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Julien Roth

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# BIHARMONIC SUBMANIFOLDS OF GENERALIZED COMPLEX SPACE FORMS

JULIEN ROTH

ABSTRACT. We investigate biharmonic submanifolds in generalized complex space forms. We first give the necessary and sufficient condition for submanifolds to be biharmonic in these spaces. Then, we obtain curvature estimates for proper-biharmonic hypersurfaces and Lagrangian surfaces in these spaces.

## 1. INTRODUCTION

A harmonic map  $\psi$  between two Riemannian manifolds  $(M, g)$  and  $(N, h)$  is defined as a critical point of the energy functional

$$E(\psi) = \frac{1}{2} \int_M |d\psi|^2 dv_g.$$

In [4], Eells and Sampson proposed a natural generalization of harmonic maps. The map  $\psi$  is called *biharmonic* if it is a critical point of the bienergy functional

$$E_2(\psi) = \frac{1}{2} \int_M |\tau(\psi)|^2 dv_g.$$

where  $\tau(\psi) = \text{tr}(\nabla d\psi)$  is the tension field which vanishes for harmonic map. The Euler-Lagrange equation for the bienergy functional is given by  $\tau_2(\psi) = 0$  (see [9]), where  $\tau_2(\psi)$  is the bitension field and is defined by

$$\tau_2(\psi) = \Delta \tau(\psi) - \text{tr}(R^N(d\psi, \tau(\psi))d\psi).$$

Here,  $\Delta$  stands for the rough Laplacian on the sections of  $\psi^{-1}(TN)$  and  $R^N$  is the curvature tensor of  $N$ .

A *biharmonic submanifold* of a Riemannian manifold is a submanifold for which the inclusion map is biharmonic. Since any harmonic map is biharmonic, we will focus on biharmonic submanifolds that are not harmonic, which are called *proper-harmonic*.

Biharmonic submanifolds of different ambient spaces have been intensively studied in the last decade. For instance we can cite the cases of real space forms [2], complex space forms [5], 3-homogeneous manifolds [8], Sasakian space forms [6] products of space forms [7, 11]. In the present note, we investigate biharmonic submanifolds

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of generalized complex space forms, which are generalizations of complex space forms that appear only in dimension 4. We will focus on hypersurfaces, as well as on complex and Lagrangian surfaces.

## 2. PRELIMINARIES

We begin by some preliminaries about generalized complex space forms. They form a particular class of Hermitian manifolds which has not been intensively studied. In 1981, Tricelli and Vanhecke [12] introduced the following generalization of the complex space forms ( $\mathbb{C}^n$ ,  $\mathbb{C}P^n$  and  $\mathbb{C}H^n$ ).

Let  $(N^{2n}, g, J)$  be an almost Hermitian manifold. We denote by  $R_1$  and  $R_2$  the generalized curvature tensors defined as follows: for  $X, Y, Z \in \Gamma(TN)$ , we set

$$R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y,$$

$$R_2(X, Y)Z = g(JY, Z)JX - g(JX, Z)JY + 2g(JY, X)JZ.$$

The manifold  $(N, g, J)$  is called *generalized complex space form* if its curvature tensor  $R$  has the following form

$$R = \alpha R_1 + \beta R_2,$$

where  $\alpha$  and  $\beta$  are smooth functions on  $N$ . The terminology comes obviously from the fact that complex space forms satisfies this property with constants  $\alpha = \beta$ .

In the same paper [12], Tricelli and Vanhecke showed that if  $N$  is of (real) dimension  $2n \geq 6$ , then  $(N, g, J)$  is a complex space form. They also showed that  $\alpha + \beta$  is necessarily constant. This implies that  $\alpha = \beta$  are constants in dimension  $2n \geq 6$ , but this is not the case in dimension 4. After that, Olszak [10] constructed examples in dimension 4 with  $\alpha$  and  $\beta$  non-constant. Hence, the notion of generalized complex space form is of interest only in dimension 4. From now on, we will denote by  $N(\alpha, \beta)$  a (4-dimensional) generalized complex space form with curvature given by  $R = \alpha R_1 + \beta R_2$ . Note that these spaces are Einstein, with constant scalar curvature equal to  $12(\alpha + \beta)$ . Of course, they are not Kählerian because if they were, they would be complex space forms.

