



HAL
open science

Concavity properties of extensions of the parallel volume

Arnaud Marsiglietti

► **To cite this version:**

| Arnaud Marsiglietti. Concavity properties of extensions of the parallel volume. 2013. hal-00843200v1

HAL Id: hal-00843200

<https://hal.science/hal-00843200v1>

Preprint submitted on 10 Jul 2013 (v1), last revised 1 Oct 2014 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Concavity properties of extensions of the parallel volume

Arnaud Marsiglietti

Université Paris-Est, LAMA (UMR 8050), UPEMLV, UPEC, CNRS, F-77454, Marne-la-Vallée, France

arnaud.marsiglietti@univ-mlv.fr

Abstract

In this paper, we establish concavity properties of two extensions of the classical notion of the outer parallel volume. On the one hand, we replace the Lebesgue measure by more general measures. On the other hand, we consider a functional version of the outer parallel sets.

Keywords: parallel volume, Brunn-Minkowski inequality, s -concave measure, Hamilton-Jacobi equation

1 Introduction

As an analogue of the famous *concavity of entropy power* in Information theory (see *e.g.* [13], [29]), Costa and Cover in [14] conjectured that the $\frac{1}{n}$ -power of the *parallel volume* $|A + tB_2^n|$ is a concave function, where B_2^n denotes the Euclidean closed unit ball.

Conjecture 1.1 (Costa-Cover [14]). *Let A be a bounded measurable set in \mathbb{R}^n then the function $t \mapsto |A + tB_2^n|^{\frac{1}{n}}$ is concave on \mathbb{R}_+ .*

This conjecture has been studied in [19], where it was shown that it is true in dimension 1 for any sets and in dimension 2 for any connected sets, but it is false for arbitrary sets in dimension 2 and for arbitrary connected sets in dimension greater than or equal to 3.

The notion of parallel volume can be extended by considering more general measures than the Lebesgue measure. An extension, provided by Borell in [7], follows from the Brunn-Minkowski inequality, which states that for every $\lambda \in [0, 1]$ and for every compact subsets A, B of \mathbb{R}^n ,

$$|(1 - \lambda)A + \lambda B|^{\frac{1}{n}} \geq (1 - \lambda)|A|^{\frac{1}{n}} + \lambda|B|^{\frac{1}{n}}. \quad (1)$$

In [7], Borell defined for $s \in [-\infty, +\infty]$ the s -concave measures, satisfying a similar inequality as (1):

$$\mu((1-\lambda)A + \lambda B) \geq ((1-\lambda)\mu(A)^s + \lambda\mu(B)^s)^{\frac{1}{s}}$$

for every $\lambda \in [0, 1]$ and for every compact subsets A, B of \mathbb{R}^n such that $\mu(A)\mu(B) > 0$. From the Brunn-Minkowski inequality (1), the Lebesgue measure is a $\frac{1}{n}$ -concave measure. Details on s -concave measures are given in the next section. The case $s = 0$ corresponds to log-concave measures. The most famous example of a log-concave measure is the *standard multivariate Gaussian measure*

$$d\gamma_n(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2}} dx,$$

where $|\cdot|$ stands for the Euclidean norm. These measures have particular interests. For example, isoperimetric inequalities have been established for the Gaussian measure $d\gamma_n$ by Borell in [9] and independently by Sudakov and Cirel'son in [28], which states that among sets of given Gauss measure, half-spaces minimize the Gauss surface area. Thereafter, Kannan-Lovász-Simonovits in [23] formulated their famous conjecture: for the uniform measure on a convex body (which can be extended to arbitrary log-concave measures) half-spaces are *nearly* minimizers. In dimension 1, Bobkov (see [3] and reference therein) proved that among sets of given measure, half-lines minimize the surface area. More recently, a refined statement has been established by Cianchi *et al.* in [11]: in dimension n , a set of given Gauss measure and *almost* minimal Gauss surface area is necessarily *close* to be a half-space. This result has been extended by De Castro in [15] for all log-concave probability measures in dimension 1.

In this paper, we pursue the study of these measures by considering the following problem, which extends Conjecture 1.1:

Problem A. *Let $s \in [-\infty, +\infty]$. Let μ be a s -concave measure in \mathbb{R}^n and A be a compact subset of \mathbb{R}^n . Is the function $t \mapsto \mu(A + tB_2^n)$ s -concave on \mathbb{R}_+ ?*

Another extensions of geometric inequalities can be set up by considering functional versions. The most famous extension of this type in the Brunn-Minkowski theory is certainly the Prékopa-Leindler inequality (see [25], [27], [8]). Functional versions provide new proofs of geometric inequalities and provide new applications. Another examples of such extensions is a functional version of the Blaschke-Santalò inequality and a functional version of the Mahler conjecture (see *e.g.* [2], [1], [20], [21], [24]).

To do so, we consider a functional version of *parallel sets* $A + tB_2^n$. We set up the following problem (the notion of γ -concave functions is defined in the next section):

Problem B. Let $\gamma \geq -\frac{1}{n}$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a bounded non-negative function and $g : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a γ -concave function. Let

$$h_t^{(\gamma)}(z) = \sup_{\substack{z=x+ty \\ f(x)>0; g(y)>0}} (f(x)^\gamma + tg(y)^\gamma)^{\frac{1}{\gamma}} \quad \text{and} \quad s = \frac{\gamma}{1 + \gamma n}.$$

Is the function $t \mapsto \int_{\mathbb{R}^n} h_t^{(\gamma)}(z) \, dz$ s -concave on \mathbb{R}_+ ?

For $\gamma = 0$, the function $h_t^{(0)}$ is interpreted by continuity, i.e. for every $z \in \mathbb{R}^n$,

$$h_t^{(0)}(z) = \sup_{z=x+ty} f(x)g(y)^t.$$

The Costa-Cover conjecture is a particular case of Problem B by taking $f = 1_A$, $g = 1_{B_2^n}$ and $\gamma = 0$. For $\gamma < 0$, which extends to $\gamma = 0$, the function $V = g^\gamma$ is convex by assumption and one can naturally connect the function $h_t^{(\gamma)}$ with the Hopf-Lax solution of the Hamilton-Jacobi equation:

$$h_t^{(\gamma)}(z) = \sup_{x \in \mathbb{R}^n} \left(f(x)^\gamma + tV \left(\frac{z-x}{t} \right) \right)^{\frac{1}{\gamma}} = \left(Q_t^{(V)} f^\gamma(z) \right)^{\frac{1}{\gamma}},$$

where for arbitrary convex function V and for arbitrary function u ,

$$Q_t^{(V)} u(z) = \inf_{x \in \mathbb{R}^n} \left(u(x) + tV \left(\frac{z-x}{t} \right) \right).$$

The Hopf-Lax solution have a particular interest. For example, it can be used to show that *hypercontractivity* of this solution is equivalent to get log-Sobolev inequalities (see e.g. [6], [22]). Through Problem B, we pursue the study of this solution by asking for concavity properties in time of the Hopf-Lax solution of the Hamilton-Jacobi equation.

We will prove that both Problem A and Problem B have positive answers in dimension 1. However, since the Costa-Cover conjecture is false in dimension $n \geq 2$ in such a generality, we won't expect other positive answers of these stronger problems. Using the geometric localization theorem of Kannan-Lovász-Simonovits in [23] in the form established by Fradelizi-Guédon in [18], we prove:

Theorem A. Let $s \in [-\infty, \frac{1}{2}] \cup [1, +\infty]$. Let A be an arbitrary compact subset of \mathbb{R} and μ be a s -concave measure in \mathbb{R} . Then, $t \mapsto \mu(A + tB_2^1)$ is s -concave on \mathbb{R}_+ . Moreover, for $s \in (\frac{1}{2}; 1)$ there exists a compact subset A of \mathbb{R} such that $t \mapsto \mu(A + tB_2^1)$ is not s -concave on \mathbb{R}_+ .

Using a precise analysis of the Hopf-Lax solution, we prove in dimension 1 a better concavity as asked in Problem B:

Theorem B. Let $\gamma \in (-1; 0]$. Let $f : \mathbb{R} \rightarrow \mathbb{R}_+$ be a Lipschitz continuous non-negative function. Define, for every $y \in \mathbb{R}$, $V(y) = \frac{|y|^p}{p}$, with $p \geq 1$. Then the function $t \mapsto \int_{\mathbb{R}} h_t^{(\gamma)}(z) dz$ is concave on \mathbb{R}_+ , where

$$h_t^{(\gamma)}(z) = \sup_{\substack{z=x+ty \\ f(x)>0; V(y)>0}} (f(x)^\gamma + tV(y))^\frac{1}{\gamma} \text{ and } h_t^{(0)}(z) = \sup_{z=x+ty} f(x)e^{-tV(y)}.$$

In the next section, we explain some notations and we present some results about s -concave measures. In the third part, we extend the classical notion of the parallel volume by considering a general s -concave measure instead of the Lebesgue measure and we give a complete answer to Problem A. In the fourth part, we extend the classical notion of parallel sets by considering a functional version and we partially answer to Problem B. To conclude this paper, we derive a weighted Brascamp-Lieb-type inequality from our functional version.

