Then we have*

$$\mathrm{Op}(a)^*\,\mathrm{Op}(b) - \mathrm{Op}(\overline{a}b + h) \in \Psi^{m+m'-2}_{\mathsf{T},\tau} \ (\textit{resp. } \Psi^{m+m'-2}_{\mathsf{T},\tau,\mathrm{cl}}),$$

or equivalently $a^*\#b - \overline{a}b - h \in S^{m+m'-2}_{\mathsf{T},\tau}$ (resp. $S^{m+m'-2}_{\mathsf{T},\tau,\mathrm{cl}}$).

For semi-classical operators in the half space with symbols that are polynomial in ξ_n we also provide a notion of formal adjoint.

Definition 2.6. *Let $b \in S^{m,r}_{\tau}$ (resp. $S^{m,r}_{\tau,\mathrm{cl}}$), with*

$$b(x, D, \tau) = \sum_{j=0}^{m} b_j(x, D', \tau)D_n^j, \qquad b_j \in S^{m+r-j}_{\mathsf{T},\tau} \ (\textit{resp. } S^{m+r-j}_{\mathsf{T},\tau,\mathrm{cl}}).$$

We set

$$b(x, D, \tau)^* = \sum_{j=0}^{m} D_n^j b_j(x, D', \tau)^*.$$

In other words, in this definition we ignore the possible occurrence of boundary terms when performing the operator transposition.

Note that for $a \in S^m_{\mathsf{T},\tau,\mathrm{cl}}$ we have $[D_n, \mathrm{Op}(a)] = \mathrm{Op}(D_n a) \in \Psi^m_{\mathsf{T},\tau,\mathrm{cl}}$ and more generally, for $j \geq 1$, we have

$$\big[D_n^j, \mathrm{Op}(a)\big] = \sum_{k=0}^{j-1} \mathrm{Op}(\alpha_k)D_n^k, \qquad \alpha_k \in S^m_{\mathsf{T},\tau,\mathrm{cl}},$$

where the symbols α_k involve various derivatives of a in the x_n-direction. As an application we see that if we consider $a_j \in S^{m-j+r}_{\mathsf{T},\tau,\mathrm{cl}}$ then we have

$$\sum_{j=0}^{m} D_n^j a_j(x, D', \tau) = \sum_{j=0}^{m} \tilde{a}_j(x, D', \tau)D_n^j,$$

where $\tilde{a}_j \in S^{m-j+r}_{\mathsf{T},\tau,\mathrm{cl}}$ and its principal part satisfies $\sigma(\tilde{a}_j) \equiv a_j$ in $S^{m-j+r}_{\mathsf{T},\tau}/S^{m-j+r-1}_{\mathsf{T},\tau}$. Hence

$$\sigma\Big(\sum_{j=0}^{m} D_n^j a_j(x, D', \tau)\Big) = \sum_{j=0}^{m} a_j(x, \xi', \tau)\xi_n^j \mod S^{m,r-1}_{\tau}.$$

From the calculus rules given above for the tangential operators and the above observation we have the following results on the principal symbols.

Proposition 2.7. *Let $a \in S^{m,r}_{\tau}$ (resp. $S^{m,r}_{\tau,\mathrm{cl}}$) and $b \in S^{m',r'}_{\tau}$ (resp. $S^{m',r'}_{\tau,\mathrm{cl}}$) with*

$$a(\varrho) = \sum_{j=0}^{m} a_j(\varrho')\xi_n^j, \quad b(\varrho) = \sum_{j=0}^{m'} b_j(\varrho')\xi_n^j, \quad \varrho = (\varrho', \xi_n),\ \varrho' = (x, \xi', \tau).$$

(1) *We have $a(x, D, \tau)^* \in \Psi^{m,r}_{\tau}$ (resp. $\Psi^{m,r}_{\tau,\mathrm{cl}}$) and*

$$\sigma\big(a(x, D, \tau)^*\big) \equiv \sum_{j=0}^{m} \overline{a}_j(\varrho')\xi_n^j \in S^{m,r}_{\tau}/S^{m,r-1}_{\tau} \text{ (resp. } S^{m,r}_{\tau,\mathrm{cl}}/S^{m,r-1}_{\tau,\mathrm{cl}}).$$

Moreover we have $\mathrm{Op}(a)^* - \mathrm{Op}(\overline{a}) \in \Psi^{m,r-1}_{\tau}$ (resp. $\Psi^{m,r-1}_{\tau,\mathrm{cl}}$).

(2) *$a(x, D, \tau)b(x, D, \tau) \in \Psi^{m+m',r+r'}_{\tau}$ (resp. $\Psi^{m+m',r+r'}_{\tau,\mathrm{cl}}$) and*

$$\sigma\big(a(x, D, \tau)b(x, D, \tau)\big) \equiv \sum_{\substack{0\le j\le m\\ 0\le k\le m'}} a_j(\varrho')b_k(\varrho')\xi_n^{j+k} \in S^{m+m',r+r'}_{\tau}/S^{m+m',r+r'-1}_{\tau}$$

$$\text{(resp. } S^{m+m',r+r'}_{\tau,\mathrm{cl}}/S^{m+m',r+r'-1}_{\tau,\mathrm{cl}}).$$

We have $\mathrm{Op}(a)\,\mathrm{Op}(b)u - \mathrm{Op}(ab)u \in \Psi^{m+m',r+r'-1}_{\tau}$ (resp. $\Psi^{m+m',r+r'-1}_{\tau,\mathrm{cl}}$).

2.3. **Sobolev continuity results.** Here we state continuity results for the operators defined above using the Sobolev norms with parameters introduced in Section 1.4. Such results can be obtained from their standard counterparts.

Let $\lambda_{\mathsf{T}}(\xi', \tau) = \left(\tau^2 + |\xi'|^2\right)^{1/2}$ and $\Lambda_{\mathsf{T}} := \mathrm{Op}(\lambda_{\mathsf{T}})$. For a given real number s, the boundary norm given by (1.9) is equivalent to the following norms:

$$|\mathbf{u}|^2_{m,s,\tau} = \sum_{k=0}^{m} |\Lambda_{\mathsf{T}}^s u_k|^2_{m-k,\tau}, \quad \mathbf{u} = (u_0, \dots, u_m). \tag{2.5}$$

Moreover, we define the following semi-classical interior norm

$$\|u\|^2_{m,s,\tau} = \|\Lambda_{\mathsf{T}}^s u\|^2_{m,\tau}, \tag{2.6}$$

Proposition 2.8. *If $a(\varrho) \in S^{m,r}_{\tau}$, with $m \in \mathbb{N}$ and $r \in \mathbb{R}$, then for $m' \in \mathbb{N}$ and $r' \in \mathbb{R}$ there exists $C > 0$ such that*

$$\|\mathrm{Op}(a)u\|_{m',r',\tau} \le C\,\|u\|_{m+m',r+r',\tau}, \qquad u \in \mathscr{S}(\overline{\mathbb{R}}^n_+).$$

A consequence of this results and Proposition 2.7 is the following property.

Corollary 2.9. *Let $a \in S^{m,r}_{\tau}$ and $m' \in \mathbb{N}$ and $s \in \mathbb{R}$. We have*

$$\|a(x, D, \tau)^* u - \overline{a}(x, D, \tau)u\|_{m',s,\tau} \le C\,\|u\|_{m+m',r+s-1,\tau}, \qquad u \in \mathscr{S}(\overline{\mathbb{R}}^n_+).$$

The following simple inequality will be used implicitly at many places in what follows when we invoke the parameter τ to be chosen sufficiently large. This will then allow us to absorb semi-classical norms of lower order.

Corollary 2.10. *Let $m \in \mathbb{N}$ and $s \in \mathbb{R}$ and $\ell \ge 0$. For some $C > 0$, we have*

$$\|u\|_{m,s,\tau} \le C\tau^{-\ell}\,\|u\|_{m,s+\ell,\tau}, \qquad u \in \mathscr{S}(\overline{\mathbb{R}}^n_+).$$

3. Interior and boundary quadratic forms

In this section, we present quadratic forms involving pseudo-differential operators that are differential in the normal direction and some of their properties.

3.1. **Interior quadratic forms.**

Definition 3.1. *Let $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$. We say that*

$$Q(u) = \sum_{s=1}^{N} (A^s u, B^s u)_+ , \qquad A^s = a^s(x, D, \tau),\ B^s = b^s(x, D, \tau), \tag{3.1}$$

is a quadratic form of type (m, σ) with $\mathscr{C}^\infty$ coefficients, if for each $s = 1, \dots N$, we have $a^s(\varrho) \in S^{m,\sigma'}_{\tau,\mathrm{cl}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$, $b^s(\varrho) \in S^{m,\sigma''}_{\tau,\mathrm{cl}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$, with $\sigma' + \sigma'' = 2\sigma$, $\varrho = (x, \xi, \tau)$.

The symbol of the quadratic form Q is defined by

$$q(\varrho) = \sum_{s=1}^{N} a^s(\varrho)\overline{b^s}(\varrho) \in S^{2m,2\sigma}_{\tau,\mathrm{cl}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n). \tag{3.2}$$

Remark 3.2. *Note that σ' and σ'' can vary with $s \in \{1, \dots, N\}$. Their sum yet remains constant equal to 2σ. In what follows we shall not write this dependency explicitly for concision.*

Clearly, this definition raises an ambiguity as one symbol can be associated with several quadratic forms. As an example, in one dimension, for $N = 1$ we can choose $A = D_n^2 \in \Psi^{2,0}_\tau$ and $B = \Lambda^2_{\mathsf{T}} \in \Psi^{0,2}_\tau \subset \Psi^{2,0}_\tau$ yielding to $|\xi'|^2\xi_n^2$ for the symbol. The choice $A = B = \Lambda_{\mathsf{T}} D_n \in \Psi^{1,1}_\tau \subset \Psi^{2,0}_\tau$ leads to the same symbol.

In fact if $u \in \mathscr{C}^\infty_c(\mathbb{R}^n_+)$ then

$$Q(u) = \sum_{s=1}^{N} ((B^s)^* \circ A^s u, u)_+ .$$

The symbol of Q thus coincides with the principal symbol of $\sum_{s=1}^N (B^s)^* \circ A^s$. Note that considering test functions with non-vanishing traces at the boundary $x_n = 0^+$ will naturally generate boundary terms when performing such operator transpositions. Such questions will be dealt with bellow.

For $s = 1, \dots, N$, as we have

$$a^s(\varrho) = \sum_{j=0}^{m} a^s_j(\varrho')\xi_n^j, \quad b^s(\varrho) = \sum_{j=0}^{m} b^s_j(\varrho')\xi_n^j, \quad \varrho = (\varrho', \xi_n),\ \varrho' = (x, \xi', \tau),$$

with $a^s_j \in S^{m-j+\sigma'}_{\tau,\mathrm{cl}}$ and $b^s_j \in S^{m-j+\sigma''}_{\tau,\mathrm{cl}}$, we write

$$A^s = \sum_{j=0}^{m} A^s_j D_n^j, \quad B^s = \sum_{j=0}^{m} B^s_j D_n^j, \qquad A^s_j = a^s_j(x, D', \tau),\ B^s_j = b^s_j(x, D', \tau).$$

Then, for $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$, the quadratic form given by (3.1) can be written as

$$Q(u) = \sum_{j=0}^{m}\sum_{k=0}^{m} \left(C_{j,k} D_n^j u, D_n^k u\right)_+,$$

where $C_{j,k}$ are tangential operators given by

$$C_{j,k} = \sum_{s=1}^{N} (B^s_k)^* A^s_j,$$

with symbols

$$c_{jk}(\varrho') = \sum_{s=1}^{N} (b^s_k)^* \# a^s_j(\varrho') \in S^{2(m+\sigma)-(j+k)}_{\mathsf{T},\tau,\mathrm{cl}}.$$

We have the following lemma whose proof is left to the reader.

Lemma 3.3. *We consider the interior quadratic form of type* (m,σ)*, as above,*

$$Q(u)=\sum_{j=0}^{m}\sum_{k=0}^{m}\big(C_{j,k}D_n^j u, D_n^k u\big)_+,\qquad C_{j,k}=c_{j,k}(x,D',\tau),\quad c_{j,k}\in S^{2(m+\sigma)-(j+k)}_{\mathsf{T},\tau,\mathrm{cl}}.$$

We have

$$|Q(u)|\le C\,\|u\|^2_{m,\sigma,\tau}\,,\quad u\in\mathscr{S}(\overline{\mathbb{R}}^n_+).$$

Next we consider the case of a quadratic form with a vanishing symbol. Such a result will be usefull when comparing quadratic forms associated with the same symbol.

Lemma 3.4. *We consider the interior quadratic form of type* (m,σ)*, as above,*

$$Q(u)=\sum_{j=0}^{m}\sum_{k=0}^{m}\big(C_{j,k}D_n^j u, D_n^k u\big)_+,\qquad C_{j,k}=c_{j,k}(x,D',\tau),\quad c_{j,k}\in S^{2(m+\sigma)-(j+k)}_{\mathsf{T},\tau,\mathrm{cl}},$$

and we further assume that the principal part of its symbol vanishes, that is,

$$\sum_{\substack{1\le j,k\le m\\ j+k=\ell}} c_{j,k}(\varrho')\equiv 0\mod S^{2(m+\sigma)-\ell-1}_{\mathsf{T},\tau,\mathrm{cl}},\quad \forall\ell\in\{0,\dots,2m\}\,,\ \varrho'=(x,\xi',\tau).$$

Then the following estimate holds

$$|Q(u)|\le C\big(\,\|u\|^2_{m,\sigma-1/2,\tau}+|T(u)|^2_{m-1,\sigma+1/2,\tau}\,\big),\qquad u\in\mathscr{S}(\overline{\mathbb{R}}^n_+).$$

Proof. Let $\ell\in\{0,\dots,2m\}$. We introduce $\alpha_\ell=\max(0,\ell-m)$ and $\beta_\ell=\min(m,\ell)$. Note that $\beta_\ell=\ell-\alpha_\ell$. We set

$$I_\ell=\sum_{\substack{1\le j,k\le m\\ j+k=\ell}}\big(C_{j,k}D_n^j u, D_n^k u\big)_+=\sum_{k=\alpha_\ell}^{\beta_\ell}\big(C_{\ell-k,k}D_n^{\ell-k}u, D_n^k u\big)_+$$

We first consider $0<\ell<2m$. For $k>\alpha_\ell$ we write

$$\begin{aligned}\big(C_{\ell-k,k}D_n^{\ell-k}u, D_n^k u\big)_+&=\big(C_{\ell-k,k}D_n^{\ell-k+1}u, D_n^{k-1}u\big)_+ +\big(\operatorname{Op}(D_nc_{\ell-k,k})D_n^{\ell-k}u, D_n^{k-1}u\big)_+\\&\quad -i\big(C_{\ell-k,k}D_n^{\ell-k}u_{|x_n=0^+}, D_n^{k-1}u_{|x_n=0^+}\big)_\partial,\end{aligned}$$

which by induction yields,

$$\begin{aligned}\big(C_{\ell-k,k}D_n^{\ell-k}u, D_n^k u\big)_+&=\big(C_{\ell-k,k}D_n^{\beta_\ell}u, D_n^{\alpha_\ell}u\big)_+ +\sum_{s=1}^{k-\alpha_\ell}\big(\operatorname{Op}(D_nc_{\ell-k,k})D_n^{\ell-k+s-1}u, D_n^{k-s}u\big)_+\\&\quad -i\sum_{s=1}^{k-\alpha_\ell}\big(C_{\ell-k,k}D_n^{\ell-k+s-1}u_{|x_n=0^+}, D_n^{k-s}u_{|x_n=0^+}\big)_\partial,\end{aligned}$$

As $D_nc_{\ell-k,k}\in S^{2(m+\sigma)-\ell}_{\mathsf{T},\tau,\mathrm{cl}}$ we note that

$$\begin{aligned}&\big|\big(\operatorname{Op}(D_nc_{\ell-k,k})D_n^{\ell-k+s-1}u, D_n^{k-s}u\big)_+\big|\\&\quad\le C\big\|\Lambda_{\mathsf{T}}^{m+\sigma+k-\ell-s+\frac12}D_n^{\ell-k+s-1}u\big\|_{L^2}\big\|\Lambda_{\mathsf{T}}^{m+\sigma-k+s-\frac12}D_n^{k-s}u\big\|_{L^2}\\&\quad\le C\big\|\Lambda_{\mathsf{T}}^{m+\sigma+k-\ell-s+\frac12}u\big\|_{\ell-k+s-1,\tau}\big\|\Lambda_{\mathsf{T}}^{m+\sigma-k+s-\frac12}u\big\|_{k-s,\tau}\\&\quad\le C\,\|u\|_{\ell-k+s-1,m+\sigma+k-\ell-s+\frac12,\tau}\,\|u\|_{k-s,m+\sigma-k+s-\frac12,\tau}\\&\quad\le C\,\|u\|^2_{m-1,\sigma+\frac12,\tau}\,,\end{aligned}\tag{3.3}$$

as $m+k-\ell-s\ge 0$ and $m-1-k+s\ge 0$.

Similarly we write

$$\big|\big(C_{\ell-k,k}D_n^{\ell-k+s-1}u_{|x_n=0^+}, D_n^{k-s}u_{|x_n=0^+}\big)_\partial\big| \leq C\,|T(u)|^2_{m-1,\sigma+\frac12,\tau}\,.$$

We thus obtain

$$|I_\ell| \leq \Big|\sum_{k=\alpha_\ell}^{\beta_\ell}\big(C_{\ell-k,k}D_n^{\beta_\ell}u, D_n^{\alpha_\ell}u\big)_+\Big| + C\big(\,\|u\|^2_{m-1,\sigma+\frac12,\tau} + |T(u)|^2_{m-1,\sigma+\frac12,\tau}\,\big).$$

As by assumption we have

$$\sum_{k=\alpha_\ell}^{\beta_\ell} C_{\ell-k,k} = \sum_{\substack{1\leq j,k\leq m\\ j+k=\ell}} C_{\ell-k,k} \in \Psi^{2(m+\sigma)-\ell-1}_{\mathsf{T},\tau,\mathrm{cl}},$$

we find

$$\begin{aligned}\Big|\sum_{k=\alpha_\ell}^{\beta_\ell}\big(C_{\ell-k,k}D_n^{\beta_\ell}u, D_n^{\alpha_\ell}u\big)_+\Big| &\leq C\big\|\Lambda_{\mathsf{T}}^{m+\sigma-\beta_\ell-\frac12}D_n^{\beta_\ell}u\big\|_{L^2}\big\|\Lambda_{\mathsf{T}}^{m+\sigma-\alpha_\ell-\frac12}D_n^{\alpha_\ell}u\big\|_{L^2}\\ &\leq C\,\|u\|_{\beta_\ell,m+\sigma-\beta_\ell-\frac12,\tau}\,\|u\|_{\alpha_\ell,m+\sigma-\alpha_\ell-\frac12,\tau}\\ &\leq C\,\|u\|^2_{m,\sigma-\frac12,\tau}\,,\end{aligned}$$

since $m-\alpha_\ell \geq 0$ and $m-\beta_\ell \geq 0$. In the case $0<\ell<2m$ we have thus obtained

$$|I_\ell| \leq C\big(\,\|u\|^2_{m,\sigma-\frac12,\tau} + |T(u)|^2_{m-1,\sigma+\frac12,\tau}\,\big). \tag{3.4}$$

Let now $\ell = 0$. Then $I_0 = (C_{0,0}u, u)_+$ and as $C_{0,0} \in \Psi^{2(m+\sigma)-1}_{\mathsf{T},\tau,\mathrm{cl}}$ we find $|I_0| \leq C\,\|u\|^2_{m,\sigma-\frac12,\tau}$. Similarly for $\ell = 2m$ we have $I_{2m} = (C_{m,m}D_n^m u, D_n^m u)_+$ with $C_{m,m} \in \Psi^{2\sigma-1}_{\mathsf{T},\tau,\mathrm{cl}}$ yielding $|I_{2m}| \leq C\,\|u\|^2_{m,\sigma-\frac12,\tau}$. This concludes the proof. □

We shall need a Gårding inequality for the quadratic forms we have introduced.

Proposition 3.5 (Gårding inequality)**.** *Let $\mathscr{U}$ be an open conic set in $\overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1}\times\mathbb{R}_+$ and let Q be an interior quadratic form of type $(m,0)$ with its symbol $q \in S^{2m,0}_{\tau,\mathrm{cl}}$ satisfying, for some $C>0$ and $R_0>0$,*

$$\operatorname{Re} q(\varrho) \geq C\lambda^{2m}, \quad \textit{for } \lambda = |(\xi,\tau)| \geq R_0,\ \ \varrho = (\varrho', \xi_n),\ \varrho' = (x,\xi',\tau) \in \mathscr{U},\ \xi_n \in \mathbb{R}.$$

Let then $\chi \in S^0_{\mathsf{T},\tau}$ be such that $\operatorname{supp}(\chi) \subset \mathscr{U}$. For $0<C_0<C$ and $N\in\mathbb{N}$ there exist τ_*, $C'>0$, and $C''_N>0$ such that the following inequality holds

$$\operatorname{Re} Q(\operatorname{Op}(\chi)u) \geq C_0\,\|\operatorname{Op}(\chi)u\|^2_{m,\tau} - C'\,|T(\operatorname{Op}(\chi)u)|^2_{m-1,1/2,\tau} - C''_N\,\|u\|^2_{m,-N,\tau}\,,$$

for $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$ and $\tau \geq \tau_*$.

The important feature of this version of the Gårding inequality is that it concerns functions defined on a half space. Such an inequality can be found in [41, 11]. Here we give a microlocal version of the inequality.

Remark 3.6. *In the case $\mathscr{U} = U_0 \times \mathbb{R}^n \times \mathbb{R}^+$, with U_0 open subset of $\overline{\mathbb{R}}^n_+$ then, by continuity, there exists U_1 open subset of $\overline{\mathbb{R}}^n_+$ such that U_1 is a neighborhood of $\overline{U_0}$ and*

$$\operatorname{Re} q(\varrho) \geq C'_0\lambda^{2m}, \quad \textit{for } \lambda = |(\xi,\tau)| \geq R_0,\ \ \varrho = (\varrho', \xi_n),\ \varrho' = (x,\xi',\tau) \in U_1 \times \mathbb{R}^n \times \mathbb{R}^+,\ \xi_n \in \mathbb{R}.$$

for $C_0 < C'_0 < C$. Then there exist C' and $\tau_* > 0$ such that

$$\operatorname{Re} Q(u) \geq C_0\,\|u\|^2_{m,\tau} - C'\,|T(u)|^2_{m-1,1/2,\tau}\,,$$

for $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$ *with* $\operatorname{supp}(u) \subset U_0$. *This is obtained from Proposition 3.5 by choosing* $\chi = \chi(x) \in \mathscr{C}^\infty(\mathbb{R}^n)$ *with* $\operatorname{supp}(\chi_{|x_n>0}) \subset U_1$ *and* $\chi \equiv 1$ *on* U_0 *and by taking* τ *sufficiently large.*

Proof. Let $\tilde\chi \in S^0_{\mathsf{T},\tau}$ have the same properties as χ with moreover $0 \le \tilde\chi \le 1$ and $\tilde\chi = 1$ on $\operatorname{supp}(\chi)$.

We introduce the interior quadratic form

$$\tilde Q(u) = \operatorname{Re} Q(u) = \frac{1}{2}\sum_{s=1}^N \big((A^s u, B^s u)_+ + (B^s u, A^s u)_+ \big),$$

that we may write in the form of (3.1) with $2N$ terms in the sum. Its symbol is given by (see (3.2))

$$\frac{1}{2}\sum_{s=1}^N \big(\overline{b}^s a^s + \overline{a}_s b^s\big) = \operatorname{Re}\sum_{s=1}^N \overline{a}_s b^s \in S^{2m,0}_{\tau,\mathrm{cl}}.$$

Without any loss of generality we may thus assume that the interior quadratic form Q has a real symbol $q(\varrho)$.

The symbol $q(\varrho)$ is in $S^{2m,0}_{\tau,\mathrm{cl}}$ and thus is written as

$$q(\varrho) = \sum_{j=0}^{2m} q_j(\varrho')\xi_n^j, \quad q_j \in S^{2m-j}_{\mathsf{T},\tau,\mathrm{cl}}, \ \ \varrho = (\varrho', \xi_n), \ \varrho' = (x, \xi', \tau).$$

Each symbol q_j takes the form $q_j \sim \sum_{k\ge 0} q_{j,k}$ with $q_{j,k}$ homogeneous of degree $2m - j - k$ in (ξ', τ) for $|(\xi', \tau)| \ge r_0$ with $r_0 \ge 0$ (see Definition 2.2). We set q^0 as the principal part of q:

$$q^0(\varrho) = \sum_{j=0}^{2m} q_{j,0}(\varrho')\xi_n^j. \tag{3.5}$$

Observe that q^0 satisfies, for $C_0 < C_1 < C$,

$$\operatorname{Re} q^0(\varrho) \ge C_1\lambda^{2m}, \quad \varrho' \in \mathscr{U}, \quad \xi_n \in \mathbb{R}.$$

With $C_0 < C_2 < C_1$, we see that $q^0(\varrho) - C_2\lambda^{2m}$ is a real polynomial function in the variable ξ_n of order $2m$, that takes positive values on the real line for $\varrho' \in \mathscr{U}$. The leading coefficient $a_0(\varrho') \in S^0_\tau$ is homogeneous of degree 0 in (ξ', τ), for $|(\xi', \tau)| \ge r_0$, and is positive. The roots of the polynomial come into conjugated pairs and are functions of the other variables $\varrho' \in \mathscr{U}$. We may thus write

$$q^0(\varrho) - C_2\lambda^{2m} = f(\varrho)\overline{f}(\varrho), \qquad \varrho = (\varrho', \xi_n), \ \varrho' \in \mathscr{U}, \ \xi_n \in \mathbb{R},$$

with

$$f(\varrho) = \sqrt{a_0(\varrho')}\prod_{i=1}^m \big(\xi_n - \rho_i^+(\varrho')\big),$$

where ρ_i^+, $i = 1, \dots, m$, denote the roots with positive imaginary parts. For all $\varrho'_0 = (x_0, \xi'_0, \tau_0) \in \mathscr{U}$, there exists a neighborhood $\mathscr{U}_{\varrho'_0}$ of ϱ'_0 in $\mathscr{U}$ such that, with the Rouché theorem, arguing as in Appendix A.2 we find that $f(\varrho) \in S^{m,0}_\tau$ for $\varrho' \in \mathscr{U}_{\varrho'_0}$, more precisely a polynomial in ξ_n with smooth homogeneous coefficients $|(\xi', \tau)| \ge r_0$. Note in particular that this uses the homogeneity of the functions $q_{j,0}$ in (3.5).

We pick $\mathscr{V}$ a conic open set such that $\overline{\mathscr{V}} \subset \mathscr{U}$ and $\operatorname{supp}(\tilde\chi) \subset \mathscr{V}$. Making use of the conic structure in the variables (ξ', τ), as above we can then pick $\varrho'_j \in \mathscr{V}$ and conic neighborhoods $\mathscr{U}_j$, $j \in J$, such that we obtain a *locally finite* covering of $\mathscr{V}$. We then associate a partition of unity of the form

$$\tilde\chi = \sum_{j\in J}\chi_j, \quad \operatorname{supp}(\chi_j) \subset \mathscr{U}_j.$$

and $f_j(\varrho) = \chi_j(\varrho')f(\varrho) \in S^{m,0}_{\mathsf{T},\tau}$. Since the supports of the χ_j are locally finite $\tilde{\chi}f = \sum_j f_j \in S^{m,0}_{\mathsf{T},\tau}$. We have

$$\tilde{\chi}^2(\varrho')\big(q^0(\varrho) - C_2\lambda^{2m}\big) = \tilde{\chi}^2(\varrho')|f|^2(\varrho), \tag{3.6}$$

for $\varrho = (\varrho', \xi_n)$ with $\varrho' \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ and $\xi_n \in \mathbb{R}$. We now take $\tau \geq r_0$. Observe that $v \to \|\mathrm{Op}(\chi)v\|^2_{m,\tau}$ is an interior quadratic form of type $(m,0)$ with symbol $\tilde{\chi}^2|(\xi,\tau)|^{2m}$. We thus see that $Q(\mathrm{Op}(\tilde{\chi})v) - C_2\,\|\mathrm{Op}(\tilde{\chi})v\|^2_{m,\tau} - \|\mathrm{Op}(\tilde{\chi}f)v\|^2_{L^2}$ is an interior quadratic form of type $(m,0)$ with a symbol r with vanishing principal part:

$$r(\varrho) = \sum_{j=0}^{m} r_j(\varrho')\xi_n^j, \qquad \text{with } r_j \in S^{2m-j-1}_{\tau}.$$

Lemma 3.4 (with $\sigma = 0$) then yields

$$\big|\operatorname{Re} Q(\mathrm{Op}(\tilde{\chi})v) - C_2\,\|\mathrm{Op}(\tilde{\chi})v\|^2_{m,\tau} - \|\mathrm{Op}(\tilde{\chi}f)v\|^2_{L^2}\big| \leq C\big(\,\|v\|^2_{m,-1/2,\tau} + |T(v)|^2_{m-1,1/2,\tau}\big), \tag{3.7}$$

for $v \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$. The triangular inequality then yields

$$\operatorname{Re} Q(\mathrm{Op}(\tilde{\chi})v) \geq C_2\,\|\mathrm{Op}(\tilde{\chi})v\|^2_{m,\tau} - C'\big(\,\|v\|^2_{m,-1/2,\tau} + |T(v)|^2_{m-1,1/2,\tau}\big), \qquad v \in \mathscr{S}(\overline{\mathbb{R}}^n_+),$$

by taking τ sufficiently large. We now set $v = \mathrm{Op}(\chi)u$. We have $\mathrm{Op}(\tilde{\chi})v = \mathrm{Op}(\chi)u + Ru$ with $R \in \cap_{N\in\mathbb{N}}\Psi^{-N}_{\mathsf{T},\tau}$ by pseudo-differential calculus. We then obtain the sought estimate by taking τ sufficiently large. □

3.2. **Boundary quadratic forms.**

Definition 3.7. *Let $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$. We say that*

$$\mathscr{B}(u) = \sum_{s=1}^{N}\big(A^s u_{|x_n=0^+}, B^s u_{|x_n=0^+}\big)_{\partial}, \quad A^s = a^s(x,D,\tau),\ B^s = b^s(x,D,\tau),$$

is a boundary quadratic form of type $(m-1,\sigma)$ with $\mathscr{C}^\infty$ coefficients, if for each $s = 1,\dots N$, we have $a^s(\varrho) \in S^{m-1,\sigma'}_{\tau,\mathrm{cl}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$, $b^s(\varrho) \in S^{m-1,\sigma''}_{\tau,\mathrm{cl}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$ with $\sigma' + \sigma'' = 2\sigma$, $\varrho = (\varrho', \xi_n)$ with $\varrho' = (x, \xi', \tau)$. The symbol of the boundary quadratic form $\mathscr{B}$ is defined by

$$B(\varrho', \xi_n, \tilde{\xi}_n) = \sum_{s=1}^{N} a^s(\varrho', \xi_n)\overline{b^s}(\varrho', \tilde{\xi}_n).$$

For $\mathbf{z} = (z_0, \dots, z_{m-1}) \in \mathbb{C}^m$ and $a(\varrho) \in S^{m-1,r}_{\tau,\mathrm{cl}}$, of the form $a(\varrho', \xi_n) = \sum_{j=0}^{m-1} a_j(\varrho')\xi_n^j$ with $a_j(\varrho') \in S^{m-1+r-j}_{\mathsf{T},\tau,\mathrm{cl}}$ we set

$$\Sigma_a(\varrho', \mathbf{z}) = \sum_{j=0}^{m-1} a_j(\varrho')z_j.$$

From the boundary quadratic form $\mathscr{B}$ we introduce the following bilinear symbol $\Sigma_{\mathscr{B}} : \mathbb{C}^m \times \mathbb{C}^m \longrightarrow \mathbb{C}$

$$\Sigma_{\mathscr{B}}(\varrho', \mathbf{z}, \mathbf{z}') = \sum_{s=1}^{N} \Sigma^s_a(\varrho', \mathbf{z})\overline{\Sigma^s_b}(\varrho', \overline{\mathbf{z}}'), \quad \mathbf{z}, \mathbf{z}' \in \mathbb{C}^m.$$

We let $\mathscr{W}$ be an open conic set in $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$.