Now, let  $M$  be a submanifold of the generalized complex space form  $N(\alpha, \beta)$ , the almost complex structure  $J$  on  $N(\alpha, \beta)$  induces the existence of four operators on  $M$ , namely

$$j : TM \longrightarrow TM, k : TM \longrightarrow NM, l : NM \longrightarrow TM \text{ and } m : NM \longrightarrow NM,$$

defined for all  $X \in TM$  and all  $\xi \in NM$  by

$$(1) \quad JX = jX + kX \quad \text{et} \quad J\xi = l\xi + m\xi.$$

Since  $J$  is an almost complex structure, it satisfies  $J^2 = -Id$  and for  $X, Y$  tangent to  $N(\alpha, \beta)$ , we have  $g(JX, Y) = -g(X, JY)$ . Then, we deduce that the operators

$j, k, l, m$  satisfy the following relations

$$\begin{aligned}
(2) \quad & j^2 X + lkX = -X, \\
(3) \quad & m^2 \xi + kl\xi = -\xi, \\
(4) \quad & jl\xi + lm\xi = 0, \\
(5) \quad & kjX + mkX = 0, \\
(6) \quad & g(kX, \xi) = -g(X, l\xi),
\end{aligned}$$

for all  $X \in \Gamma(TM)$  and all  $\xi \in \Gamma(NM)$ . Moreover  $j$  and  $m$  are skew-symmetric.

### 3. BIHARMONIC SUBMANIFOLDS IN GENERALIZED COMPLEX SPACE FORMS

First of all, we give the following theorem which is a characterization of biharmonic submanifolds in generalized complex space forms.

**Theorem 3.1.** *Let  $N(\alpha, \beta)$  be a generalized complex space form and  $M^n$ ,  $n < 4$ , a submanifold of  $N(\alpha, \beta)$  with second fundamental form  $B$ , shape operator  $A$  and mean curvature  $H$ . Then  $M$  is biharmonic if and only if the following two equations are satisfied*

$$\begin{cases} -\Delta^\perp H + \text{tr}(B(\cdot, A_H \cdot)) - n\alpha H + 3\beta klH = 0, \\ \frac{n}{2} \text{grad}|H|^2 + 2\text{tr}(A_{\nabla^\perp H}(\cdot)) + 6\beta jlH = 0. \end{cases}$$

**Proof:** The equations of biharmonicity are well known (voir [1, 3, 7] for instance). After projection of the equation  $\tau_2(\psi) = 0$  on both tangent and normal bundles, we get the two following equations

$$\begin{cases} -\Delta^\perp H + \text{tr}(B(\cdot, A_H \cdot)) + \text{tr}(R(\cdot, H)\cdot)^\perp = 0, \\ \frac{n}{2} \text{grad}|H|^2 + 2\text{tr}(A_{\nabla^\perp H}(\cdot)) + 2\text{tr}(R(\cdot, H)\cdot)^\top = 0. \end{cases}$$

Recall that the curvature tensor of  $N(\alpha, \beta)$  is given by

$$\text{tr}(R(\cdot, H)\cdot) = \alpha \text{tr}(R_1(\cdot, H)\cdot) + \beta \text{tr}(R_2(\cdot, H)\cdot).$$

Let compute the two terms of the right had side. For this, let  $\{X_i\}_{i=1}^n$  be a local orthonormal frame of  $TM$ . First, we have

$$\begin{aligned}
\text{tr}(R_1(\cdot, H)\cdot) &= \sum_{i=1}^n R_1(X_i, H)X_i \\
&= \sum_{i=1}^n [g(H, X_i)X_i - g(X_i, X_i)H] \\
&= -nH.
\end{aligned}$$