2 Preliminaries

We work in the Euclidean space \mathbb{R}^n , $n \geq 1$, equipped with the usual scalar product $\langle \cdot, \cdot \rangle$ and the ℓ_2^n norm $|\cdot|$, whose closed unit ball is denoted by B_2^n and the canonical basis by (e_1, \dots, e_n) . We also denote $|\cdot|$ the Lebesgue measure in \mathbb{R}^n . For non-empty sets A, B in \mathbb{R}^n we define their *Minkowski sum*

$$A + B = \{a + b; a \in A, b \in B\}.$$

We denote by $\text{int}(A)$ the *interior* of the set A , by \bar{A} the *closure* of A and by $\partial A = \bar{A} \setminus \text{int}(A)$ the *boundary* of A .

In this paper, we only consider non-negative measures. For an arbitrary measure μ , we call (*outer*) *parallel μ -volume* of a set A the function defined on \mathbb{R}_+ by $t \mapsto V_A^\mu(t) = \mu(A + tB_2^n)$. We simply call (*outer*) *parallel volume* if μ is the Lebesgue measure.

Let us recall some terminologies and results about s -concave measures introduced by Borell in [7], [8]. One says that a measure μ in \mathbb{R}^n is *s -concave*, $s \in [-\infty, +\infty]$, if the inequality

$$\mu((1 - \lambda)A + \lambda B) \geq ((1 - \lambda)\mu(A)^s + \lambda\mu(B)^s)^\frac{1}{s}$$

holds for all compact subsets $A, B \subset \mathbb{R}^n$ such that $\mu(A)\mu(B) > 0$ and for all $\lambda \in [0, 1]$. The limit cases are interpreted by continuity, *i.e.* the right-hand side of this inequality is equal to $\mu(A)^{1-\lambda}\mu(B)^\lambda$ for $s = 0$, which corresponds to log-concave measures, to $\min(\mu(A), \mu(B))$ for $s = -\infty$ and to $\max(\mu(A), \mu(B))$ for $s = +\infty$. Notice that a s -concave measure is r -concave for all $r \leq s$.

For $s \leq \frac{1}{n}$, Borell shows that any s -concave measure μ absolutely continuous with respect to the Lebesgue measure admits a γ -concave density function, with

$$\gamma = \frac{s}{1 - sn} \in [-\frac{1}{n}, +\infty],$$

where a function f is said to be γ -concave, with $\gamma \in [-\infty, +\infty]$, if the inequality

$$f((1 - \lambda)x + \lambda y) \geq ((1 - \lambda)f(x)^\gamma + \lambda f(y)^\gamma)^{\frac{1}{\gamma}}$$

holds for every $x, y \in \mathbb{R}^n$ such that $f(x)f(y) > 0$ and for every $\lambda \in [0, 1]$. As for the s -concave measures, the limit cases are interpreted by continuity. Notice that a 1-concave function is concave on its support, that a $-\infty$ -concave function has its level sets $\{x; f(x) \geq t\}$ convex, and that a $+\infty$ -concave function is constant on its support. For $s > \frac{1}{n}$, Borell shows that any s -concave measure μ absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^n is the Null measure.

3 The s -concavity of the parallel μ -volume

In this section, we generalize the Costa-Cover conjecture in the more general context of s -concave measures instead of the Lebesgue measure. Let us recall the new problem:

Problem A. *Let $s \in [-\infty, +\infty]$. Let μ be a s -concave measure in \mathbb{R}^n and A be a compact subset of \mathbb{R}^n . Is the function $t \mapsto \mu(A + tB_2^n)$ s -concave on \mathbb{R}_+ ?*

By the Brunn-Minkowski inequality, the n -dimensional Lebesgue measure is $\frac{1}{n}$ -concave. Thus the problem A generalizes the conjecture 1.1.

Let $a \in \mathbb{R}^n$. The Dirac measure $\delta_{\{a\}}$ is $+\infty$ -concave and we notice that the function $t \mapsto \delta_{\{a\}}(A + tB_2^n)$ is constant on its support and thus is $+\infty$ -concave on \mathbb{R}_+ , which solves Problem A for $s = +\infty$.

Since the function $t \mapsto \mu(A + tB_2^n)$ is non-decreasing, it follows that the answer to Problem A is positive for $s = -\infty$.

Notice that Problem A is solved for convex sets. Indeed, let A be a compact convex subset of \mathbb{R}^n , then for every $\lambda \in [0, 1]$ and every $t_1, t_2 \in \mathbb{R}_+$, we get

$$\begin{aligned} \mu(A + ((1 - \lambda)t_1 + \lambda t_2)B_2^n) &= \mu((1 - \lambda)(A + t_1B_2^n) + \lambda(A + t_2B_2^n)) \\ &\geq ((1 - \lambda)\mu(A + t_1B_2^n)^s + \lambda\mu(A + t_2B_2^n)^s)^{\frac{1}{s}}. \end{aligned}$$

In the sequel, μ will denote a s -concave measure which admits a density with respect to the n -dimensional Lebesgue measure. For $\mu \neq 0$, one has $s \leq \frac{1}{n}$.

We first establish a preliminary lemma in dimension 1.

Lemma 3.1. *Let $s \leq 1$. Let A be an arbitrary compact subset of \mathbb{R} and μ be a s -concave measure in \mathbb{R} . Then $(V_A^\mu)^s$ admits left and right derivatives on \mathbb{R}_+^* . We denote $((V_A^\mu)^s)'_-$ (resp. $((V_A^\mu)^s)'_+$) the left (resp. right) derivative of $(V_A^\mu)^s$. If $s \geq 0$, then*

$$((V_A^\mu)^s)'_- \geq ((V_A^\mu)^s)'_+$$

and if $s < 0$, then

$$((V_A^\mu)^s)'_- \leq ((V_A^\mu)^s)'_+.$$

Proof. We denote ψ the density of μ . Notice that for every $t_0 > 0$, the set $A + t_0 B_2^1$ is a disjoint finite union of intervals. Then, setting $A + t_0 B_2^1$ instead of A , we can assume that $A = \cup_{i=1}^N [a_i, b_i]$, with $a_1 < b_1 < \dots < a_N < b_N$. If $N = 1$, we get for every $t \in \mathbb{R}_+$,

$$V_A^\mu(t) = \int_{a_1-t}^{b_1+t} \psi(x) dx.$$

Hence, V_A^μ is differentiable on \mathbb{R}_+ . It follows that $((V_A^\mu)^s)'_- = ((V_A^\mu)^s)'_+$. If $N \geq 2$, we denote $t_i = \frac{b_i + a_{i+1}}{2}$, for $i \in \{1, \dots, N-1\}$. We notice that V_A^μ is differentiable on $\mathbb{R}_+ \setminus \{t_1, \dots, t_{N-1}\}$ and for every $t \in \mathbb{R}_+ \setminus \{t_1, \dots, t_{N-1}\}$, we get

$$\begin{aligned} (V_A^\mu)'_+(t) &= \sum_{a \in \partial(A+tB_2^1)} \psi(a) \\ (V_A^\mu)'_-(t) &= \sum_{a \in \partial(A+t \text{int}(B_2^1))} \psi(a). \end{aligned}$$

Notice that $\overline{A + tB_2^1} = \overline{A + t \text{int}(B_2^1)}$ thus $\partial(A + tB_2^1) \subset \partial(A + t \text{int}(B_2^1))$. Hence for every $i \in \{1, \dots, N-1\}$, we get

$$(V_A^\mu)'_-(t_i) \geq (V_A^\mu)'_+(t_i).$$

For every $t \neq t_i$ and $s \neq 0$, one has

$$((V_A^\mu)^s)'(t) = s(V_A^\mu)'(t)(V_A^\mu)^{s-1}(t),$$

thus we conclude that if $s > 0$ then $((V_A^\mu)^s)'_- \geq ((V_A^\mu)^s)'_+$, and if $s < 0$ then $((V_A^\mu)^s)'_- \leq ((V_A^\mu)^s)'_+$. For every $t \neq t_i$ and $s = 0$, one has

$$(\log(V_A^\mu))'(t) = \frac{(V_A^\mu)'(t)}{V_A^\mu(t)},$$

thus $(\log(V_A^\mu))'_- \geq (\log(V_A^\mu))'_+$. □

Let us solve the problem A in dimension 1 for $s \leq \frac{1}{2}$.