Definition 3.8. *Let $\mathscr{B}$ be a boundary quadratic form of type $(m-1,\sigma)$ associated with the bilinear symbol $\Sigma_{\mathscr{B}}(\varrho', \mathbf{z}, \mathbf{z}')$. We say that $\mathscr{B}$ is positive definite in $\mathscr{W}$ if there exist $C > 0$ and $R > 0$ such that*

$$\operatorname{Re}\Sigma_{\mathscr{B}}(\varrho'', x_n = 0^+, \mathbf{z}, \mathbf{z}) \geq C\sum_{j=0}^{m-1} \lambda_{\mathsf{T}}^{2(m-1-j+\sigma)}|z_j|^2,$$

for $\lambda_{\mathsf{T}} = |(\xi', \tau)| \geq R$, $\varrho'' = (x', \xi', \tau) \in \mathscr{W}$, *and* $\mathbf{z} = (z_0, \dots, z_{m-1}) \in \mathbb{C}^m$.

We have the following Lemma.

Lemma 3.9. *Let* $\mathscr{B}$ *be a boundary quadratic form of type* $(m-1, \sigma)$*, positive definite in* $\mathscr{W}$*, an open conic set in* $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$*, with bilinear symbol* $\Sigma_{\mathscr{B}}(\varrho', \mathbf{z}, \mathbf{z}')$*. Let* $\chi \in S^0_{\mathsf{T},\tau}$ *be homogeneous of degree 0, with* $\operatorname{supp}(\chi_{|x_n=0^+}) \subset \mathscr{W}$ *and let* $N \in \mathbb{N}$*. Then there exist* $\tau_* \geq 1$*,* $C > 0$*,* $C_N > 0$ *such that*

$$\operatorname{Re} \mathscr{B}(\operatorname{Op}(\chi)u) \geq C \left|T(\operatorname{Op}(\chi)u)\right|^2_{m-1,\sigma,\tau} - C_N \left|T(u)\right|^2_{m-1,\sigma-N,\tau},$$

for $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$ *and* $\tau \geq \tau_*$.

Proof. The boundary quadratic form can be written as

$$\mathscr{B}(v) = \sum_{j,k=0}^{m-1} \left(G_{jk} \Lambda_{\mathsf{T}}^{m+\sigma-1-j} D_n^j v_{|x_n=0^+}, \Lambda_{\mathsf{T}}^{m+\sigma-1-k} D_n^k v_{|x_n=0^+}\right)_\partial$$

where $G_{jk} = \operatorname{Op}(g_{jk}) \in \Psi^0_{\mathsf{T},\tau,\mathrm{cl}}$.

We introduce $\tilde{\chi} \in S^0_{\mathsf{T},\tau}$ that has the same properties as χ with moreover $0 \leq \tilde{\chi} \leq 1$ and $\tilde{\chi} = 1$ in a neighborhood of $\operatorname{supp} \chi$. We then set $\mathbf{g}(\varrho') = (g_{ij}(\varrho'))_{0 \leq i,j, \leq m-1}$ and $\tilde{\mathbf{g}}(\varrho') = (\tilde{g}_{ij}(\varrho'))_{0 \leq i,j, \leq m-1}$ with

$$\tilde{\mathbf{g}}(\varrho') = \tilde{\chi}(\varrho')\mathbf{g}(\varrho') + (1 - \tilde{\chi}(\varrho'))I_m.$$

As $\mathscr{B}$ is positive definite we have $\operatorname{Re}\left(\mathbf{g}(\varrho'', x_n = 0^+)\mathbf{z}, \mathbf{z}\right) \geq C \left|\mathbf{z}\right|^2_{\mathbb{C}^m}$ with $C > 0$ for $\varrho'' = (x', \xi', \tau) \in \mathscr{W}$ with λ_t sufficiently large. Thus we have $\operatorname{Re}\left(\tilde{\mathbf{g}}(\varrho'', x_n = 0^+)\mathbf{z}, \mathbf{z}\right) \geq C' \left|\mathbf{z}\right|^2_{\mathbb{C}^m}$ with $C' > 0$ for $\varrho'' \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ with λ_t sufficiently large.

For a function v we define the m-tuple functions $V = (v_0, \dots, v_{m-1})$ by

$$v_k = \Lambda_{\mathsf{T}}^{m+\sigma-1-k} D_n^k v_{|x_n=0^+}, \quad k = 0, \dots, m-1.$$

We then have, for $N \in \mathbb{Z}$,

$$\begin{aligned} |V|^2_{N,\tau} &= \sum_{k=0}^{m-1} |v_k|^2_{N,\tau} = \sum_{k=0}^{m-1} \left|\Lambda_{\mathsf{T}}^{m+\sigma-1-k} D_n^k v_{|x_n=0^+}\right|^2_{N,\tau} \\ &\asymp \sum_{k=0}^{m-1} \left|\Lambda_{\mathsf{T}}^{\sigma+N} D_n^k v\right|^2_{m-1-k,\tau} = |T(v)|^2_{m-1,\sigma+N,\tau}. \end{aligned} \tag{3.8}$$

We set $\underline{u} = \operatorname{Op}(\chi)u$ and introduce $U = (u_0, \dots, u_{m-1})$ and $\underline{U} = (\underline{u}_0, \dots, \underline{u}_{m-1})$ as above:

$$u_k = \Lambda_{\mathsf{T}}^{m+\sigma-1-k} D_n^k u_{|x_n=0^+}, \quad \underline{u}_k = \Lambda_{\mathsf{T}}^{m+\sigma-1-k} D_n^k \underline{u}_{|x_n=0^+}, \quad k = 0, \dots, m-1.$$

We have

$$\mathscr{B}(\underline{u}) = \sum_{j,k=0}^{m-1} \left(G_{jk|x_n=0^+} \underline{u}_j, \underline{u}_k\right)_\partial.$$

Writing $g_{ij} = \tilde{g}_{ij} + r_{ij}$ with $r_{ij} = (g_{ij} - \delta_{ij})(1 - \tilde{\chi})$, with $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, we find

$$\mathscr{B}(\underline{u}) = \sum_{j,k=0}^{m-1} \left(\operatorname{Op}(\tilde{g}_{ij}|_{x_n=0^+})\underline{u}_j, \underline{u}_k\right)_\partial + \sum_{j,k=0}^{m-1} \left(\operatorname{Op}(r_{ij}|_{x_n=0^+})\underline{u}_j, \underline{u}_k\right)_\partial$$

As the supports of $1 - \tilde{\chi}$ and χ are disjoint, with the pseudo-differential calculus and with the Gårding inequality in the transverse direction, for any $N \in \mathbb{N}$ we find $C > 0$ and $C_N > 0$ such that

$$\operatorname{Re} \mathscr{B}(\underline{u}) \geq C \left|\underline{U}\right|^2_{0,\tau} - C_N \left|U\right|^2_{-N,\tau},$$

for τ sufficiently large. Combined with (3.8) this yields the conclusion. □

Lemma 3.10. *Let $h_k(\zeta)$, $k=0,\dots,m'-1$, be a set of polynomials of degree less than or equal to $(m-1)$ with $m'\geq m$. Consider the following bilinear form $\Sigma_{\mathscr{B}}(\mathbf{z},\mathbf{z}')=\sum_{k=0}^{m}\Sigma_{h_k}(\mathbf{z})\overline{\Sigma_{h_k}(\mathbf{z}')}$, for $\mathbf{z},\mathbf{z}'\in\mathbb{C}^m$. Then the following statements are equivalent:*

(1) *the set of polynomials is complete;*
(2) *the quadratic form given by $\Sigma_{\mathscr{B}}(\mathbf{z},\mathbf{z})$ is definite positive: there exists $C>0$ such that*

$$\Sigma_{\mathscr{B}}(\mathbf{z},\mathbf{z})\geq C\,|\mathbf{z}|^2_{\mathbb{C}^m}\,,\quad \mathbf{z}=(z_0,\dots,z_{m-1})\in\mathbb{C}^m.$$

Proof. Writing $h_k(\zeta)=\sum_{j=0}^{m-1}h_{kj}\zeta^j$, the completeness of the set of polynomials means that the matrix

$$H=(h_{kj})_{\substack{0\leq k\leq m'-1\\ 0\leq j\leq m-1}}\,,$$

is of maximal rank, that is of rank m. As we have $\operatorname{rank}{}^t\overline{H}H=\operatorname{rank}H$ and

$$\Sigma_{\mathscr{B}}(\mathbf{z},\mathbf{z})=|H\mathbf{z}|^2_{\mathbb{C}^{m'}}=\left({}^t\overline{H}H\mathbf{z},\mathbf{z}\right)_{\mathbb{C}^m}\,,$$

the conclusion follows. □

3.3. **Bézout matrices.**

Definition 3.11. *Given two univariate polynomials $a(\zeta)=\sum\limits_{j=0}^{m}a_j\zeta^j$, $b(\zeta)=\sum\limits_{j=0}^{m}b_j\zeta^j$ of degree less than or equal to m (note that any coefficient could be zero), we build the following bivariate polynomial*

$$B_{a,b}(\zeta,\widetilde{\zeta})=\frac{a(\zeta)b(\widetilde{\zeta})-a(\widetilde{\zeta})b(\zeta)}{\zeta-\widetilde{\zeta}}=\sum_{j,k=0}^{m-1}g_{j,k}\,\zeta^j\widetilde{\zeta}^k,$$

called the Bézoutian of a and b, and the corresponding symmetric matrix $g_{a,b}=(g_{j,k})$ of size $m\times m$ with entries $g_{j,k}$, bilinear in the coefficients of a and b, is called the Bézout matrix and given by (see [15]):

$$g_{j,k}=\sum_{\ell=0}^{\min(j,k)}\left(b_\ell a_{j+k-\ell+1}-b_{j+k-\ell+1}a_\ell\right),\tag{3.9}$$

upon letting $a_k=b_k:=0$ for $k>m$ and $k<0$. With this Bézout matrix we associate the following bilinear form

$$\Sigma_{B_{a,b}}(\mathbf{z},\mathbf{z}')=\sum_{j,k=0}^{m-1}g_{j,k}z_j\overline{z}'_k,\quad \mathbf{z}=(z_0,\dots,z_{m-1}),\ \mathbf{z}'=(z'_0,\dots,z'_{m-1})\in\mathbb{C}^m.$$

Lemma 3.12. *Given two univariate polynomials $a(\zeta)=\sum\limits_{j=0}^{m}a_j\zeta^j$, $b(\zeta)=\sum\limits_{j=0}^{m}b_j\zeta^j$ of degree less than or equal to m, we have the following identity*

$$\Sigma_{B_{a,b}}(\mathbf{z},\mathbf{z}')=-\sum_{j<k}\sum_{r=0}^{k-j-1}g'_{j,k}z_{j+r}\overline{z}'_{k-1-r},\quad \mathbf{z},\mathbf{z}'\in\mathbb{C}^m,\tag{3.10}$$

where

$$g'_{j,k}=(a_jb_k-a_kb_j)=-g'_{k,j}.\tag{3.11}$$

Moreover if $a=a_1a_2$ and $b=\overline{a}$ we have the following property

$$B_{a,b}(\zeta,\widetilde{\zeta})=a_2(\zeta)\overline{a}_2(\widetilde{\zeta})B_{a_1,\overline{a}_1}(\zeta,\widetilde{\zeta})+\overline{a}_1(\zeta)a_1(\widetilde{\zeta})B_{a_2,\overline{a}_2}(\zeta,\widetilde{\zeta}).\tag{3.12}$$

Remark that this expression is not symmetric in a_1 and a_2.

Proof. For $t, s \in \mathbb{R}$, we write

$$a(t)b(s) - a(s)b(t) = \sum_{k,j=0}^{m} (a_j b_k - a_k b_j)\, t^j s^k = \sum_{j,k=0}^{m} g'_{j,k} t^j s^k,$$

and using the anti-symmetry of g', viz. $g'_{k,j} = -g'_{j,k}$, we find

$$a(t)b(s) - a(s)b(t) = \sum_{j<k} g'_{j,k}\big(t^j s^k - t^k s^j\big) = (s-t) \sum_{j<k} g'_{j,k} \sum_{r=0}^{k-j-1} t^{j+r} s^{k-1-r}.$$

By continuity we then obtain the following identities

$$B_{a,b}(t,s) = B'_{a,b}(t,s) := -\sum_{j<k} g'_{j,k} \sum_{r=0}^{k-j-1} t^{j+r} s^{k-1-r}. \tag{3.13}$$

With the matrix $g' = (g'_{j,k})$ we associate the following bilinear form

$$\Sigma_{B'_{a,b}}(\mathbf{z}, \mathbf{z}') = -\sum_{j<k} \sum_{r=0}^{k-j-1} g'_{j,k} z_{j+r} \overline{z}'_{k-1-r}, \quad \mathbf{z}, \mathbf{z}' \in \mathbb{C}^m.$$

To prove the first result, i.e., $\Sigma_{B'_{a,b}} = \Sigma_{B_{a,b}}$, it is sufficient to have $\Sigma_{B'_{a,b}}(\mathbf{v}_p, \mathbf{v}_q) = \Sigma_{B_{a,b}}(\mathbf{v}_p, \mathbf{v}_q)$, $p, q \in \{1, \dots, m\}$, for any basis $(\mathbf{v}_1, \dots, \mathbf{v}_m)$ of $\mathbb{C}^m$.

Let then $\omega_1, \dots, \omega_m \in \mathbb{C}$ be such that $\omega_i \neq \omega_j$ for $i \neq j$. Setting $\mathbf{v}_j = (1, \omega_j, \dots, \omega_j^{m-1})$, $j = 1, \dots, m$, yields a basis of $\mathbb{C}^m$, as we have the Vandermonde determinant

$$\det(\mathbf{v}_1, \dots, \mathbf{v}_m) = \prod_{1 \le i < j \le m} (\omega_j - \omega_i) \neq 0.$$

Observe then that we have

$$\Sigma_{B_{a,b}}(\mathbf{v}_p, \mathbf{v}_q) = B_{a,b}(\omega_p, \omega_q), \quad \Sigma_{B'_{a,b}}(\mathbf{v}_p, \mathbf{v}_q) = B'_{a,b}(\omega_p, \omega_q).$$

We then deduce the first result from (3.13).

Finally the proof of (3.12) is a simple algebraic manipulation that is left to the reader. □

The following Hermite Theorem provides a relation between the position of the roots of a polynomial and the Bézout matrix associated with the real and imaginary parts of the polynomial. We give an elementary proof in Appendix A.3.

Proposition 3.13 (Hermite Theorem)**.** *Let $h(\zeta) = a(\zeta) + ib(\zeta)$ be a polynomial of degree $k \ge 1$, where $a(\zeta)$ and $b(\zeta)$ are polynomials with real coefficients. Assume that all the roots of $h(\zeta)$ are in the lower complex half-plane $\{\operatorname{Im}\zeta < 0\}$. Then the roots of $a(\zeta)$ and $b(\zeta)$ are real and distinct. Moreover, the bilinear form $\Sigma_{B_{a,b}}(\mathbf{z}, \mathbf{z}')$ is positive, i.e., there exists $C > 0$ such that*

$$\Sigma_{B_{a,b}}(\mathbf{z}, \mathbf{z}) \ge C\, |\mathbf{z}|^2, \quad \mathbf{z} \in \mathbb{C}^k.$$

3.4. **A generalized Green formula.** Consider two symbols of $a \in S^{m,0}_{\tau,\mathrm{cl}}$ and $b \in S^{m-1,1}_{\tau,\mathrm{cl}} \subset S^{m,0}_{\tau}$,

$$a(\varrho) = \sum_{j=0}^{m} a_j(\varrho')\xi_n^j, \quad b(\varrho) = \sum_{j=0}^{m-1} b_j(\varrho')\xi_n^k, \quad a_j \in S^{m-j}_{\mathsf{T},\tau,\mathrm{cl}}, \quad b_j \in S^{m-k}_{\mathsf{T},\tau,\mathrm{cl}}, \tag{3.14}$$

with $\varrho = (\varrho', \xi_n)$ and $\varrho' = (x, \xi', \tau)$. Considering them as polynomials in ξ_n, we introduce a quadratic form (Bézout form)

$$B_{a,b}(\xi_n, \tilde{\xi}_n) = \frac{a(\varrho', \xi_n) b(\varrho', \tilde{\xi}_n) - a(\varrho', \tilde{\xi}_n) b(\varrho', \xi_n)}{\xi_n - \tilde{\xi}_n} = \sum_{j,k=1}^{m-1} g_{j,k}(\varrho') \xi_n^j \tilde{\xi}_n^k,$$

where according to (3.9)

$$g_{j,k} = \sum_{\ell=0}^{\min(j,k)} \left(b_\ell a_{j+k-\ell+1} - b_{j+k-\ell+1} a_\ell\right) \in S^{2m-1-(j+k)}_{\mathsf{T},\tau,\mathrm{cl}}.$$

With a and b we associate the following boundary quadratic form

$$\mathscr{B}_{a,b}(u) = \sum_{j,k=0}^{m-1} \left(G_{j,k} D_n^j u_{|x_n=0^+}, D_n^k u_{|x_n=0^+}\right)_\partial \tag{3.15}$$

where $G_{j,k} = \mathrm{Op}(g_{j,k})$. By Lemma 3.12 we deduce that

$$\mathscr{B}_{a,b}(u) = -\sum_{j<k} \sum_{\ell=0}^{k-j-1} \left(\mathrm{Op}(g'_{j,k}) D_n^{j+\ell} u_{|x_n=0^+}, D_n^{k-1-\ell} u_{|x_n=0^+}\right)_\partial$$

where

$$g'_{j,k}(\varrho') = (a_j b_k - a_k b_j)(\varrho') \in S^{2m-j-k}_{\mathsf{T},\tau,\mathrm{cl}}.$$

For any a and b as given by (3.14), we introduce

$$\begin{aligned} \mathrm{sub}(a,b) &= \sum_{|\alpha|=1} \partial_x^\alpha (b \partial_\xi^\alpha a - a \partial_\xi^\alpha b) \\ &= \{a,b\} + \sum_{|\alpha|=1} \left(b \partial_\xi^\alpha \partial_x^\alpha a - a \partial_\xi^\alpha \partial_x^\alpha b\right) \in S^{2m-1,0}_\tau. \end{aligned} \tag{3.16}$$

We have the following lemma.

Lemma 3.14. *We have*

$$\mathrm{sub}(a,b)(\varrho) = -\sum_{j,k=0}^{m} h_{jk}(\varrho') \xi_n^{j+k} - \frac{1}{2} \sum_{j,k=0}^{m} \partial_n (g'_{jk})(\varrho')(k-j) \xi_n^{k+j-1} \in S^{2m-1,0}_\tau.$$

where $\varrho = (\varrho', \xi_n)$ and

$$h_{j,k} = \sum_{|\beta|=1} \partial_{x'}^\beta \left(a_j \partial_{\xi'}^\beta b_k - b_j \partial_{\xi'}^\beta a_k\right) \in S^{2m-1-j-k}_{\mathsf{T},\tau,\mathrm{cl}}.$$

We refer to Appendix A.4 for a proof.

We shall now prove the following proposition.

Proposition 3.15 (Generalized Green's formula)**.** *Consider two smooth and real symbols $a \in S^{m,0}_{\tau,\mathrm{cl}}$ and $b \in S^{m-1,1}_{\tau,\mathrm{cl}}$. The following identity holds true*

$$2\,\mathrm{Re}\,(Au, iBu) = H_{a,b}(u) + \mathscr{B}_{a,b}(u) + R(u), \quad A = a(x,D,\tau),\ B = b(x,D,\tau), \tag{3.17}$$

for any $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$. Here, $\mathscr{B}_{a,b}$ is the boundary quadratic form of type $(m-1, 1/2)$ given by (3.15) and $H_{a,b}$ is an interior quadratic form of type $(m, -1/2)$ with real symbol

$$h_{a,b}(\varrho) = \mathrm{sub}(a,b)(\varrho).$$

Finally, the remainder term $R(u)$ is a quadratic form that satisfies

$$|R(u)| \le C \|u\|^2_{m,-1,\tau}.$$

Proof. We write the first term in the l.h.s. of (3.17) as the following interior quadratic form of type $(m,0)$

$$Q(u) = 2\,\mathrm{Re}\,(Au, iBu)_+ = -i\big(\,(Au,Bu)_+ - (Bu,Au)_+\,\big) = \sum_{j,k=0}^{m} I_{jk}(u),$$

where the $I_{jk}(u)$ are given by

$$I_{jk}(u) = -i\big((A_jD_n^ju, B_kD_n^ku)_+ - (B_jD_n^ju, A_kD_n^ku)_+\big),$$

with the tangential operators $A_j = \mathrm{Op}(a_j)$, $B_j = \mathrm{Op}(b_j)$, $j = 1,\dots,m$. We write the interior quadratic form I_{jk} in the form

$$I_{jk}(u) = -i\big(\,(B_k^*\circ A_j - A_k^*\circ B_j)\,D_n^ju, D_n^ku\big)_+.$$

From symbolic calculus (Proposition 2.5) we have

$$B_k^*\circ A_j - A_k^*\circ B_j = G'_{j,k} - iH_{j,k} + R_{j,k},$$

with $G'_{j,k} = \mathrm{Op}(g'_{jk})$, $H_{j,k} = \mathrm{Op}(h_{jk})$, where

$$g'_{j,k} = a_jb_k - a_kb_j = -g'_{k,j} \in S^{2m-j-k}_{\mathsf{T},\tau,\mathrm{cl}}, \quad h_{j,k} = \sum_{|\beta|=1} \partial_{x'}^\beta(a_j\partial_{\xi'}^\beta b_k - b_j\partial_{\xi'}^\beta a_k) \in S^{2m-1-j-k}_{\mathsf{T},\tau,\mathrm{cl}},$$

and the remainder term $R_{j,k} \in \Psi^{2m-2-j-k}_{\mathsf{T},\tau,\mathrm{cl}}$. We thus have

$$I_{jk}(u) = \underbrace{-i\big(G'_{j,k}D_n^ju, D_n^ku\big)_+}_{=:J_{j,k}(u)} - \big(H_{j,k}D_n^ju, D_n^ku\big)_+ + R_{j,k}(u), \quad |R_{j,k}(u)| \le C\,\|u\|^2_{m,-1,\tau}\,.$$

We consider the term $J_{j,k}$ for $j<k$. With an integration by parts, we obtain

$$\begin{aligned} J_{j,k}(u) = &-i\big(\,\mathrm{Op}(D_ng'_{j,k})D_n^ju, D_n^{k-1}u\big)_+ - i\big(G'_{j,k}D_n^{j+1}u, D_n^{k-1}u\big)_+ \\ &- \big(G'_{j,k}D_n^ju_{|x_n=0^+}, D_n^{k-1}u_{|x_n=0^+}\big)_\partial. \end{aligned}$$

Therefore, by induction, we find

$$\begin{aligned} J_{j,k}(u) = &-i\sum_{\ell=0}^{k-j-1}\big(\,\mathrm{Op}(D_ng'_{j,k})D_n^{j+\ell}u, D_n^{k-1-\ell}u\big)_+ \overbrace{-i\big(G'_{j,k}D_n^ku, D_n^ju\big)_+}^{=-J_{k,j}(u)}. \\ &- \sum_{\ell=0}^{k-j-1}\big(G'_{j,k}D_n^{j+\ell}u_{|x_n=0^+}, D_n^{k-1-\ell}u_{|x_n=0^+}\big)_\partial. \end{aligned}$$

We thus obtain

$$\begin{aligned} \sum_{j,k=0}^{m} J_{j,k}(u) &= \sum_{j<k} J_{j,k}(u) + \sum_{j<k} J_{k,j}(u) \\ &= -i\sum_{j<k}\sum_{\ell=0}^{k-j-1}\big(\,\mathrm{Op}(D_ng'_{j,k})D_n^{j+\ell}u, D_n^{k-1-\ell}u\big)_+ \\ &\quad - \sum_{j<k}\sum_{\ell=0}^{k-j-1}\big(G'_{j,k}D_n^{j+\ell}u_{|x_n=0^+}, D_n^{k-1-\ell}u_{|x_n=0^+}\big)_\partial. \end{aligned}$$

Using Lemma 3.12, we find

$$\sum_{j,k=0}^{m} J_{j,k}(u) = \mathscr{B}_{a,b}(u) - \sum_{j<k}\sum_{\ell=0}^{k-j-1}\big(\,\mathrm{Op}(\partial_ng'_{j,k})D_n^{j+\ell}u, D_n^{k-1-\ell}u\big)_+.$$

We then obtain

$$Q(u) = \mathscr{B}_{a,b}(u) + H_{a,b}(u) + R(u), \qquad R(u) = \sum_{j,k=0}^{m} R_{j,k}(u),$$

where

$$H_{a,b}(u) = -\sum_{j,k=0}^{m} \big(H_{j,k} D_n^j u, D_n^k u\big)_+ - \sum_{j<k} \sum_{\ell=0}^{k-j-1} \big(\operatorname{Op}(\partial_n g'_{j,k}) D_n^{j+\ell} u, D_n^{k-1-\ell} u\big)_+,$$

with symbol, in the sense of Definition 3.1, given by

$$h_{a,b}(\varrho) = -\sum_{j,k=0}^{m} h_{jk}(\varrho')\xi_n^{j+k} - \frac{1}{2}\sum_{j,k=0}^{m} (\partial_n g'_{jk})(\varrho')(k-j)\xi_n^{k+j-1} = \operatorname{sub}(a,b),$$

using Lemma 3.14. □

4. Proof of the Carleman estimate

As is usual in the proof of Carleman estimates we consider the following conjugated operator

$$P_\varphi = e^{\tau\varphi} P e^{-\tau\varphi}.$$

As $e^{\tau\varphi} D_j e^{-\tau\varphi} = D_j + i\tau\partial_j\varphi$ we see that $P_\varphi \in \Psi^{2m,0}_{\tau,\mathrm{cl}}$. Its principal symbol is given by $p_\varphi(\varrho) = p(x, \xi + i\tau\varphi'(x)) \in S^{2m,0}_{\tau,\mathrm{cl}}$.

Similarly we set

$$B_\varphi^k = e^{\tau\varphi} B^k e^{-\tau\varphi} \in \Psi^{\beta_k,0}_{\tau,\mathrm{cl}},$$

with principal symbol $b_\varphi^k(\varrho) = b^k(x, \xi + i\tau\varphi'(x)) \in S^{\beta_k,0}_{\tau,\mathrm{cl}}$.

We start the proof of the main theorem with a microlocal elliptic estimate that will be exploited below through the strong Lopatinskii condition.

4.1. Elliptic estimate.

Here we consider a polynomial function with roots with negative imaginary parts in a microlocal region. Then, we can obtain a perfect microlocal elliptic estimate.

Lemma 4.1. *Let $\ell(\varrho', \xi_n) \in S^{k,0}_\tau$, $\varrho' = (x, \xi', \tau)$, with $k \geq 1$, be polynomial in ξ_n with homogeneous coefficients in (ξ', τ) and $L = \ell(x, D, \tau)$. When viewed as a polynomial in ξ_n the leading coefficient is 1. Let $\mathscr{U}$ be a conic open subset of $\overline{V_+} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$. We assume that all the roots of $\ell(\varrho', \xi_n) = 0$ have negative imaginary part for $\varrho' = (x, \xi', \tau) \in \mathscr{U}$. Letting $\chi(\varrho') \in S^0_{\mathsf{T},\tau}$ be such that $\operatorname{supp}(\chi) \subset \mathscr{U}$, and $N \in \mathbb{N}$, there exist $C > 0$, $C_N > 0$, and $\tau_* > 0$ such that*

$$\|\operatorname{Op}(\chi)w\|^2_{k,\tau} + |T(\operatorname{Op}(\chi)w)|^2_{k-1,1/2,\tau} \leq C\,\|L\operatorname{Op}(\chi)w\|^2_{L^2} + C_N\big(\,\|w\|^2_{k,-N,\tau} + |T(w)|^2_{k-1,-N,\tau}\big),$$

for $w \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$ and $\tau \geq \tau_*$.

Here we recall that V_+ is bounded (see Section 1.6).

Proof. Let $\mathscr{V}$ be a conic open set of $\overline{V_+} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ such that $\overline{\mathscr{V}} \subset \mathscr{U}$ and $\operatorname{supp}(\chi) \subset \mathscr{V}$.

We write $\ell(\varrho) = a(\varrho) + ib(\varrho)$, where a and b are both real and homogeneous, with $a \in S^{k,0}_\tau$ and $b \in S^{k-1,1}_\tau$. We set $A = \operatorname{Op}(a)$ and $B = \operatorname{Op}(b)$ and we introduce the following quadratic form of type $(k,0)$ $Q(v) = \|Av\|^2_{L^2} + \|Bv\|^2_{L^2}$ with symbol

$$q(\varrho) = |a(\varrho)|^2 + |b(\varrho)|^2 \in S^{2k,0}_\tau.$$

The Hermite theorem (Proposition 3.13) implies that $a(\varrho', \xi_n)$ and $b(\varrho', \xi_n)$ have distinct real roots for all $\varrho' \in \mathscr{U}$. Thus, on the compact set $\mathcal{K} = \{\varrho = (x, \xi, \tau); \varrho' = (x, \xi', \tau) \in \overline{\mathscr{V}},\ \xi_n \in \mathbb{R},\ |\xi|^2 + \tau^2 = 1\}$, we have $q \neq 0$ yielding by homogeneity

$$q(\varrho) \geq C\,|(\xi, \tau)|^{2k}\,, \quad \varrho' \in \mathscr{V}, \quad \xi_n \in \mathbb{R}.$$

Setting $\underline{w} = \mathrm{Op}(\chi)w$, the Gårding inequality of Proposition 3.5 gives, for any $N \in \mathbb{N}$,

$$Q(\underline{w}) \geq C\,\|\underline{w}\|^2_{k,\tau} - C'\,|T(\underline{w})|^2_{k-1,1/2,\tau} - C''_N\,\|w\|^2_{k,-N,\tau}\,. \tag{4.1}$$

Next, by the generalized Green formula of Proposition 3.15 we obtain

$$|2\,\mathrm{Re}\,(A\underline{w}, iB\underline{w}) - \mathscr{B}_{a,b}(\underline{w})| \leq |H_{a,b}(\underline{w})| + C\,\|\underline{w}\|^2_{k,-1,\tau} \leq C'\,\|\underline{w}\|^2_{k,-1/2,\tau}\,,$$

by Lemma 3.3 as here $H_{a,b}$ is an interior quadratic form of type $(k, -1/2)$. Here $\mathscr{B}_{a,b}(\underline{w})$ is a boundary quadratic form of type $(k-1, 1/2)$. Then we deduce

$$2\,\mathrm{Re}\,(A\underline{w}, iB\underline{w})_+ \geq \mathscr{B}_{a,b}(\underline{w}) - C\,\|\underline{w}\|^2_{k,-1/2,\tau}\,.$$

By the Hermite theorem (Proposition 3.13) the bilinear Bézout form[4] $\Sigma_{\mathscr{B}_{a,b}}$ is positive. With the homogeneity we find

$$\Sigma_{\mathscr{B}_{a,b}}(\varrho', \mathbf{z}, \mathbf{z}) \geq C \sum_{j=0}^{k-1} \lambda_{\mathsf{T}}^{2(k-1-j+\frac{1}{2})} |z_j|^2, \quad \varrho' \in \overline{\mathscr{V}},\ \mathbf{z} = (z_0, \ldots, z_{m-1}) \in \mathbb{C}^m,\ \lambda_{\mathsf{T}} = |(\xi', \tau)|.$$

Then the Gårding inequality of Lemma 3.9 gives, for any $N \in \mathbb{N}$,

$$2\,\mathrm{Re}\,(A\underline{w}, iB\underline{w})_+ \geq C\,|T(\underline{w})|^2_{k-1,1/2,\tau} - C'\,\|\underline{w}\|^2_{k,-1/2,\tau} - C''_N\,|T(w)|^2_{k-1,-N,\tau} \tag{4.2}$$

Then from (4.1) and (4.2) we have

$$\begin{aligned} \|L\underline{w}\|^2_{L^2} &= Q(\underline{w}) + 2\,\mathrm{Re}\,(A\underline{w}, iB\underline{w})_+ \\ &\geq C\,\|\underline{w}\|^2_{k,\tau} - C'\,|T(\underline{w})|^2_{k-1,1/2,\tau} - C_N\big(\,\|w\|^2_{k,-N,\tau} + |T(w)|^2_{k-1,-N,\tau}\,\big), \end{aligned} \tag{4.3}$$

for τ chosen sufficiently large.

