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# Spectral discretization of the Navier-Stokes problem coupled with the heat equation

by Rahma Agroum<sup>1</sup>, Saloua Mani Aouadi<sup>2</sup>, Christine Bernardi<sup>3</sup>, and Jamil Satouri<sup>4</sup>

**Abstract:** We consider the spectral discretization of the Navier-Stokes equations coupled with the heat equation where the viscosity depends on the temperature, with boundary conditions which involve the velocity and the temperature. This problem admits a variational formulation with three independent unknowns, the velocity, the pressure and the temperature. We prove optimal error estimates and present some numerical experiments which confirm the interest of the discretization.

**Résumé:** Nous considérons une discrétisation par méthode spectrale des équations de Navier-Stokes couplées avec l'équation de la chaleur là où la viscosité dépend de la température, avec des conditions aux limites portant sur la vitesse et la température. Ce système admet une formulation variationnelle qui comporte trois inconnues indépendantes: la vitesse, la pression et la température. Nous prouvons des estimations d'erreur optimales et présentons quelques expériences numériques qui confirment l'intérêt de la discrétisation.

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# 1 Introduction.

Let  $\Omega$  be a connected bounded open set in  $\mathbb{R}^d$ ,  $d = 2$  or  $3$ , with a Lipschitz-continuous boundary  $\partial\Omega$ . The following system models the stationary flow of a viscous incompressible fluid, in the case where the viscosity of the fluid depends on the temperature

$$\left\{ \begin{array}{ll} -\operatorname{div}(\nu(T)\nabla\mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \operatorname{grad} p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ -\alpha \Delta T + (\mathbf{u} \cdot \nabla)T = g & \text{in } \Omega, \\ \mathbf{u} = \mathbf{u}_0 & \text{on } \partial\Omega, \\ T = T_0 & \text{on } \partial\Omega. \end{array} \right. \quad (1.1)$$

The unknowns are the velocity  $\mathbf{u}$ , the pressure  $p$ , and the temperature  $T$  of the fluid, while the data are the distributions  $\mathbf{f}$  and  $g$ . The function  $\nu$  is positive and bounded, while the coefficient  $\alpha$  is a positive constant. A similar but slightly more complex model has been derived and analyzed in [1], [6].

We do think that the equations in (1.1) are a very realistic model for a number of incompressible fluids when the temperature presents high variations, for instance induced by the boundary condition. The solution  $(\mathbf{u}, p)$  of the first two lines of the system behaves like the viscosity solution of Euler's equations. We refer to [4] for the first study of such a simplification.

We propose a spectral discretization in the basic situation where the domain is a square or a cube. More complex geometries can be treated thanks to the arguments in [11], however we prefer to avoid them for simplicity. The numerical analysis of the nonlinear discrete problem makes use of the approach of Brezzi, Rappaz and Raviart [5], the main difficulty being the lack of compactness of the nonlinear term linked to the viscosity. Nevertheless, we prove the existence of a solution for the discrete problem. We establish a priori error estimates for this discretization.

In a final step, we describe the Newton type iterative algorithm that is used to solve the nonlinear discrete problem. We refer to [5] for the arguments needed for its proof of convergence. We conclude with some numerical experiments where the viscosity of the fluid  $\nu$  is either a constant, or a function dependent of the space variable, or a function that depends on the temperature  $T$ . All of them confirm the optimality of the discretization and justify the choice of this formulation.

An outline of the paper is as follows:

- In Section 2, we prove the existence of a solution for problem (1.1).

- The discrete problem is described in Section 3, and we prove optimal a priori error estimates for the error.
- The Newton algorithm are described in Section 4 and some numerical experiments are presented.

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## 2 The continuous problem.

We intend to write a variational formulation of system (1.1). We first make precise the assumptions on the function  $\nu$  : It belongs to  $L^\infty(\mathbb{R})$  and satisfies, for two positive constants  $\nu_1$  and  $\nu_2$ ,

$$\text{for a.e. } \tau \in \mathbb{R}, \quad \nu_1 \leq \nu(\tau) \leq \nu_2. \quad (2.1)$$

Note that these assumptions are not at all restrictive.

In order to avoid the technical results linked to the Hopf lemma, (see [9, Chap. IV, Lemma 2.3]) for the analysis of the problem, we assume that  $\mathbf{u}_0 = \mathbf{0}$ .

To go further, for any subset  $O$  of  $\Omega$  with a Lipschitz-continuous boundary  $\partial O$ , we consider the full scale of Sobolev spaces  $H^s(O)$ ,  $s \in \mathbb{R}$ , and also the analogous spaces  $H^s(\partial O)$  on its boundary. We need the spaces  $W^{m,p}(O)$ , for any nonnegative integer  $m$  and  $1 < p < +\infty$ , equipped with the norm  $\|\cdot\|_{W^{m,p}(O)}$  and seminorm  $|\cdot|_{W^{m,p}(O)}$ . We denote by  $W_0^{m,p}(O)$  the closure in  $W^{m,p}(O)$  of the space  $\mathcal{D}(O)$  of infinitely differentiable functions with a compact support in  $O$ , by  $W^{-m,p'}(O)$  its dual space (with  $\frac{1}{p} + \frac{1}{p'} = 1$ ), and by  $W^{m-\frac{1}{p},p}(\partial O)$  the space of traces of functions in  $W^{m,p}(O)$  on  $\partial O$ . We also introduce the space

$$L_0^2(O) = \{q \in L^2(O); \int_O q(\mathbf{x}) \, d\mathbf{x} = 0\}. \quad (2.2)$$