Theorem 3.2. *Let $s \leq \frac{1}{2}$. Let A be an arbitrary compact subset of \mathbb{R} and μ be a s -concave measure in \mathbb{R} . Then, the function $t \mapsto V_A^\mu(t) = \mu(A + tB_2^1)$ is s -concave on \mathbb{R}_+ .*

Proof. Let $s \leq \frac{1}{2}$. For $s = -\infty$, we noticed above that the result holds true. We assume $-\infty < s \leq \frac{1}{2}$. We also assume $s \neq 0$, the case $s = 0$ follows by continuity. Let μ_0 be a s -concave measure on \mathbb{R} and A be a compact subset of \mathbb{R} . Notice that for every $t > 0$, $A + tB_2^1$ is a disjoint finite union of intervals. Thus, by setting $A + tB_2^1$ for arbitrary $t > 0$ instead of A , we can assume that $A = \cup_{i=1}^N [a_i, b_i]$, with $a_i < b_i$ and $N \in \mathbb{N}^*$. We also assume $N \geq 2$, otherwise A is convex and we immediately conclude. Notice that for

$$t_0 = \frac{1}{2} \sup_{i=1, \dots, N-1} |a_{i+1} - b_i|,$$

then $A + t_0B_2^1$ is convex and for every $t < t_0$, the set $A + tB_2^1$ is not convex. Thus, $t \mapsto \mu_0(A + tB_2^1)$ is s -concave on $[t_0, +\infty)$.

Now, let us show that $t \mapsto \mu_0(A + tB_2^1)$ is s -concave on $(0; t_0)$. We use a geometric localization theorem due to Kannan-Lovász-Simonovits [23] in the more precise form established by Fradelizi-Guédon [18]. We denote $K = A + t_0B_2^1$, then K is a convex body. We consider the restriction of μ_0 over K , then it is a finite measure that we can assume to be a probability measure without loss of generality. For convenience, we always denote this measure μ_0 . We call $\mathcal{P}(K)$ the set of all probabilities whose support is included in K . We have $\mu_0 \in \mathcal{P}(K)$.

Step 1: Reduction to extremal measures

Let $t_1, t_2 \in (0, t_0)$ such that $\mu_0(A + t_1B_2^1)\mu_0(A + t_2B_2^1) > 0$. We want to show that

$$\mu_0\left(A + \frac{t_1 + t_2}{2}B_2^1\right) \geq \left(\frac{\mu_0(A + t_1B_2^1)^s}{2} + \frac{\mu_0(A + t_2B_2^1)^s}{2}\right)^{\frac{1}{s}} \quad (2)$$

which is sufficient because of the continuity property of $t \mapsto \mu(A + tB_2^1)$. We assume $t_1 < t_2$. We set

$$\alpha = \left(\frac{1}{2} \left(\frac{\mu_0(A + t_2B_2^1)^s}{\mu_0(A + t_1B_2^1)^s} + 1\right)\right)^{\frac{1}{s}}$$

and

$$\beta = \frac{\mu_0(A + t_2B_2^1)}{\mu_0(A + t_1B_2^1)}.$$

Notice that $\beta \geq 1$. If $\beta = 1$, then $t \mapsto \mu(A + tB_2^1)$ is constant on $[t_1, t_2]$ and thus s -concave. We assume thereafter that $\beta > 1$. We set

$$f = 1_{A+t_2B_2^1} - \beta 1_{A+t_1B_2^1}$$

and

$$P_f = \left\{ \mu \in \mathcal{P}(K); \mu \text{ } s\text{-concave such that } \int f \, d\mu \geq 0 \right\}.$$

Notice that $\mu_0 \in P_f$. At last, we set

$$\Phi(\mu) = \alpha\mu(A + t_1 B_2^1) - \mu \left(A + \frac{t_1 + t_2}{2} B_2^1 \right).$$

Inequality (2) is equivalent to

$$\Phi(\mu_0) \leq 0.$$

We shall prove that for every $\mu \in P_f$, $\Phi(\mu) \leq 0$. By the geometric localization theorem [18], we get

$$\sup_{\mu \in P_f} \Phi(\mu) = \Phi(\nu)$$

where ν is either a Dirac measure at a point x such that $f(x) \geq 0$, or either a probability measure which admits a s -affine density supported on a segment $[a, b]$, such that $\int f \, d\nu = 0$ and $\forall x \in (a, b), \int_{[x, b]} f \, d\nu < 0$.

Step 2: s -concavity for extremal measures

- We assume that $\nu = \delta_x$ with x such that $f(x) \geq 0$. The condition $f(x) \geq 0$ says that

$$1_{A+t_2 B_2^1}(x) \geq \beta 1_{A+t_1 B_2^1}(x).$$

Since $\beta > 1$, it follows that $x \notin A + t_1 B_2^1$. Hence,

$$\Phi(\delta_x) = -\delta_x \left(A + \frac{t_1 + t_2}{2} B_2^1 \right) \leq 0.$$

- We assume that ν admits a density ψ γ -affine with

$$\gamma = \frac{s}{1-s},$$

supported in a segment $[a, b]$, *i.e.* for every x in \mathbb{R} , $\psi(x) = (mx + p)^{\frac{1}{\gamma}} 1_{[a, b]}(x)$, with m and p such that for every $x \in [a, b]$, $mx + p \geq 0$. Without loss of generality, we can assume that $m = 1$. We also assume that ν satisfies $\int f \, d\nu = 0$ and $\int_{[x, b]} f \, d\nu < 0$ on (a, b) . We will show that

$$\nu \left(A + \frac{t_1 + t_2}{2} B_2^1 \right) \geq \left(\frac{\nu(A + t_1 B_2^1)^s}{2} + \frac{\nu(A + t_2 B_2^1)^s}{2} \right)^{\frac{1}{s}}.$$

It will follow that

$$\Phi(\nu) \leq 0.$$

In fact, we will prove the s -concavity of $t \mapsto \nu(A + tB_2^1)$ on $[t_1, t_2]$ by differentiation. The proof will be local on $[t_1, t_2]$. The condition $\int_{[x,b]} f \, d\nu < 0$ on (a, b) says that for every $x \in (a, b)$

$$\nu((A + t_2B_2^1) \cap [x, b]) < \beta\nu((A + t_1B_2^1) \cap [x, b]). \quad (3)$$

If $b \notin A + t_1B_2^1$, then there exists $x \in (a, b)$ such that $(A + t_1B_2^1) \cap [x, b] = \emptyset$. This contradicts (3). It follows that $b \in A + t_1B_2^1$. For convenience, we denote A for $A + t_1B_2^1$. Notice that $1 - \gamma \geq 0$ and $1 + \gamma \geq 0$.

Case 1: The case $\gamma > 0$.

Sub-case 1: The case $a \notin A$.

Recall that $b \in A$. The set A is a disjoint finite union of intervals, then we can assume that $A = \cup_{i=1}^{N-1} [a_i, b_i] \cup [a_N, b]$, with $a < a_1 < b_1 < \dots < a_N < b$. We denote $V_A^\nu(t) = \nu(A + tB_2^1)$. For t small enough, we get

$$\begin{aligned} V_A^\nu(t) &= \sum_{i=1}^{N-1} \int_{a_i-t}^{b_i+t} (x+p)^{\frac{1}{\gamma}} \, dx + \int_{a_N-t}^b (x+p)^{\frac{1}{\gamma}} \, dx \\ (V_A^\nu)'(t) &= \sum_{i=1}^{N-1} \left((b_i+t+p)^{\frac{1}{\gamma}} + (a_i-t+p)^{\frac{1}{\gamma}} \right) + (a_N-t+p)^{\frac{1}{\gamma}} \\ (V_A^\nu)''(t) &= \frac{1}{\gamma} \left(\sum_{i=1}^{N-1} \left((b_i+t+p)^{\frac{1-\gamma}{\gamma}} - (a_i-t+p)^{\frac{1-\gamma}{\gamma}} \right) \right. \\ &\quad \left. - (a_N-t+p)^{\frac{1-\gamma}{\gamma}} \right) \\ &= \frac{1}{\gamma} \left(-(a_1-t+p)^{\frac{1-\gamma}{\gamma}} + \sum_{i=2}^N \left((b_{i-1}+t+p)^{\frac{1-\gamma}{\gamma}} \right. \right. \\ &\quad \left. \left. - (a_i-t+p)^{\frac{1-\gamma}{\gamma}} \right) \right) \\ &\leq 0. \end{aligned}$$

Hence, V_A^ν is concave, which is an improvement of the result expected.

Sub-case 2: The case $a \in A$.