Secondly, we have

$$\begin{aligned}
\operatorname{tr}(R_2(\cdot, H)\cdot) &= \sum_{i=1}^n R_2(X_i, H)X_i. \\
&= \sum_{i=1}^n [g(JH, X_i)JX_i - g(JX_i, X_i)JH + 2g(JH, X_i)JX_i] \\
&= 3 \sum_{i=1}^n g(lH, X_i)JX_i \\
&= 3JlH \\
&= 3jlH + 3klH.
\end{aligned}$$

Finally, since  $jlH$  is tangent and  $klH$  is normal, by identification of tangnt and normal parts, we get the equations of the theorem.  $\square$

We have the following corollary for the particular cases of hypersurfaces, Lagrangian or complex surfaces and curves.

**Corollary 3.2.** *Let  $N(\alpha, \beta)$  be a generalized complex space form and  $M$  a submanifold of  $N(\alpha, \beta)$  with second fundamental form  $B$ , Shape operator  $A$  and mean curvature  $H$ .*

(1) *If  $M$  is a hypersurface, then  $M$  is biharmonic if and only if*

$$\begin{cases} -\Delta^\perp H + \operatorname{tr}(B(\cdot, A_H\cdot)) - 3(\alpha + \beta)H = 0, \\ \frac{3}{2}\operatorname{grad}|H|^2 + 2\operatorname{tr}(A_{\nabla^\perp H}(\cdot)) = 0. \end{cases}$$

(2) *If  $M$  is a complex surface, then  $M$  is biharmonic if and only if*

$$\begin{cases} -\Delta^\perp H + \operatorname{tr}(B(\cdot, A_H\cdot)) - 2\alpha H = 0, \\ \operatorname{grad}|H|^2 + 2\operatorname{tr}(A_{\nabla^\perp H}(\cdot)) = 0. \end{cases}$$

(3) *If  $M$  is a Lagrangian surface, then  $M$  is biharmonic if and only if*

$$\begin{cases} -\Delta^\perp H + \operatorname{tr}(B(\cdot, A_H\cdot)) - 2\alpha H - 3\beta H = 0, \\ \operatorname{grad}|H|^2 + 2\operatorname{tr}(A_{\nabla^\perp H}(\cdot)) = 0. \end{cases}$$

(4) *If  $M$  is a curve, then  $M$  is biharmonic if and only if*

$$\begin{cases} -\Delta^\perp H + \operatorname{tr}(B(\cdot, A_H\cdot)) - \alpha H - 3\beta(H + m^2 H) = 0, \\ \frac{1}{2}\operatorname{grad}|H|^2 + 2\operatorname{tr}(A_{\nabla^\perp H}(\cdot)) = 0. \end{cases}$$

**Proof:** The proof is a direct consequence of Theorem 3.1.

(1) If  $M$  is a hypersurface, then  $J$  maps normal vectors on tangent vectors, that is,  $m = 0$ . Hence, by relation (3), we have  $klH = -H$  and by relation (4),  $jlH = 0$ , which gives the result by Theorem 3.1.

- (2) If  $M$  is a complex surface, then,  $k = 0$  and  $l = 0$ .
- (3) If  $M$  is a Lagrangian surface, then  $j = 0$  and  $m = 0$ . Moreover, since  $m = 0$ , as for hypersurfaces, we have  $klH = -H$  by relation (3).
- (4) If  $M$  is a curve, then  $j = 0$ . Hence, by relation (3),  $klH = -(H + m^2H)$ .

□

We obtain immediately the following corollaries for cruves and complex or Lagranian surfaces with parallel mean curvature.