Note however that with (4.2) we also find, as $Q(\underline{w}) \geq 0$,

$$\|L\underline{w}\|^2_{L^2} \geq C\,|T(\underline{w})|^2_{k-1,1/2,\tau} - C'\,\|\underline{w}\|^2_{k,-1/2,\tau} - C''_N\,|T(w)|^2_{k-1,-N,\tau}\,. \tag{4.4}$$

Combining (4.3) and (4.4) and taking τ sufficiently large we obtain the sought result. □

4.2. **Estimate with the strong Lopatinskii condition.** Here, we consider a point in the cotangent bundle, at the boundary, where the strong Lopatinskii condition holds. We then obtain an estimate of a boundary norm.

Lemma 4.2. *Assume that the strong Lopatinskii condition is satisfied at $\varrho'_0 = (x_0, \xi'_0, \tau_0) \in \mathbb{S}^*_{\mathsf{T},\tau}(V)$ with $x_0 \in \partial\Omega \cap V$. Then there exists $\mathscr{U}$ a conic open neighborhood of ϱ'_0 in $\overline{V_+} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ such that for $\chi \in S^0_{\mathsf{T},\tau}$ with $\mathrm{supp}(\chi) \subset \mathscr{U}$, there exist $C > 0$ and $\tau_* > 0$ such that*

$$C\,|T(\mathrm{Op}(\chi)v)|^2_{m-1,1/2,\tau} \leq \sum_{k=1}^{\mu} \big|B^k_\varphi v_{|x_n=0^+}\big|^2_{m-1/2-\beta_k,\tau} + \|P_\varphi v\|^2_{L^2} + \|v\|^2_{m,-1,\tau} + |T(v)|^2_{m-1,-1/2,\tau}\,,$$

for $\tau \geq \tau_*$, $v \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$.

[4] In the notation of Sections 3.3 and 3.4 we have $\Sigma_{\mathscr{B}_{a,b}} = \Sigma_{B_{a,b}}$.

Proof. We consider the factorization of $p_\varphi(\varrho', \xi_n)$ in a conic open set $\mathscr{U}_0$ neighborhood of ϱ'_0 in $\overline{V_+} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ introduced in Section 1.6 by means of Lemma A.2 in Appendix A.2:

$$p_\varphi(\varrho) = p^+_\varphi(\varrho) p^-_\varphi(\varrho) p^0_\varphi(\varrho), \quad \varrho = (\varrho', \xi_n) \quad \varrho' \in \mathscr{U}_0,\ \xi_n \in \mathbb{R},$$

and we set $\kappa_\varphi = p^+_\varphi p^0_\varphi$. The polynomials (in ξ_n) $p^-_\varphi(\varrho', \xi_n)$ and $\kappa_\varphi(\varrho', \xi_n)$ are of constant degree for $\varrho' \in \mathscr{U}_0$. We have $m^- = d^\circ p^-_\varphi(\varrho', \xi_n) = d^\circ p^-_\varphi(\varrho'_0, \xi_n)$.

As in Section 1.6 we introduce the following polynomial functions in ξ_n (with ϱ' as a smooth parameter)

$$e^k_\varphi(\varrho', \xi_n) = \begin{cases} b^k_\varphi(\varrho', \xi_n) & k = 1, \dots, \mu, \\ \kappa_\varphi(\varrho', \xi_n) \xi_n^{k-(\mu+1)} & k = \mu+1, \dots, m' = m^- + \mu. \end{cases}$$

Then by Proposition 1.8 we have $m' \geq m$. The strong Lopatinskii condition at ϱ'_0 remains valid in a conic open set $\mathscr{U}_1$ neighborhood of ϱ'_0 with $\overline{\mathscr{U}_1} \subset \mathscr{U}_0$. Precisely, this means that the set of polynomials $(e^k_\varphi)_{1 \leq k \leq m'}$ is complete in the class of polynomials in ξ_n of degree less than or equal to $m-1$ for $\varrho' \in \mathscr{U}_1$. Observe that $\mathcal{K} = \overline{\mathscr{U}_1} \cap \overline{\mathbb{S}^*_{\mathsf{T},\tau}(V)}$ is compact, recalling that V_+ is bounded.

By Lemma 3.10, for $\varrho'_1 \in \mathcal{K}$ we have (using the notation of Section 3.2)

$$\sum_{k=0}^{m'} |\Sigma_{e^k_\varphi}(\varrho'_1, \mathbf{z})|^2 \gtrsim |\mathbf{z}|^2_{\mathbb{C}^m}, \quad \mathbf{z} = (z_0, \dots, z_{m-1}) \in \mathbb{C}^m.$$

By continuity this inequality remains true in a small neighborhood of ϱ'_1 in $\mathcal{K}$. Using the compactness of $\mathcal{K}$ we thus find that there exists $C > 0$ such that

$$\sum_{k=0}^{\mu} |\Sigma_{b^k_\varphi}(\varrho', \mathbf{z})|^2 + \sum_{k=\mu+1}^{m'} |\Sigma_{e^k_\varphi}(\varrho', \mathbf{z})|^2 = \sum_{k=0}^{m'} |\Sigma_{e^k_\varphi}(\varrho', \mathbf{z})|^2 \geq C\, |\mathbf{z}|^2_{\mathbb{C}^m}, \quad \mathbf{z} = (z_0, \dots, z_{m-1}) \in \mathbb{C}^m,\ \varrho' \in \mathcal{K}.$$

Introducing the map

$$M_t \varrho' = (x, t\eta), \qquad \varrho' = (x, \eta) \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+, \quad t > 0,$$

as we have $\overline{\mathscr{U}_1} = \{M_t \varrho';\ t > 0,\ \varrho' \in \mathcal{K}\}$, we find

$$\sum_{k=0}^{\mu} |\Sigma_{b^k_\varphi}(M_t \varrho', \mathbf{z}')|^2 + \sum_{k=\mu+1}^{m'} |\Sigma_{e^k_\varphi}(M_t \varrho', \mathbf{z}')|^2 \gtrsim |\mathbf{z}'|^2_{\mathbb{C}^m}, \quad \varrho' \in \overline{\mathscr{U}_1},$$

where $t = \lambda_{\mathsf{T}}^{-1} = |(\xi', \tau)|^{-1}$ and $\mathbf{z}' = (z'_0, \dots, z'_{m-1}) \in \mathbb{C}^m$ with $z'_j = t^{-m+1/2+j} z_j$, yielding

$$\sum_{k=0}^{\mu} \lambda_{\mathsf{T}}^{2(m-1/2-\beta_k)} |\Sigma_{b^k_\varphi}(\varrho', \mathbf{z})|^2 + \sum_{k=\mu+1}^{m'} \lambda_{\mathsf{T}}^{2(m^- -1/2-k+\mu+1)} |\Sigma_{e^k_\varphi}(\varrho', \mathbf{z})|^2 \gtrsim \sum_{j=0}^{m-1} \lambda_{\mathsf{T}}^{2(m-1/2-j)} |z_j|^2,$$

for all $\mathbf{z} = (z_0, \dots, z_{m-1}) \in \mathbb{C}^m$ and $\varrho' \in \overline{\mathscr{U}_1}$, using the homogeneity of the symbols.

We now choose $\mathscr{U}$ a conic open subset, neighborhood of ϱ'_0, such that $\overline{\mathscr{U}} \subset \mathscr{U}_1$. We let χ be as in the statement of the lemma. We also choose $\tilde{\chi} \in S^0_{\mathsf{T},\tau}$ with $\operatorname{supp}(\tilde{\chi}) \subset \mathscr{U}_1$ and $\tilde{\chi} = 1$ in a neighborhood of $\overline{\mathscr{U}}$. Then,

(4.5)
$$\sum_{k=0}^{\mu} \lambda_{\mathsf{T}}^{2(m-1/2-\beta_k)} |\Sigma_{b^k_\varphi}(\varrho', \mathbf{z})|^2 + \sum_{k=\mu+1}^{m'} \lambda_{\mathsf{T}}^{2(m^- -1/2-k+\mu+1)} |\tilde{\chi}(\varrho') \Sigma_{e^k_\varphi}(\varrho', \mathbf{z})|^2 \gtrsim \sum_{j=0}^{m-1} \lambda_{\mathsf{T}}^{2(m-1/2-j)} |z_j|^2,$$

for all $\mathbf{z} = (z_0, \dots, z_{m-1}) \in \mathbb{C}^m$ and $\varrho' \in \overline{\mathscr{U}}$,

As b^k_φ is the principal symbol of the conjugated operator B^k_φ, according to the Gårding inequality of Lemma 3.9 for a boundary quadratic forms of type $(m-1,1/2)$, there exists $\tau_*>0$ such that

$$\sum_{k=1}^{\mu}\left|B^k_\varphi \underline{v}_{|x_n=0^+}\right|^2_{m-1/2-\beta_k,\tau}+\sum_{k=\mu+1}^{m'}\left|E^k_\varphi \underline{v}_{|x_n=0^+}\right|^2_{m^--1/2-k+\mu+1,\tau} \geq C\left|T(\underline{v})\right|^2_{m-1,1/2,\tau}-C_N\left|T(v)\right|^2_{m-1,-N,\tau}, \tag{4.6}$$

with $\underline{v}=\mathrm{Op}(\chi)v$ and $N\in\mathbb{N}$, for $\tau\geq\tau_*$, with $E^k_\varphi=\mathrm{Op}(\tilde\chi e^k_\varphi)$. The introduction of $\tilde\chi$ is made so that $\tilde\chi e^k_\varphi$ is defined on the whole tangential phase-space.

The function $p^-_\varphi(\varrho',\xi_n)$ is polynomial in ξ_n with homogeneous coefficients in $\varrho'\in\mathscr{U}$ and leading coefficient equal to 1. Its degree is constant and equal to m^- for $\varrho'\in\mathscr{U}$. We smoothly extend $p^-_\varphi(\varrho',\xi_n)$ for ϱ' outside of $\mathscr{U}$ keeping the leading coefficient equal to 1 and we denote this extension by $\underline{p}^-_\varphi$. In fact we have $\chi p_\varphi=\chi\kappa_\varphi p^-_\varphi=\chi\tilde\chi\kappa_\varphi\underline{p}^-_\varphi$. We thus obtain $\mathrm{Op}(\chi)P_\varphi=\mathrm{Op}(\underline{p}^-_\varphi)\,\mathrm{Op}(\chi)\,\mathrm{Op}(\tilde\chi\kappa_\varphi)+R$ with R in $\Psi^{m,-1}_\tau$ by the last point of Proposition 2.7. Observe that $\tilde\chi\kappa_\varphi$ is a well defined symbol.

Applying Lemma 4.1 to $\mathrm{Op}(\underline{p}^-_\varphi)$ and $w=\mathrm{Op}(\tilde\chi\kappa_\varphi)v$ we obtain

$$\begin{aligned}
&\|\mathrm{Op}(\chi)w\|^2_{m^-,\tau}+|T(\mathrm{Op}(\chi)w)|^2_{m^--1,1/2,\tau}\\
&\quad\lesssim\left\|\mathrm{Op}(\underline{p}^-_\varphi)\,\mathrm{Op}(\chi)w\right\|^2_{L^2}+\|w\|^2_{m^-,-N,\tau}+|T(w)|^2_{m^--1,-N,\tau}\\
&\quad\lesssim\|\mathrm{Op}(\chi)P_\varphi v\|^2_{L^2}+\|v\|^2_{m,-1,\tau}+\|v\|^2_{m,-N,\tau}+|T(v)|^2_{m-1,-N,\tau}\\
&\quad\lesssim\|P_\varphi v\|^2_{L^2}+\|v\|^2_{m,-1,\tau}+|T(v)|^2_{m-1,-N,\tau},
\end{aligned}$$

yielding

$$\sum_{j=0}^{m^--1}\left|D^j_n\,\mathrm{Op}(\chi)w_{|x_n=0^+}\right|^2_{m^--1/2-j,\tau}\lesssim\|P_\varphi v\|^2_{L^2}+\|v\|^2_{m,-1,\tau}+|T(v)|^2_{m-1,-N,\tau}.$$

Recalling that $e^{j+\mu+1}_\varphi=\kappa_\varphi\xi^j_n$, $j=0,\dots,m^--1$ in $\mathscr{U}_1$ we have $D^j_n\,\mathrm{Op}(\chi)\,\mathrm{Op}(\tilde\chi\kappa_\varphi)v=E^{j+\mu+1}_\varphi\underline{v}+R_jv$ with $R_j\in\Psi^{m-m^-+j,-1}_\tau$ by the last point of Proposition 2.7. We then obtain, for τ sufficiently large

$$\sum_{j=0}^{m^--1}\left|E^{j+\mu+1}_\varphi\underline{v}_{|x_n=0^+}\right|^2_{m^--1/2-j,\tau}\lesssim\|P_\varphi v\|^2_{L^2}+\|v\|^2_{m,-1,\tau}+|T(v)|^2_{m-1,-1/2,\tau}. \tag{4.7}$$

Observing that

$$\begin{aligned}
\left|B^k_\varphi\underline{v}_{|x_n=0^+}\right|_{m-1/2-\beta_k,\tau}&\lesssim\left|B^k_\varphi v_{|x_n=0^+}\right|_{m-1/2-\beta_k,\tau}+|T(v)|_{\beta_k,m-1/2-\beta_k-1,\tau}\\
&\lesssim\left|B^k_\varphi v_{|x_n=0^+}\right|_{m-1/2-\beta_k,\tau}+|T(v)|_{m-1,-1/2,\tau}.
\end{aligned}$$

we obtain the result of Lemma 4.2 by collecting estimates (4.6) and (4.7). □

4.3. **Estimate with a positive Poisson bracket on the characteristic set.** Here we consider the case of two symbols a,b such that their Poisson bracket $\{a,b\}$ is positive of the characterisitic set $\{a=b=0\}$. This allows us to derive an estimate with the control of a volume norm.

Lemma 4.3. *Let U be an open set of $\overline{V_+}$. Let $a\in S^{m,0}_\tau$ and $b\in S^{m-1,1}_\tau$ be real symbols homogeneous of degree m in (τ,ξ), and set*

$$Q_{a,b}(v)=2Re\,(Av,iBv)_+,\quad A=a(x,D,\tau),\ B=b(x,D,\tau).$$

We assume that

$$a(\varrho)=b(\varrho)=0\ \Rightarrow\ \{a,b\}>0,\quad \varrho=(x,\xi,\tau),$$

for $x \in \overline{U}$, $(\xi,\tau) \neq (0,0)$. Then there exist $C>0$, $C'>0$, and $\tau_*>0$ such that

$$C\,\|v\|^2_{m,\tau} \leq C'\big(\,\|Av\|^2_{L^2} + \|Bv\|^2_{L^2} + |T(v)|^2_{m-1,1/2,\tau}\,\big) + \tau\,\mathrm{Re}\,\big(Q_{a,b}(v) - \mathscr{B}_{a,b}(v)\big),$$

for $\tau > \tau_*$ and for $v \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$ with $\mathrm{supp}(v) \subset U$.

Proof. Note that with the definition of $h_{a,b}(\varrho) = \mathrm{sub}(a,b)(\varrho)$ in Section 3.4 we have

$$a(\varrho) = b(\varrho) = 0 \;\;\Rightarrow\;\; \mathrm{sub}(a,b)(\varrho) > 0, \quad \varrho = (x,\xi,\tau),$$

Observe that $h_{a,b}(\varrho)$ is homogeneous of degree $2m-1$ in (ξ,τ). On the compact $\mathbb{S}^*_{\mathsf{T},\tau}\big(\overline{U}\big)$ we have

$$\tau h_{a,b} + \nu\big(|a|^2 + |b|^2\big) \geq C_0 > 0,$$

for $\nu > 0$ sufficiently large. Then by homogeneity we obtain

$$\tau h_{a,b}(\varrho) + \nu\big(|a|^2 + |b|^2\big)(\varrho) \geq C_0|(\xi,\tau)|^{2m}, \quad \varrho = (x,\xi,\tau),\; x \in \overline{U},\; \xi \in \mathbb{R}^n,\; \tau \geq 0,$$

By the Gårding inequality of Proposition 3.5 for interior quadratic forms of type $(m,0)$ and Remark 3.6 we have

$$\tau\,\mathrm{Re}\,H_{a,b}(v) + \nu(\|Av\|^2_{L^2} + \|Bv\|^2_{L^2}) + C'\,|T(v)|^2_{m-1,1/2,\tau} \geq C\,\|v\|^2_{m,\tau}\,,$$

where $H_{a,b}$ is a quadratic form of type $(m,0)$ with symbol $h_{a,b}$. Such a form is for instance given in the proof of Proposition 3.15.

The generalized Green formula of Proposition 3.15 gives

$$Q_{a,b}(v) - \mathscr{B}_{a,b}(v) + C\,\|v\|^2_{m,-1,\tau} \geq H_{a,b}(v),$$

yielding

$$\tau\,\mathrm{Re}\,\big(Q_{a,b}(v) - \mathscr{B}_{a,b}(v)\big) + \nu(\|Av\|^2_{L^2} + \|Bv\|^2_{L^2}) + C\tau\,\|v\|^2_{m,-1,\tau} + C'\,|T(v)|^2_{m-1,1/2,\tau} \geq C''\,\|v\|^2_{m,\tau}\,,$$

which gives the result by choosing τ sufficiently large. □

4.4. **A microlocal Carleman estimate.** With the previous results if the strong Lopatinskii condition holds at one point of the cotangent bundle at the boundary we can then derive a Carleman estimate that holds microlocally, that is, with a cut-off in phase-space applied through a tangential pseudo-differential operator.

Theorem 4.4. *Let $x_0 \in \partial\Omega \cap V$. Assume that $\{P,\varphi\}$ satisfies the sub-ellipticity condition on a neighborhood of x_0 in $\overline{V_+}$. Assume moreover that $\{P, B^k, \varphi,\; k = 1,\dots,\mu\}$ satisfies the strong Lopatinskii condition at $\varrho_0' = (x_0,\xi_0',\tau_0) \in \mathbb{S}^*_{\mathsf{T},\tau}(\overline{V_+})$. Then there exists $\mathscr{U}$ a conic open neighborhood of ϱ_0' in $\overline{V_+} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ such that for $\chi \in S^0_{\mathsf{T},\tau}$ with $\mathrm{supp}(\chi) \subset \mathscr{U}$, there exist $C>0$ and $\tau_*>0$ such that*

$$\begin{aligned}(4.8)\quad \|P_\varphi v\|^2_{L^2} + \sum_{k=1}^{\mu} \big|B^k_\varphi v_{|x_n=0^+}\big|^2_{m-\beta_k-1/2,\tau} \\ + \|v\|^2_{m,-1,\tau} + |T(v)|^2_{m-1,-1/2,\tau} \geq C\big(\tau^{-1}\,\|\mathrm{Op}(\chi)v\|^2_{m,\tau} + |T(\mathrm{Op}(\chi)v)|^2_{m-1,1/2,\tau}\big),\end{aligned}$$

for $\tau \geq \tau_*$, $v \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$.

Note that there are remainder terms, viz.

$$\|v\|^2_{m,-1,\tau} + |T(v)|^2_{m-1,-1/2,\tau}$$

that concern the unknown function v everywhere and not only in the microlocal region $\mathscr{U}$ we consider here. The norms of these remainder terms are weaker that those in the r.h.s. of the estimates. When patching microlocal estimates of the form of (4.8) together these remainder terms can be dealt with; see Section 4.5 below.

Proof. Let U_0 be a open neighborhood of x_0 in $\overline{V_+}$ with the sub-ellipticity condition holding in $\overline{U_0}$.

In the local coordinates we have chosen we have

$$P = P(x, D) = \sum_{j=1}^{m} P_j(x, D')D_n^j,$$

with $P_m = 1$ (see Section 1.6). We decompose the conjugated operator $P_\varphi = e^{\tau\varphi} P e^{-\tau\varphi}$ as

$$P_\varphi = P_2 + iP_1, \quad P_2 = \frac{1}{2}(P_\varphi + P_\varphi^*), \;\; P_1 = \frac{1}{2i}(P_\varphi - P_\varphi^*).$$

The operators P_2 and P_1 are thus formally self-adjoint. Their respective principal symbols $a(x, \xi, \tau) \in S_\tau^{m,0}$ and $b(x, \xi, \tau) \in S_\tau^{m-1,1}$ are both real and homogeneous. We set $p_\varphi = a + ib$. We then consider the following interior quadratic form of type $(m, 0)$

$$Q_{a,b}(v) = 2\,\mathrm{Re}(Av, iBv)_+, \qquad A = \mathrm{Op}(a),\; B = \mathrm{Op}(b).$$

Note that we have

$$P_\varphi = A + iB + R, \qquad R \in \Psi_\tau^{m,-1}. \tag{4.9}$$

The sub-ellipticity condition of Definition 1.1 reads

$$p_\varphi(x, \xi, \tau) = 0 \;\; \Rightarrow \;\; \{a, b\}(x, \xi, \tau) > 0,$$

for $x \in U_0$ and $(\xi, \tau) \neq (0, 0)$. Note that the case $\tau = 0$ is achieved because of the ellipticity of P (see Definition 1.1 and Remark 1.2).

Let now $\mathscr{U}$ be as given by Lemma 4.2, possibly reduced so that $\mathscr{U} \subset U_0 \times \mathbb{R}^{n-1} \times \mathbb{R}_+$, and let χ be as in the statement of the theorem. By Lemma 4.3 we then have, for $\underline{v} = \mathrm{Op}(\chi)v$,

$$\mathrm{Re}\left(Q_{a,b}(\underline{v}) - \mathscr{B}_{a,b}(\underline{v})\right) \geq C\tau^{-1}\,\|\underline{v}\|_{m,\tau}^2 - C'\tau^{-1}\big(\,\|A\underline{v}\|_{L^2}^2 + \|B\underline{v}\|_{L^2}^2 + |T(\underline{v})|_{m-1,1/2,\tau}^2\big), \tag{4.10}$$

with $\mathscr{B}_{a,b}(\underline{v})$ given by (3.15). As $\mathscr{B}_{a,b}$ is of type $(m-1, 1/2)$ we have

$$|\mathscr{B}_{a,b}(\underline{v})| \lesssim |T(\underline{v})|_{m-1,1/2,\tau}^2\,.$$

With Lemma 4.2, making use of the strong Lopatinskii condition, we obtain for M chosen sufficiently large

$$\begin{aligned}\mathrm{Re}\,\mathscr{B}_{a,b}(\underline{v}) + M\sum_{k=1}^{\mu}\big|B_\varphi^k v_{|x_n=0^+}\big|_{m-\beta_k-1/2,\tau}^2 \\ \geq C\,|T(\underline{v})|_{m-1,1/2,\tau}^2 - C'\big(\,\|v\|_{m,-1,\tau}^2 + |T(v)|_{m-1,-1/2,\tau}^2 + \|P_\varphi v\|_{L^2}^2\,\big).\end{aligned} \tag{4.11}$$

Summing (4.10) and (4.11) we find, by taking τ sufficiently large,

$$\begin{aligned}\mathrm{Re}\,Q_{a,b}(\underline{v}) + \|P_\varphi v\|_{L^2}^2 + \tau^{-1}(\|A\underline{v}\|_{L^2}^2 + \|B\underline{v}\|_{L^2}^2) + \sum_{k=1}^{\mu}\big|B_\varphi^k v_{|x_n=0^+}\big|_{m-\beta_k-1/2,\tau}^2 \\ + \|v\|_{m,-1,\tau}^2 + |T(v)|_{m-1,-1/2,\tau}^2 \gtrsim \tau^{-1}\,\|\underline{v}\|_{m,\tau}^2 + |T(\underline{v})|_{m-1,1/2,\tau}^2\,.\end{aligned}$$

Finally, noting that

$$\|A\underline{v}\|_{L^2}^2 + \|B\underline{v}\|_{L^2}^2 + \mathrm{Re}\,Q_{a,b}(\underline{v}) \leq \|(A + iB)\underline{v}\|_{L^2}^2 \lesssim \|P_\varphi\underline{v}\|_{L^2}^2 + \|\underline{v}\|_{m,-1,\tau}^2 \lesssim \|P_\varphi v\|_{L^2}^2 + \|v\|_{m,-1,\tau}^2\,,$$

by (4.9) and pseudo-differential calculus (last point of Proposition 2.7), we obtain the sought microlocal estimate. □

4.5. **Proof of Theorem 1.6.** We shall patch together estimates of the form given in Theorem 4.4.

With x_0 as in the statement of Theorem 1.6 the strong Lopatinskii condition holds for all boundary quadruples $\omega = (x_0, Y, N, \tau)$ with $Y \in T^*_{x_0}(\partial\Omega)$, $N \in N^*_{x_0}(\partial\Omega)$, $\tau \geq 0$. In the local coordinates that we use here this means that this property is satisfied for N equal to the unit conormal to $\{x_n = 0\}$ and all $\varrho' = (x_0, \xi', \tau)$ with $\xi' \in \mathbb{R}^{n-1}$ and $\tau \geq 0$. (See Section 1.6.) It is fact sufficient to consider $(\xi', \tau) \in \mathbb{S}^{n-1}_+ = \{(\xi', \tau) \in \mathbb{R}^n,\ \tau \geq 0,\ |(\xi', \tau)| = 1\}$.

By Theorem 4.4 for all $(\xi'_0, \tau_0) \in \mathbb{S}^{n-1}_+$ there exists a conic open set $\mathscr{U}_{\varrho'_0}$ of $\varrho'_0 = (x_0, \xi'_0, \tau_0)$ in $\overline{V_+} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ such that the estimate (4.8) holds. In fact by reducing $\mathscr{U}_{\varrho'_0}$ we can choose $\mathscr{U}_{\varrho'_0} = \mathcal{O}_{\varrho'_0} \times \Gamma_{\varrho'_0}$ where $\mathcal{O}_{\varrho'_0}$ is an open set in $\overline{V_+}$ and $\Gamma_{\varrho'_0}$ is a conic open set in $\mathbb{R}^{n-1} \times \mathbb{R}_+$. With the compactness of $\mathbb{S}^{n-1}_+$ we can thus find finitely many such open sets $\mathscr{U}_j = \mathcal{O}_j \times \Gamma_j$, $j \in J$, such that $\mathbb{S}^{n-1}_+ \subset \cup_{j\in J}\Gamma_j$. We then set $\mathcal{O} = \cap_{j\in J}\mathcal{O}_j$ that is an open neighborhood of x_0 in $\overline{V_+}$ and we set $\mathscr{V}_j = \mathcal{O} \times \Gamma_j \subset \mathscr{U}_j$. We also choose an open neighborhood W of x_0 in $\mathbb{R}^n$ such that $W^+ = W \cap \overline{V_+} \Subset \mathcal{O}$.

We then choose a partition of unity $\chi_j \in S^0_{\mathsf{T},\tau}$, $j \in J$, on $\overline{W_+} \times \mathbb{R}^{n-1} \times R_+$ subordinated by the covering by the open sets $\mathscr{V}_j$:

$$\sum_{j\in J} \chi_j(\varrho') = 1, \text{ for } \varrho' = (x, \xi', \tau) \in \overline{W_+} \times \mathbb{R}^{n-1} \times R_+ \text{ and } |(\xi', \tau)| \geq r_0 > 0, \qquad \operatorname{supp}(\chi_j) \subset \mathscr{V}_j.$$

The symbols χ_j are chosen homogeneous of degree 0 for $|(\xi', \tau)| \geq r_0 > 0$. We set $\underline{\chi} = 1 - \sum_{j\in J}\chi_j$ and have $\underline{\chi} \in \cap_{N\in\mathbb{N}} S^{-N}_{\mathsf{T},\tau}$.

As $\operatorname{supp}(\chi_j) \subset \mathscr{U}_j$, we can apply the microlocal estimate of Theorem 4.4:

$$\begin{aligned}(4.12)\qquad \|P_\varphi v\|^2_{L^2} &+ \sum_{k=1}^{\mu} \left|B^k_\varphi v_{|x_n=0^+}\right|^2_{m-\beta_k-1/2,\tau} \\ &+ \|v\|^2_{m,-1,\tau} + |T(v)|^2_{m-1,-1/2,\tau} \gtrsim \tau^{-1}\|\operatorname{Op}(\chi_j)v\|^2_{m,\tau} + |T(\operatorname{Op}(\chi_j)v)|^2_{m-1,1/2,\tau},\end{aligned}$$

for τ chosen sufficiently large and for $v = e^{\tau\varphi}u$ with $u = w_{|\overline{\mathbb{R}}^n_+}$ with $w \in \mathscr{C}^\infty_c(W)$.

Observe then that, for any $N \in \mathbb{N}$,

$$\|v\|_{m,\tau} \leq \sum_{j\in J}\|\operatorname{Op}(\chi_j)v\|_{m,\tau} + \left\|\operatorname{Op}(\underline{\chi})v\right\|_{m,\tau} \lesssim \sum_{j\in J}\|\operatorname{Op}(\chi_j)v\|_{m,\tau} + \|v\|_{m,-N,\tau},$$

and

$$\begin{aligned}|T(v)|_{m-1,1/2,\tau} &\leq \sum_{j\in J}|T(\operatorname{Op}(\chi_j)v)|_{m-1,1/2,\tau} + \left|T(\operatorname{Op}(\underline{\chi})v)\right|_{m-1,1/2,\tau} \\ &\lesssim \sum_{j\in J}|T(\operatorname{Op}(\chi_j)v)|_{m-1,1/2,\tau} + |T(v)|_{m-1,-N,\tau}.\end{aligned}$$

Summing the estimates 4.12 for each χ_j we thus obtain

$$\|P_\varphi v\|^2_{L^2} + \sum_{k=1}^{\mu}\left|B^k_\varphi v_{|x_n=0^+}\right|^2_{m-\beta_k-1/2,\tau} + \|v\|^2_{m,-1,\tau} + |T(v)|^2_{m-1,-1/2,\tau} \gtrsim \tau^{-1}\|v\|^2_{m,\tau} + |T(v)|^2_{m-1,1/2,\tau}.$$

Choosing now τ sufficiently large we obtain

$$(4.13)\qquad \|P_\varphi v\|^2_{L^2} + \sum_{k=1}^{\mu}\left|B^k_\varphi v_{|x_n=0^+}\right|^2_{m-\beta_k-1/2,\tau} \gtrsim \tau^{-1}\|v\|^2_{m,\tau} + |T(v)|^2_{m-1,1/2,\tau}.$$

Setting $v = e^{\tau\varphi}u$ the conclusion of the proof of Theorem 1.6 is then classical. □

4.6. **Shifted estimates.** It may be interesting to consider shifted esimates in the Sobolev scales. Namely we may wish to have an estimate of the following form.