We can assume that $A = [a, b_1] \cup \dots \cup [a_N, b]$, with $a < b_1 < \dots < a_N < b$. For t small enough, we get

$$\begin{aligned} V_A^\nu(t) &= \int_a^{b_1+t} (x+p)^{\frac{1}{\gamma}} \, dx + \sum_{i=2}^{N-1} \int_{a_i-t}^{b_i+t} (x+p)^{\frac{1}{\gamma}} \, dx + \int_{a_N-t}^b (x+p)^{\frac{1}{\gamma}} \, dx \\ (V_A^\nu)'(t) &= (b_1+t+p)^{\frac{1}{\gamma}} + \dots + (a_N-t+p)^{\frac{1}{\gamma}} \\ (V_A^\nu)''(t) &= \frac{1}{\gamma} \left(\sum_{i=2}^N \left((b_{i-1}+t+p)^{\frac{1-\gamma}{\gamma}} - (a_i-t+p)^{\frac{1-\gamma}{\gamma}} \right) \right) \\ &\leq 0. \end{aligned}$$

Hence, V_A^ν is concave.

Case 2: The case $\gamma < 0$.

In the following, we use the notations

$$a_i(t) = (a_i - t + p)^{\frac{1}{\gamma}}, \quad b_i(t) = (b_i + t + p)^{\frac{1}{\gamma}}, \quad b = (b + p)^{\frac{1}{\gamma}}.$$

Notice that

$$0 \leq b \leq a_N(t) \leq \dots \leq b_1(t) \leq a_1(t).$$

Sub-case 1: The case $a \notin A$.

For t small enough, we get

$$\begin{aligned} V_A^\nu(t) &= \sum_{i=1}^{N-1} \int_{a_i-t}^{b_i+t} (x+p)^{\frac{1}{\gamma}} dx + \int_{a_N-t}^b (x+p)^{\frac{1}{\gamma}} dx \\ &= \frac{\gamma}{\gamma+1} \left(\sum_{i=1}^{N-1} (b_i(t)^{\gamma+1} - a_i(t)^{\gamma+1}) + b^{\gamma+1} - a_N(t)^{\gamma+1} \right) \\ (V_A^\nu)'(t) &= \sum_{i=1}^{N-1} (b_i(t) + a_i(t)) + a_N(t) \\ (V_A^\nu)''(t) &= \frac{1}{\gamma} \left(\sum_{i=1}^{N-1} (b_i(t)^{1-\gamma} - a_i(t)^{1-\gamma}) - a_N(t)^{1-\gamma} \right). \end{aligned}$$

We have V_A^ν s -concave if and only if $V_A^\nu(t)(V_A^\nu)''(t) \leq (1-s)(V_A^\nu)'(t)^2$ if and only if

$$\begin{aligned} &\left(\sum_{i=1}^{N-1} (b_i(t)^{\gamma+1} - a_i(t)^{\gamma+1}) + b^{\gamma+1} - a_N(t)^{\gamma+1} \right) \\ &\quad \times \left(\sum_{i=1}^{N-1} (b_i(t)^{1-\gamma} - a_i(t)^{1-\gamma}) - a_N(t)^{1-\gamma} \right) \\ &\quad \leq \left(\sum_{i=1}^{N-1} (b_i(t) + a_i(t)) + a_N(t) \right)^2. \end{aligned}$$

For convenience, we write b_i for $b_i(t)$ and a_i for $a_i(t)$. In fact, we prove a stronger inequality:

$$\begin{aligned} &\left(\sum_{i=1}^{N-1} (b_i^{\gamma+1} - a_i^{\gamma+1}) + b^{\gamma+1} - a_N^{\gamma+1} \right) \left(\sum_{i=1}^{N-1} (b_i^{1-\gamma} - a_i^{1-\gamma}) - a_N^{1-\gamma} \right) \\ &\quad \leq \sum_{i=1}^{N-1} (b_i^2 + a_i^2) + a_N^2. \end{aligned}$$

We prove this inequality by induction on $N \geq 2$. For $N = 2$, we have to prove

$$\left(b_1^{\gamma+1} - a_1^{\gamma+1} + b^{\gamma+1} - a_2^{\gamma+1}\right) \left(b_1^{1-\gamma} - a_1^{1-\gamma} - a_2^{1-\gamma}\right) \leq b_1^2 + a_1^2 + a_2^2. \quad (4)$$

We get

$$\begin{aligned} (4) &\iff b_1^{1+\gamma}(-a_1^{1-\gamma} - a_2^{1-\gamma}) - a_1^{1+\gamma}(b_1^{1-\gamma} - a_2^{1-\gamma}) \\ &\quad + b(b_1^{1-\gamma} - a_1^{1-\gamma} - a_2^{1-\gamma}) - a_2^{1+\gamma}(b_1^{1-\gamma} - a_1^{1-\gamma}) \leq 0 \\ &\iff -a_2^{1-\gamma}b_1^{1+\gamma} - a_1^{1+\gamma}(b_1^{1-\gamma} - a_2^{1-\gamma}) + b(b_1^{1-\gamma} - a_1^{1-\gamma} - a_2^{1-\gamma}) \\ &\quad - a_2^{1+\gamma}b_1^{1-\gamma} + a_1^{1-\gamma}(a_2^{1+\gamma} - b_1^{1+\gamma}) \leq 0, \end{aligned}$$

and each term is non-positive.

Let $N \geq 2$. We assume that

$$\begin{aligned} \left(\sum_{i=1}^{N-1} (b_i^{\gamma+1} - a_i^{\gamma+1}) + b^{\gamma+1} - a_N^{\gamma+1}\right) \left(\sum_{i=1}^{N-1} (b_i^{1-\gamma} - a_i^{1-\gamma}) - a_N^{1-\gamma}\right) \\ \leq \sum_{i=1}^{N-1} (b_i^2 + a_i^2) + a_N^2. \end{aligned}$$

and we want to show that

$$\begin{aligned} \left(\sum_{i=1}^N (b_i^{\gamma+1} - a_i^{\gamma+1}) + b^{\gamma+1} - a_{N+1}^{\gamma+1}\right) \left(\sum_{i=1}^N (b_i^{1-\gamma} - a_i^{1-\gamma}) - a_{N+1}^{1-\gamma}\right) \\ \leq \sum_{i=1}^N (b_i^2 + a_i^2) + a_{N+1}^2. \end{aligned}$$

Using the induction hypothesis, it is sufficient to show that

$$\begin{aligned} \left(\sum_{i=1}^{N-1} (b_i^{\gamma+1} - a_i^{\gamma+1}) + b^{\gamma+1} - a_N^{\gamma+1}\right) (b_N^{1-\gamma} - a_{N+1}^{1-\gamma}) + (b_N^{1+\gamma} - a_{N+1}^{1+\gamma}) \\ \times \left(\sum_{i=1}^{N-1} (b_i^{1-\gamma} - a_i^{1-\gamma}) - a_N^{1-\gamma} + b_N^{1-\gamma} - a_{N+1}^{1-\gamma}\right) \leq b_N^2 + a_{N+1}^2. \end{aligned}$$

This is equivalent to

$$\begin{aligned} \left(\sum_{i=1}^{N-1} (b_i^{\gamma+1} - a_i^{\gamma+1}) + b^{\gamma+1} - a_N^{\gamma+1}\right) (b_N^{1-\gamma} - a_{N+1}^{1-\gamma}) + (b_N^{1+\gamma} - a_{N+1}^{1+\gamma}) \\ \times \left(\sum_{i=1}^{N-1} (b_i^{1-\gamma} - a_i^{1-\gamma}) - a_N^{1-\gamma}\right) - b_N^{1+\gamma}a_{N+1}^{1-\gamma} - a_{N+1}^{1+\gamma}b_N^{1-\gamma} \leq 0, \end{aligned}$$

and each term is non-positive.

Sub-case 2: The case $a \in A$.

We have seen in case 1, sub-case 2, that for t small enough

$$(V_A^\nu)''(t) = \frac{1}{\gamma} \left(\sum_{i=2}^N (b_{i-1}(t)^{1-\gamma} - a_i(t)^{1-\gamma}) \right).$$

This quantity is non-positive. Hence V_A^ν is concave.

It follows that V_A^ν is piecewise s -concave on $[t_1, t_2]$. From Lemma 3.1, we deduce that V_A^ν is s -concave on $[t_1, t_2]$. Hence,

$$\Phi(\nu) \leq 0.$$

We conclude that $V_A^{\mu_0}$ is s -concave on $(0, t_0)$.