**Corollary 3.3.** (1) *If  $M$  be a Lagrangian surface of  $N(\alpha, \beta)$  with parallel mean curvature, then  $M$  is biharmonic if and only if*

$$\text{tr}(B(\cdot, A_H \cdot)) = (2\alpha + 3\beta)H.$$

- (2) *If  $M$  be a complex surface of  $N(\alpha, \beta)$  with parallel mean curvature. then  $M$  is biharmonic if and only if*

$$\text{tr}(B(\cdot, A_H \cdot)) = 2\alpha H.$$

- (3) *If  $M$  is a curve in  $N(\alpha, \beta)$  with parallel mean curvature, then  $M$  is biharmonic if and only if*

$$\text{tr}(B(\cdot, A_H \cdot)) = \alpha H + 3\beta(H + m^2H) = 0.$$

#### 4. CURVATURE PROPERTIES OF BIHARMONIC SUBMANFOLDS

In this section, we will give some curvature properties of constant mean curvature submanifolds in  $N(\alpha, \beta)$ . We have this first proposition for hypersurfaces.

**Proposition 4.1.** *Let  $N(\alpha, \beta)$  be a generalized complex space form and  $M^3$  a hypersurface of  $N(\alpha, \beta)$  with non zero constant mean curvature  $H$ . Then,  $M$  is proper-biharmonic if and only if*

$$\|B\|^2 = 3(\alpha + \beta),$$

*of equivalently if the scalar curvature of  $M$  satisfies*

$$\text{Scal}_M = 3(\alpha + \beta) + 9H^2.$$

**Remark 4.2.** *In particular, the norm of the second fundamental form and the scalar of  $M$  are constant.*

**Proof:** Since  $M$  is a hypersurface, by Corollary 3.2,  $M$  is biharmonic if and only if

$$\begin{cases} -\Delta^\perp H + \text{tr}B(\cdot, A_H \cdot) - 3\alpha H - 3\beta H = 0, \\ \frac{3}{2}\text{grad}|H|^2 + 2\text{tr}A_{\nabla^\perp H}(\cdot) = 0. \end{cases}$$

Since  $M$  has constant mean curvature, the second equation is trivially satisfied and the second becomes

$$\text{tr}(B(\cdot, A_H(\cdot))) = (3\alpha + 3\beta)H.$$

Moreover, for hypersurfaces, we have  $A_H = HA$  which implies

$$\operatorname{tr}(B(\cdot, A_H(\cdot))) = H \operatorname{tr}(B(\cdot, A(\cdot))) = H \|B\|^2.$$

Finally, since  $H$  is a non-zero constant, we get the desired identity  $|B|^2 = 3(\alpha + \beta)$ . For the second equivalence, by the Gauss equation, we have

$$\operatorname{Scal}_M = \sum_{i,j=1}^3 \langle R^N(X_i, X_j)X_j, X_i \rangle - \|B\|^2 + 9H^2,$$

where  $\{X_1, X_2, X_3\}$  is a local orthonormal frame of  $M$ . From the expression of the curvature tensor of  $N(\alpha, \beta)$ , we get

$$\operatorname{Scal}_M = 6(\alpha + \beta) - \|B\|^2 + 9H^2.$$

Hence, we deduce that  $M$  is proper biharmonic if and only if  $\|B\|^2 = 3(\alpha + \beta)$ , that is, if and only if  $\operatorname{Scal}_M = 3(\alpha + \beta) + 9H^2$ .  $\square$

An immediate consequence of this proposition is the following corollary.

**Corollary 4.3.** *There exists no biharmonic hypersurface with constant mean curvature in a generalized complex space form  $N(\alpha, \beta)$  of negative scalar curvature.*

**Proof:** From Proposition 4.1, a constant mean curvature hypersurface of  $N(\alpha, \beta)$  is biharmonic if and only if  $|B|^2 = 3(\alpha + \beta)$ , which is possible only if  $\alpha + \beta$  is positive, that is if  $N(\alpha, \beta)$  has positive scalar curvature.  $\square$

Finally, we give this last proposition which give an estimate of the mean curvature for a biharmonic Lagrangian surface.