Corollary 4.5. *Let $x_0 \in \partial\Omega$ and let $\varphi \in \mathscr{C}^\infty(\Omega)$ be such that the pair $\{P, \varphi\}$ has the sub-ellipticity property of Definition 1.1 in a neighborhood of x_0 in $\overline{\Omega}$. Moreover, assume that $\{P, \varphi, B^k,\ k = 1, \dots, \mu\}$ satisfies the strong Lopatinskii condition at x_0. Let $\ell \in \mathbb{N}$. Then there exist a neighborhood W of x_0 in $\mathbb{R}^n$ and two constants C and $\tau_* > 0$ such that*

(4.14)
$$\tau^{-1} \left\|e^{\tau\varphi}u\right\|^2_{m+\ell,\tau} + \left|e^{\tau\varphi}T(u)\right|^2_{m+\ell-1,1/2,\tau} \leq C\Big(\left\|e^{\tau\varphi}P(x,D)u\right\|^2_{\ell,\tau} + \sum_{k=1}^{\mu} \left|e^{\tau\varphi}T(B^k(x,D)u)\right|^2_{\ell,m-1/2-\beta_k,\tau}\Big),$$

for all $u = w_{|\Omega}$ with $w \in \mathscr{C}^\infty_c(W)$ and $\tau \geq \tau_*$.

Proof. We proceed by induction on ℓ. As the result holds for $\ell = 0$ we assume it holds for some $\ell \in \mathbb{N}$; we then have the counterpart of (4.13):

$$\|P_\varphi v\|^2_{\ell,\tau} + \sum_{k=1}^{\mu} \left|T(B^k_\varphi v)\right|^2_{\ell,m-\beta_k-1/2,\tau} \gtrsim \tau^{-1} \|v\|^2_{m+\ell,\tau} + |T(v)|^2_{\ell,m-1,1/2,\tau},$$

which we shall apply to $D_{x_n}v$ and $D^\alpha_{x'}v$ for $|\alpha| = 1$. We have

$$\begin{aligned}\|P_\varphi D_{x_n}v\|_{\ell,\tau} + \|P_\varphi D^\alpha_{x'}v\|_{\ell,\tau} &\leq \|P_\varphi v\|_{\ell+1,\tau} + \|[P_\varphi, D_{x_n}]v\|_{\ell,\tau} + \|[P_\varphi, D^\alpha_{x'}]v\|_{\ell,\tau}\\ &\lesssim \|P_\varphi v\|_{\ell+1,\tau} + \|v\|_{m+\ell,\tau}.\end{aligned}$$

We also have

$$\begin{aligned}&\left|T(B^k_\varphi D_{x_n}v)\right|_{\ell,m-\beta_k-1/2,\tau} + \left|T(B^k_\varphi D^\alpha_{x'}v)\right|_{\ell,m-\beta_k-1/2,\tau}\\ &\quad\leq \left|T(D_{x_n}B^k_\varphi v)\right|_{\ell,m-\beta_k-1/2,\tau} + \left|T(D^\alpha_{x'}B^k_\varphi v)\right|_{\ell,m-\beta_k-1/2,\tau} + \left|T(v)\right|_{\ell+\beta_k,m-\beta_k-1/2,\tau}\\ &\quad\leq \left|T(B^k_\varphi v)\right|_{\ell+1,m-\beta_k-1/2,\tau} + \left|T(v)\right|_{\ell+\beta_k,m-\beta_k-1/2,\tau}.\end{aligned}$$

We thus have

$$\begin{aligned}&\|P_\varphi v\|_{\ell+1,\tau} + \left|T(B^k_\varphi v)\right|_{\ell+1,m-\beta_k-1/2,\tau} + \|v\|_{m+\ell,\tau} + \left|T(v)\right|_{\ell+\beta_k,m-\beta_k-1/2,\tau}\\ &\quad\gtrsim \tau \|P_\varphi v\|_{\ell,\tau} + \|P_\varphi D_{x_n}v\|_{\ell,\tau} + \|P_\varphi D^\alpha_{x'}v\|_{\ell,\tau}\\ &\qquad + \tau\left|T(B^k_\varphi v)\right|_{\ell,m-\beta_k-1/2,\tau} + \left|T(B^k_\varphi D_{x_n}v)\right|_{\ell,m-\beta_k-1/2,\tau} + \left|T(B^k_\varphi D^\alpha_{x'}v)\right|_{\ell,m-\beta_k-1/2,\tau}.\end{aligned}$$

This yields, by induction,

$$\begin{aligned}&\|P_\varphi v\|_{\ell+1,\tau} + \left|T(B^k_\varphi v)\right|_{\ell+1,m-\beta_k-1/2,\tau} + \|v\|_{m+\ell,\tau} + \left|T(v)\right|_{\ell+\beta_k,m-\beta_k-1/2,\tau}\\ &\quad\gtrsim \tau^{\frac12} \|v\|_{m+\ell,\tau} + \tau^{-\frac12} \sum_{1\leq j\leq n} \left\|D_{x_j}v\right\|_{m+\ell,\tau} + \tau\,|T(v)|_{\ell+m-1,1/2,\tau} + \sum_{1\leq j\leq n} \left|T(D_{x_j}v)\right|_{\ell+m-1,1/2,\tau}\\ &\quad\gtrsim \tau^{-1/2} \|v\|_{m+\ell+1,\tau} + |T(v)|_{\ell+m,1/2,\tau},\end{aligned}$$

which then implies the result. □

5. A PSEUDO-DIFFERENTIAL CALCULUS WITH TWO LARGE PARAMETERS

The weight function we shall consider below is of the form $\varphi(x) = \exp(\gamma\psi(x))$. The function ψ is assumed to be $\mathscr{C}^\infty$ and to satisfy

$$|\psi'| > 0, \quad \psi \geq C > 0 \quad \text{and } \|\psi\|_{L^\infty} < \infty.$$

We take $\gamma \geq 1$. The goal of what follows is to achieve estimates as in Theorem 1.6 with the explicit dependency upon the additional parameter γ. This can be done by the introduction of an appropriate pseudo-differential calculus. Assumption of the function ψ will be made in Section 6.1, namely, the strong pseudo-convexity conditions, to obtain a Carleman estimate.

Here, by ϱ and ϱ' we shall denote $\varrho = (x, \xi, \tau, \gamma) \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+$ and $\varrho' = (x, \xi', \tau, \gamma) \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+$.

5.1. **Metric, symbols, operators and Sobolev norms.** We set $\tilde{\tau}(x) = \tau\gamma\varphi(x)$. We consider the metrics on phase-space

$$g = \gamma^2|dx|^2 + \frac{|d\xi|^2}{\mu^2}, \qquad \text{with } \mu^2 = \mu^2(\varrho) = |(\tilde{\tau}(x), \xi)|^2 = \tilde{\tau}(x)^2 + |\xi|^2,$$

and on tangent phase space

$$g_{\mathsf{T}} = \gamma^2|dx|^2 + \frac{|d\xi'|^2}{{\mu_{\mathsf{T}}}^2}, \qquad \text{with } {\mu_{\mathsf{T}}}^2 = {\mu_{\mathsf{T}}}^2(\varrho') = |(\tilde{\tau}(x), \xi')|^2 = \tilde{\tau}(x)^2 + |\xi'|^2,$$

for $\tau \geq 1$ and $\gamma \geq 1$. Below, the explicit dependencies of μ and μ_{T} upon ϱ and ϱ' are dropped to ease notation.

The metric g (resp. g_{T}) along with the order function μ (resp. μ_{T}) generates a (resp. tangential) Weyl-Hörmander pseudo-differential calculus as proven in [30, Proposition 2.2]. For a presentation of the Weyl-Hörmander calculus we refer to [36], [19, Sections 18.4–6] and [18].

Let $a(x, \xi, \tau, \gamma) \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, with τ, γ as parameter in $[\tau_{\min}, +\infty)$ and $[\gamma_{\min}, +\infty)$, $\tau_{\min} > 0$, $\gamma_{\min} > 0$, and $m \in \mathbb{R}$, be such that for all multi-indices $\alpha, \beta \in \mathbb{N}^n$ we have

$$\left|\partial_x^\alpha \partial_\xi^\beta a(\varrho)\right| \leq C_{\alpha,\beta}\gamma^\alpha \mu^{m-|\beta|}, \qquad \varrho \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^n \times [\tau_{\min}, +\infty) \times [\gamma_{\min}, +\infty). \tag{5.1}$$

With the notation of [19, Sections 18.4-18.6] we then have $a(\varrho) \in S(\mu^m, g)$. For simplicity we shall write $a \in S^m_{\tilde{\tau}}$.

The associated class of pseudo-differential operators, as given by (2.3), is denoted by $\Psi^m_{\tilde{\tau}}$. If a is polynomial in ξ, τ, γ and $\varphi(x)$ then we write $\mathrm{Op}(a) \in \mathscr{D}^m_{\tilde{\tau}}$.

Similarly, let $a(x, \xi', \tau, \gamma) \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$ and $m \in \mathbb{R}$, be such that for all multi-indices $\alpha, \beta \in \mathbb{N}^n$ we have

$$\left|\partial_x^\alpha \partial_{\xi'}^\beta a(\varrho')\right| \leq C_{\alpha,\beta}\gamma^\alpha {\mu_{\mathsf{T}}}^{m-|\beta|}, \qquad \varrho' \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times [\tau_{\min}, +\infty) \times [\gamma_{\min}, +\infty). \tag{5.2}$$

We then have $a(\varrho') \in S^m_{\mathsf{T},\tilde{\tau}} = S({\mu_{\mathsf{T}}}^m, g_{\mathsf{T}})$.

The associated class of pseudo-differential operators, as given by (2.4), is denoted by $\Psi^m_{\mathsf{T},\tilde{\tau}}$.

With $\varrho = (x, \xi, \tau, \gamma) \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+$ (resp. $\varrho' = (x, \xi', \tau, \gamma) \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+$) we shall associate $\tilde{\varrho} = (x, \xi, \tilde{\tau}(x)) \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^n \times \mathbb{R}_+$ (resp. $\tilde{\varrho}' = (x, \xi', \tilde{\tau}(x)) \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+$).

Note that if $\hat{a}(x, \xi, \hat{\tau}) \in S^m_\tau$, with the notation of Section 2.1, satisfying moreover, for all multi-indices $\alpha, \beta', \beta'' \in \mathbb{N}^n$, with $\beta = \beta' + \beta''$,

$$\left|\partial_x^\alpha \partial_\xi^{\beta'} \partial_{\hat{\tau}}^{\beta''} \hat{a}(x, \xi, \hat{\tau})\right| \leq C_{\alpha,\beta',\beta''} \left|(\xi, \hat{\tau})\right|^{m-|\beta|}, \quad x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n,\ \hat{\tau} \in [\tau_{\min}, +\infty), \tag{5.3}$$

i.e., differentiation w.r.t. $\hat{\tau}$ yields the same additional decay as a differentiation w.r.t. ξ, then

$$a(x, \xi, \tau, \gamma) = \hat{a}(x, \xi, \tilde{\tau}(x)) \in S^m_{\tilde{\tau}},$$

which we shall write $a(\varrho) = \hat{a}(\tilde{\varrho})$. Similarly if $\hat{a}(x, \xi', \hat{\tau}) \in S^m_{\mathsf{T},\tilde{\tau}}$ with the same additional property regarding differentiation w.r.t. $\hat{\tau}$ we have $a(\varrho') = \hat{a}(\tilde{\varrho}') \in S^m_{\mathsf{T},\tilde{\tau}}$.

In what follows we shall assume that symbol in $S^m_{\tilde\tau}$ and $S^m_{\mathsf{T},\tilde\tau}$ have this additional regularity property.

We then say that $a \in S^m_{\tilde\tau}$ (resp. $S^m_{\mathsf{T},\tilde\tau}$) is homogeneous of degree m with respect to $(\xi,\tilde\tau)$ (resp. $(\xi',\tilde\tau)$) if we have $a(\varrho) = \hat a(\tilde\varrho))$ (resp. $a(\varrho') = \hat a(\tilde\varrho')$) with $\hat a(x,\xi,\hat\tau) \in S^m_\tau$ (resp. $\hat a(x,\xi',\hat\tau) \in S^m_{\mathsf{T},\tau}$) homogeneous of degree m in $(\xi,\hat\tau)$ (resp. $(\xi',\hat\tau)$).

We shall also use the following classes of symbols $S(\tilde\tau^r {\mu_\mathsf{T}}^m, g_\mathsf{T}) = \tilde\tau^r S^m_{\mathsf{T},\tilde\tau}$ on $\overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1}$, for $r,m \in \mathbb{R}$. The associated class of tangential pseudo-differential operators is denoted by $\tilde\tau^r \Psi({\mu_\mathsf{T}}^m, g_\mathsf{T}) = \tilde\tau^r \Psi^m_{\mathsf{T},\tilde\tau}$.

We shall say that $a(\varrho) \in \tilde\tau^r S^{m,\sigma}_{\tilde\tau}$ if

$$a(\varrho) = \sum_{j=0}^m a_j(\varrho')\xi_n^j, \quad \text{with } a_j \in \tilde\tau^r S^{m-j+\sigma}_{\mathsf{T},\tilde\tau}.$$

The principal part is given by $\sum_{j=0}^m \sigma(a_j)(\varrho')\xi_n^j$. The associated class of pseudo-differential operators is denoted by $\tilde\tau^r \Psi^{m,\sigma}_{\tilde\tau}$.

We have the following lemma whose proof is similar to that of Lemma 2.7 in [30].

Lemma 5.1. *Let $r,m \in \mathbb{R}$ and $a \in \tilde\tau^r S^m_{\mathsf{T},\tilde\tau}$. There exists $C>0$ such that for τ sufficiently large*

$$|\,(\operatorname{Op}(a)u,v)_{L^2} \le C\big\|\operatorname{Op}(\tilde\tau^{r'}{\mu_\mathsf{T}}^{m'})u\big\|_{L^2}\big\|\operatorname{Op}(\tilde\tau^{r''}{\mu_\mathsf{T}}^{m''})v\big\|_{L^2}, \qquad u,v \in \mathscr{S}(\overline{\mathbb{R}}^n_+).$$

for $r = r'+r''$, $m = m'+m''$.

This contains the estimate

$$\big\|\operatorname{Op}(\tilde\tau^s{\mu_\mathsf{T}}^p)\operatorname{Op}(a)u\big\|_{L^2} \le C\big\|\operatorname{Op}(\tilde\tau^{s+r}{\mu_\mathsf{T}}^{p+m})u\big\|_{L^2}, \qquad u \in \mathscr{S}(\overline{\mathbb{R}}^n_+),$$

for $s,p \in \mathbb{R}$.

Note also that we have

$$\big\|\operatorname{Op}(\tilde\tau^r{\mu_\mathsf{T}}^m)u\big\|_{L^2} \asymp \big\|\operatorname{Op}({\mu_\mathsf{T}}^m)\tilde\tau^r u\big\|_{L^2}, \qquad u \in \mathscr{S}(\overline{\mathbb{R}}^n_+), \tag{5.4}$$

for τ chosen sufficiently large.

Next we say that $a(x,\xi',\tau,\gamma) \in \tilde\tau^r S^m_{\mathsf{T},\tilde\tau,\mathrm{cl}}$ if there exists $a_j \in \gamma^j\tilde\tau^r S^{m-j}_{\mathsf{T},\tau}$, with $\gamma^{-j}a_j$ homogeneous of degree $m+r-j$ in $(\xi',\tilde\tau)$ for $|(\xi',\tilde\tau)| \ge r_0$, with $r_0 \ge 0$, such that

$$a \sim \sum_{j\ge 0} a^{(j)}, \quad \text{in the sense that} \quad a - \sum_{j=0}^N a^{(j)} \in \gamma^{N+1}\tilde\tau^r S^{m-N-1}_{\mathsf{T},\tau}. \tag{5.5}$$

A representative of the principal part, denoted by $\sigma(a)$, is then given by the first term in the expansion.

Then, we shall say that $a(\varrho) \in \tilde\tau^r S^{m,\sigma}_{\tilde\tau,\mathrm{cl}}$ if

$$a(\varrho) = \sum_{j=0}^m a_j(\varrho')\xi_n^j, \quad \text{with } a_j \in \tilde\tau^r S^{m-j+\sigma}_{\mathsf{T},\tilde\tau,\mathrm{cl}}.$$

The principal part is given by $\sum_{j=0}^m \sigma(a_j)(\varrho')\xi_n^j$.

With these symbol classes we associate classes of pseudo-differential operators, $\tilde\tau^r\Psi^m_{\mathsf{T},\tilde\tau,\mathrm{cl}}$ and $\tilde\tau^r\Psi^{m,\sigma}_{\tilde\tau,\mathrm{cl}}$, as is done in Section 2.2.

We define the following semi-classical interior norm

$$|u|^2_{m,\tilde\tau} = |\operatorname{Op}({\mu_\mathsf{T}}^m)u|^2_{L^2(\mathbb{R}^{n-1})}.$$

$$\|u\|^2_{m,\tilde\tau} = \sum_{j=0}^m \big\|\operatorname{Op}({\mu_\mathsf{T}}^{m-j})D_n^j u\big\|^2_{L^2}, \quad m \in \mathbb{N}.$$

We also set

$$\|u\|^2_{m,\sigma,\tilde\tau} = \left\| \operatorname{Op}(\mu_{\mathsf{T}}{}^{\sigma})u\right\|^2_{m,\tilde\tau} \sim \sum_{j=0}^{m} \left\| \operatorname{Op}(\mu_{\mathsf{T}}{}^{m-j+\sigma})D_n^j u\right\|^2_{L^2}, \quad m\in\mathbb{N}, \sigma\in\mathbb{R}.$$

At the boundary $\{x_n = 0^+\}$ we define the following norms:

$$|T(u)|^2_{m,\sigma,\tilde\tau} = \sum_{j=0}^{m} |\operatorname{Op}(\mu_{\mathsf{T}}{}^{m-j+\sigma})T_j(u)|^2_{L^2}, \quad m\in\mathbb{N}, \sigma\in\mathbb{R}.$$

Proposition 5.2. *Let $r, m \in \mathbb{R}$, and $a \in \tilde\tau^r S^m_{\mathsf{T},\tilde\tau}$. Then, for $r', m' \in \mathbb{R}$, there exists $C > 0$ such that*

$$\left|\tilde\tau^{r'} \operatorname{Op}(a)u_{|x_n=0^+}\right|_{m',\tilde\tau} \le C \left|\tilde\tau^{r+r'} u_{|x_n=0^+}\right|_{m+m',\tilde\tau}, \quad u\in\mathscr{S}(\overline{\mathbb{R}}^n_+).$$

Proposition 5.3. *Let $r, \sigma \in \mathbb{R}$, $m \in \mathbb{N}$, and $a \in \tilde\tau^r S^{m,\sigma}_{\tilde\tau}$. Then, for $r', \sigma' \in \mathbb{R}$ and $m' \in \mathbb{N}$, there exists $C > 0$ such that*

$$\left\|\tilde\tau^{r'} \operatorname{Op}(a)u\right\|_{m',\sigma',\tilde\tau} \le C \left\|\tilde\tau^{r+r'} u\right\|_{m+m',\sigma+\sigma',\tilde\tau}, \quad u\in\mathscr{S}(\overline{\mathbb{R}}^n_+).$$

5.2. **Differential forms.**

5.2.1. *Interior quadratic forms.*

Definition 5.4. *Let $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$. We say that*

$$Q(u) = \sum_{s=1}^{N} (A^s u, B^s u)_+, \qquad A^s = a^s(x, D, \tau, \gamma),\ B^s = b^s(x, D, \tau, \gamma), \tag{5.6}$$

is a quadratic form of type (r, m, σ) with $\mathscr{C}^\infty$ coefficients, if for each $s = 1, \dots N$, we have $a^s(\varrho) \in \tilde\tau^{r'} S^{m,\sigma'}_{\tilde\tau,\mathrm{cl}}$, $b^s(\varrho) \in \tilde\tau^{r''} S^{m,\sigma''}_{\tilde\tau,\mathrm{cl}}$, with $r' + r'' = 2r$ and $\sigma' + \sigma'' = 2\sigma$.

The symbol of the quadratic form Q is defined by

$$q(\varrho) = \sum_{s=1}^{N} a^s(\varrho)\overline{b^s}(\varrho) \in \tilde\tau^{2r} S^{2m,2\sigma}_{\tilde\tau,\mathrm{cl}}. \tag{5.7}$$

As in Section 3.1 an interior quadratic from can be written in the form

$$Q(u) = \sum_{j=0}^{m}\sum_{k=0}^{m} \left(C_{j,k} D_n^j u, D_n^k u\right)_+,$$

where $C_{j,k}$ are tangential operators with symbol $c_{jk}(\varrho') \in \tilde\tau^{2r} S^{2(m+\sigma)-(j+k)}_{\mathsf{T},\tilde\tau}$.

Lemma 3.3 is then changed into the following lemma.

Lemma 5.5. *We consider the interior quadratic form of type (r, m, σ) as above. We have*

$$|Q(u)| \le C \|\tilde\tau^r u\|^2_{m,\sigma,\tau}, \quad u\in\mathscr{S}(\overline{\mathbb{R}}^n_+).$$

Lemma 3.4 is changed into the following lemma.

Lemma 5.6. *We consider the interior quadratic form of type (r, m, σ) as above and we further assume that the principal part of its symbol vanishes, that is,*

$$\sum_{\substack{1\le j,k\le m\\ j+k=\ell}} c_{j,k}(\varrho') \equiv 0 \mod \gamma\tilde\tau^{2r} S^{2(m+\sigma)-\ell-1}_{\mathsf{T},\tilde\tau}, \quad \forall \ell \in \{0, \dots, 2m\}.$$

Then, for τ sufficiently large, the following estimate holds

$$|Q(u)| \le C\big(\gamma \|\tilde\tau^r u\|^2_{m,\sigma-1/2,\tau} + \left|\tilde\tau^{2r} T(u)\right|^2_{m-1,\sigma+1/2,\tau}\big), \qquad u\in\mathscr{S}(\overline{\mathbb{R}}^n_+).$$

Proof. We only point out differences from the proof of Lemma 3.4. Estimate (3.3) is modified. As here $D_n c_{\ell-k,k} \in \gamma\tilde{\tau}^{2r} S^{2(m+\sigma)-\ell}_{\mathsf{T},\tilde{\tau}}$, with Lemma 5.1 and (5.4) (and additional commutator arguments) we write

$$\begin{aligned}
&\big|\big(\operatorname{Op}(D_n c_{\ell-k,k}) D_n^{\ell-k+s-1} u, D_n^{k-s} u\big)_+\big| \\
&\quad \le C\gamma \big\|\operatorname{Op}(\tilde{\tau}^r {\mu_{\mathsf{T}}}^{m+\sigma+k-\ell-s+\frac{1}{2}}) D_n^{\ell-k+s-1} u\big\|_{L^2} \big\|\operatorname{Op}(\tilde{\tau}^r {\mu_{\mathsf{T}}}^{m+\sigma-k+s-\frac{1}{2}}) D_n^{k-s} u\big\|_{L^2} \\
&\quad \le C\gamma \, \|\tilde{\tau}^r u\|_{\ell-k+s-1,m+\sigma+k-\ell-s+\frac{1}{2},\tilde{\tau}} \, \|\tilde{\tau}^r u\|_{k-s,m+\sigma-k+s-\frac{1}{2},\tilde{\tau}} \\
&\quad \le C\gamma \, \|\tilde{\tau}^r u\|^2_{m-1,\sigma+\frac{1}{2},\tilde{\tau}} \,,
\end{aligned}$$

for τ sufficiently large, as $m+k-\ell-s \ge 0$ and $m-1-k+s \ge 0$. Using also that

$$\sum_{k=\alpha_\ell}^{\beta_\ell} C_{\ell-k,k} = \sum_{\substack{1\le j,k\le m \\ j+k=\ell}} C_{\ell-k,k} \in \gamma\tilde{\tau}^{2r} \Psi^{2(m+\sigma)-\ell-1}_{\mathsf{T},\tilde{\tau}},$$

estimate (3.4) then becomes

$$|I_\ell| \le C\big(\gamma \, \|\tilde{\tau}^r u\|^2_{m,\sigma-\frac{1}{2},\tilde{\tau}} + |\tilde{\tau}^r T(u)|^2_{m-1,\sigma+\frac{1}{2},\tilde{\tau}}\big),$$

and the result follows. □

The Gårding inequality for interior quadratic forms reads as follows.

Proposition 5.7 (Gårding inequality)**.** *Let $\mathscr{U}$ be an open conic set in $\overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ and let Q be an interior quadratic form of type $(r,m,0)$ with its symbol $q(x,\xi,\tau,\gamma) \in \tilde{\tau}^{2r} S^{2m,0}_{\tilde{\tau},\mathrm{cl}}$ satisfying, for some $C>0$ and $R_0>0$,*

$$\operatorname{Re} q(\varrho) \ge C\tilde{\tau}^{2r}\mu^{2m}, \quad \text{for } \tilde{\varrho}' = (x,\xi',\tilde{\tau}(x)) \in \mathscr{U} \text{ and } \mu = |(\xi,\tilde{\tau}(x))| \ge R_0,$$

with $\varrho = (\varrho', \xi_n)$, $\varrho' = (x,\xi',\tau,\gamma)$, and $\xi_n \in \mathbb{R}$. Let then $\hat{\chi} \in S^0_{\mathsf{T},\tau}$ be such that $\operatorname{supp}(\hat{\chi}) \subset \mathscr{U}$ and set $\chi(\varrho') = \hat{\chi}(\tilde{\varrho}') \in S^0_{\mathsf{T},\tilde{\tau}}$. For $0 < C_0 < C$ and $N \in \mathbb{N}$ there exist τ_*, $C' > 0$, and $C''_N > 0$ such that the following inequality holds

$$\operatorname{Re} Q(\operatorname{Op}(\chi)u) \ge C_0 \, \|\tilde{\tau}^r \operatorname{Op}(\chi)u\|^2_{m,\tilde{\tau}} - C' \, |T(\tilde{\tau}^r \operatorname{Op}(\chi)u)|^2_{m-1,1/2,\tilde{\tau}} - C''_N \, \|u\|^2_{m,-N,\tilde{\tau}} \,,$$

for $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$ and $\tau \ge \tau_*$.

Remark 5.8. *With the same proof as Remark 3.6 if $\mathscr{U} = U_0 \times \mathbb{R}^n \times \mathbb{R}_+$ then*

$$\operatorname{Re} Q(u) \ge C_0 \, \|\tilde{\tau}^r u\|^2_{m,\tilde{\tau}} - C' \, |T(\tilde{\tau}^r u)|^2_{m-1,1/2,\tilde{\tau}} \,,$$

for $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$ with $\operatorname{supp}(u) \subset U_0$.

Proof. The proof follows that of Proposition 3.5. Here homogeneity of the symbols is to be understood with respect to $(\tilde{\tau},\xi)$ or $(\tilde{\tau},\xi')$ (as presented in Section 5.1).

We let $\hat{\hat{\chi}} \in S^0_{\mathsf{T},\tau}$ have the same properties as $\hat{\chi}$ with moreover $0 \le \hat{\hat{\chi}} \le 1$ and $\hat{\hat{\chi}} = 1$ on $\operatorname{supp}(\hat{\chi})$. We then set $\tilde{\chi}(\varrho') = \hat{\hat{\chi}}(\tilde{\varrho}') \in S^0_{\mathsf{T},\tilde{\tau}}$.

We define q_0 as the principal part of $\tilde{\tau}^{-2r} q$. It is homogeneous of degree $2m$ in $(\tilde{\tau},\xi)$. We have $q_0(\varrho) = \hat{q}_0(\tilde{\varrho})$ with $\hat{q}_0(x,\xi,\hat{\tau})$ homogeneous of degree $2m$ in $(\xi,\hat{\tau})$ with moreover

$$\hat{q}_0(x,\xi,\hat{\tau}) \ge C_1 |(\xi,\hat{\tau})|^{2m}, \qquad \text{for } C_0 < C_1 < C.$$

Similarly to (3.6), for $C_0 < C_2 < C_1$, we have

$$\hat{\hat{\chi}}^2(x,\xi',\hat{\tau})\big(\hat{q}^0(x,\xi,\hat{\tau}) - C_2|(\xi,\hat{\tau})|^{2m}\big) = \hat{\hat{\chi}}^2(x,\xi',\hat{\tau})|\hat{f}|^2(x,\xi,\hat{\tau}), \qquad \hat{f} \in S^{m,0}_{\mathsf{T},\tau},$$

leading to

$$\tilde{\chi}^2(\varrho')\big(q^0(\varrho) - C_2\mu^{2m}\big) = \tilde{\chi}^2(\varrho')|f|^2(\varrho),$$

with $f(\varrho) = \hat{f}(\tilde{\varrho}) \in S^{m,0}_{\mathsf{T},\tilde{\tau}}$, polynomial in ξ_n and with smooth homogeneous coefficients.

With Lemma 5.6 estimate (3.7) becomes

$$\big|\operatorname{Re} Q(\operatorname{Op}(\tilde{\chi})v) - C_2\,\|\tilde{\tau}^r \operatorname{Op}(\tilde{\chi})v\|^2_{m,\tilde{\tau}} - \|\operatorname{Op}(\tilde{\tau}^r\tilde{\chi}f)v\|^2_{L^2}\big| \le C\big(\gamma\,\|\tilde{\tau}^r v\|^2_{m,-1/2,\tilde{\tau}} + |\tilde{\tau}^r T(v)|^2_{m-1,1/2,\tilde{\tau}}\big),$$

for $v \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$ and we conclude the proof by setting $v = \operatorname{Op}(\chi)u$ and by taking τ sufficiently large. □

5.2.2. *Boundary quadratic forms and generalized Green formula.* Boundary quadratic forms can be introduced as in Section 3.2

Definition 5.9. *Let $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$. We say that*

$$\mathscr{B}(u) = \sum_{s=1}^{N}\big(A^s u_{|x_n=0^+}, B^s u_{|x_n=0^+}\big)_\partial, \quad A^s = a^s(x,D,\tau,\gamma),\ B^s = b^s(x,D,\tau,\gamma),$$

is a boundary quadratic form of type $(r, m-1, \sigma)$ with $\mathscr{C}^\infty$ coefficients, if for each $s = 1,\dots N$, we have $a^s(\varrho) \in \tilde{\tau}^{r'} S^{m-1,\sigma'}_{\tilde{\tau},\mathrm{cl}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$, $b^s(\varrho) \in \tilde{\tau}^{r''} S^{m-1,\sigma''}_{\tilde{\tau},\mathrm{cl}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$, with $r' + r'' = 2r$ and $\sigma' + \sigma'' = \sigma$.

The symbol of the boundary quadratic form $\mathscr{B}$ is defined by

$$B(\varrho', \xi_n, \tilde{\xi}_n) = \sum_{s=1}^{N} a^s(\varrho',\xi_n)\overline{b^s}(\varrho',\tilde{\xi}_n), \qquad \varrho' = (x,\xi',\tau,\gamma).$$

As in Section 3.2 we associate to $\mathscr{B}$ a bilinear symbol $\Sigma_{\mathscr{B}}(\varrho', \mathbf{z}, \mathbf{z}')$.

We let $\mathscr{W}$ be an open conic set in $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$.