We have already seen the s -concavity of $V_A^{\mu_0}$ on $[t_0, +\infty)$. Once again we use Lemma 3.1 to conclude that $V_A^{\mu_0}$ is s -concave on \mathbb{R}_+^* . Finally, by the non-decreasing property of $V_A^{\mu_0}$, it follows that $V_A^{\mu_0}$ is s -concave on \mathbb{R}_+ . \square

Remark. The result obviously holds true if we replace the Euclidean ball by any symmetric convex body of \mathbb{R} . But it is not necessarily true for arbitrary convex body B . For example, let $0 < s \leq \frac{1}{2}$, and consider $B = [0, 1]$, $A = [0, 1] \cup [2, 3]$ and $d\mu(x) = x^{\frac{1}{\gamma}} 1_{[0,3]}(x) dx$, with $\gamma = \frac{s}{1-s}$. Then, μ is a s -concave measure. For $t \in [0, \frac{1}{2})$ we get

$$V_A^\mu(t) = \mu(A + tB) = \frac{\gamma}{\gamma+1} \left((1+t)^{\frac{\gamma+1}{\gamma}} + 3^{\frac{\gamma+1}{\gamma}} - 2^{\frac{\gamma+1}{\gamma}} \right).$$

Thus

$$V_A^\mu(0)(V_A^\mu)''(0) - (1-s)(V_A^\mu)'(0)^2 = \frac{1}{\gamma+1} \left(3^{\frac{\gamma+1}{\gamma}} - 2^{\frac{\gamma+1}{\gamma}} \right) > 0.$$

Hence V_A^μ is not s -concave on \mathbb{R}_+ . For $s = 0$, the same example works. For $s < 0$, one can take $B = [-1, 0]$, $A = [0, 1] \cup [2, 3]$ and $d\mu(x) = x^{\frac{1}{\gamma}} 1_{[a,3]}(x) dx$, with $\gamma = \frac{s}{1-s}$ and a sufficiently small.

We can't use the geometric localization theorem for $s \in (\frac{1}{2}, 1)$, see [18]. In fact, for $s \in (\frac{1}{2}, 1)$, the answer to Problem A is negative in general but under particular conditions, we can show a positive answer which improve the concavity. First, let us show that the answer to Problem A is negative in dimension 1 for $s \in (\frac{1}{2}, 1)$.

Proposition 3.3. *Let $s \in (\frac{1}{2}, 1)$, thus $\gamma = \frac{s}{1-s} > 1$. Let $b = 10(1 - 2^{\frac{1-\gamma}{\gamma}})^{-1}$ and μ be a measure such that $d\mu(x) = x^{\frac{1}{\gamma}} 1_{[0,b]}(x) dx$. We set $A = [0, 1] \cup [2, b]$. Then, $t \mapsto V_A^\mu(t) = \mu(A + tB_2^1)$ is not s -concave on \mathbb{R}_+ .*

Proof. For every $t \in [0, \frac{1}{2})$,

$$\begin{aligned} V_A^\mu(t) &= \frac{\gamma}{\gamma+1} \left((1+t)^{\frac{1+\gamma}{\gamma}} + b^{\frac{1+\gamma}{\gamma}} - (2-t)^{\frac{1+\gamma}{\gamma}} \right), \\ (V_A^\mu)'(t) &= (1+t)^{\frac{1}{\gamma}} + (2-t)^{\frac{1}{\gamma}}, \\ (V_A^\mu)''(t) &= \frac{1}{\gamma} \left((1+t)^{\frac{1-\gamma}{\gamma}} - (2-t)^{\frac{1-\gamma}{\gamma}} \right). \end{aligned}$$

Hence,

$$V_A^\mu(0)(V_A^\mu)''(0) - (1-s)(V_A^\mu)'(0)^2 = \frac{1}{\gamma+1} \left(b^{\frac{1+\gamma}{\gamma}} (1 - 2^{\frac{1-\gamma}{\gamma}}) - 2^{\frac{1-\gamma}{\gamma}} - 2^{\frac{1+2\gamma}{\gamma}} \right).$$

But $\gamma > 1$, thus $1 - 2^{\frac{1-\gamma}{\gamma}} > 0$ and since

$$b > \left(\frac{2^{\frac{1+2\gamma}{\gamma}} + 2^{\frac{1-\gamma}{\gamma}}}{1 - 2^{\frac{1-\gamma}{\gamma}}} \right)^{\frac{\gamma}{\gamma+1}}$$

it follows that

$$V_A^\mu(0)(V_A^\mu)''(0) - (1-s)(V_A^\mu)'(0)^2 > 0.$$

We conclude that V_A^μ is not s -concave on \mathbb{R}_+ . \square

We denote by $\text{supp}(\mu)$ the support of μ and by $\text{dist}(A, \text{supp}(\mu)^c)$ the distance between A and the complement of the support of μ which equal to $+\infty$ if the support of μ is \mathbb{R} .

Proposition 3.4. *Let $s \geq \frac{1}{2}$. Let μ be a s -concave measure in \mathbb{R} . Let A be a compact subset of \mathbb{R} such that $\text{dist}(A, \text{supp}(\mu)^c) > 0$. Then the function $t \mapsto V_A^\mu(t) = \mu(A + tB_2^1)$ is concave on $[0, \text{dist}(A, \text{supp}(\mu)^c)]$.*

Proof. First, we assume that $s = \frac{1}{2}$. Hence μ admits a 1-concave density ψ . Then, for $t \in [0, \text{dist}(A, \text{supp}(\mu)^c))$ we get

$$\begin{aligned} V_A^\mu(t) &= \sum_{i=1}^N \int_{a_i-t}^{b_i+t} \psi(x) dx, \\ (V_A^\mu)'(t) &= \sum_{i=1}^N (\psi(b_i+t) + \psi(a_i-t)). \end{aligned}$$

Since ψ is concave, it follows that for every $i \in \{1, \dots, N\}$, the function $t \mapsto \psi(b_i+t) + \psi(a_i-t)$ is non-increasing. Thus $(V_A^\mu)'$ is non-increasing. We conclude that V_A^μ is concave on $[0, \text{dist}(A, \text{supp}(\mu)^c)]$.

Finally, if μ is s -concave with $s \geq \frac{1}{2}$, then μ is $\frac{1}{2}$ -concave and we conclude from the first part of the proof that V_A^μ is concave on $[0, \text{dist}(A, \text{supp}(\mu)^c)]$. \square

We finish the study in dimension 1 with the 1-concave measures. We assume that μ is 1-concave. Hence,

$$\forall x \in \mathbb{R}, \quad d\mu(x) = C1_{[a,b]}(x)dx$$

with $C > 0$ is a constant and $[a, b]$ an interval of \mathbb{R} , with $a < b$. Then it follows by a direct computation that $t \mapsto \mu(A + tB_2^1)$ is 1-concave on \mathbb{R}_+ .

Now we study the problem A in dimension $n \geq 2$. It was shown in [19] that the Costa-Cover conjecture 1.1 is false in dimension $n \geq 2$, and thus the answer to Problem A is negative in general. Let us recall the counterexample in the following remark:

Remark. Let $n \geq 2$. We set $A = B_2^n \cup \{2e_1\}$. Then, the function $V_A(t) = |A + tB_2^n|$ is not $\frac{1}{n}$ -concave on \mathbb{R}_+ .

Proof. For every $t \in [0, \frac{1}{2})$, we get

$$|A + tB_2^n| = |B_2^n \cup \{2e_1\} + tB_2^n| = |B_2^n + tB_2^n| + |tB_2^n| = |B_2^n|((1+t)^n + t^n).$$

Since the $\frac{1}{n}$ -power of this function is not concave (it is strictly convex), V_A is not $\frac{1}{n}$ -concave on \mathbb{R}_+ for $n \geq 2$. \square

This could appear surprising since we get positive results in dimension 1 with the geometric localization theorem. In general, this localization theorem is used to reduce a n -dimensional problem to dimension 1 (see *e.g.* [17] and references therein). Let us explain why we can't exit from dimension 1 here. The reduction done in dimension 1 with localization works the same way in dimension n and we get the following equivalence for every compact set A of \mathbb{R}^n :

- i) V_A^μ is s -concave for every μ s -concave.
- ii) V_A^ν is s -concave for every ν s -affine on a segment $[a, b]$.

However, ii) is not true in dimension $n \geq 2$ since we can construct an explicit counterexample to show that in fact the function $t \mapsto |(A + tB_2^n) \cap [a, b]|_1$ is not continuous and hence not s -concave. For example, consider $A = \{(0, 0)\} \cup \{(3, 0)\} \cup \{(x, 1); x \in [1, 2]\}$ and $[a, b] = \{(x, 0); x \in [0, 3]\}$.

In [19], it was shown that in dimension 2, $t \mapsto |A + tB_2^2|$ is $\frac{1}{2}$ -concave on \mathbb{R}_+ , if A is a connected subset of \mathbb{R}^2 . However, the next proposition shows that this is false in the general case of s -concave measures.