**Proposition 4.4.** *Let  $M$ , be a Lagrangian surface of  $N(\alpha, \beta)$  ( $\alpha + \beta > 0$ ) with non-zero constant mean curvature. Then we have*

- (1) *If  $M$  is proper-biharmonic, then  $0 < |H|^2 \leq \inf_M \left( \frac{2\alpha+3\beta}{2} \right)$ .*
- (2) *If  $|H|^2 = \inf_M \left( \frac{2\alpha+3\beta}{2} \right)$ , then  $M$  is proper-bihramonic if and only if  $\alpha = \beta$  are constant,  $N(\alpha, \beta)$  is the complex projective plan  $\mathbb{C}P^2(4\alpha)$ ,  $M$  is pseudo-umbilical and  $\nabla^\perp H = 0$ .*

**Proof:** Let  $M$  be a biharmonic submanifold of  $N(\alpha, \beta)$  with non-zero constant mean curvature. Since  $M$  is a hypersurface, by the first assertion of Corollary 3.2, we have

$$-\Delta^\perp H + \operatorname{tr}(B(\cdot, A_H(\cdot))) - (2\alpha + 3\beta)H = 0.$$

Hence, by taking the scalar product with  $H$ , we have

$$-\langle \Delta^\perp H, H \rangle = (2\alpha + 3\beta)|H|^2 - \operatorname{tr}B(\cdot, A_H(\cdot)).$$

Using the Bochner formula and the fact that  $|H|$  is constant, we get

$$(2\alpha + 3\beta)|H|^2 = \operatorname{tr}(B(\cdot, A_H(\cdot))) + |\nabla^\perp H|^2.$$

Moreover, by Cauchy-Schwarz inequality, we get  $\text{tr}(B(\cdot, A_H \cdot)) \geq 2|H|^4$ . Therefore, we have

$$(2\alpha + 3\beta)|H|^2 \geq 2|H|^4 + |\nabla^\perp H|^2 \geq 2|H|^4.$$

Since  $|H|$  is a non-zero constant, we have  $0 < H^2 \leq \inf_M \left( \frac{2\alpha+3\beta}{2} \right)$ .

Now, assume that  $|H|^2 = \inf_M \left( \frac{2\alpha+3\beta}{2} \right)$ . If  $M$  is proper biharmonic, then all the inequalities above become equality. First,  $2\alpha + 3\beta$  is constant. But  $\alpha + \beta$  is also constant, then  $\alpha$  and  $\beta$  are constant. Hence, necessarily,  $\alpha = \beta$  and  $N(\alpha, \beta)$  is the complex projective plane  $\mathbb{C}P^2(4\alpha)$ . Moreover, equality occurs in the Cauchy-Schwarz inequality, that is  $M$  is pseudo-umbilical. Finally, we also have  $\nabla^\perp H = 0$ .

Conversely, if  $\alpha = \beta$ ,  $N(\alpha, \beta)$  is the complex projective plane  $\mathbb{C}P^2(4\alpha)$ ,  $M$  is pseudo-umbilical and  $\nabla^\perp H = 0$ , then all the above inequalities are trivially equalities and therefore we have

$$-\Delta^\perp H + \text{tr}(B(\cdot, A_H \cdot)) - (2\alpha + 3\beta)H = 0.$$

Moreover, since  $|H|$  is constant and  $\nabla^\perp H = 0$ , we have obviously

$$\text{grad}|H|^2 + 2\text{tr}A_{\nabla^\perp H}(\cdot) = 0.$$

Hence, by Corollary 3.2,  $M$  is biharmonic. This concludes the proof.  $\square$

Note, that the results of these section contains the particular case of the complex projective planes  $\mathbb{C}P^2(4\alpha)$ , proved in [5].

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(J. Roth) LAMA, UNIVERSITÉ PARIS-EST MARNE-LA-VALLÉE, CITÉ DESCARTES, CHAMPS SUR  
MARNE, 77454 MARNE-LA-VALLÉ CEDEX 2, FRANCE

*E-mail address:* julien.roth@univ-mlv.fr