Definition 5.10. *Let $\mathscr{B}$ be a boundary quadratic form of type $(0, m-1, \sigma)$ associated with the bilinear symbol $\Sigma_{\mathscr{B}}(\varrho', \mathbf{z}, \mathbf{z}')$. We say that $\mathscr{B}$ is positive definite in $\mathscr{W}$ if there exist $C > 0$ and $R > 0$ such that*

$$\Sigma_{\mathscr{B}}(\varrho'', x_n = 0^+, \mathbf{z}, \mathbf{z}) \ge C\sum_{j=0}^{m-1} \mu_{\mathsf{T}|x_n=0^+}^{2(m-1-j+\sigma)}|z_j|^2, \quad \tilde{\varrho}'' \in \mathscr{V},\ \ \mathbf{z} = (z_0,\dots,z_{m-1}) \in \mathbb{C}^m,$$

for $\mu_{\mathsf{T}} \ge R$, $\varrho'' = (x',\xi',\tau,\gamma)$, and $\tilde{\varrho}'' = (x',\xi',\tilde{\tau}(x', x_n = 0^+))$.

Lemma 5.11. *Let $\mathscr{B}$ be a boundary quadratic form of type $(0, m-1, \sigma)$ with bilinear symbol $\Sigma_{\mathscr{B}}(\varrho', \mathbf{z}, \mathbf{z}')$. If $\mathscr{B}$ is positive definite in $\mathscr{W}$, an open subset of $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$, and $\hat{\chi} \in S^0_{\mathsf{T},\tau}$ homogeneous of degree 0, with $\operatorname{supp}(\hat{\chi}_{|x_n=0^+}) \subset \mathscr{W}$. Let $N \in \mathbb{N}$. Then there exist $\tau_* \ge 1$, $\gamma_* \ge 1$, $C > 0$, $C_N > 0$ such that*

$$\mathscr{B}(\operatorname{Op}(\chi)u) \ge C\,|T(\operatorname{Op}(\chi)u)|^2_{m-1,\sigma,\tilde{\tau}} - C_N\,|T(u)|^2_{m-1,\sigma-N,\tilde{\tau}},$$

for $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$, $\tau \ge \tau_*$, $\gamma \ge \gamma_*$, and $\chi(\varrho') = \hat{\chi}(\tilde{\varrho}') \in S^0_{\mathsf{T},\tilde{\tau}}$, with $\varrho' = (x,\xi',\tau,\gamma)$ and $\tilde{\varrho}' = (x,\xi',\tilde{\tau}(x))$.

Proof. We highlight modifications from the arguments in the proof of Lemma 3.9. Here $\operatorname{Op}(\mu_{\mathsf{T}}{}^s)$ is not simply invertible (as it is not a Fourier multiplier). Yet, by Lemma 2.4 in [30], there exists $v_j \in L^2$ such that

$$D^j_n \underline{u}_{|x_n=0^+} = \operatorname{Op}(\mu_{\mathsf{T}}{}^{-(m+\sigma-1-j)})v_j.$$

Then the boundary quadratic form can be written as

$$\mathscr{B}(\underline{u}) = \sum_{j,k=0}^{m-1}\big(\operatorname{Op}(g_{jk|x_n=0^+})v_j, v_k\big)_\partial,$$

with $g_{jk} \in S^0_{\mathsf{T},\tilde{\tau},\mathrm{cl}}$ and we have $(g_{jk|x_n=0^+}) \ge C > 0$. The Gårding inequality in the tangential directions then yields the conclusion as in the proof of Lemma 3.9. □

The generalized Green formula, counterpart of that of Proposition 3.15, with a similar proof, reads as follows.

Proposition 5.12 (Generalized Green's formula)**.** *Consider two smooth and real symbols $a \in S^{m,0}_{\tilde\tau,\mathrm{cl}}$ and $b \in S^{m-1,1}_{\tilde\tau,\mathrm{cl}}$. The following identity holds true*

$$2\,\mathrm{Re}\,(Au, iBu) = H_{a,b}(u) + \mathscr{B}_{a,b}(u) + R(u), \quad A = a(x,D,\tau,\gamma),\ B = b(x,D,\tau,\gamma), \tag{5.8}$$

for any $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$. Here, $\mathscr{B}_{a,b}$ is the boundary quadratic form of type $(0, m-1, 1/2)$ given by

$$\mathscr{B}_{a,b}(u) = \sum_{j,k=0}^{m-1} \big(G_{j,k} D_n^j u_{|x_n=0^+}, D_n^k u_{|x_n=0^+}\big)_\partial$$

where $G_{j,k} = \mathrm{Op}(g_{j,k})$ with

$$g_{j,k} = \sum_{\ell=0}^{\min(j,k)} (b_\ell a_{j+k-\ell+1} - b_{j+k-\ell+1} a_\ell) \ \in S^{2m-1-(j+k)}_{\mathsf{T},\tilde\tau,\mathrm{cl}},$$

and $\gamma^{-1} H_{a,b}$ is an interior quadratic form of type $(0, m, -1/2)$ with real symbol

$$\gamma^{-1} h_{a,b}(x,\xi,\tau,\gamma) = \gamma^{-1}\,\mathrm{sub}(a,b)(x,\xi,\tau,\gamma) \in S^{2m-1,0}_{\tilde\tau,\mathrm{cl}}$$

Finally, the remainder term $R(u)$ is a quadratic form that satisfies

$$|R(u)| \le C\gamma^2 \|u\|^2_{m,-1,\tilde\tau}\,.$$

6. Carleman estimate with two large parameters

With a weight function of the form $\varphi(x) = \exp(\gamma\psi(x))$, some condition on ψ can yield φ to fulfill the sub-ellipticity condition of Definition 1.1. Those are the strong pseudo-convexity conditions introduced by L. Hörmander (see [16], [17, Section 8.6] and [20, Section 28.3]). We shall see that along with the strong Lopatinskii condition they are sufficient to derive Carleman estimates with an explicit dependency upon the additional parameter γ. In fact the strong pseudo-convexity condition is also necessary if one considers a weight function of this form; for such question we refer to [30].

6.1. Strong pseudo-convexity.

In the present article, we restrict ourselves to elliptic operators. The notion of strong pseudo-convexity then reduces to the following (the reader can compare with Section 28.3 in [20]).

Definition 6.1 (strong pseudo-convexity)**.** *We say that ψ is strongly pseudo-convex at $x \in \overline{\Omega}$ w.r.t. p if $\psi'(x) \neq 0$ and if for all $\xi \in \mathbb{R}^n$ and $\hat\tau > 0$,*

$$\text{(s-}\Psi\text{c)} \quad p(x,\xi + i\hat\tau\psi'(x)) = 0 \text{ and } \big\{p,\psi\big\}(x,\xi+i\hat\tau\psi'(x)) = 0 \\ \Rightarrow \ \frac{1}{2i}\big\{\overline{p}(x,\xi - i\hat\tau\psi'(x)), p(x,\xi+i\hat\tau\psi'(x))\big\} > 0.$$

Let U be an open subset of Ω. The function ψ is said to be strongly pseudo-convex w.r.t. $\overline{U}$ and p up to the boundary if (s-Ψc) is valid for all $x \in \overline{U}$.

Proposition 28.3.3 in [20] shows that this property imply the sub-ellipticity condition for φ for γ chosen sufficiently large, which in turn yields a Carleman estimate in an open set away from the boundary. In this section our goal is to derive a Carleman estimate at the boundary that keeps track of the dependency of the two large parameters, τ and γ, as is done in [30] away from the boundary.

Setting $\hat a = \mathrm{Re}\,p(x,\xi+i\hat\tau\psi'(x))$ and $\hat b = \mathrm{Im}\,p(x,\xi+i\hat\tau\psi'(x))$ we have

$$\frac{1}{2i}\big\{\overline{p}(x,\xi - i\hat\tau\psi'(x)), p(x,\xi+i\hat\tau\psi'(x))\big\} = \{\hat a, \hat b\}.$$

In fact we recall that we have (see (3.16))

$$\frac{1}{2i}\,\mathrm{sub}\,\big(\overline{p}(x,\xi-i\hat{\tau}\psi'(x)),p(x,\xi+i\hat{\tau}\psi'(x))\big)=\mathrm{sub}(\hat{a},\hat{b})=\sum_{|\alpha|=1}\partial_x^\alpha(\hat{b}\partial_\xi^\alpha\hat{a}-\hat{a}\partial_\xi^\alpha\hat{b})$$
$$=\{\hat{a},\hat{b}\}+\sum_{|\alpha|=1}\Big(\hat{b}\partial_\xi^\alpha\partial_x^\alpha\hat{a}-\hat{a}\partial_\xi^\alpha\partial_x^\alpha\hat{b}\Big).$$

Property (s-Ψc) may thus be written

$$p(x,\xi+i\hat{\tau}\psi'(x))=0 \text{ and } \big\{p,\psi\big\}(x,\xi+i\hat{\tau}\psi'(x))=0 \quad\Rightarrow\quad \mathrm{sub}\,\big(\overline{p}(x,\xi-i\hat{\tau}\psi'(x)),p(x,\xi+i\hat{\tau}\psi'(x))\big)>0, \tag{6.1}$$

for all $\xi\in\mathbb{R}^n$, $\hat{\tau}>0$, and $x\in\overline{U}$.

We set

$$\Theta_{p,\psi}(x,\xi,\hat{\tau})=\hat{\tau}\sum_{j,k}\partial^2_{x_jx_k}\psi(x)\,\big(\partial_{\xi_j}p(x,\zeta)\,\partial_{\xi_k}\overline{p}(x,\overline{\zeta})-\mathrm{Re}\,p(x,\zeta)\partial^2_{\xi_j\xi_k}\overline{p}(x,\overline{\zeta})\big)$$
$$+\mathrm{Im}\sum_j\big((\partial_{x_j}p)(x,\zeta)\,\partial_{\xi_j}\overline{p}(x,\overline{\zeta})+p(x,\zeta)(\partial^2_{x_j\xi_j}\overline{p})(x,\overline{\zeta})\big),$$

where $\zeta=\xi+i\hat{\tau}\psi'$. Observe that $\Theta_{p,\psi}(x,\xi,\hat{\tau})$ is homogeneous of degree $2m-1$ in $(\xi,\hat{\tau})$ and that we have

$$\mathrm{sub}\,\big(\overline{p}(x,\xi-i\hat{\tau}\psi'(x)),p(x,\xi+i\hat{\tau}\psi'(x))\big)=\Theta_{p,\psi}(x,\xi,\hat{\tau}).$$

We set $p_\varphi(x,\xi,\tau,\gamma)=p(x,\xi+i\tau\varphi')\in S^{m,0}_{\tilde{\tau}}$. We write $P_\varphi=A+iB+R$, with $A=\mathrm{Op}(a)\in\Psi^{m,0}_{\tilde{\tau}}$, $B=\mathrm{Op}(b)\in\Psi^{m-1,1}_{\tilde{\tau}}$, where $a=\mathrm{Re}\,p_\varphi$ and $b=\mathrm{Im}\,p_\varphi$, and with $R\in\gamma\Psi^{m,-1}_{\tilde{\tau}}$. As in Section 4, part of the analysis relies on the properties of the symbol $\mathrm{sub}\,(a,b)$. Here $\varphi=\exp(\gamma\psi)$.

We compute

$$\begin{aligned}\mathrm{sub}\,(a,b)&=\Theta_{p,\varphi}(x,\xi,\tau)\\&=\tilde{\tau}(x)\sum_{j,k}\partial^2_{x_jx_k}\psi(x)\,\big(\partial_{\xi_j}p(x,\zeta)\,\partial_{\xi_k}\overline{p}(x,\overline{\zeta})-\mathrm{Re}\,p(x,\zeta)\partial^2_{\xi_j\xi_k}\overline{p}(x,\overline{\zeta})\big)\\&\quad+\gamma\tilde{\tau}(x)\big(\big|\langle p'_\xi(x,\zeta),\psi'(x)\rangle\big|^2-\mathrm{Re}\,p(x,\zeta)\sum_{j,k}\partial_{x_j}\psi(x)\partial_{x_k}\psi(x)\partial^2_{\xi_j\xi_k}\overline{p}(x,\overline{\zeta})\big)\\&\quad+\mathrm{Im}\sum_j\big((\partial_{x_j}p)(x,\zeta)\,\partial_{\xi_j}\overline{p}(x,\overline{\zeta})+p(x,\eta)(\partial^2_{x_j\xi_j}\overline{p})(x,\overline{\zeta})\big),\\&=\Theta_{p,\psi}(x,\xi,\tilde{\tau}(x))+\gamma\tilde{\tau}(x)\big(\big|\langle p'_\xi(x,\zeta),\psi'(x)\rangle\big|^2\\&\quad-\mathrm{Re}\,p(x,\zeta)\sum_{j,k}\partial_{x_j}\psi(x)\partial_{x_k}\psi(x)\partial^2_{\xi_j\xi_k}\overline{p}(x,\overline{\zeta})\big).\end{aligned}\tag{6.2}$$

where here $\tilde{\tau}(x)=\tau\gamma\varphi(x)$ and $\zeta=\xi+i\tau\varphi'(x)=\xi+i\tilde{\tau}(x)\psi'(x)$. Note that the first term is homogeneous of degree $2m-1$ in $(\xi,\tilde{\tau})$ and that the two other terms do not satisfy this homogeneity. In the present article, our positivity arguments rely on the classical Gårding inequality for homogeneous polynomial symbols. In what follows some adjusting will be performed on the symbol level to obtain the desired homogeneity.

We start with the following symbol inequality.

Proposition 6.2. *Let P be elliptic on Ω and let ψ have the strong pseudo-convexity property of Definition 6.1) for all $x\in\overline{U}$, U an open subset of Ω. We set $\varphi=e^{\gamma\psi}$ and*

$$\zeta=\zeta(x,\xi,\tau,\gamma)=\xi+i\tau\varphi'(x)=\xi+i\tilde{\tau}(x)\psi'(x),\qquad\tilde{\tau}(x)=\tau\gamma\varphi(x).$$

There exist $C > 0$, $\tau_* \geq 1$, $\gamma_* \geq 1$, and $\nu > 0$ such that

$$C\mu^{2m} \leq \nu|p(x,\zeta)|^2 + \tilde{\tau}(x)\Theta_{p,\psi}(x,\xi,\tilde{\tau}(x)) + \nu\tilde{\tau}(x)^2|\langle p'_\xi(x,\zeta),\psi'(x)\rangle|^2, \qquad \tau \geq \tau_*,\ \gamma \geq \gamma_*,\ (x,\xi) \in \overline{U} \times \mathbb{R}^n.$$

Note that symbol on the r.h.s. of (6.3) is homogeneous of degree $2m$ in $(\xi,\tilde{\tau})$.

Proof. We shall in fact prove that there exist $C > 0$, $\nu > 0$, and $\hat{\tau}_0 > 0$ such that

$$C|(\hat{\tau},\xi)|^{2m} \leq \nu|p(x,\xi+i\hat{\tau}\psi'(x)|^2 + \hat{\tau}\Theta_{p,\psi}(x,\xi,\hat{\tau}) + \nu\hat{\tau}^2|\langle p'_\xi(x,\xi+i\hat{\tau}\psi'(x)),\psi'(x)\rangle|^2,$$

for $\hat{\tau} \geq \hat{\tau}_0$ and $(x,\xi) \in \overline{U} \times \mathbb{R}^n$. Then, substituting $\tilde{\tau}(x)$ for $\hat{\tau}$ and letting γ and τ be sufficiently large yields the result.

Because of homogeneity it suffices the prove

$$0 < C \leq \nu|p(x,\xi+i\hat{\tau}\psi'(x)|^2 + \hat{\tau}\Theta_{p,\psi}(x,\xi,\hat{\tau}) + \nu\hat{\tau}^2|\langle p'_\xi(x,\xi+i\hat{\tau}\psi'(x)),\psi'(x)\rangle|^2, \tag{6.3}$$

on the compact set $\mathcal{K} = \{(x,\xi,\hat{\tau}); x \in \overline{U},\ \xi \in \mathbb{R}^n,\ \hat{\tau} \geq 0,\ |(\xi,\hat{\tau})| = 1\}$. The ellipticity of P reads $|p(x,\xi)| \geq C|\xi|^2$, for some $C > 0$. By continuity we see that (6.3) holds for $\hat{\tau} \ll |\xi|$ and some $\nu_0 > 0$. Moreover it remains true for $\nu \geq \nu_0$.

We now treat the case $|\xi| \leq \delta\hat{\tau}$, that is we consider the compact set

$$\mathcal{K}_\delta = \{(x,\xi,\hat{\tau}); x \in \overline{U},\ \xi \in \mathbb{R}^n,\ \hat{\tau} \geq 0,\ |(\xi,\hat{\tau})| = 1,\ |\xi| \leq \delta\hat{\tau}\}.$$

We have $\Theta_{p,\psi}(x,\xi,\hat{\tau}) = \frac{1}{2i}\operatorname{sub}\big(\overline{p}(x,\xi-i\hat{\tau}\psi'(x)),p(x,\xi+i\hat{\tau}\psi'(x))\big)$. Hence condition (6.1) that follows from the strong pseudo-convexity condition reads, for $\hat{\tau} > 0$,

$$p(x,\xi+i\hat{\tau}\psi'(x)) = 0 \text{ and } \big\{p,\psi\big\}(x,\xi+i\hat{\tau}\psi'(x)) = 0 \ \Rightarrow\ \Theta_{p,\psi}(x,\xi,\hat{\tau}) > 0. \tag{6.4}$$

Then on the compact set $\mathcal{K}_\delta$ the result follows from Lemma 6.3 below, by choosing ν sufficiently large. □

Lemma 6.3. *Consider two continuous functions, f and g, defined in a compact set $\mathcal{K}$, and assume that $f \geq 0$ and $f(y) = 0 \Rightarrow g(y) > 0$. Setting $h_\nu = \nu f + g$ we have $h_\nu \geq C > 0$ for $\nu > 0$ chosen sufficiently large.*

The proof is left to the reader.

6.2. **Conjugated operators and strong Lopatinskii condition.** The principal symbol of $P_\varphi = e^{\tau\varphi}Pe^{-\tau\varphi} \in \Psi^{m,0}_{\tilde{\tau},\mathrm{cl}}$ in the present calculus is

$$p_\varphi(x,\xi,\tau) = p(x,\xi+i\tau\varphi'(x)) = p(x,\xi+i\tilde{\tau}(x)\psi'(x)) = p_\psi(x,\xi,\tilde{\tau}) \in S^{m,0}_{\tilde{\tau},\mathrm{cl}},$$

Similarly, the principal symbol of $B^k_\varphi = e^{\tau\varphi}B^ke^{-\tau\varphi} \in \Psi^{\beta_k,0}_{\tilde{\tau},\mathrm{cl}}$, $k = 1,\dots,\mu$, is

$$b^k_\varphi(x,\xi,\tau) = b^k(x,\xi+i\tau\varphi'(x)) = b^k(x,\xi+i\tilde{\tau}\psi'(x)) = b^k_\psi(x,\xi,\tilde{\tau}) \in S^{\beta_k,0}_{\tilde{\tau},\mathrm{cl}}.$$

Above the dependency upon γ is hidden either in φ or in $\tilde{\tau}$. By abuse of notation we shall write $p_\varphi(\varrho)$ (resp. $p_\psi(\varrho)$) and $b^k_\varphi(\varrho)$ (resp. $b^k_\psi(\varrho)$) with $\varrho = (x,\xi,\tau,\gamma)$.

Recalling the notation of Section 1.3 for the boundary quadruplet $\omega = (x,Y,N,\tau)$ we set $\tilde{\omega} = (x,Y,N,\tilde{\tau})$, where $\tilde{\tau} = \tau\gamma\varphi(x)$. Observe then that $\tilde{p}_\varphi(\omega,\lambda) = \tilde{p}_\psi(\tilde{\omega},\lambda)$.

Setting $\tilde{\kappa}_\varphi = \tilde{p}^+_\varphi\tilde{p}^0_\varphi$ and $\tilde{\kappa}_\psi = \tilde{p}^+_\psi\tilde{p}^0_\psi$, we then find

$$\tilde{\kappa}_\varphi(\omega,\lambda) = \tilde{\kappa}_\psi(\tilde{\omega},\lambda).$$

Similarly, for $B = \{B^k\}_{k=1,\dots,\mu}$ the set of boundary operators and $b^k(x,\xi)$ their principal symbols, we have

$$\tilde{b}^k_\varphi(\omega,\lambda) = \tilde{b}^k_\psi(\tilde{\omega},\lambda).$$

From these simple observations we thus conclude that $\{P, B^k, \varphi,\ k=1,\dots,\mu\}$ satisfies the strong Lopatinskii condition at the boundary quadruple $\omega = (x, Y, N, \tau)$ *if and only if* $\{P, B^k, \psi,\ k=1,\dots,\mu\}$ satisfies the strong Lopatinskii condition at the boundary quadruple $\tilde{\omega} = (x, Y, N, \tilde{\tau})$.

6.3. **Statement of the Carleman estimate with two large parameters.** We shall prove the following theorem, counterpart of Theorem 1.6 in the case of a weight function of the form $\varphi = \exp(\gamma\psi)$, with an explicit dependency with respect to the second large parameter γ.

Theorem 6.4. *Let $x_0 \in \partial\Omega$ and let $\psi \in \mathscr{C}^\infty(\overline{\Omega})$ have the strong pseudo-convexity property of Definition 6.1 with respect to P in a neighborhood of x_0 in $\overline{\Omega}$. Moreover, assume that $\{P, \psi, B^k,\ k=1,\dots,\mu\}$ satisfies the strong Lopatinskii condition at x_0. Then there exist a neighborhood W of x_0 in $\mathbb{R}^n$ and three constants C, $\tau_* > 0$, and $\gamma_* > 0$ such that for $\varphi = \exp(\gamma\psi)$ and $\tilde{\tau} = \tau\gamma\varphi$:*

(6.5)
$$\left\|\tilde{\tau}^{-\frac{1}{2}} e^{\tau\varphi} u\right\|^2_{m,\tilde{\tau}} + |e^{\tau\varphi} T(u)|^2_{m-1,1/2,\tilde{\tau}} \le C\big(\|e^{\tau\varphi} P(x,D)u\|^2_{L^2} + \sum_{k=1}^{\mu} |e^{\tau\varphi} B^k(x,D) u_{|\partial\Omega}|^2_{m-1/2-\beta_k,\tilde{\tau}}\big),$$

for all $u = w_{|\Omega}$ with $w \in \mathscr{C}^\infty_c(W)$, $\tau \ge \tau_*$ and $\gamma \ge \gamma_*$.

6.4. **Preliminary estimates.** The following lemma is the counterpart of Lemma 4.1, that is, an elliptic estimate.

As above with $\varrho' = (x, \xi', \tau, \gamma) \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+$ we shall associate $\tilde{\varrho}' = (x, \xi', \tilde{\tau}(x)) \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+$.

Lemma 6.5. *Let $\ell(\varrho) \in S^{k,0}_{\tilde{\tau}}$, with $\varrho = (x, \xi, \tau, \gamma)$ and $k \ge 1$, be polynomial in ξ_n with homogeneous coefficients in $(\xi', \tilde{\tau})$ and $L = \ell(x, D, \tau, \gamma)$. When viewed as a polynomial in ξ_n the leading coefficient is 1. Let $\mathscr{U}$ be a conic open subset of $\overline{V_+} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$. We assume that all roots of $\ell(\varrho', \xi_n) = 0$ have negative imaginary part for $\tilde{\varrho}' \in \mathscr{U}$. Letting $\hat{\chi}(\hat{\varrho}') \in S^0_{\mathsf{T},\tau}$, $\hat{\varrho}' = (x, \xi', \hat{\tau})$, be such that $\operatorname{supp}(\hat{\chi}) \subset \mathscr{U}$, and $N \in \mathbb{N}$, there exist $C > 0$, $C_N > 0$, $\tau_* > 0$ and γ_*, such that*

$$\|\operatorname{Op}(\chi)w\|^2_{k,\tilde{\tau}} + |T(\operatorname{Op}(\chi)w)|^2_{k-1,1/2,\tilde{\tau}} \le C\,\|L\operatorname{Op}(\chi)w\|^2_{L^2} + C_N\big(\|w\|^2_{k,-N,\tilde{\tau}} + |T(w)|^2_{k-1,-N,\tilde{\tau}}\big),$$

for $w \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$ and $\tau \ge \tau_*$, $\gamma \ge \gamma_*$ and $\chi(\varrho') = \hat{\chi}(\tilde{\varrho}') \in S^0_{\mathsf{T},\tilde{\tau}}$.

Proof. The proof is very similar to that of Lemma 4.1. We highlight differences that mainly involve factors γ and norm indices. We write $\ell(\varrho) = a(\varrho) + ib(\varrho)$, where a and b are both real and homogeneous in $(\xi, \tilde{\tau})$, with $a \in S^{k,0}_{\tilde{\tau}}$ and $b \in S^{k-1,1}_{\tilde{\tau}}$. We set $A = \operatorname{Op}(a)$ and $B = \operatorname{Op}(b)$ and we introduce the following quadratic form, of type $(0,k,0)$, $Q(w) = \|Aw\|^2_{L^2} + \|Bw\|^2_{L^2}$ with symbol

$$q(\varrho) = |a(\varrho)|^2 + |b(\varrho)|^2 \in S^{2k,0}_{\tilde{\tau}}.$$

Setting $\underline{w} = \operatorname{Op}(\chi)w$, the Hermite theorem and the Gårding inequality of Proposition 5.7 give

$$Q(\underline{w}) \ge C\,\|\underline{w}\|^2_{k,\tilde{\tau}} - C'\,|T(\underline{w})|^2_{k-1,1/2,\tilde{\tau}}\,, -C''_N\,\|w\|^2_{k,-N,\tilde{\tau}}\,,$$

and the generalized Green formula of Proposition 5.12 gives

$$\left|2\operatorname{Re}\left(A\underline{w}, iB\underline{w}\right)_+ - \mathscr{B}_{a,b}(\underline{w})\right| \le |H_{a,b}(\underline{w})| + C\gamma^2\,\|\underline{w}\|^2_{k,-1,\tilde{\tau}} \le C'\gamma\,\|\underline{w}\|^2_{k,-1/2,\tilde{\tau}}\,,$$

for $\tau \geq 1$. Note that Lemma 5.5 is used as $\gamma^{-1}H_{a,b}$ is an interior quadratic form of type $(0,k,-1/2)$. Here $\mathscr{B}_{a,b}(\underline{w})$ is a boundary quadratic form of type $(0,k-1,1/2)$. Then we deduce

$$2\,\mathrm{Re}\,(A\underline{w}, iB\underline{w})_+ \geq \mathscr{B}_{a,b}(\underline{w}) - C\gamma\,\|\underline{w}\|^2_{k,-1/2,\tau}\,.$$

With the Gårding inequality of Lemma 5.11 we obtain

$$2\,\mathrm{Re}\,(A\underline{w}, iB\underline{w})_+ \geq C\,|T(\underline{w})|^2_{k-1,1/2,\tilde{\tau}} - C'\gamma\,\|\underline{w}\|^2_{k,-1/2,\tilde{\tau}} - C''_N\,|T(w)|^2_{k-1,-N,\tilde{\tau}}\,.$$

Arguing as in the end of the proof of Lemma 4.1 the result follows by choosing τ and γ sufficiently large. □

The following lemma is the counterpart of Lemma 4.2, that is, an estimate exploiting the strong Lopatinskii condition, yielding an estimate of a boundary norm.

Lemma 6.6. *Assume that $\{P, B^k, \psi,\ k=1,\dots,\mu\}$ satisfies the strong Lopatinskii condition at $(x_0,\xi'_0,\hat{\tau}_0) \in \mathbb{S}^*_{\mathsf{T},\tau}(\overline{V_+})$ with $x_0 \in \partial\Omega\cap V$. Then there exists $\mathscr{U}$ a conic open neighborhood of $(x_0,\xi'_0,\hat{\tau}_0)$ in $\overline{V_+}\times\mathbb{R}^{n-1}\times\mathbb{R}_+$ such that for $\hat{\chi}\in S^0_{\mathsf{T},\tau}$ with $\mathrm{supp}(\hat{\chi})\subset\mathscr{U}$, there exist $C>0$, $\tau_*>0$ and $\gamma_*>0$ such that*

$$C\,|T(\mathrm{Op}(\chi)v)|^2_{m-1,1/2,\tilde{\tau}} \leq \sum_{k=1}^{\mu}\left|B^k_\varphi v_{|x_n=0^+}\right|^2_{m-1/2-\beta_k,\tilde{\tau}} + \|P_\varphi v\|^2_{L^2} + \gamma^2\,\|v\|^2_{m,-1,\tilde{\tau}} + \gamma^2\,|T(v)|^2_{m-1,-1/2,\tilde{\tau}}\,,$$

for $v\in\mathscr{S}(\overline{\mathbb{R}}^n_+)$, $\tau\geq\tau_*$, $\gamma\geq\gamma_*$ and $\chi(\varrho')=\hat{\chi}(\tilde{\varrho}')\in S^0_{\mathsf{T},\tilde{\tau}}$, with $\varrho'=(x,\xi',\tau,\gamma)$ and $\tilde{\varrho}'=(x,\xi',\tilde{\tau}(x))$.

Proof. The beginning of the proof is nearly identical to that of Lemma 4.2. Inequality (4.5) becomes (using the notation of Section 3.2)

$$\sum_{k=0}^{\mu}\lambda_{\mathsf{T}}^{2(m-1/2-\beta_k)}|\Sigma_{b^k_\psi}(\hat{\varrho}',\mathbf{z})|^2 + \sum_{k=\mu+1}^{m'}\lambda_{\mathsf{T}}^{2(m^--1/2-k+\mu+1)}|\hat{\hat{\chi}}(\hat{\varrho}')\Sigma_{e^k_\psi}(\hat{\varrho}',\mathbf{z})|^2 \gtrsim \sum_{j=0}^{m-1}\lambda_{\mathsf{T}}^{2(m-1/2-j)}|z_j|^2,$$

for all $\mathbf{z}=(z_0,\dots,z_{m-1})\in\mathbb{C}^m$ and $\hat{\varrho}'=(x,\xi',\hat{\tau})\in\overline{\mathscr{U}}$, with $\lambda^2_{\mathsf{T}}=|\xi|'^2+\hat{\tau}^2$ and

$$e^k_\psi(\hat{\varrho}',\xi_n)=\kappa_\psi(\hat{\varrho}',\xi_n)\xi_n^{k-(\mu+1)}, \qquad k=\mu+1,\dots,m'=m^-+\mu.$$

Here, $\hat{\hat{\chi}}\in S^0_{\mathsf{T},\tau}$ is such that $\hat{\hat{\chi}}=1$ in a neighborhood of $\mathscr{U}$.

We then obtain, taking $\hat{\varrho}'=\tilde{\varrho}'=(x,\xi',\tilde{\tau}(x))$,

$$\sum_{k=0}^{\mu}{\mu_{\mathsf{T}}}^{2(m-1/2-\beta_k)}|\Sigma_{b^k_\varphi}(\varrho',\mathbf{z})|^2 + \sum_{k=\mu+1}^{m'}{\mu_{\mathsf{T}}}^{2(m^--1/2-k+\mu+1)}|\tilde{\chi}(\varrho')\Sigma_{e^k_\varphi}(\varrho',\mathbf{z})|^2 \gtrsim \sum_{j=0}^{m-1}{\mu_{\mathsf{T}}}^{2(m-1/2-j)}|z_j|^2,$$

for all $\mathbf{z}=(z_0,\dots,z_{m-1})\in\mathbb{C}^m$ and $\varrho'=(x,\xi',\tau,\gamma)$ such that $\tilde{\varrho}'\in\overline{\mathscr{U}}$. Here $\tilde{\chi}(\varrho')=\hat{\hat{\chi}}(\tilde{\varrho}')\in S^0_{\mathsf{T},\tilde{\tau}}$.