Proposition 3.5. *In dimension 2, there exists a connected set A and a $\frac{1}{2}$ -concave measure μ such that $t \mapsto \mu(A + tB_2^2)$ is not $\frac{1}{2}$ -concave on \mathbb{R}_+ .*

Proof. We set $d\mu(x) = 1_{B_1^2}(x)dx$, where B_1^2 denotes the unit ball for the ℓ_1^2 norm. Hence, μ is $\frac{1}{2}$ -concave. We construct the points $B = (-1, 0)$,

$C = (-0.5, -0.5)$, $D = (0.5, 0.5)$, $E = (0, 1)$, $F = (-2, 0)$, $G = (0, -2)$, $H = (0, -1)$, $I = (2, 0)$, $J = (1, 0)$. We set

$$A = \text{conv}(BCDE) \cup [FB] \cup [FG] \cup [GH] \cup [GI] \cup [IJ].$$

Then A is connected and for every $t \in [0, \frac{1}{8}]$, we get

$$V_A^\mu(t) = \mu(A + tB_2^2) = \frac{\sqrt{2}}{2} + \sqrt{2}t + \frac{\pi}{2}t^2.$$

It follows that

$$\left(\sqrt{V_A^\mu}\right)''(0) > 0.$$

We conclude that $t \mapsto \mu(A + tB_2^2)$ is not $\frac{1}{2}$ -concave on \mathbb{R}_+ . \square

Remark. Notice that we can adapt the counterexample of Proposition 3.5 to show that there exists, in dimension $n \geq 2$, a s -concave measure μ such that for every $r \in (-\infty; s)$ there exists a compact connected subset A of \mathbb{R}^n such that $t \mapsto \mu(A + tB_2^n)$ is not r -concave on \mathbb{R}_+ .

Question. Let μ be a s -concave measure in \mathbb{R}^2 and A be a compact set in \mathbb{R}^2 . Is the function $\mu(A + tB_2^2)$ s -concave for the t 's such that the set $\text{supp}(\mu) \cap (A + tB_2^2)$ is connected?

4 Functional version

In this section, we set the Costa-Cover conjecture into a functional version. Let us recall the problem:

Problem B. Let $\gamma \geq -\frac{1}{n}$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a bounded non-negative function and $g : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a γ -concave function. Let

$$h_t^{(\gamma)}(z) = \sup_{\substack{z=x+ty \\ f(x)>0; g(y)>0}} (f(x)^\gamma + tg(y)^\gamma)^{\frac{1}{\gamma}} \quad \text{and} \quad s = \frac{\gamma}{1 + \gamma n}.$$

Is the function $t \mapsto \int_{\mathbb{R}^n} h_t^{(\gamma)}(z) dz$ s -concave on \mathbb{R}_+ ?

For $\gamma = 0$, the function $h_t^{(0)}$ is interpreted by continuity, i.e. for every $z \in \mathbb{R}^n$,

$$h_t^{(0)}(z) = \sup_{z=x+ty} f(x)g(y)^t.$$

Let us notice that the problem B is a functional version of the Costa-Cover conjecture 1.1. Indeed, let A be a non-empty compact subset of \mathbb{R}^n . By considering $f = 1_A$ and $g = 1_{B_2^n}$, we get

$$1_A(x)1_{B_2^n}(y) = \begin{cases} 1 & \text{if } x \in A \text{ and } y \in B_2^n \\ 0 & \text{otherwise} \end{cases}$$

so for every $z \in \mathbb{R}^n$, $h_t^{(0)}(z) = 1_{A+tB_2^n}(z)$. Then,

$$\int_{\mathbb{R}^n} h_t^{(0)}(z) dz = |A + tB_2^n|.$$

As in section 3 where Problem A was solved for convex sets, the next proposition shows that Problem B is solved for γ -concave functions, as it is natural to expect.

Proposition 4.1. *Let $\gamma \geq -\frac{1}{n}$. Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be two γ -concave functions. Then the function $t \mapsto \int_{\mathbb{R}^n} h_t^{(\gamma)}(z) dz$ is s -concave on \mathbb{R}_+ , where*

$$h_t^{(\gamma)}(z) = \sup_{\substack{z=x+ty \\ f(x)>0; g(y)>0}} (f(x)^\gamma + tg(y)^\gamma)^{\frac{1}{\gamma}} \quad \text{and} \quad s = \frac{\gamma}{1 + \gamma n}.$$

Proof. For convenience, we denote $h_t = h_t^{(\gamma)}$. Let $\lambda \in [0, 1]$ and $t_1, t_2 \in \mathbb{R}_+$. We want to show that

$$\int_{\mathbb{R}^n} h_{(1-\lambda)t_1 + \lambda t_2} \geq \left((1-\lambda) \left(\int_{\mathbb{R}^n} h_{t_1} \right)^s + \lambda \left(\int_{\mathbb{R}^n} h_{t_2} \right)^s \right)^{\frac{1}{s}}.$$

From the Borell-Brascamp-Lieb inequality [8],[10] (dimensional Prékopa's inequality), it is sufficient to show that

$$\forall y_1, y_2 \in \mathbb{R}^n, h_{(1-\lambda)t_1 + \lambda t_2}((1-\lambda)y_1 + \lambda y_2) \geq ((1-\lambda)h_{t_1}(y_1)^\gamma + \lambda h_{t_2}(y_2)^\gamma)^{\frac{1}{\gamma}}.$$

Let $y_1, y_2 \in \mathbb{R}^n$ and $\lambda \in [0, 1]$. We write for $i \in \{1, 2\}$,

$$h_{t_i}(y_i) = \sup_{y_i = x + t_i y} (f(x)^\gamma + t_i g(y)^\gamma)^{\frac{1}{\gamma}} = \sup_{x \in \mathbb{R}^n} \left(f(x)^\gamma + t_i g\left(\frac{y_i - x}{t_i}\right)^\gamma \right)^{\frac{1}{\gamma}}.$$

Let $x_1, x_2 \in \mathbb{R}^n$ such that

$$\forall i \in \{1, 2\}, h_{t_i}(y_i) = \left(f(x_i)^\gamma + t_i g\left(\frac{y_i - x_i}{t_i}\right)^\gamma \right)^{\frac{1}{\gamma}}.$$

We denote

$$h = h_{(1-\lambda)t_1 + \lambda t_2}((1-\lambda)y_1 + \lambda y_2) \quad \text{and} \quad t = (1-\lambda)t_1 + \lambda t_2.$$

We get

$$\begin{aligned}
h &= \sup_{x \in \mathbb{R}^n} \left(f(x)^\gamma + tg \left(\frac{(1-\lambda)y_1 + \lambda y_2 - x}{t} \right)^\gamma \right)^{\frac{1}{\gamma}} \\
&\geq \left(f((1-\lambda)x_1 + \lambda x_2)^\gamma + tg \left(\frac{(1-\lambda)y_1 + \lambda y_2 - ((1-\lambda)x_1 + \lambda x_2)}{t} \right)^\gamma \right)^{\frac{1}{\gamma}} \\
&= \left(f((1-\lambda)x_1 + \lambda x_2)^\gamma + tg \left(\frac{(1-\lambda)t_1 y_1 - x_1}{t} + \frac{\lambda t_2 y_2 - x_2}{t} \right)^\gamma \right)^{\frac{1}{\gamma}} \\
&\geq \left((1-\lambda)f(x_1)^\gamma + \lambda f(x_2)^\gamma + (1-\lambda)t_1 g \left(\frac{y_1 - x_1}{t_1} \right)^\gamma + \lambda t_2 g \left(\frac{y_2 - x_2}{t_2} \right)^\gamma \right)^{\frac{1}{\gamma}} \\
&= ((1-\lambda)h_{t_1}(y_1)^\gamma + \lambda h_{t_2}(y_2)^\gamma)^{\frac{1}{\gamma}}.
\end{aligned}$$

□

Borell showed in [8] that if $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is β -concave and $g : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is γ -concave, then fg is α -concave for every $\alpha, \beta, \gamma \in \mathbb{R} \cup \{+\infty\}$ such that $\beta + \gamma \geq 0$ and $\frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{\alpha}$. A generalized form of Proposition 4.1 follows:

Proposition 4.2. *Let $\gamma \geq -\frac{1}{n}$. If a measure μ has a β -concave density, with $\beta \geq -\gamma$, and if $f, g : \mathbb{R}^n \rightarrow \mathbb{R}_+$ are two γ -concave functions, then $t \mapsto \int_{\mathbb{R}^n} h_t^{(\gamma)}(z) d\mu(z)$ is s -concave on \mathbb{R}_+ , with $s = \frac{\alpha}{1+\alpha n}$, where $\frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{\alpha}$.*

Before starting the study of Problem B for general function f , let us rely $h_t^{(\gamma)}$ with the Hopf-Lax solution of the Hamilton-Jacobi equation. Since by assumption g is γ -concave, hence for $\gamma \in (-\frac{1}{n}; 0)$, $g^\gamma = V$, with V a convex function, which extends to $\gamma = 0$. It follows that

$$h_t^{(\gamma)}(z) = \sup_{x \in \mathbb{R}^n} \left(f(x)^\gamma + tV \left(\frac{z-x}{t} \right) \right)^{\frac{1}{\gamma}} = \left(Q_t^{(V)} f^\gamma(z) \right)^{\frac{1}{\gamma}},$$

where for arbitrary convex function V and arbitrary function u ,

$$Q_t^{(V)} u(z) = \inf_{x \in \mathbb{R}^n} \left(u(x) + tV \left(\frac{z-x}{t} \right) \right).$$

We assume that

$$\lim_{|z| \rightarrow +\infty} \frac{V(z)}{|z|} = +\infty.$$

For Lipschitz continuous function u , it is known (see *e.g.* [16]) that $Q_t^{(V)} u$ is the solution, called Hopf-Lax solution, of the following partial differential equation, called Hamilton-Jacobi equation:

$$\begin{cases} \frac{\partial}{\partial t} h(t, z) + V^*(\nabla h(t, z)) = 0 & \text{on } (0, +\infty) \times \mathbb{R}^n \\ h(t, z) = u(z) & \text{on } \{t = 0\} \times \mathbb{R}^n \end{cases},$$

where V^* is the Legendre transform of V defined on \mathbb{R}^n by

$$V^*(y) = \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - V(x)).$$

It is shown in [16] that if u is Lipschitz continuous on \mathbb{R}^n then $Q_t^{(V)}u$ is Lipschitz continuous on $[0, +\infty) \times \mathbb{R}^n$. However, for arbitrary convex function V , $t \mapsto Q_t^{(V)}u$ is not necessarily continuous on 0.

Let us show partial positive answer to Problem B in dimension 1. In fact, in dimension 1, we can improve the concavity:

Theorem 4.3. *Let $\gamma \in (-1; 0]$. Let $f : \mathbb{R} \rightarrow \mathbb{R}_+$ be a Lipschitz continuous non-negative function. Define, for every $y \in \mathbb{R}$, $V(y) = \frac{|y|^p}{p}$, with $p \geq 1$. Then the function $t \mapsto \int_{\mathbb{R}} h_t^{(\gamma)}(z) dz$ is concave on \mathbb{R}_+ , where*

$$h_t^{(\gamma)}(z) = \sup_{\substack{z=x+ty \\ f(x)>0; V(y)>0}} (f(x)^\gamma + tV(y))^\frac{1}{\gamma} \text{ and } h_t^{(0)}(z) = \sup_{z=x+ty} f(x)e^{-tV(y)}.$$

Proof. We denote for $t \in \mathbb{R}_+$,

$$F(t) = \int_{\mathbb{R}} h_t^{(\gamma)}(z) dz.$$

We assume $\gamma \neq 0$, the case $\gamma = 0$ follows by continuity. For $p = 1$, the function F is constant. We then consider $p > 1$. We have seen above that

$$h_t^{(\gamma)}(z) = \left(Q_t^{(V)} f^\gamma(z) \right)^\frac{1}{\gamma}.$$

For convenience, we set $h(t, z) = Q_t^{(V)} f^\gamma(z)$ and $h' = \frac{\partial}{\partial z} g$. Under regularity assumption, we get by a direct computation that

$$F'(t) = -\frac{1}{\gamma} \int_{\mathbb{R}} V^*(h') h^\frac{1-\gamma}{\gamma},$$

$$F''(t) = \frac{1}{\gamma} \int_{\mathbb{R}} h'' ((V^*)'(h'))^2 h^\frac{1-\gamma}{\gamma} + \frac{1-\gamma}{\gamma^2} \int_{\mathbb{R}} (V^*(h'))^2 h^\frac{1-2\gamma}{\gamma}.$$

We assumed that $V(u) = \frac{|u|^p}{p}$. Hence $V^*(u) = \frac{|u|^q}{q}$, with $\frac{1}{p} + \frac{1}{q} = 1$. It follows that

$$F''(t) = \frac{1}{\gamma} \int_{\mathbb{R}} h''(h')^{2q-2} h^\frac{1-\gamma}{\gamma} + \frac{1-\gamma}{\gamma^2} \int_{\mathbb{R}} \frac{(h')^2}{q^2} h^\frac{1-2\gamma}{\gamma}.$$

Integration by parts gives

$$\frac{1}{\gamma} \int_{\mathbb{R}} h''(h')^{2q-2} h^\frac{1-\gamma}{\gamma} = -\frac{2q-2}{\gamma} \int_{\mathbb{R}} h''(h')^{2q-2} h^\frac{1-\gamma}{\gamma} - \frac{1-\gamma}{\gamma^2} \int_{\mathbb{R}} (h')^{2q} h^\frac{1-2\gamma}{\gamma}.$$

Then

$$\frac{2q-1}{\gamma} \int_{\mathbb{R}} h''(h')^{2q-2} h^{\frac{1-\gamma}{\gamma}} = -\frac{1-\gamma}{\gamma^2} \int_{\mathbb{R}} (h')^{2q} h^{\frac{1-2\gamma}{\gamma}}.$$

Finally,

$$F''(t) = -\frac{1-\gamma}{\gamma^2} (q-1)^2 \int_{\mathbb{R}} (h')^{2q} h^{\frac{1-2\gamma}{\gamma}} \leq 0.$$

We conclude that $t \mapsto \int_{\mathbb{R}} h_t^{(\gamma)}(z) dz$ is concave on \mathbb{R}_+ . \square

Question. Problem B is open in dimension 1 for arbitrary γ -concave function g .

5 Links with weighted Brascamp-Lieb-type inequalities

Recall that for $\gamma < 0$, $h_t^{(\gamma)}(z) = \left(Q_t^{(g^\gamma)} f^\gamma(z)\right)^{\frac{1}{\gamma}}$, where for arbitrary convex function V and arbitrary function u ,

$$Q_t^{(V)} u(z) = \inf_{x \in \mathbb{R}^n} \left(u(x) + tV \left(\frac{z-x}{t} \right) \right).$$

Proposition 5.1. *Let $\gamma \in (-\frac{1}{n}, 0)$ and $s = \frac{\gamma}{1+\gamma n}$. Denote $V = g^\gamma$. The function $F : t \mapsto \int_{\mathbb{R}^n} h_t^{(\gamma)}(z) dz$ is s -concave if and only if*

$$\begin{aligned} \text{Var}_\mu(G) \leq & -\frac{\gamma}{1-\gamma} \int \frac{\langle \text{Hess } Q_t^{(V)} f^\gamma, (\nabla V^*) (\nabla Q_t^{(V)} f^\gamma), (\nabla V^*) (\nabla Q_t^{(V)} f^\gamma) \rangle}{Q_t^{(V)} f^\gamma} d\mu \\ & + \frac{\gamma-s}{1-\gamma} \left(\int G d\mu \right)^2, \end{aligned}$$

where

$$d\mu(z) = \frac{(Q_t^{(V)} f^\gamma(z))^{\frac{1}{\gamma}}}{\int (Q_t^{(V)} f^\gamma)^{\frac{1}{\gamma}}} dz \text{ and } G = \frac{V^* (\nabla Q_t^{(V)} f^\gamma)}{Q_t^{(V)} f^\gamma}.$$

Proof. To prove the s -concavity of the function $F : t \mapsto \int_{\mathbb{R}^n} h_t^{(\gamma)}(z) dz$, it is a natural idea to proceed by differentiation. We have seen above that

$$h_t^{(\gamma)}(z) = \left(Q_t^{(V)} f^\gamma(z)\right)^{\frac{1}{\gamma}},$$

where for arbitrary convex function V and arbitrary function u ,

$$Q_t^{(V)} u(z) = \inf_{x \in \mathbb{R}^n} \left(u(x) + tV \left(\frac{z-x}{t} \right) \right).$$

For convenience, we set $\phi = f^\gamma$ and $Q_t = Q_t^{(V)}$. Under regularity assumption, we get by a direct computation

$$\frac{\partial}{\partial t} h_t^{(\gamma)} = -\frac{1}{\gamma} V^*(\nabla Q_t \phi) (Q_t \phi)^{\frac{1-\gamma}{\gamma}},$$

$$\begin{aligned} \frac{\partial^2}{\partial t^2} h_t^{(\gamma)} &= \frac{1}{\gamma} \langle (\text{Hess } Q_t \phi)(\nabla V^*)(\nabla Q_t \phi), (\nabla V^*)(\nabla Q_t \phi) \rangle (Q_t \phi)^{\frac{1-\gamma}{\gamma}} \\ &\quad + \frac{1-\gamma}{\gamma^2} (V^*(\nabla Q_t \phi))^2 (Q_t \phi)^{\frac{1-2\gamma}{\gamma}}. \end{aligned}$$