Then, according to Gårding inequality of Lemma 5.11 for a boundary quadratic forms of type $(0,m-1,1/2)$, there exists $\tau_*>0$ and γ_* such that

$$\sum_{k=1}^{\mu}\left|B^k_\varphi \underline{v}_{|x_n=0^+}\right|^2_{m-1/2-\beta_k,\tilde{\tau}} + \sum_{k=\mu+1}^{m'}\left|E^k_\varphi \underline{v}_{|x_n=0^+}\right|^2_{m^--1/2-k+\mu+1,\tilde{\tau}} \gtrsim |T(\underline{v})|^2_{m-1,1/2,\tilde{\tau}} - C_N\,|T(v)|^2_{m-1,-N,\tilde{\tau}}\,, \tag{6.6}$$

with $\underline{v}=\mathrm{Op}(\chi)v$ and $N\in\mathbb{N}$, for $\tau\geq\tau_*$ and $\gamma\geq\gamma_*$, with $B^k_\varphi=\mathrm{Op}(b^k_\varphi)$ and $E^k_\varphi=\mathrm{Op}(\tilde{\chi}e^k_\varphi)$.

Arguing as in the proof of Lemma 4.2 we write $\chi p_\varphi = \chi\kappa_\varphi p_\varphi^- = \chi\tilde\chi\kappa_\varphi \underline{p}_\varphi^-$ where $\underline{p}_\varphi^-(\varrho', \xi_n)$ denotes an extension of $p_\varphi^-(\varrho', \xi_n)$ to the whole phase space. Then $\operatorname{Op}(\chi)P_\varphi = \operatorname{Op}(\underline{p}_\varphi^-)\operatorname{Op}(\chi)\operatorname{Op}(\tilde\chi\kappa_\varphi) + R$ with $R \in \gamma\Psi_{\tilde\tau}^{m,-1}$. Applying Lemma 6.5 to $\operatorname{Op}(\underline{p}_\varphi^-)$ and $w = \operatorname{Op}(\tilde\chi\kappa_\varphi)v$ we obtain

$$\|\operatorname{Op}(\chi)w\|^2_{m^-,\tilde\tau} + |T(\operatorname{Op}(\chi)w)|^2_{m^- -1,1/2,\tilde\tau} \lesssim \|P_\varphi v\|^2_{L^2} + \gamma^2\|v\|^2_{m,-1,\tilde\tau} + |T(v)|^2_{m-1,-N,\tilde\tau},$$

yielding

$$\sum_{j=0}^{m^- -1} \left|D_n^j \operatorname{Op}(\chi)w_{|x_n=0^+}\right|^2_{m^- -1/2-j,\tilde\tau} \lesssim \|P_\varphi v\|^2_{L^2} + \gamma^2\|v\|^2_{m,-1,\tilde\tau} + |T(v)|^2_{m-1,-N,\tilde\tau}.$$

Recalling that $e_\varphi^{j+\mu+1} = \kappa_\varphi\xi_n^j$, $j = 0, \dots, m^- - 1$ we have $D_n^j \operatorname{Op}(\chi)\operatorname{Op}(\tilde\chi\kappa_\varphi)v = E_\varphi^{j+\mu+1}v + R_j v$ with $R_j \in \gamma\Psi_{\tilde\tau}^{m-m^-+j,-1}$. We then obtain

$$\sum_{j=0}^{m^- -1} \left|E_\varphi^{j+\mu+1}v_{|x_n=0^+}\right|^2_{m^- -1/2-j,\tilde\tau} \lesssim \|P_\varphi v\|^2_{L^2} + \gamma^2\|v\|^2_{m,-1,\tilde\tau} + \gamma^2|T(v)|^2_{m-1,-1/2,\tilde\tau}. \tag{6.7}$$

Collecting (6.6) and (6.7) we obtain the result of Lemma 6.6, for τ and γ chosen sufficiently large. □

6.5. **Proof of the Carleman estimate with two-large parameters.** We prove a microlocal result, counterpart of that of Theorem 4.4. Patching microlocal estimates of this type, arguing as in Section 4.5 we can then obtain the local Carleman estimate of Theorem 6.4, which proof is left to the reader. Remainder terms are absorbed using that, for $N \in \mathbb{N}$ we have $\gamma^N \ll \tilde\tau(x) = \gamma\tau\exp(\gamma\psi)$ for γ large since $\psi \geq C > 0$.

Theorem 6.7. *Let $x_0 \in \partial\Omega \cap V$ and let ψ have the strong pseudo-convexity property of Definition 6.1 with respect to P in a neighborhood of x_0 in $\overline{V_+}$. Moreover, assume that $\{P, B^k, \psi,\ k = 1, \dots, \mu\}$ satisfies the strong Lopatinskii condition at $(x_0, \xi_0', \hat\tau_0) \in \mathbb{S}^*_{\mathsf{T},\tau}(\overline{V_+})$. Then there exists $\mathscr{U}$ a conic open neighborhood of $(x_0, \xi_0', \hat\tau_0)$ in $\overline{V_+} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ such that for $\hat\chi \in S^0_{\mathsf{T},\tau}$ with $\operatorname{supp}(\hat\chi) \subset \mathscr{U}$, there exist $C > 0$, $\tau_* > 0$, and $\gamma_* > 0$ such that*

$$\begin{aligned}\|P_\varphi v\|^2_{L^2} &+ \sum_{k=1}^{\mu}\left|B_\varphi^k v_{|x_n=0^+}\right|^2_{m-\beta_k-1/2,\tilde\tau} \\ &+ \gamma^2\|v\|^2_{m,-1,\tilde\tau} + \gamma^2|T(v)|^2_{m-1,-1/2,\tilde\tau} \geq C\big(\big\|\tilde\tau^{-\frac12}\operatorname{Op}(\chi)v\big\|^2_{m,\tilde\tau} + |T(\operatorname{Op}(\chi)v)|^2_{m-1,1/2,\tilde\tau}\big),\end{aligned} \tag{6.8}$$

for $\tau \geq \tau_*$, $\gamma \geq \gamma_*$, $v \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$, and $\chi(\varrho') = \hat\chi(\tilde\varrho') \in S^0_{\mathsf{T},\tilde\tau}$.

The main argument lays in (the proof of) the following lemma.

Lemma 6.8. *Under the assumptions of Theorem 6.7 there exists $\mathscr{U}$ a conic open neighborhood of $(x_0, \xi_0', \hat\tau_0)$ in $\overline{V_+} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ such that for $\hat\chi \in S^0_{\mathsf{T},\tau}$ with $\operatorname{supp}(\hat\chi) \subset \mathscr{U}$, there exist $C > 0$, $\tau_* > 0$, and $\gamma_* > 0$ such that*

$$\begin{aligned}C\big\|P_\varphi v\big\|^2_{L^2} - \operatorname{Re}\mathscr{B}_{a,b}(\operatorname{Op}(\chi)v) &\geq C'\big\|\tilde\tau^{-\frac12}\operatorname{Op}(\chi)v\big\|^2_{m,\tilde\tau} \\ &- C''\big(\gamma^2\|v\|^2_{m,-1,\tilde\tau} + \big|\tilde\tau^{-\frac12}T(\operatorname{Op}(\chi)v)\big|^2_{m-1,1/2,\tilde\tau} + \gamma|T(\operatorname{Op}(\chi)v)|^2_{m-1,0,\tilde\tau}\big),\end{aligned} \tag{6.9}$$

for $\tau \geq \tau_*$, $\gamma \geq \gamma_*$, $v \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$, and $\chi(\varrho') = \hat\chi(\tilde\varrho') \in S^0_{\mathsf{T},\tilde\tau}$.

Proof. Let U_0 be an open neighborhood of x_0 in $\overline{V_+}$ where the result of Proposition 6.2 holds.

Similarly to Section 4.5, we write

$$P_\varphi = A + iB + R, \qquad R \in \gamma\Psi_{\tilde\tau}^{m,-1}.$$

where $A = \mathrm{Op}(a)$ and $B = \mathrm{Op}(a)$, with $a = \operatorname{Re} p_\varphi \in S^{m,0}_{\tilde\tau}$ and $b = \operatorname{Im} p_\varphi \in S^{m-1,1}_{\tilde\tau}$. We set

$$\begin{aligned} e(x,\xi,\tau,\gamma) &= \gamma^{\frac12}\tilde\tau(x)^{\frac12}\langle p'_\xi(x,\zeta),\psi'(x)\rangle \in \gamma^{\frac12}\tilde\tau^{\frac12}S^{m-1,0}_{\tilde\tau},\\ f(x,\xi,\tau,\gamma) &= \gamma^{-1}\Theta_{p,\psi}(x,\xi,\tilde\tau(x)) + \tilde\tau(x)|\langle p'_\xi(x,\zeta),\psi'(x)\rangle|^2 \in S^{2m-1,0}_{\tilde\tau},\\ g(x,\xi,\tau,\gamma) &= \gamma\tilde\tau(x)\sum_{j,k}\partial_{x_j}\psi(x)\partial_{x_k}\psi(x)\partial^2_{\xi_j\xi_k}p(x,\zeta) \in \gamma\tilde\tau S^{m-2,0}_{\tilde\tau}, \end{aligned}$$

with $\zeta = \xi + i\tau\varphi'(x) = \xi + i\tilde\tau(x)\psi'(x)$. We let $F(v)$ be an interior quadratic form of type $(0,m,-\frac12))$ with symbol f. With ν as given by Proposition 6.2 and γ and τ large so that $\gamma^{-1}\nu \le 1$ and $\tilde\tau(x)^{-1}\nu \le 1$, we have

$$\begin{aligned} &\nu\big(\|\tilde\tau^{-\frac12}Av\|^2_{L^2} + \|\tilde\tau^{-\frac12}Bv\|^2_{L^2}\big) + \gamma\operatorname{Re}F(v)\\ &\qquad\ge \nu\big(\|\tilde\tau^{-\frac12}Av\|^2_{L^2} + \|\tilde\tau^{-\frac12}Bv\|^2_{L^2}\big) + \gamma\operatorname{Re}F(v) + (\nu\gamma^{-1}-1)\,\|\mathrm{Op}(e)v\|^2_{L^2}. \end{aligned}$$

Observe that the interior quadratic form in the r.h.s., of type $(-\frac12,m,0)$, is of symbol

$$\nu\tilde\tau^{-1}|p_\varphi|^2 + \Theta_{p,\psi}(x,\xi,\tilde\tau(x)) + \nu\tilde\tau(x)|\langle p'_\xi(x,\zeta),\psi'(x)\rangle|^2 \ge C\tilde\tau^{-1}\mu^{2m},$$

for $x \in U_0$ and $(\xi,\tau,\gamma) \in \mathbb{R}^n \times R_+ \times \mathbb{R}_+$, with τ and γ sufficiently large by Proposition 6.2. This symbol is polynomial in ξ and homogeneous of degree $2m-1$ in $(\xi,\tilde\tau)$. We let $\mathscr{U}$, possibly reduced so that $\mathscr{U} \subset U_0 \times \mathbb{R}^{n-1} \times \mathbb{R}_+$, and χ be as given by Lemma 6.6. The Gårding inequality of Proposition 5.7 and Remark 5.8 then yields

$$\nu\big(\|\tilde\tau^{-\frac12}A\underline{v}\|^2_{L^2} + \|\tilde\tau^{-\frac12}B\underline{v}\|^2_{L^2}\big) + \gamma\operatorname{Re}F(\underline{v}) \ge C\|\tilde\tau^{-\frac12}\underline{v}\|^2_{m,\tilde\tau} - C'|\tilde\tau^{-\frac12}T(\underline{v})|^2_{m-1,1/2,\tilde\tau}, \tag{6.10}$$

with $\underline{v} = \mathrm{Op}(\chi)v$. Recall that $\gamma^{-1}H_{a,b}$ is an interior quadratic form of type $(0,m,-\frac12)$ with symbol equal to $\gamma^{-1}\operatorname{sub}(a,b)$. Since by (6.2) the interior quadratic form, of type $(0,m,-\frac12)$,

$$\gamma^{-1}H_{a,b}(w) - F(w) + \gamma^{-1}\operatorname{Re}(P_\varphi w, \mathrm{Op}(g)w)_+$$

has a vanishing symbol, by Lemma 5.6 we obtain

$$\operatorname{Re}\Big(H_{a,b}(\underline{v}) - \gamma F(\underline{v}) - \operatorname{Re}(P_\varphi\underline{v},\mathrm{Op}(g)\underline{v})_+\Big) \ge -C\big(\gamma^2\|\underline{v}\|^2_{m,-1,\tau} + \gamma\,|T(\underline{v})|^2_{m-1,0,\tau}\big). \tag{6.11}$$

We also have,

$$\operatorname{Re}(P_\varphi\underline{v},\mathrm{Op}(g)\underline{v})_+ \ge -C\|P_\varphi\underline{v}\|^2_{L^2} - C\gamma^2\|\tilde\tau\underline{v}\|^2_{m-2,\tilde\tau}. \tag{6.12}$$

The sum of (6.10), (6.11), and (6.12) yields, for τ and γ sufficiently large,

$$\begin{aligned} &C\|P_\varphi\underline{v}\|^2_{L^2} + \nu\big(\|\tilde\tau^{-\frac12}A\underline{v}\|^2_{L^2} + \|\tilde\tau^{-\frac12}B\underline{v}\|^2_{L^2}\big) + \operatorname{Re}H_{a,b}(\underline{v})\\ &\qquad\ge C'\|\tilde\tau^{-\frac12}\underline{v}\|^2_{m,\tilde\tau} - C''\big(|\tilde\tau^{-\frac12}T(\underline{v})|^2_{m-1,1/2,\tilde\tau} + \gamma\,|T(\underline{v})|^2_{m-1,0,\tilde\tau}\big). \end{aligned} \tag{6.13}$$

Next, the generalized Green formula of Proposition 5.12 gives

$$2\operatorname{Re}(A\underline{v},iB\underline{v})_+ - \operatorname{Re}\mathscr{B}_{a,b}(\underline{v}) + C\gamma^2\|\underline{v}\|^2_{m,-1,\tilde\tau} \ge \operatorname{Re}H_{a,b}(\underline{v}). \tag{6.14}$$

The sum of (6.13) and (6.14) yields the counterpart of Lemma 4.3, for τ and γ sufficiently large,

$$\begin{aligned} &C\|P_\varphi\underline{v}\|^2_{L^2} + \nu\big(\|\tilde\tau^{-\frac12}A\underline{v}\|^2_{L^2} + \|\tilde\tau^{-\frac12}B\underline{v}\|^2_{L^2}\big) + 2\operatorname{Re}(A\underline{v},iB\underline{v})_+ - \operatorname{Re}\mathscr{B}_{a,b}(\underline{v})\\ &\qquad\ge C'\|\tilde\tau^{-\frac12}\underline{v}\|^2_{m,\tilde\tau} - C''\big(|\tilde\tau^{-\frac12}T(\underline{v})|^2_{m-1,1/2,\tilde\tau} + \gamma\,|T(\underline{v})|^2_{m-1,0,\tilde\tau}\big). \end{aligned}$$

We have

$$\nu\big(\|\tilde\tau^{-\frac12}A\underline{v}\|^2_{L^2} + \|\tilde\tau^{-\frac12}B\underline{v}\|^2_{L^2}\big) + 2\operatorname{Re}(A\underline{v},iB\underline{v})_+ \le \|(A+iB)\underline{v}\|^2_{L^2} \lesssim \|P_\varphi\underline{v}\|^2_{L^2} + \gamma^2\|\underline{v}\|^2_{m,-1,\tilde\tau},$$

which gives, for τ and γ chosen sufficiently large,

$$C\big\|P_\varphi \underline{v}\big\|_{L^2}^2 - \operatorname{Re}\mathscr{B}_{a,b}(\underline{v}) \geq C'\big\|\tilde{\tau}^{-\frac{1}{2}}\underline{v}\big\|_{m,\tilde{\tau}}^2 - C''\big(\big|\tilde{\tau}^{-\frac{1}{2}}T(\underline{v})\big|_{m-1,1/2,\tilde{\tau}}^2 + \gamma\,|T(\underline{v})|_{m-1,0,\tilde{\tau}}^2\big).$$

Since $[P_\varphi, \operatorname{Op}(\chi)] \in \gamma\Psi_{\tilde{\tau}}^{m,-1}$ we obtain estimate (6.9). □

Proof of Theorem 6.7. We write $\underline{v} = \operatorname{Op}(\chi)v$. As $\mathscr{B}_{a,b}$ is of type $(0, m-1, 1/2)$ we have

$$|\mathscr{B}_{a,b}(\underline{v})| \lesssim |T(\underline{v})|_{m-1,1/2,\tilde{\tau}}^2\,.$$

With Lemma 6.6, making use of the strong Lopatinskii condition, we obtain for M chosen sufficiently large

$$\begin{aligned}\operatorname{Re}\mathscr{B}_{a,b}(\underline{v}) + M\sum_{k=1}^{\mu}\big|B_\varphi^k v_{|x_n=0^+}\big|_{m-\beta_k-1/2,\tilde{\tau}}^2 &\geq C\,|T(\underline{v})|_{m-1,1/2,\tilde{\tau}}^2 \\ &\quad - C'\big(\,\|P_\varphi v\|_{L^2}^2 + \gamma^2\,\|v\|_{m,-1,\tilde{\tau}}^2 + \gamma^2\,|T(v)|_{m-1,-1/2,\tilde{\tau}}^2\,\big).\end{aligned} \tag{6.15}$$

Summing (6.9) and (6.15) we find the result, by taking τ and γ sufficiently large. □

6.6. **Estimate with the simple characterisitic property.** A stronger estimate with two parameters can be achieved if one assumes that the operator P and the weight function ψ fulfills the so-called simple characterisitic property. This is proven in [30] for estimates away from a boundary. Here we show that this can be extended at the boundary if the strong Lopatinskii condition is also assumed.

6.6.1. *The simple characterisitic property.* We introduce the map

$$\begin{aligned}\rho_{x,\xi} : \mathbb{R}^+ &\to \mathbb{C}, \\ \hat{\tau} &\mapsto p(x, \xi + i\hat{\tau}\psi'(x)),\end{aligned} \tag{6.16}$$

where $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n$.

Definition 6.9. *Let U be an open subset of Ω. Given a weight function ψ and an operator P we say that the simple-characteristic property is satisfied in $\overline{U}$ if, for all $x \in \overline{U}$, we have $\xi = 0$ and $\hat{\tau} = 0$ when the map $\rho_{x,\xi}$ has a double root.*

Note that the case $\xi = 0$ is particular, as the root $\hat{\tau} = 0$ has of course multiplicity m. Note also that we have

$$\rho'_{x,\xi}(\hat{\tau}) = i\langle p'_\xi(x, \xi + i\hat{\tau}\psi'(x)), \psi'(x)\rangle = i\{p, \psi\}(x, \xi + i\hat{\tau}\psi'(x)). \tag{6.17}$$

We can thus formulate the condition of Definition 6.9 as

$$p(x, \xi + i\hat{\tau}\psi'(x)) = \{p, \psi\}(x, \xi + i\hat{\tau}\psi'(x)) = 0 \quad \Rightarrow \quad \xi = 0,\ \hat{\tau} = 0.$$

or equivalently

$$p(x, \xi + i\hat{\tau}\psi'(x)) = 0 \text{ and } (\xi, \hat{\tau}) \neq (0, 0) \quad \Rightarrow \quad \{p, \psi\}(x, \xi + i\hat{\tau}\psi'(x)) \neq 0. \tag{6.18}$$

Observe that the simple-characteristic property (6.18) implies that ψ is strongly pseudo-convex with respect to $\overline{\Omega}$ and P.

We have the following lemma.

Lemma 6.10. *Assume that P and ψ satisfy the simple characteristic property in $\overline{U}$. Then there exist $C > 0$, $\tau_* \geq 1$, $\gamma_* \geq 1$, and $\nu > 0$ such that*

$$C\mu^{2m} \leq \nu\,|p(x, \zeta)|^2 + \tilde{\tau}(x)^2|\langle p'_\xi(x, \zeta), \psi'(x)\rangle|^2, \quad \tau \geq \tau_*,\ \gamma \geq \gamma_*,\ (x, \xi) \in \overline{U} \times \mathbb{R}^n.$$

The proof follows from Lemma 6.3, the simple characteristic property and homogeneity.

6.6.2. *Carleman estimate with the simple characteristic property.*

Theorem 6.11. *Let $x_0 \in \partial\Omega$ and let $\psi \in \mathscr{C}^\infty(\overline{\Omega})$ be such that P and ψ have the simple characteristic property of Definition 6.9 in a neighborhood of x_0 in $\overline{\Omega}$. Moreover, assume that $\{P, \psi, B^k,\ k = 1, \dots, \mu\}$ satisfies the strong Lopatinskii condition at x_0. Then there exist a neighborhood W of x_0 in $\mathbb{R}^n$ and three constants C, $\tau_* > 0$, and $\gamma_* > 0$ such that for $\varphi = \exp(\gamma\psi)$ and $\tilde{\tau} = \tau\gamma\varphi$:*

$$\gamma\big\|\tilde{\tau}^{-\frac{1}{2}} e^{\tau\varphi} u\big\|^2_{m,\tilde{\tau}} + |e^{\tau\varphi} T(u)|^2_{m-1,1/2,\tilde{\tau}} \le C\big(\|e^{\tau\varphi} P(x,D)u\|^2_{L^2} + \sum_{k=1}^{\mu} |e^{\tau\varphi} B^k(x,D) u_{|\partial\Omega}|^2_{m-1/2-\beta_k,\tilde{\tau}}\big), \tag{6.19}$$

for all $u = w_{|\Omega}$ with $w \in \mathscr{C}^\infty_c(W)$, $\tau \ge \tau_*$, and $\gamma \ge \gamma_*$.

Observe that the first norm in the l.h.s. bears an additional factor γ as compared to the estimate of Theorem 6.4. Conversely, this additional factor implies that P and ψ have the simple characteristic property; moreover one cannot expect to have an additional factor γ^θ with $\theta > 1$ unless the conjugated operator is elliptic, i.e., $p_\psi(x, \xi, \hat{\tau}) = p(x, \xi + i\hat{\tau}\psi'(x)) \neq 0$ for $(\xi, \hat{\tau}) \neq (0, 0)$ [30, Section 5].

As above, we only prove a microlocal estimate and we leave to the reader the adaptation of Section 4.5 for the patching of those estimates to obtain estimate (6.19).

Theorem 6.12. *Let $x_0 \in \partial\Omega \cap V$ and let ψ be such that P and ψ have the simple characteristic property of Definition 6.9 in a neighborhood of x_0 in $\overline{V_+}$. Moreover, assume that $\{P, B_k, \psi,\ k = 1, \dots, \mu\}$ satisfies the strong Lopatinskii condition at $(x_0, \xi_0', \hat{\tau}_0) \in \mathbb{S}^*_{\mathsf{T},\tau}(\overline{V_+})$. Then there exists $\mathscr{U}$ a conic open neighborhood of $(x_0, \xi_0', \hat{\tau}_0)$ in $\overline{V_+} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ such that for $\hat{\chi} \in S^0_{\mathsf{T},\tau}$ with $\operatorname{supp}(\hat{\chi}) \subset \mathscr{U}$, there exist $C > 0$, $\tau_* > 0$, and $\gamma_* > 0$ such that*

$$\|P_\varphi v\|^2_{L^2} + \sum_{k=1}^{\mu} \big|B^k_\varphi v_{|x_n=0^+}\big|^2_{m-\beta_k-1/2,\tilde{\tau}} + \gamma^2 \|v\|^2_{m,-1,\tilde{\tau}} + \gamma^2 |T(v)|^2_{m-1,-1/2,\tilde{\tau}} \ge C\big(\gamma\big\|\tilde{\tau}^{-\frac{1}{2}} \operatorname{Op}(\chi) v\big\|^2_{m,\tilde{\tau}} + |T(\operatorname{Op}(\chi)v)|^2_{m-1,1/2,\tilde{\tau}}\big) \tag{6.20}$$

for $\tau \ge \tau_*$, $\gamma \ge \gamma_*$, $v \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$, and $\chi(\varrho') = \hat{\chi}(\tilde{\varrho}') \in S^0_{\mathsf{T},\tilde{\tau}}$.

The main argument lays in the following lemme which is the counterpart of Lemma 6.8.

Lemma 6.13. *Under the assumptions of Theorem 6.12 there exists $\mathscr{U}$ a conic open neighborhood of $(x_0, \xi_0', \hat{\tau}_0)$ in $\overline{V_+} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ such that for $\hat{\chi} \in S^0_{\mathsf{T},\tau}$ with $\operatorname{supp}(\hat{\chi}) \subset \mathscr{U}$, there exist $C > 0$, $\tau_* > 0$, and $\gamma_* > 0$ such that*

$$C\big\|P_\varphi v\big\|^2_{L^2} - \operatorname{Re}\mathscr{B}_{a,b}(\operatorname{Op}(\chi)v) \ge C\gamma\big\|\tilde{\tau}^{-\frac{1}{2}} \operatorname{Op}(\chi)v\big\|^2_{m,\tilde{\tau}} - C'\big(\gamma^2 \|v\|^2_{m,-1,\tilde{\tau}} + \gamma\big|\tilde{\tau}^{-\frac{1}{2}} T(\operatorname{Op}(\chi)v)\big|^2_{m-1,1/2,\tilde{\tau}}\big), \tag{6.21}$$

for $\tau \ge \tau_*$, $\gamma \ge \gamma_*$, $v \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$, and $\chi(\varrho') = \hat{\chi}(\tilde{\varrho}') \in S^0_{\mathsf{T},\tilde{\tau}}$.

Proof. Let U_0 be an open neighborhood of x_0 in $\overline{V_+}$ where the result of Lemma 6.10 holds.

Similarly to Section 4.5, we write

$$P_\varphi = A + iB + R, \qquad R \in \gamma\Psi^{m,-1}_{\tilde{\tau}}.$$

where $A = \mathrm{Op}(a)$ and $B = \mathrm{Op}(a)$, with $a = \mathrm{Re}\, p_\varphi \in S^{m,0}_{\tilde\tau}$ and $b = \mathrm{Im}\, p_\varphi \in S^{m-1,1}_{\tilde\tau}$. We set

$$\begin{aligned} e(x,\xi,\tau,\gamma) &= \tilde\tau(x)^{\frac12}\langle p'_\xi(x,\zeta),\psi'(x)\rangle \in \tilde\tau^{\frac12} S^{m-1,0}_{\tilde\tau},\\ f(x,\xi,\tau,\gamma) &= \Theta_{p,\psi}(x,\xi,\tilde\tau(x)) \in S^{2m-1,0}_{\tilde\tau},\\ g(x,\xi,\tau,\gamma) &= \gamma\tilde\tau(x)\sum_{j,k}\partial_{x_j}\psi(x)\partial_{x_k}\psi(x)\partial^2_{\xi_j\xi_k}p(x,\zeta) \in \gamma\tilde\tau S^{m-2,0}_{\tilde\tau}, \end{aligned}$$

with $\zeta = \xi + i\tau\varphi'(x) = \xi + i\tilde\tau(x)\psi'(x)$. We let $\mathscr{U}$, possibly reduced so that $\mathscr{U} \subset U_0 \times \mathbb{R}^{n-1}\times\mathbb{R}_+$, and χ be as given by Lemma 6.6. With ν as given by Proposition 6.10 and γ and τ large so that $\tilde\tau(x)^{-1}\nu \le 1$, we have by Propositon 5.7

$$\nu\big(\|\tilde\tau^{-\frac12}A\underline{v}\|^2_{L^2} + \|\tilde\tau^{-\frac12}B\underline{v}\|^2_{L^2}\big) + \|\mathrm{Op}(e)\underline{v}\|^2_{L^2} \ge C\|\tilde\tau^{-\frac12}\underline{v}\|^2_{m,\tilde\tau} - C'|\tilde\tau^{-\frac12}T(\underline{v})|^2_{m-1,1/2,\tilde\tau}, \tag{6.22}$$

with $\underline{v} = \mathrm{Op}(\chi)v$. In fact, the symbol of the interior quadratic form in the l.h.s., of type $(-\frac12, m, 0)$ satisfies by Lemma 6.10

$$\nu\tilde\tau^{-1}|p_\varphi|^2(\varrho',\xi_n) + \tilde\tau|\langle p'_\xi(x,\zeta),\psi'(x)\rangle|^2(\varrho',\xi_n) \ge C\tilde\tau^{-1}\mu^{2m}, \quad \tilde\varrho' = (x,\xi',\tilde\tau(x)) \in \mathscr{U},\ \xi_n\in\mathbb{R}.$$

This symbol is polynomial in ξ and homogeneous of degree $2m-1$ in $(\xi,\tilde\tau)$. Hence, the Gårding inequality of Proposition 5.7 applies.

Let $F(w)$ be an interior quadratic form of type $(0,m,-\frac12))$ with symbol f. We have by Lemma 5.5

$$\mathrm{Re}\, F(\underline{v}) \ge -C\,\|\underline{v}\|^2_{m,-\frac12,\tau} \tag{6.23}$$

Recall that $\gamma^{-1}H_{a,b}$ is an interior quadratic form of type $(0,m,-\frac12)$ with symbol equal to $\gamma^{-1}\,\mathrm{sub}(a,b)$. Since by (6.2) the interior quadratic form, of type $(0,m,-\frac12)$,

$$\gamma^{-1}H_{a,b}(w) - \gamma^{-1}F(w) - \|\mathrm{Op}(e)w\|^2_{L^2} + \gamma^{-1}\,\mathrm{Re}\,(P_\varphi w, \mathrm{Op}(g)w)_+$$

has a vanishing symbol, by Lemma 5.6 we obtain

$$\begin{aligned} &\mathrm{Re}\Big(H_{a,b}(\underline{v}) - F(\underline{v}) - \gamma\,\|\mathrm{Op}(e)\underline{v}\|^2_{L^2} + \mathrm{Re}\,(P_\varphi\underline{v},\mathrm{Op}(g)\underline{v})_+\Big)\\ &\qquad\qquad \ge -C\big(\gamma^2\,\|\underline{v}\|^2_{m,-1,\tau} + \gamma\,|T(\underline{v})|^2_{m-1,0,\tau}\big). \end{aligned} \tag{6.24}$$

We also have

$$-\mathrm{Re}\,(P_\varphi\underline{v},\mathrm{Op}(g)\underline{v})_+ \ge -C\|P_\varphi\underline{v}\|^2_{L^2} - C\gamma^2\|\tilde\tau\underline{v}\|^2_{m-2,\tilde\tau}. \tag{6.25}$$

The sum γ(6.22) + (6.23) + (6.24) + (6.25) gives, taking τ and γ sufficiently large,

$$\begin{aligned} &C\|P_\varphi\underline{v}\|^2_{L^2} + \nu\big(\|(\tau\varphi)^{-\frac12}A\underline{v}\|^2_{L^2} + \|(\tau\varphi)^{-\frac12}B\underline{v}\|^2_{L^2}\big) + \mathrm{Re}\,H_{a,b}(\underline{v})\\ &\qquad\qquad \ge C\gamma\|\tilde\tau^{-\frac12}\underline{v}\|^2_{m,\tilde\tau} - C'\gamma|\tilde\tau^{-\frac12}T(\underline{v})|^2_{m-1,1/2,\tilde\tau}, \end{aligned} \tag{6.26}$$

Summing (6.14) and (6.26) gives, for τ and γ sufficiently large,

$$\begin{aligned} &C\|P_\varphi\underline{v}\|^2_{L^2} + \nu\big(\|(\tau\varphi)^{-\frac12}A\underline{v}\|^2_{L^2} + \|(\tau\varphi)^{-\frac12}B\underline{v}\|^2_{L^2}\big) + 2\,\mathrm{Re}\,(A\underline{v}, iB\underline{v})_+ - \mathrm{Re}\,\mathscr{B}_{a,b}(\underline{v})\\ &\qquad\qquad \ge C'\gamma\|\tilde\tau^{-\frac12}\underline{v}\|^2_{m,\tilde\tau} - C''\gamma|\tilde\tau^{-\frac12}T(\underline{v})|^2_{m-1,1/2,\tilde\tau}. \end{aligned}$$

We have

$$\begin{aligned} &\nu\big(\|(\tau\varphi)^{-\frac12}A\underline{v}\|^2_{L^2} + \|(\tau\varphi)^{-\frac12}B\underline{v}\|^2_{L^2}\big) + 2\,\mathrm{Re}\,(A\underline{v}, iB\underline{v})_+\\ &\qquad\qquad \le \|(A+iB)\underline{v}\|^2_{L^2} \lesssim \|P_\varphi\underline{v}\|^2_{L^2} + \gamma^2\,\|\underline{v}\|^2_{m,-1,\tilde\tau}\,, \end{aligned}$$

which gives, for τ and γ sufficiently large,

$$C\|P_\varphi \underline{v}\|_{L^2}^2 - \operatorname{Re}\mathscr{B}_{a,b}(\underline{v}) \geq C\gamma\|\tilde\tau^{-\frac12}\underline{v}\|_{m,\tilde\tau}^2 - C'\gamma|\tilde\tau^{-\frac12}T(\underline{v})|_{m-1,1/2,\tilde\tau}^2,$$

Since $[P_\varphi, \operatorname{Op}(\chi)] \in \gamma\Psi_{\tilde\tau}^{m,-1}$ we obtain (6.21). □

Proof of Theorem 6.12. Summing (6.21) and (6.15) with $\underline{v} = \operatorname{Op}(\chi)v$ we find the result, by taking τ and γ sufficiently large. □

6.7. **Shifted estimates.** As in Section 4.6 we can also derive shifted estimates. The result of Theorems 6.4 and 6.11 then read as follows.