Thus the function F is s -concave if and only if $F(t)F''(t) \leq (1-s)F'(t)^2$ if and only if

$$\begin{aligned} \text{Var}_\mu(G) &\leq -\frac{\gamma}{1-\gamma} \int \frac{\langle (\text{Hess } Q_t \phi)(\nabla V^*)(\nabla Q_t \phi), (\nabla V^*)(\nabla Q_t \phi) \rangle}{Q_t \phi} d\mu \\ &\quad + \frac{\gamma-s}{1-\gamma} \left(\int G d\mu \right)^2, \end{aligned}$$

where

$$d\mu(z) = \frac{(Q_t \phi(z))^{\frac{1}{\gamma}}}{\int (Q_t \phi)^{\frac{1}{\gamma}} dz} dz \text{ and } G = \frac{V^*(\nabla Q_t \phi)}{Q_t \phi}.$$

□

Remark. For $\gamma = 0$ and $V(u) = \frac{|u|^2}{2}$, we get that $t \mapsto \int_{\mathbb{R}^n} h_t^{(0)}(z) dz$ is log-concave if and only if

$$\text{Var}_\mu(|\nabla Q_t \phi|^2) \leq 4 \int \langle (\text{Hess } Q_t \phi) \nabla Q_t \phi, \nabla Q_t \phi \rangle d\mu,$$

where $\phi = -\log f$ and

$$d\mu(z) = \frac{e^{-Q_t \phi(z)}}{\int e^{-Q_t \phi} dz} dz.$$

Question. For which functions u the following inequality holds?

$$\text{Var}_\mu(|\nabla u|^2) \leq 4 \int \langle (\text{Hess } u) \nabla u, \nabla u \rangle d\mu$$

where

$$d\mu(z) = \frac{e^{-u(z)}}{\int e^{-u} dz} dz.$$

From Proposition 4.1, if f is γ -concave then the inequality of Proposition 5.1 is true and we get the following weighted Brascamp-Lieb-type inequality by tending t to 0:

Corollary 5.2. *Let $\gamma \in (-\frac{1}{n}, 0)$ and $s = \frac{\gamma}{1+\gamma n}$. For every $V, \phi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ convex such that $\lim_{|z| \rightarrow +\infty} V(z)/z = +\infty$, we get*

$$\text{Var}_\mu(G) \leq -\frac{\gamma}{1-\gamma} \int \frac{\langle (\text{Hess}\phi)^{-1} \nabla G \phi, \nabla G \phi \rangle}{\phi} d\mu + \frac{\gamma-s}{1-\gamma} \left(\int G d\mu \right)^2, \quad (5)$$

where $d\mu(z) = \frac{\phi^{\frac{1}{\gamma}}(z)}{\int \phi^{\frac{1}{\gamma}}} dz$ and $G = \frac{V(\nabla\phi)}{\phi}$.

We reproved a result of Bobkov and Ledoux in [5] (for a smaller class of function G) who used the same idea since the inequality (5) is derived from the Borell-Brascamp-Lieb inequality (dimensional Prékopa inequality) but using a more suitable function instead of the Hopf-Lax solution we used. In fact, Bobkov and Ledoux already noticed in [4] that one can deduce the classical Brascamp-Lieb inequality from the classical Prékopa inequality (corresponding to the log-concave case). This idea has been explored by Cordero-Erausquin and Klartag in [12] where they showed that in fact the converse is true, *i.e.* we can derive the Prékopa inequality from the Brascamp-Lieb inequality. More recently, Nguyen in [26] generalized the work of Cordero-Erausquin and Klartag in the case of γ -concave measure (even for $\gamma \geq 0$) and improved the Brascamp-Lieb-Type inequality of Bobkov-Ledoux (inequality (5)).

Acknowledgement.

I thank my adviser Matthieu Fradelizi for his helpful remarks and suggestions.

References

- [1] S. Artstein, B. Klartag, V. Milman, *The Santalò point of a function and a functional form of Santalò inequality*, *Mathematika* 51, 33-48 (2004)
- [2] K. Ball, *Isometric problems in ℓ_p and sections of convex sets*, PhD Dissertation, Cambridge (1986)
- [3] S. Bobkov, *Isoperimetric and analytic inequalities for log-concave probability measures*, *Ann. Probab.* 27 (1999), no. 4, 1903-1921
- [4] S.G. Bobkov, M. Ledoux, *From Brunn-Minkowski to Brascamp-Lieb and to logarithmic Sobolev inequalities*, *Geom. Funct. Anal.* 10 (2000), no. 5, 1028-1052
- [5] S.G. Bobkov, M. Ledoux, *Weighted Poincaré-type inequalities for Cauchy and other convex measures*, *Ann. Probab.* 37 (2009), no. 2, 403-427

- [6] S. Bobkov, I. Gentil, M. Ledoux, *Hypercontractivity of Hamilton-Jacobi equations*, J. Math. Pures Appl. (9) 80 (2001), no. 7, 669-696
- [7] C. Borell, *Convex measures on locally convex spaces*, Ark.Mat. 12 (1974), 239-252
- [8] C. Borell, *Convex set functions in d -space*, Periodica Mathematica Hungarica Vol. 6, 111-136, 1975
- [9] C. Borell, *The Brunn-Minkowski inequality in Gauss space*, Invent. Math. 30 (1975), no. 2, 207-216
- [10] H.J. Brascamp, E.H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, (1976) J. Funct. Anal. 22 366-389.
- [11] A. Cianchi, N. Fusco, F. Maggi, A. Pratelli, *On the isoperimetric deficit in Gauss space*, Amer. J. Math. 133 (2011), no. 1, 131-186
- [12] D. Cordero-Erausquin, B. Klartag, *Interpolations, convexity and geometric inequalities*, Geometric aspects of functional analysis, 151-168, Lecture Notes in Math., 2050, Springer, Heidelberg, 2012
- [13] M. Costa, *A new entropy power inequality*, IEEE Trans. Inform. Theory 31 (1985) 751-760
- [14] M. Costa, T.M. Cover, *On the similarity of the entropy power inequality and the Brunn-Minkowski inequality*, IEEE Trans. Inform. Theory 30 (1984), no. 6, 837-839
- [15] Y. De Castro, *Quantitative isoperimetric inequalities on the real line*, Ann. Math. Blaise Pascal 18 (2011), no. 2, 251-271
- [16] L.C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998. xviii+662 pp
- [17] M. Fradelizi, *Concentration inequalities for s -concave measures of dilations of Borel sets and applications*, Electron. J. Probab. 14 (2009), no. 71, 2068-2090
- [18] M. Fradelizi, O. Guédon, *The extreme points of subsets of s -concave probabilities and a geometric localization theorem*, Discrete Comput. Geom. 31 (2004), no. 2, 327-335
- [19] M. Fradelizi, A. Marsiglietti, *On the analogue of the concavity of entropy power in the Brunn-Minkowski theory*, arXiv:1302.6093 [math.FA]

- [20] M. Fradelizi, M. Meyer, *Some functional forms of Blaschke-Santalò inequality*, Math. Z. 256 (2007), no. 2, 379-395
- [21] M. Fradelizi, M. Meyer, *Functional inequalities related to Mahler's conjecture*, Monatsh. Math. 159 (2010), no. 1-2, 13-25
- [22] N. Gozlan, C. Roberto, P-M. Samson, *Hamilton Jacobi equations on metric spaces and transport-entropy inequalities*, arXiv:1203.2783 [math.PR]
- [23] R. Kannan, L. Lovász, M. Simonovits, *Isoperimetric problems for convex bodies and a localization lemma*, (English summary) Discrete Comput. Geom. 13 (1995), no. 3-4, 541-559.
- [24] J. Lehec, *A direct proof of the functional Santalò inequality*, C. R. Math. Acad. Sci. Paris 347 (2009), no. 1-2, 55-58
- [25] L. Leindler, *On a certain converse of Hölder's inequality*, II, Acta Sci. Math., 33 (1972), 217-223
- [26] V.H. Nguyen, *Dimensional variance inequalities of Brascamp-Lieb type and a local approach to dimensional Prékopa's theorem*, arXiv:1302.4589 [math.FA]
- [27] A. Prékopa, *On logarithmic concave measures and functions*, Acta Sci. Math. (Szeged) 34 (1973), 335-343
- [28] V. N. Sudakov, B. S. Cirel'son, *Extremal properties of half-spaces for spherically invariant measures* (Russian) Problems in the theory of probability distributions, II. Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 41 (1974), 14-24, 165
- [29] C. Villani, *A short proof of the 'concavity of entropy power'*, IEEE Trans. Inform. Theory 46 (2000) no. 4, 1695-1696