Corollary 6.14 (Shifted estimate under strong pseudo-convexity)**.** *Let $x_0 \in \partial\Omega$ and let $\psi \in \mathscr{C}^\infty(\overline{\Omega})$ have the strong pseudo-convexity property of Definition 6.1 with respect to P in a neighborhood of x_0 in $\overline{\Omega}$. Moreover, assume that $\{P, \psi, B^k,\ k = 1, \dots, \mu\}$ satisfies the strong Lopatinskii condition at x_0. Let $\ell \in \mathbb{N}$ and $s \in \mathbb{R}$. Then there exist a neighborhood W of x_0 in $\mathbb{R}^n$ and three constants C, $\tau_* > 0$, and $\gamma_* > 0$ such that for $\varphi = \exp(\gamma\psi)$ and $\tilde\tau = \tau\gamma\varphi$:*

$$\begin{aligned}(6.27)\quad &\|\tilde\tau^{s-\frac12}e^{\tau\varphi}u\|_{m+\ell,\tilde\tau}^2 + |\tilde\tau^s e^{\tau\varphi}T(u)|_{m+\ell-1,1/2,\tilde\tau}^2 \\ &\qquad \leq C\big(\|\tilde\tau^s e^{\tau\varphi}P(x,D)u\|_{\ell,\tilde\tau}^2 + \sum_{k=1}^{\mu}\left|\tilde\tau^s e^{\tau\varphi}T(B^k(x,D)u)\right|_{\ell,m-1/2-\beta_k,\tilde\tau}^2\big),\end{aligned}$$

for all $u = w_{|\Omega}$ with $w \in \mathscr{C}_c^\infty(W)$, $\tau \geq \tau_*$ and $\gamma \geq \gamma_*$.

Corollary 6.15. *Let $x_0 \in \partial\Omega$ and let $\psi \in \mathscr{C}^\infty(\overline{\Omega})$ be such that P and ψ have the simple characteristic property of Definition 6.9 in a neighborhood of x_0 in $\overline{\Omega}$. Moreover, assume that $\{P, \psi, B^k,\ k = 1, \dots, \mu\}$ satisfies the strong Lopatinskii condition at x_0. Let $\ell \in \mathbb{N}$ and $s \in \mathbb{R}$. Then there exist a neighborhood W of x_0 in $\mathbb{R}^n$ and three constants C, $\tau_* > 0$, and $\gamma_* > 0$ such that for $\varphi = \exp(\gamma\psi)$ and $\tilde\tau = \tau\gamma\varphi$:*

$$\begin{aligned}(6.28)\quad &\gamma\|\tilde\tau^{s-\frac12}e^{\tau\varphi}u\|_{m+\ell,\tilde\tau}^2 + |\tilde\tau^s e^{\tau\varphi}T(u)|_{m+\ell-1,1/2,\tilde\tau}^2 \\ &\qquad \leq C\big(\|\tilde\tau^s e^{\tau\varphi}P(x,D)u\|_{\ell,\tilde\tau}^2 + \sum_{k=1}^{\mu}\left|\tilde\tau^s e^{\tau\varphi}B^k(x,D)u_{|\partial\Omega}\right|_{\ell,m-1/2-\beta_k,\tilde\tau}^2\big),\end{aligned}$$

for all $u = w_{|\Omega}$ with $w \in \mathscr{C}_c^\infty(W)$, $\tau \geq \tau_*$, and $\gamma \geq \gamma_*$.

We provide the proof of Corollary 6.14 here. That of Corollary 6.15 can be written similarly.

Proof of Corollary 6.14. We first prove the result for $\ell = 0$ and $s \in \mathbb{R}$. We start from the following form of the Carleman estimate:

$$\|\tilde\tau^{-\frac12}v\|_{m,\tilde\tau}^2 + |T(v)|_{m-1,1/2,\tilde\tau}^2 \lesssim \big(\|P_\varphi v\|_{L^2}^2 + \sum_{k=1}^{\mu}\left|T(B_\varphi^k v)\right|_{m-1/2-\beta_k,\tilde\tau}^2\big),$$

and we shall apply it to $\tilde\tau^s v$ in place of v. In fact

$$\begin{aligned}\|P_\varphi\tilde\tau^s v\|_{L^2} &\leq \|\tilde\tau^s P_\varphi v\|_{L^2} + \|[P_\varphi, \tilde\tau^s]v\|_{L^2} \\ &\lesssim \|\tilde\tau^s P_\varphi v\|_{L^2} + \gamma\,\|\tilde\tau^s v\|_{m-1,\tilde\tau},\end{aligned}$$

as $[\tilde\tau^s, P_\varphi] \in \gamma\tilde\tau^s\mathscr{D}_{\tilde\tau}^{m-1}$ (see Section 5.1). Similarly

$$|B_\varphi^k\tilde\tau^s v|_{m-\beta_k-1/2,\tilde\tau} \leq |\tilde\tau^s B_\varphi^k v|_{m-\beta_k-1/2,\tilde\tau} + \gamma|\tilde\tau^s T(v)|_{\beta_k-1,m-\beta_k-1/2,\tilde\tau},$$

as $[\tilde\tau^s, B^k_\varphi] \in \gamma\tilde\tau^s \mathscr{D}^{\beta_k-1}_{\tilde\tau}$. We thus find

$$\|\tilde\tau^s P_\varphi v\|_{L^2} + \left|\tilde\tau^s B^k_\varphi v\right|_{m-\beta_k-1/2,\tilde\tau} + \gamma\left\|\tilde\tau^s v\right\|_{m-1,\tilde\tau} + \gamma\left|\tilde\tau^s T(v)\right|_{\beta_k-1,m-\beta_k-1/2,\tilde\tau} \gtrsim \left\|\tilde\tau^{s-\frac12} v\right\|_{m,\tilde\tau} + |T(\tilde\tau^s v)|_{m-1,1/2,\tilde\tau}.$$

As $[D^j_{x_n}, \tilde\tau^s]v \in \gamma\tilde\tau^s\mathscr{D}^{j-1}_{\tilde\tau}$ we have

$$|\tilde\tau^s T(v)|_{m-1,1/2,\tilde\tau} \lesssim |T(\tilde\tau^s v)|_{m-1,1/2,\tilde\tau} + \gamma|\tilde\tau^s T(v)|_{m-2,1/2,\tilde\tau},$$

yielding for τ large

$$|\tilde\tau^s T(v)|_{m-1,1/2,\tilde\tau} \lesssim |T(\tilde\tau^s v)|_{m-1,1/2,\tilde\tau}.$$

We thus find

$$\|\tilde\tau^s P_\varphi v\|_{L^2} + \left|\tilde\tau^s B^k_\varphi v\right|_{m-\beta_k-1/2,\tilde\tau} + \gamma\left\|\tilde\tau^s v\right\|_{m-1,\tilde\tau} + \gamma\left|\tilde\tau^s T(v)\right|_{\beta_k-1,m-\beta_k-1/2,\tilde\tau} \gtrsim \left\|\tilde\tau^{s-\frac12} v\right\|_{m,\tilde\tau} + |\tilde\tau^s T(v)|_{m-1,1/2,\tilde\tau}.$$

For τ chosen sufficiently large we obtain

$$\|\tilde\tau^s P_\varphi v\|_{L^2} + \left|\tilde\tau^s B^k_\varphi v\right|_{m-\beta_k-1/2,\tilde\tau} \gtrsim \left\|\tilde\tau^{s-\frac12} v\right\|_{m,\tilde\tau} + |\tilde\tau^s T(v)|_{m-1,1/2,\tilde\tau},$$

that is, the result for $\ell = 0$.

Next, we proceed by induction on ℓ. As the result holds for $\ell = 0$ we assume it holds for some $\ell \in \mathbb{N}$; we then have

$$\left\|\tilde\tau^{s-\frac12} v\right\|^2_{m+\ell,\tilde\tau} + |\tilde\tau^s T(v)|^2_{m+\ell-1,1/2,\tilde\tau} \lesssim \big(\tilde\tau^s \|P_\varphi v\|^2_{\ell,\tilde\tau} + \sum_{k=1}^{\mu}\left|\tilde\tau^s T(B^k_\varphi v)\right|^2_{\ell,m-1/2-\beta_k,\tilde\tau}\big),$$

which we shall apply to $D_{x_n}v$ and $D^\alpha_{x'}v$ for $|\alpha| = 1$. We have

$$\begin{aligned}\|\tilde\tau^s P_\varphi D_{x_n} v\|_{\ell,\tilde\tau} + \|\tilde\tau^s P_\varphi D^\alpha_{x'} v\|_{\ell,\tilde\tau} &\lesssim \|\tilde\tau^s P_\varphi v\|_{\ell+1,\tilde\tau} + \|\tilde\tau^s [P_\varphi, D_{x_n}]v\|_{\ell,\tilde\tau} + \|\tilde\tau^s [P_\varphi, D^\alpha_{x'}]v\|_{\ell,\tilde\tau}\\ &\lesssim \|\tilde\tau^s P_\varphi v\|_{\ell+1,\tilde\tau} + \gamma\|\tilde\tau^s v\|_{m+\ell,\tilde\tau},\end{aligned}$$

as $\tilde\tau^s[P_\varphi, D_{x_j}] \in \gamma\tilde\tau^s\mathscr{D}^m_{\tilde\tau}$. We also have

$$\begin{aligned}&\left|\tilde\tau^s T(B^k_\varphi D_{x_n} v)\right|_{\ell,m-\beta_k-1/2,\tilde\tau} + \left|\tilde\tau^s T(B^k_\varphi D^\alpha_{x'} v)\right|_{\ell,m-\beta_k-1/2,\tilde\tau}\\ &\quad\le \left|\tilde\tau^s T(D_{x_n} B^k_\varphi v)\right|_{\ell,m-\beta_k-1/2,\tilde\tau} + \left|\tilde\tau^s T(D^\alpha_{x'} B^k_\varphi v)\right|_{\ell,m-\beta_k-1/2,\tilde\tau} + \gamma\left|\tilde\tau^s T(v)\right|_{\ell+\beta_k,m-\beta_k-1/2,\tilde\tau}\\ &\quad\le \left|\tilde\tau^s T(B^k_\varphi v)\right|_{\ell+1,m-\beta_k-1/2,\tilde\tau} + \gamma\left|\tilde\tau^s T(v)\right|_{\ell+\beta_k,m-\beta_k-1/2,\tilde\tau}.\end{aligned}$$

We thus have

$$\begin{aligned}&\|\tilde\tau^s P_\varphi v\|_{\ell+1,\tilde\tau} + \left|\tilde\tau^s T(B^k_\varphi v)\right|_{\ell+1,m-\beta_k-1/2,\tilde\tau} + \gamma\|\tilde\tau^s v\|_{m+\ell,\tilde\tau} + \gamma\left|\tilde\tau^s T(v)\right|_{\ell+\beta_k,m-\beta_k-1/2,\tilde\tau}\\ &\quad\gtrsim \left\|\tilde\tau^{s+1} P_\varphi v\right\|_{\ell,\tilde\tau} + \|\tilde\tau^s P_\varphi D_{x_n} v\|_{\ell,\tilde\tau} + \|\tilde\tau^s P_\varphi D^\alpha_{x'} v\|_{\ell,\tilde\tau}\\ &\qquad + \left|\tilde\tau^{s+1} T(B^k_\varphi v)\right|_{\ell,m-\beta_k-1/2,\tilde\tau} + \left|\tilde\tau^s T(B^k_\varphi D_{x_n} v)\right|_{\ell,m-\beta_k-1/2,\tilde\tau} + \left|\tilde\tau^s T(B^k_\varphi D^\alpha_{x'} v)\right|_{\ell,m-\beta_k-1/2,\tilde\tau}.\end{aligned}$$

This yields, by induction,

$$\begin{aligned}&\|\tilde\tau^s P_\varphi v\|_{\ell+1,\tilde\tau} + \left|\tilde\tau^s T(B^k_\varphi v)\right|_{\ell+1,m-\beta_k-1/2,\tilde\tau} + \gamma\|\tilde\tau^s v\|_{m+\ell,\tilde\tau} + \gamma\left|\tilde\tau^s T(v)\right|_{\ell+\beta_k,m-\beta_k-1/2,\tilde\tau}\\ &\quad\gtrsim \left\|\tilde\tau^{s+\frac12} v\right\|_{m+\ell,\tilde\tau} + \sum_{1\le j\le n}\left\|\tilde\tau^{s-\frac12} D_{x_j} v\right\|_{m+\ell,\tilde\tau}\\ &\qquad + \left|\tilde\tau^{s+1} T(v)\right|_{\ell+m-1,1/2,\tilde\tau} + \sum_{1\le j\le n}\left|\tilde\tau^s T(D_{x_j} v)\right|_{\ell+m-1,1/2,\tilde\tau}.\end{aligned}$$

For τ sufficiently large, we thus obtain

$$\|\tilde{\tau}^s P_\varphi v\|_{\ell+1,\tilde{\tau}} + \left|\tilde{\tau}^s T(B^k_\varphi v)\right|_{\ell+1,m-\beta_k-1/2,\tilde{\tau}} \gtrsim \left\|\tilde{\tau}^{s-\frac{1}{2}} v\right\|_{m+\ell+1,\tau} + |\tilde{\tau}^s T(v)|_{\ell+m,1/2,\tau}\,,$$

which then implies the result. □

7. Application to unique continuation

With the Carleman estimates we have derived here we can obtain unique continuation results near a boundary for high-order elliptic operators.

7.1. Uniqueness under strong pseudo-convexity and strong Lopatinskii conditions.

Theorem 7.1. *Let P and B^k, $k = 1, \dots, \mu = m/2$ be given as in Section 1. Let $x_0 \in \partial\Omega$, $f \in \mathscr{C}^\infty(\overline{\Omega})$, and V be a neighborhood of x_0 in $\overline{\Omega}$, be such that f has the strong pseudo-convexity property of Definition 6.1 with respect to P in $\overline{V}$. Moreover, assume that $\left\{P, f, B^k,\ k = 1, \dots, \mu\right\}$ satisfies the strong Lopatinskii condition at x_0. Assume that $u \in H^m(\Omega)$ satisfies*

-

$$|Pu(x)| \le C \sum_{|\alpha|\le m-1} |D^\alpha u(x)|, \quad \text{a.e. in } V; \tag{7.29}$$

- *for $k = 1, \dots, \mu$*

$$|B^k u(x)| \le C \sum_{|\alpha|\le \beta_k-1} |D^\alpha u(x)|, \quad \text{a.e. in a neighborhood of } V \cap \partial\Omega, \tag{7.30}$$

- *and u vanishes in $\{x \in V;\ f(x) \ge f(x_0)\}$.*

Then u vanishes in a neighborhood of x_0.

Proof. Strong pseudo-convexity is a stable notion in $\mathscr{C}^2$ (see Proposition 28.3.2 in [20]). For ε chosen sufficiently small, there exists a neighborhood V' of x_0 such that the function $\psi(x) = f(x) - \varepsilon|x - x_0|^2$ has the strong pseudo-convexity property of Definition 6.1 with respect to P in V'. Similary we saw in Section 1.6, for the proof of Proposition 1.8, that the strong Lopatinskii condition is stable. Hence if ε is chosen sufficiently small $\left\{P, \psi, B^k,\ k = 1, \dots, \mu\right\}$ will also satisfy this condition.

We set $\varphi = \exp(\gamma\psi)$. As shown in Proposition 28.3.3 in [20] the strong pseudo-convexity of the function ψ with respect to P implies the sub-ellipticity condition for φ and P for γ chosen sufficiently large. Moreover, as seen in Section 6.2, $\left\{P, \varphi, B^k,\ k = 1, \dots, \mu\right\}$ also satisfies the strong Lopatinskii condition at x_0.

We call W the region $\{x \in V;\ f(x) \ge f(x_0)\}$ (region beneath $\{f(x) = f(x_0)\}$ in Figure 1) where u vanishes by assumption. We choose V'' a neighborhood of x_0 such that $V'' \Subset V'$. The geometrical situation is illustrated in Figure 1. We pick a function $\chi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ such that $\chi = 1$ in V'' and $\operatorname{supp}(\chi) \cap V \subset V'$. We observe that the Carleman estimate of Theorem 1.6 applies to χu by density (possibly by reducing the neighborhoods V and V' of x_0):

$$\tau^{-1/2}\,\|e^{\tau\varphi}\chi u\|_{m,\tau} + |e^{\tau\varphi}T(\chi u)|_{m-1,1/2,\tau} \lesssim \|e^{\tau\varphi}P(\chi u)\|_{L^2} + \sum_{k=1}^{\mu} \left|e^{\tau\varphi}B^k(\chi u)_{|\partial\Omega}\right|_{m-1/2-\beta_k,\tau}.$$

We have

$$P(\chi u) = \chi P u + [P, \chi]u, \quad B^k(\chi u) = \chi B^k u + [B^k, \chi]u,$$

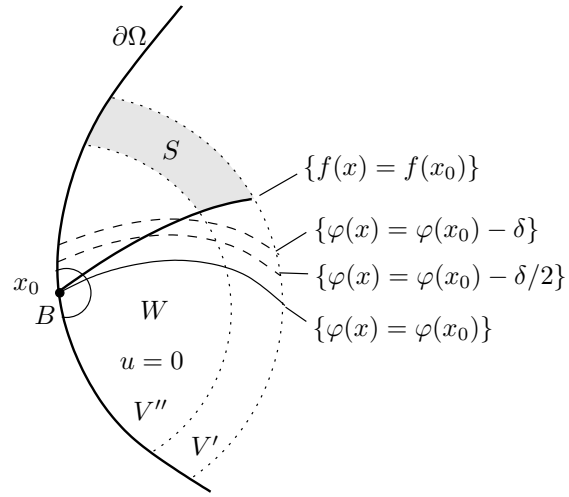


FIGURE 1. Local geometry for the unique continuation problem. The shaded region contains the supports of $[P, \chi]u$ and $[\chi, D^j_{x_k}]u$.

where the commutators are differential operators of order $m-1$ and $\beta_k - 1$ respectively. With (7.29) we have

$$\begin{aligned}
\|e^{\tau\varphi}P(\chi u)\|_{L^2} &\lesssim \sum_{|\alpha|\le m-1} \|e^{\tau\varphi}\chi D^\alpha u\|_{L^2} + \|e^{\tau\varphi}[P,\chi]u\|_{L^2} \\
&\lesssim \sum_{|\alpha|\le m-1} \|e^{\tau\varphi}D^\alpha(\chi u)\|_{L^2} + \sum_{|\alpha|\le m-1} \|e^{\tau\varphi}[\chi, D^\alpha]u\|_{L^2} + \|e^{\tau\varphi}[P,\chi]u\|_{L^2} \\
&\lesssim \|e^{\tau\varphi}\chi u\|_{m-1,\tau} + \sum_{|\alpha|\le m-1} \|e^{\tau\varphi}[\chi, D^\alpha]u\|_{L^2} + \|e^{\tau\varphi}[P,\chi]u\|_{L^2}\,.
\end{aligned}$$

With (7.30) we have

$$\begin{aligned}
&\sum_{k=1}^{\mu} \left|e^{\tau\varphi}B^k(\chi u)_{|\partial\Omega}\right|_{m-1/2-\beta_k,\tau} \\
&\quad\lesssim \sum_{k=1}^{\mu}\Big(\sum_{|\alpha|\le\beta_k-1} \left|e^{\tau\varphi}\chi D^\alpha u_{|\partial\Omega}\right|_{m-1/2-\beta_k,\tau} + \left|e^{\tau\varphi}[B^k,\chi]u_{|\partial\Omega}\right|_{m-1/2-\beta_k,\tau}\Big) \\
&\quad\lesssim |e^{\tau\varphi}(T(\chi u)|_{m-2,1/2,\tau} + \sum_{k=1}^{\mu}\Big(\sum_{|\alpha|\le\beta_k-1} \left|e^{\tau\varphi}[\chi, D^\alpha]u_{|\partial\Omega}\right|_{m-1/2-\beta_k,\tau} \\
&\qquad\qquad + \left|e^{\tau\varphi}[B^k,\chi]u_{|\partial\Omega}\right|_{m-1/2-\beta_k,\tau}\Big),
\end{aligned}$$

as $\beta_k \le m-1$. For τ chosen sufficiently large we thus obtain

$$\begin{aligned}
\tau^{-1/2}\,\|e^{\tau\varphi}\chi u\|_{m,\tau} &+ |e^{\tau\varphi}T(\chi u)|_{m-1,1/2,\tau} \\
&\lesssim \sum_{|\alpha|\le m-1} \|e^{\tau\varphi}[\chi, D^\alpha]u\|_{L^2} + \|e^{\tau\varphi}[P,\chi]u\|_{L^2} \\
&\quad + \sum_{k=1}^{\mu}\Big(\sum_{|\alpha|\le\beta_k-1} \left|e^{\tau\varphi}[\chi, D^\alpha]u_{|\partial\Omega}\right|_{m-1/2-\beta_k,\tau} + \left|e^{\tau\varphi}[B^k,\chi]u_{|\partial\Omega}\right|_{m-1/2-\beta_k,\tau}\Big).
\end{aligned} \tag{7.31}$$

We set $S := V' \setminus (V'' \cup W)$ (see the shaded region in Figure 1). We have

$$\operatorname{supp}([P,\chi]u) \subset S, \quad \operatorname{supp}([B^k,\chi]u) \subset S, \quad \operatorname{supp}([\chi, D^\alpha]u) \subset S,$$

as they are confined in the region where χ varies and u does not vanish. We thus obtain

$$\tau^{-1/2} \|e^{\tau\varphi}\chi u\|_{m,\tau} + |e^{\tau\varphi}T(\chi u)|_{m-1,1/2,\tau} \lesssim \sum_{|\alpha|\le m-1} \|e^{\tau\varphi}D^\alpha u\|_{L^2(S)} + \tau^r \sum_{|\alpha|\le m-2} \left|e^{\tau\varphi}D^\alpha u_{|\partial\Omega}\right|_{H^{m-|\alpha|-3/2}(S\cap\partial\Omega)},$$

for some $r > 0$. For all $\delta > 0$, we set $V_\delta = \{x \in V;\ \varphi(x) \le \varphi(x_0) - \delta\}$. There exists $\delta > 0$ such that $S \Subset V_\delta$. We then choose B a neighborhood of x_0 such that $\overline{B} \subset V'' \setminus V_{\delta/2}$ and obtain, as $\chi \equiv 1$ on B,

$$e^{\tau \inf_B \varphi} \|u\|_{H^m(B)} \lesssim e^{\tau(\sup_S \varphi + \delta/2)} \big(\|u\|_{H^{m-1}(S)} + \sum_{|\alpha|\le m-2} \left|D^\alpha u_{|\partial\Omega}\right|_{H^{m-|\alpha|-3/2}(S\cap\partial\Omega)} \big), \quad \tau \ge \tau_1,$$

for some $\tau_1 > 0$. Since $\inf_B \varphi > \sup_S \varphi + \delta/2$, letting τ go to ∞, we obtain $u = 0$ in B. □

7.2. **Uniqueness for product operators.** If we consider two ellitpic operators P_1 and P_2 of order m_1 and m_2, one may possibly wonder about unique continuation for the operator $P = P_1P_2$ of order $m = m_1+m_2$, in particular, in the case no Carleman estimate for P can be achieved.

Let us assume that for a function ψ and the weigth function $\varphi = \exp(\gamma\psi)$ we can derive Carleman estimates for P_1 and P_2. The estimates we prove here are characterized by the (optimal) loss of a half derivative. Applying such estimates one after another leads to an estimate with the loss of a full derivative. This can be limiting for the treatment of unique continuation problems. Here we show that if one of the operators, say P_1, satisfies the single characteristic properties (for the weight ψ) and the second one, P_2, satisfies the strong pseudo-convexity condition, and the strong Lopatinsky conditions holds for both, then uniqueness can be proven. Note that this allows for the occurence of complex roots of order 3 for the conjugated operators P_φ with symbol $p(x, \xi + i\tau\varphi'(x))$.

Theorem 7.2. *Let P_1 and P_2 be elliptic of order m_1 and m_2 respectively, and let also B_1^k be of order $\beta_{1,k}$, $k = 1,\dots,\mu_1 = m_1/2$, and B_2^k be of order $\beta_{2,k}$, $k = 1,\dots,\mu_2 = m_2/2$ be given as in Section 1. Let $x_0 \in \partial\Omega$, $f \in \mathscr{C}^\infty(\overline{\Omega})$, and V be a neighborhood of x_0 in $\overline{\Omega}$ be such that:*

(1) *f and P_1 satisfy the simpe characteristic property of of Definition 6.9 in $\overline{V}$;*
(2) *f has the strong pseudo-convexity property of Definition 6.1 with respect to P_2 in $\overline{V}$;*
(3) *$\left\{P_j, f, B_j^k,\ k = 1,\dots,\mu_j\right\}$ satisfies the strong Lopatinskii condition at x_0, $j = 1, 2$.*

Let $m = m_1 + m_2$ and assume that $u \in H^m(\Omega)$ satisfies

- $$|P_1P_2u(x)| \le C \sum_{|\alpha|\le m-1} |D^\alpha u(x)|, \quad \text{a.e. in } V; \tag{7.32}$$
- *for $k = 1,\dots,\mu_1$*
$$|B_1^k P_2 u(x)| \le C \sum_{|\alpha|\le m_2+\beta_{1,k}-1} |D^\alpha u(x)|, \quad \text{a.e. in a neighborhood of } V \cap \partial\Omega; \tag{7.33}$$
- *for $k = 1,\dots,\mu_2$ and $j = 1,\dots,m_1$*
$$|D_{x_n}^j B_2^k u(x)| \le C \sum_{|\alpha|\le j+\beta_{2,k}-1} |D^\alpha u(x)|, \quad \text{a.e. in a neighborhood of } V \cap \partial\Omega, \tag{7.34}$$
- *and u vanishes in $\{x \in V;\ f(x) \ge f(x_0)\}$.*

Then u vanishes in a neighborhood of x_0.

Proof. The proof follows that of Theorem 7.1. We set $\psi(x) = f(x) - \varepsilon|x - x_0|^2$ and conditions (1)-(3) in the statement of the theorem are also satisfied by ψ for ε chosen sufficiently small in a neighborhood $V' \subset V$ of x_0. We then set $\varphi = \exp(\gamma\psi)$.

We derive an estimate for $P = P_1P_2$. For P_1, by Theorem 6.11, there exists V_1 neighborhood of x_0 in $\mathbb{R}^n$ such that $V_1 \cap \Omega \subset V'$ and

$$\gamma^{1/2}\big\|\tilde\tau^{-\frac12}e^{\tau\varphi}v\big\|_{m_1,\tilde\tau} + |e^{\tau\varphi}T(v)|_{m_1-1,1/2,\tilde\tau} \lesssim \|e^{\tau\varphi}P_1v\|_{L^2} + \sum_{k=1}^{\mu_1}\Big|e^{\tau\varphi}B_1^kv_{|\partial\Omega}\Big|_{m_1-1/2-\beta_{1,k},\tilde\tau}, \tag{7.35}$$

for all $v = w_{|\Omega}$ with $w \in \mathscr{C}_c^\infty(V_1)$, $\tau \geq \tau_1$, and $\gamma \geq \gamma_1$, for τ_1 and γ_1 chosen sufficiently large.

For P_2, by Corollary 6.14, there exists V_2 neighborhood of x_0 in $\mathbb{R}^n$ such that $V_2 \cap \Omega \subset V'$ and

$$\begin{aligned}\big\|\tilde\tau^{-1}e^{\tau\varphi}v\big\|_{m,\tilde\tau} + \Big|\tilde\tau^{-\frac12}e^{\tau\varphi}T(v)\Big|_{m-1,1/2,\tilde\tau} \\ \lesssim \Big\|\tilde\tau^{-\frac12}e^{\tau\varphi}P_2v\Big\|_{m_1,\tilde\tau} + \sum_{k=1}^{\mu_2}\Big|\tilde\tau^{-\frac12}e^{\tau\varphi}T(B_2^kv)\Big|_{m_1,m_2-1/2-\beta_{2,k},\tilde\tau},\end{aligned} \tag{7.36}$$

with $m = m_1 + m_2$, for all $v = w_{|\Omega}$ with $w \in \mathscr{C}_c^\infty(V_2)$, $\tau \geq \tau_2$, and $\gamma \geq \gamma_2$, for τ_2 and γ_2 chosen sufficiently large.

Letting $V_3 = V_1 \cap V_2$ and $w \in \mathscr{C}_c^\infty(V_3)$ for $v = w_{|\Omega}$ with (7.35) and (7.36) we obtain

$$\begin{aligned}&\gamma^{1/2}\big\|\tilde\tau^{-1}e^{\tau\varphi}v\big\|_{m,\tilde\tau} + \gamma^{1/2}\Big|\tilde\tau^{-\frac12}e^{\tau\varphi}T(v)\Big|_{m-1,1/2,\tilde\tau} \\ &\lesssim \|e^{\tau\varphi}Pv\|_{L^2} + \sum_{k=1}^{\mu_1}\Big|e^{\tau\varphi}B_1^kP_2v_{|\partial\Omega}\Big|_{m_1-1/2-\beta_{1,k},\tilde\tau} + \gamma^{1/2}\sum_{k=1}^{\mu_2}\Big|\tilde\tau^{-\frac12}e^{\tau\varphi}T(B_2^kv)\Big|_{m_1,m_2-1/2-\beta_{2,k},\tilde\tau}.\end{aligned} \tag{7.37}$$

We choose χ as in the proof of Theorem 7.1 and we apply estimate (7.37) to $v = \chi u$ as can be done by a density argument. We now sketch how the remainder of the proof can be carried out.

We have $P(\chi u) = \chi Pu + [P,\chi]u$. The term $[P,\chi]u$ is supported in the set S introduced in the proof of Theorem 7.1 and can be handle as it is done there. For the first term, with (7.32) we have

$$\|e^{\tau\varphi}\chi Pu\|_{L^2} \lesssim \sum_{|\alpha|\leq m-1}\|e^{\tau\varphi}\chi D^\alpha u\|_{L^2} \lesssim \sum_{|\alpha|\leq m-1}\|e^{\tau\varphi}D^\alpha(\chi u)\|_{L^2} + \sum_{|\alpha|\leq m-1}\|e^{\tau\varphi}[\chi,D^\alpha]u\|_{L^2}$$

The second term in the r.h.s. concerns functions with support located in S and their treatment is done as in the proof of Theorem 7.1. For the first term we have

$$\sum_{|\alpha|\leq m-1}\|e^{\tau\varphi}D^\alpha(\chi u)\|_{L^2} \lesssim \|e^{\tau\varphi}\chi u\|_{m-1,\tilde\tau} \lesssim \big\|\tilde\tau^{-1}e^{\tau\varphi}\chi u\big\|_{m,\tilde\tau},$$

which can be absorbed by the first term in (7.37) by chosing γ sufficiently large.

Next we write $B_1^kP_2\chi u = \chi B_1^kP_2u + [B_1^kP_2,\chi]u$. The term $[B_1^kP_2,\chi]u$ is supported in the set S and can be treated as in the proof of Theorem 7.1. For the first term, with (7.33) we have

$$\begin{aligned}&\sum_{k=1}^{\mu_1}\Big|e^{\tau\varphi}\chi B_1^kP_2v_{|\partial\Omega}\Big|_{m_1-1/2-\beta_{1,k},\tilde\tau} \\ &\lesssim \sum_{|\alpha|\leq m_2+\beta_{1,k}-1}\Big|e^{\tau\varphi}\chi D^\alpha u_{|\partial\Omega}\Big|_{m_1-1/2-\beta_{1,k},\tilde\tau} \\ &\lesssim \sum_{|\alpha|\leq m_2+\beta_{1,k}-1}\Big(\Big|e^{\tau\varphi}D^\alpha(\chi u)_{|\partial\Omega}\Big|_{m_1-1/2-\beta_{1,k},\tilde\tau} + \Big|e^{\tau\varphi}[\chi,D^\alpha]u_{|\partial\Omega}\Big|_{m_1-1/2-\beta_{1,k},\tilde\tau}\Big) \\ &\lesssim |e^{\tau\varphi}T(\chi u)|_{m-2,1/2,\tilde\tau} + \sum_{|\alpha|\leq m_2+\beta_{1,k}-1}\Big|e^{\tau\varphi}[\chi,D^\alpha]u_{|\partial\Omega}\Big|_{m_1-1/2-\beta_{1,k},\tilde\tau},\end{aligned}$$

The first term can be absorbed by the second term in (7.37) by chosing τ or γ sufficiently large. The second term in the r.h.s. concerns functions with support located in S and their treatment is done as in the proof of Theorem 7.1.

We finally write

$$\sum_{k=1}^{\mu_2}\left|\tilde\tau^{-\frac12}e^{\tau\varphi}T(B_2^k\chi u)\right|_{m_1,m_2-1/2-\beta_{2,k},\tilde\tau}=\sum_{j=0}^{m_1}\sum_{k=1}^{\mu_2}\left|\tilde\tau^{-\frac12}e^{\tau\varphi}D_{x_n}^jB_2^k\chi u)\right|_{m_2-j-1/2-\beta_{2,k},\tilde\tau}.$$

We thus write $D_{x_n}^jB_2^k\chi u=\chi D_{x_n}^jB_2^kxu+[D_{x_n}^jB_2^k,\chi]u$. The term $[D_{x_n}^jB_2^k,\chi]u$ is supported in the set S and can be treated as in the proof of Theorem 7.1. For the first term, with (7.34) we have

$$\begin{aligned}&\sum_{j=0}^{m_1}\sum_{k=1}^{\mu_2}\left|\tilde\tau^{-\frac12}e^{\tau\varphi}\chi D_{x_n}^jB_2^ku)\right|_{m_2-j-1/2-\beta_{2,k},\tilde\tau}\\&\quad\lesssim\sum_{j=0}^{m_1}\sum_{k=1}^{\mu_2}\sum_{|\alpha|\le j+\beta_{2,k}-1}\left|\tilde\tau^{-\frac12}e^{\tau\varphi}\chi D^\alpha u\right|_{m_2-j-1/2-\beta_{2,k},\tilde\tau}\\&\quad\lesssim\sum_{j=0}^{m_1}\sum_{k=1}^{\mu_2}\sum_{|\alpha|\le j+\beta_{2,k}-1}\Big(\left|\tilde\tau^{-\frac12}e^{\tau\varphi}D^\alpha(\chi u)\right|_{m_2-j-1/2-\beta_{2,k},\tilde\tau}+\left|\tilde\tau^{-\frac12}e^{\tau\varphi}[\chi,D^\alpha]u\right|_{m_2-j-1/2-\beta_{2,k},\tilde\tau}\Big).\end{aligned}$$

This yields

$$\begin{aligned}&\gamma^{1/2}\sum_{j=0}^{m_1}\sum_{k=1}^{\mu_2}\left|\tilde\tau^{-\frac12}e^{\tau\varphi}\chi D_{x_n}^jB_2^ku)\right|_{m_2-j-1/2-\beta_{2,k},\tilde\tau}\\&\quad\lesssim\gamma^{1/2}\left|\tilde\tau^{-\frac12}e^{\tau\varphi}T(\chi u)\right|_{m-2,1/2,\tilde\tau}+\gamma^{1/2}\sum_{j=0}^{m_1}\sum_{k=1}^{\mu_2}\sum_{|\alpha|\le j+\beta_{2,k}-1}\left|\tilde\tau^{-\frac12}e^{\tau\varphi}[\chi,D^\alpha]u\right|_{m_2-j-1/2-\beta_{2,k},\tilde\tau}.\end{aligned}$$

The first term can be absorbed by the second term in (7.37) by chosing τ sufficiently large. The second term in the r.h.s. concerns functions with support located in S and their treatment is done as in the proof of Theorem 7.1.

The end of the proof is then similar to that of Theorem 7.1. □

Appendix A. Proofs of some technical results

A.1. **Details on the examples of Section 1.7.** We first consider $p(x,\xi)=\xi_n^2+r(x,\xi')$. We have

$$p_\varphi(x,\xi',\tau)=(\xi_n+i\tau\varphi')^2+r(x,\xi')=(\xi_n+i\tau\varphi'+i\theta(x,\xi'))(\xi_n+i\tau\varphi'-i\theta(x,\xi')),$$

with $\theta(x,\xi')\ge C|\xi'|$ and $\theta(x,\xi')^2=r(x,\xi')$. The roots of the operators are thus given by $\alpha_1=-i\tau\varphi'-i\theta(x,\xi')$ and $\alpha_2=-i\tau\varphi'+i\theta(x,\xi')$.

If $\partial_{x_n}\varphi=\varphi'<0$ we may have simultaneously $\operatorname{Im}\alpha_1>0$ and $\operatorname{Im}\alpha_2>0$ yielding $p_\varphi^-=1$ and $\kappa_\varphi=p_\varphi$, a situation that forbids the strong Lopatinskii condition (see Remark 1.5 and Proposition 1.8). This explains the assumption $\partial_{x_n}\varphi=\varphi'>0$. (In fact having φ' vanishing prevents the Carleman estimate to hold [17, 31, 30].) Then $\operatorname{Im}\alpha_1<0$ and we see that $d^\circ\kappa_\varphi\le1$. If $d^\circ\kappa_\varphi=0$, i.e., both roots have negative imaginary parts, then the strong Lopatinskii is trivially fulfilled independently of the boundary operators. This includes the low frequency regime, $|\xi'|\le\delta\tau$ for δ sufficiently small.

If now $d^\circ\kappa_\varphi=1$, then $\kappa_\varphi=\xi_n+i\tau\varphi'-i\theta(x,\xi')$ and the strong Lopatinskii condition is satisfied if $(b_\varphi,\kappa_\varphi)$ is a complete family in the space of polynomials of degree less than or equal to 1. In this second case $\xi'\neq0$.

(1) In the case $Bu=u$, then $b=b_\varphi=1$ and the result is clear.
(2) In the case $Bu=D_{x_n}u+a(x)u$ we have $b_\varphi=\xi_n+i\tau\varphi'$. Since $\xi'\neq0$ here, then b_φ and κ_φ have distinct roots and thus form a complete family.

(3) In the case $Bu = D_{x_n}u + ia(x)D_{x_1}u$ then $b_\varphi = \xi_n + i\tau\varphi' + ia\xi_1$. Assuming that b_φ and κ_φ have a common root this means $a\xi_1 = -\theta(x,\xi')$, implying $a^2\xi_1^2 = r(x,\xi')$. Yet $a^2\xi_1^2 \le a^2|\xi'|^2 < r(x,\xi')$, unless $\xi' = 0$ which is excluded here. We thus see that b_φ and κ_φ do not have a common root. They thus generate polynomials of degree less than or equal to 1.

Finally, observe that all the above remains valid if we let φ also depend on x' with a $|\partial_{x'}\varphi| \ll |\partial_{x_n}\varphi|$.

We now consider $p = \xi_1^4 + \xi_2^4$, with here $n = 2$, in $V_+ = \{x_2 > 0\}$. At first we take $\varphi = \varphi(x_2)$. Then $p_\varphi = \xi_1^4 + (\xi_2 + i\tau\varphi')^4$ that we write

$$p_\varphi = \prod_{j=1}^{4}(\xi_2 - \alpha_j), \quad \text{with } \alpha_j = -i\tau\varphi' - e^{i\pi(2j-1)/4}\xi_1,\ j = 1,2,3,4.$$

Here also we assume $\varphi' > 0$ to forbid all the roots to be in the upper complex half plane. Then $\operatorname{Im}\alpha_1 < 0$ and $\operatorname{Im}\alpha_2 < 0$. We thus have $d^\circ\kappa_\varphi \le 2$.

(1) If $B^1u = u$, $B^2u = D_{x_2}u$ then $b_\varphi^1 = 1$ and $b_\varphi^2 = \xi_2 + i\tau\varphi'$. As b_φ^1 and b_φ^2 generate the polynomials of degree less than or equal to 1, the strong Lopatinskii condition is fulfilled.

(2) If $B^1u = u$, $B^2u = \Delta u$ then $b_\varphi^1 = 1$ and

$$b_\varphi^2 = (\xi_2 + i\tau\varphi')^2 + \xi_1^2 = (\xi_2 + i\tau\varphi' + i\xi_1)(\xi_2 + i\tau\varphi' - i\xi_1).$$

As the roots of b_φ^1, b_φ^2 and κ_φ are all distinct we see that they generate the polynomials of degree less than or equal to 2. The strong Lopatinskii condition thus holds.

(3) If $B^1u = u$, $B^2u = D_{x_2}\Delta u$ then $b_\varphi^1 = 1$ and

$$b_\varphi^2 = (\xi_2 + i\tau\varphi')(\xi_2 + i\tau\varphi' + i\xi_1)(\xi_2 + i\tau\varphi' - i\xi_1).$$

- If $d^\circ\kappa_\varphi = 0$ then the strong Lopatinskii condition holds trivially.
- If $d^\circ\kappa_\varphi = 1$ then the strong Lopatinskii condition holds as b_φ^1 generates the constant polynomials.
- If $d^\circ\kappa_\varphi = 2$ then the polynomials b_φ^1, b_φ^2, κ_φ, and $\xi_2\kappa_\varphi$ are linearly independent. They thus generates the polynomials of degree less than or equal to 3 meaning that the strong Lopatinskii condition holds.

As above we observe that all the above remains valid if we let φ also depend on x_1 with a $|\partial_{x_1}\varphi| \ll |\partial_{x_2}\varphi|$.

A.2. **Regularity of the decomposition $p_\varphi = p_\varphi^- p_\varphi^+ p_\varphi^0$.** For concision, we denote by ϱ' the variable $(x,\xi',\tau) \in \overline{\mathbb{R}}_+^n \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ as is sometimes done in the main text.

Homogeneity is always understood w.r.t. the variables $\eta = (\xi',\tau)$ for $|\eta| \ge r_0$ for some $r_0 \ge 0$, leading to the introduction of the map

$$M_\lambda\varrho' = (x,\lambda\eta), \qquad \varrho' = (x,\eta) \in \overline{\mathbb{R}}_+^n \times \mathbb{R}^{n-1} \times \mathbb{R}_+.$$

A function $\varrho' \mapsto f(\varrho')$ is thus said to be homogeneous of degree k if

$$f \circ M_\lambda(\varrho') = \lambda^k f(\varrho'), \quad \varrho' = (x,\eta),\ |\eta| \ge r_0,\ \lambda \ge 1.$$

We start with a classical result stating the homogeneity of the roots of an homogeneous polynomial function.

Lemma A.1. *Let $p(\varrho',\zeta) = \sum_{j=0}^m a_j(\varrho')\zeta^j$ be a polynomial function with $\mathscr{C}^r$ coefficients a_j. Assume that the coefficients $a_j(\varrho')$ are homogeneous of degree $m-j$. Then the roots $\alpha_j(\varrho')$ of the polynomial function $p(\varrho',\zeta)$ in ζ are homogeneous functions of degree one.*

Proof. For any $\varrho' = (x, \eta) \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+$, we factorize the polynomial function $p(\varrho', \zeta)$ w.r.t. ζ

$$p(\varrho', \zeta) = \prod_{j=1}^{m} \big(\zeta - \alpha_j(\varrho')\big),$$

where $\alpha_j(\varrho')$, $j = 1, \dots, m$, denote the roots repeated with multiplicity. Let $\lambda \geq 1$ and let $\eta \in \mathbb{R}^n$, $|\eta| \geq r_0$. For $k = 1, \dots, m$, the roots of $p(M_\lambda \varrho', \zeta)$ are $\alpha_k(M_\lambda \varrho')$. We consider $\beta_k(\lambda, \varrho') = \lambda^{-1}\alpha_k(M_\lambda \varrho')$. Then, we have

$$p(\varrho', \zeta) = \lambda^{-m} p(M_\lambda \varrho', \lambda\zeta) = \lambda^{-m} \prod_{j=1}^{m} \big(\lambda\zeta - \alpha_j(M_\lambda \varrho')\big) = \prod_{j=1}^{m} \big(\zeta - \beta_j(\lambda, \varrho')\big).$$

That is for any $k = 1, \dots, m$ and any $\lambda \geq 1$, $\beta_k(\lambda, \varrho')$ is a root of $p(\varrho', \zeta)$. The roots of $p(\varrho', \zeta)$ are continuous w.r.t. ϱ', as it is a classical result that the roots depend continuously upon the coefficients (a proof is in fact given in the beginning of the proof of Lemma A.2). Hence, $\lambda \to \lambda^{-1}\alpha_k(M_\lambda \varrho')$ is a continuous function. Above we saw that it can only take a finite number of values. It follows that it is a constant function. This concludes the proof. □

Lemma A.2. *Let $p(\varrho', \zeta) = \sum_{j=0}^{m} a_{m-j}(\varrho')\zeta^j$ be a polynomial function with $\mathscr{C}^r$ homogeneous coefficients $a_{m-j}(\varrho')$ of degree $m - j$, the coefficient $a_0(\varrho')$ not vanishing. For a fixed point $\varrho'_0 = (x_0, \eta_0) \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+$, with $|\eta_0| \geq r_0$, we denote the roots of $p(\varrho'_0, \zeta)$ by $\alpha_1, \dots, \alpha_N$, with respective multiplicities $\mu_1, \dots, \mu_N$ satisfying $\mu_1 + \cdots + \mu_N = m$.*

There exist a small conic neighborhood $\mathscr{U}$ of ϱ'_0 in $\overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+$, and three polynomial functions $p^+(\varrho', \zeta)$, $p^-(\varrho', \zeta)$ and $p^0(\varrho', \zeta)$ with $\mathscr{C}^r$ homogeneous coefficients and of constant degrees in $\mathscr{U}_{r_0} = \mathscr{U} \cap \{|\eta| \geq r_0\}$, such that

$$p(\varrho', \zeta) = a_0(\varrho')p^+(\varrho', \zeta)p^-(\varrho', \zeta)p^0(\varrho', \zeta), \qquad \varrho' \in \mathscr{U}_{r_0},\ \zeta \in \mathbb{R},$$

where the roots of $p^+(\varrho', \zeta)$ (resp. $p^-(\varrho', \zeta)$) are all positive (resp. negative) and we have

$$p^{\pm}(\varrho'_0, \zeta) = \prod_{\pm \operatorname{Im} \alpha_j > 0} (\zeta - \alpha_j)^{\mu_j}, \quad p^0(\varrho'_0, \zeta) = \prod_{\operatorname{Im} \alpha_j = 0} (\zeta - \alpha_j)^{\mu_j}. \tag{A.1}$$

Note that there is no constraint on the sign of the imaginary part of the roots of $p^0(\varrho', \zeta)$ for $\varrho' \neq \varrho'_0$. The idea of this lemma is the following. At $\varrho' = \varrho'_0$ the roots can be split into three groups: those with positive imaginary parts, those with negative imaginary part, and the real roots. This splitting is preserved if ϱ' remains in a small neighborhood of ϱ'_0, apart for the third group since real roots can become complex if ϱ' changes. Moreover the three groups of roots yield smooth polynomials even if the roots themselves may not be smooth[5]. This last point is of great importance here as one needs to manipulate smooth symbols in the present work. This cannot be done at the root level.

Proof. We denote by α^0_j, $j = 1, \dots, M$, the real roots with m_j-multiplicity of the polynomial function $p(\varrho'_0, \zeta)$ where $\alpha^0_i \neq \alpha^0_j$ for $i \neq j$. We shall consider ϱ' in a small neighborhood of ϱ'_0 in $\overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+$. We consider a small closed circular curve $\gamma_j : [0, 1] \to \mathbb{C}$ in $\mathbb{C}$ with center α^0_j, such that α^0_j is the only root of the polynomial equation $p(\varrho'_0, \zeta) = 0$ in D_j, the interior disk of $\mathcal{C}_j = \gamma_j([0, 1])$.

We set $\epsilon_j = \frac{1}{2} \min_{\zeta \in \mathcal{C}_j} |p(\varrho'_0, \zeta)| > 0$. Let $z \in \mathcal{C}_j$. By continuity of p, there exists a neighborhood $U^j_z \subset \mathbb{C}$ of z and a neighborhood $Y^j_z \subset \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ of ϱ'_0 such that

$$\big|p(\varrho', \zeta) - p(\varrho'_0, z)\big| < \epsilon_j, \qquad \zeta \in U^j_z,\ \varrho' \in Y^j_z.$$

[5]Only continuity w.r.t. ϱ' is certain. Smoothness may fail if multiplicity varies.

Since $\mathcal{C}_j \subset \cup_{z\in\mathcal{C}_j} U^j_z$, and $\mathcal{C}_j$ is compact, we can extract a finite covering with such neighborhoods, viz., there exists $z_1,\dots,z_\ell \in \mathcal{C}_j$ such that

$$\mathcal{C}_j \subset \cup_{k=1}^{\ell} U^j_{z_k}.$$

Then $Y^j = \cap_{k=1}^{\ell} Y^j_{z_k} \subset \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ defines also a neighborhood of ϱ'_0 such that for all $\zeta \in \mathcal{C}_j$ and all $\varrho' \in Y^j$

$$\left|p(\varrho',\zeta) - p(\varrho'_0,\zeta)\right| < 2\epsilon_j \le \left|p(\varrho'_0,\zeta)\right|.$$

By Rouché's Theorem, for each $\varrho' \in Y^j$ the equation $p(\varrho',\zeta) = 0$ has m_j roots (counted with their multiplicity) in the disc D_j, that we denote by $\alpha_{jk}(\varrho')$, $k = 1,\dots,m_j$. Since we can reduce the circle $\mathcal{C}_j$ to the point α^0_j, we get[6] $\lim_{\varrho'\to\varrho'_0} \alpha_{jk}(\varrho') = \alpha^0_j$ for all k. Invoking Lemma A.1 we extend the function $\alpha_{jk}(\varrho')$ as an homogeneous continuous function of degree one in a small conic neighborhood $\mathscr{U}$ of ϱ'_0, for $|\eta| \ge r_0$. We set $\mathscr{U}_{r_0} = \mathscr{U} \cap \{|\eta| \ge r_0\}$. Let us consider the unitary polynomial

$$p^{(j)}(\varrho',\zeta) = \prod_{k=1}^{m_j} \big(\zeta - \alpha_{jk}(\varrho')\big),$$

whose coefficients are continuous since the roots are continuous w.r.t. ϱ' as seen above. We have $p^{(j)}(\varrho'_0,\zeta) = (\zeta-\alpha^0_j)^{m_j}$. In a similar way, we can define the polynomials $p^{\pm}(\varrho',\zeta)$ and $p^0(\varrho',\zeta)$ in a small conic neighborhood of ϱ'_0 as polynomials with continuous coefficients, constant degrees, and they moreover satisfy (A.1). Now we prove that the coefficients are $\mathscr{C}^r$.

Recalling that the polynomial $p(\varrho',\zeta)$ has exactly m_j roots $\alpha_{jk}(\varrho')$, $k = 1,\dots,m_j$ (counted with multiplicities) in the disc D_j, by the Residue theorem, we find

$$\frac{1}{2i\pi}\int_{\mathcal{C}_j} \frac{\partial p(\varrho',\zeta)}{\partial\zeta}\frac{1}{p(\varrho',\zeta)}\zeta^\ell d\zeta = \sum_{k=1}^{m_j} \alpha^\ell_{jk}(\varrho') =: s^j_\ell(\varrho'), \quad \ell = 1,\dots,m_j.$$

Since the l.h.s. is a $\mathscr{C}^r$ function w.r.t. ϱ', then the functions $s^j_1(\varrho'),\dots,s^j_{m_j}(\varrho')$ are also of class $\mathscr{C}^r$. If we write $p^{(j)}(\varrho',\zeta) = \sum_{k=0}^{m_j} b^j_k(\varrho')\zeta^k$, with $b^j_{m_j} = 1$, then we have

$$\begin{aligned}
b^j_{m_j-1} &= -s^j_1\\
2b^j_{m_j-2} &= -(s^j_2 + b_{m_j-1}s^j_1)\\
3b_{m_j-3} &= -(s^j_3 + b_{m_j-1}s^j_2 + b_{m_j-2}s^j_1)\\
&\vdots\\
m_j b^j_0 &= -(s^j_{m_j} + b^j_{m_j-1}s^j_{m_j-1} + \cdots + b^j_1 s^j_1).
\end{aligned}$$

We deduce that the coefficients of $p^{(j)}(\varrho',\zeta)$ are in $\mathscr{C}^r(\mathscr{U}_{r_0})$.

Consider now the polynomial function, of degree $m - m_j$,

$$H(\varrho',\zeta) = \frac{p(\varrho',\zeta)}{p^{(j)}(\varrho',\zeta)}, \quad \varrho' \in \mathscr{U}_{r_0}.$$

We have $p(\varrho'_0,\zeta) = (\zeta-\alpha^0_j)^{m_j} H(\varrho'_0,\zeta)$ with $H(\varrho'_0,\alpha^0_j) \neq 0$. Write $H(\varrho',\zeta) = \sum_{k=0}^{m-m_j} h_k(\varrho')(\zeta-\alpha^0_j)^k$. By the Cauchy formula, we obtain

$$h_k(\varrho') = \frac{1}{2i\pi}\int_{\mathcal{C}_j} \frac{p(\varrho',\zeta)}{p^{(j)}(\varrho',\zeta)}\frac{d\zeta}{(\zeta-\alpha^0_j)^{k+1}}, \quad \varrho' \in \mathscr{U}_{r_0}.$$

[6] In particular at this point this prove the continuity of the roots with respect to the parameter ϱ'.

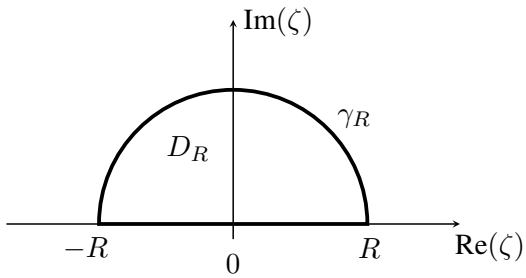
Since the coefficients of $p(\varrho', \zeta)$ and $p^{(j)}(\varrho', \zeta)$ are of class $\mathscr{C}^r$ then, the coefficients $h_k(\varrho')$ is of class $\mathscr{C}^r$. We may now repeat the previous arguments for the polynomial $H(\varrho', \zeta)$ yielding the $\mathscr{C}^r$ regularity of the coefficients of $p^{\pm}(\varrho', \zeta)$ and $p^0(\varrho', \zeta)$ w.r.t. ϱ'. The proof is complete. □

A.3. **Proof of the Hermite Theorem (Proposition 3.13).** All the roots of h are assumed to be in the lower complex half-plane $\{\operatorname{Im}\zeta < 0\}$. In particular h cannot have real coefficients. We claim that we have

$$\forall \zeta \in \mathbb{C}, \quad |h(\zeta)| = |\overline{h}(\zeta)| \;\Leftrightarrow\; \zeta \in \mathbb{R}. \tag{A.2}$$

Let f be the holomorphic function in $\{\operatorname{Im}\zeta > 0\}$ given by $f(\zeta) = \overline{h}(\zeta)/h(\zeta)$. Clearly if $\zeta \in \mathbb{R}$ then $|f(\zeta)| = 1$. Observe that neither f nor $|f|$ can be constant in $\{\operatorname{Im}\zeta > 0\}$ since $\overline{h}$ has roots in this set.

We let $R > 0$ and consider the domain D_R inside the contour γ_R formed by the interval $[-R, R]$ on the real axis and the following half circle in the upper complex half-plane $\{|\zeta| = R;\ \operatorname{Im}\zeta > 0\}$.



Let $\zeta_0 \in \mathbb{C}$ be such that $\operatorname{Im}\zeta_0 > 0$. Letting $\varepsilon \geq 0$, we choose $R > |\zeta_0|$ sufficiently large so that $\max_{\gamma_R}|f| \leq 1 + \varepsilon$, observing that $\lim_{|\zeta|\to\infty}|f(\zeta)| = 1$. Then the maximum modulus principle yields $|f(\zeta_0)| \leq \max_{\overline{D}_R}|f| = \max_{\gamma_R}|f| \leq 1 + \varepsilon$. Since ε is arbitrary we obtain that $|f(\zeta_0)| \leq 1$. Hence $|f| \leq 1$ in the upper complex half-plane. This now yields $\max_{\gamma_R}|f| = 1$ for any $R > 0$ since $|f| = 1$ on the real axis. Considering again an arbitrary $\zeta_0 \in \mathbb{C}$ with $\operatorname{Im}\zeta_0 > 0$ and $R > |\zeta_0|$ we find $|f(\zeta_0)| < 1$ since the maximum modulus cannot be reached in the interior of D_R since f is not constant. We have thus obtained that

$$|f| < 1 \;\text{ in }\; \{\operatorname{Im}\zeta > 0\}. \tag{A.3}$$

The same analysis can be carried out with the holomorphic function $g(\zeta) = h(\zeta)/\overline{h}(\zeta)$ in the lower complex half-plane since the roots of $\overline{h}$ have positive imaginary parts:

$$|g| < 1 \;\text{ in }\; \{\operatorname{Im}\zeta < 0\}. \tag{A.4}$$

Together (A.3) and (A.4) yield the claim (A.2), as the case $h(\zeta) = \overline{h}(\zeta) = 0$ is to be excluded since it yields both $\operatorname{Im}\zeta < 0$ and $\operatorname{Im}\zeta > 0$.

Let now ζ be a root of a. Then $|h(\zeta)| = |\overline{h}(\zeta)|$ implying that ζ is real. The same applies for the roots of b. Moreover, if ζ is a root of a then $b(\zeta) \neq 0$, as otherwise $h(\zeta) = \overline{h}(\zeta) = 0$, which is excluded (see above). The roots of a and b are thus distinct and real.

We denote the roots of h by α_i, $i = 1, \ldots, k$, and we introduce

$$h_j(\zeta) = (\zeta - \alpha_j), \qquad h(\zeta) = \nu \prod_{j=1}^{k} h_j(\zeta), \quad q_j(\zeta) = \prod_{i=j+1}^{k} h_i(\zeta), \quad q_k(\zeta) = 1,$$

where $\nu \in \mathbb{C}$, $\nu \neq 0$, is the leading-order coefficient of h. We observe that $iB_{h,\overline{h}}(\zeta,\tilde{\zeta}) = 2B_{a,b}(\zeta,\tilde{\zeta})$ and by (3.12), we obtain

$$\begin{aligned}|\nu|^{-2}B_{h,\overline{h}}(\zeta,\tilde{\zeta}) &= B_{h_1q_1,\overline{h}_1\overline{q}_1}(\zeta,\tilde{\zeta})\\ &= q_1(\zeta)\overline{q}_1(\tilde{\zeta})B_{h_1,\overline{h}_1}(\zeta,\tilde{\zeta}) + \overline{h}_1(\zeta)h_1(\zeta)B_{q_1,\overline{q}_1}(\zeta,\tilde{\zeta})\\ &= 2i\,\mathrm{Im}(\alpha_1)q_1(\zeta)\overline{q}_1(\tilde{\zeta}) + \overline{h}_1(\zeta)h_1(\zeta)B_{q_1,\overline{q}_1}(\zeta,\tilde{\zeta})\end{aligned}$$

By induction we then find

$$|\nu|^{-2}B_{h,\overline{h}}(\zeta,\tilde{\zeta}) = 2i\sum_{j=1}^{k}\mathrm{Im}(\alpha_j)R_j(\zeta)\overline{R}_j(\tilde{\zeta})$$

where R_j is a polynomial of degree $k-1$ given by

$$R_j = q_j\prod_{i=1}^{j-1}\overline{h}_i.$$

Note that the roots of R_j are $\overline{\alpha}_1,\dots,\overline{\alpha}_{j-1}$ and $\alpha_{j+1},\dots,\alpha_k$. Assuming the $\sum_{j=1}^{k}\lambda_jR_j(\zeta) = 0$, by successively estimating this sum for $\zeta = \overline{\alpha}_1,\overline{\alpha}_2,\dots,\overline{\alpha}_k$ we find $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$. The family of polynomials R_j, $j = 1,\dots,k$ is thus linearly independent.

We have

$$\Sigma_{\mathscr{B}_{a,b}}(\mathbf{z},\mathbf{z}') = -|\nu|^2\sum_{j=1}^{k}\mathrm{Im}(\alpha_j)\Sigma_{R_j}(\mathbf{z})\overline{\Sigma_{R_j}}(\overline{\mathbf{z}'}),$$

with $\mathrm{Im}\,\alpha_j < 0$. Lemma 3.10 yields the conclusion of Proposition 3.13. □

A.4. **Proof of Lemma 3.14.** We write

$$\mathrm{sub}(a,b) = \sum_{|\alpha|=1}\partial_x^\alpha(b\partial_\xi^\alpha a - a\partial_\xi^\alpha b) = \sum_{|\alpha|=1}\partial_{x'}^\alpha(b\partial_{\xi'}^\alpha a - a\partial_{\xi'}^\alpha b) + \partial_{x_n}(b\partial_{\xi_n}a - a\partial_{\xi_n}b),$$

and we have on the one hand

$$\sum_{|\alpha|=1}\partial_{x'}^\alpha(b\partial_{\xi'}^\alpha a - a\partial_{\xi'}^\alpha b) = \sum_{j,k=0}^{m}\underbrace{\sum_{|\alpha|=1}\partial_{x'}^\alpha(b_j\partial_{\xi'}^\alpha a_k - a_j\partial_{\xi'}^\alpha b_k)}_{=-h_{j,k}}\xi_n^{j+k},$$

and on the other hand

$$\begin{aligned}\partial_{x_n}(b\partial_{\xi_n}a - a\partial_{\xi_n}b) &= \sum_{j,k=0}^{m}\partial_{x_n}(kb_ja_k\xi_n^{j+k-1} - ja_kb_j\xi_n^{j+k-1})\\ &= \sum_{j,k=0}^{m}(k-j)\xi_n^{j+k-1}\partial_{x_n}(b_ja_k)\\ &= \frac{1}{2}\sum_{j,k=0}^{m}(k-j)\underbrace{\partial_{x_n}(b_ja_k - b_ka_j)}_{-\partial_ng'_{j,k}}\xi_n^{j+k-1},\end{aligned}$$

which gives the result. □

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