We thus consider the variational problem

Find  $(\mathbf{u}, p, T)$  in  $H_0^1(\Omega)^d \times L_0^2(\Omega) \times H^1(\Omega)$  such that

$$T = T_0 \quad \text{on } \partial\Omega, \quad (2.3)$$

and that,

$$\begin{aligned} \forall \mathbf{v} \in H_0^1(\Omega)^d, \quad & \int_\Omega \nu(T)(\mathbf{x})(\mathbf{grad} \mathbf{u})(\mathbf{x}) : (\mathbf{grad} \mathbf{v})(\mathbf{x}) \, d\mathbf{x} \\ & + \int_\Omega ((\mathbf{u} \cdot \nabla) \mathbf{u})(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \, d\mathbf{x} - \int_\Omega (\text{div} \mathbf{v})(\mathbf{x}) p(\mathbf{x}) \, d\mathbf{x} = \langle \mathbf{f}, \mathbf{v} \rangle_\Omega, \\ \forall q \in L_0^2(\Omega), \quad & - \int_\Omega (\text{div} \mathbf{u})(\mathbf{x}) q(\mathbf{x}) \, d\mathbf{x} = 0, \\ \forall S \in H_0^1(\Omega), \quad & \alpha \int_\Omega (\mathbf{grad} T)(\mathbf{x}) \cdot (\mathbf{grad} S)(\mathbf{x}) \, d\mathbf{x} \\ & + \int_\Omega ((\mathbf{u} \cdot \nabla) T)(\mathbf{x}) S(\mathbf{x}) \, d\mathbf{x} = \langle g, S \rangle_\Omega, \end{aligned} \quad (2.4)$$

where  $\langle \cdot, \cdot \rangle_\Omega$  denotes the duality pairing between  $H^{-1}(\Omega)$  and  $H_0^1(\Omega)$  and also between  $H^{-1}(\Omega)^d$  and  $H_0^1(\Omega)^d$ .

Standard arguments relying on the density of  $\mathcal{D}(\Omega)$  in  $H_0^1(\Omega)$  lead to the following result.

**Proposition 2.1** *Problems (1.1) with  $\mathbf{u}_0 = \mathbf{0}$  and (2.3) – (2.4) are equivalent: Any triple  $(\mathbf{u}, p, T)$  in  $H_0^1(\Omega)^d \times L_0^2(\Omega) \times H^1(\Omega)$  is a solution of (1.1) (in the distribution sense) if and only if it is a solution of (2.3) – (2.4).*

The existence of a solution can be established owing to a fixed-point theorem. Its proof requires the kernel

$$\mathbb{V} = \{ \mathbf{v} \in H_0^1(\Omega)^d; \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega \}. \quad (2.5)$$

**Theorem 2.2** *For any data  $(\mathbf{f}, g)$  in  $H^{-1}(\Omega)^d \times H^{-1}(\Omega)$  and  $T_0$  in  $H^{\frac{1}{2}}(\partial\Omega)$ , problem (2.3) – (2.4) admits at least a solution  $(\mathbf{u}, p, T)$  in  $H_0^1(\Omega)^d \times L_0^2(\Omega) \times H^1(\Omega)$ . Moreover, this solution satisfies, for a constant  $c$  only depending on  $\nu_1$  and  $\alpha$ ,*

$$\| \mathbf{u} \|_{H^1(\Omega)^d} + \| T \|_{H^1(\Omega)} \leq c ( \| \mathbf{f} \|_{H^{-1}(\Omega)^d} + \| g \|_{H^{-1}(\Omega)} + \| T_0 \|_{H^{\frac{1}{2}}(\partial\Omega)}). \quad (2.6)$$

**Proof:** It is performed in several steps.

1) We refer to the Hopf lemma, see [9, Chap. IV, Lemma 2.3] for the following result: For any  $\varepsilon > 0$ , there exists a lifting  $\bar{T}_0$  of  $T_0$  which satisfies

$$\| \bar{T}_0 \|_{H^1(\Omega)} \leq c \| T_0 \|_{H^{\frac{1}{2}}(\partial\Omega)} \quad \text{and} \quad \| \bar{T}_0 \|_{L^4(\Omega)} \leq \varepsilon \| T_0 \|_{H^{\frac{1}{2}}(\partial\Omega)}, \quad (2.7)$$

where the constant  $c$  is independent of  $\varepsilon$ .

2) Setting  $U = (\mathbf{u}, T)$  and  $V = (\mathbf{v}, S)$ , we define the mapping  $\Phi$  from  $\mathbb{V} \times H_0^1(\Omega)$  into its dual space by

$$\begin{aligned} \langle \Phi(U), V \rangle = & \int_{\Omega} \nu(T + \bar{T}_0)(\mathbf{grad} \mathbf{u})(\mathbf{x}) : (\mathbf{grad} \mathbf{v})(\mathbf{x}) d\mathbf{x} + \int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{u})(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) d\mathbf{x} \\ & + \alpha \int_{\Omega} \mathbf{grad} (T + \bar{T}_0)(\mathbf{x}) \cdot (\mathbf{grad} S)(\mathbf{x}) d\mathbf{x} \\ & + \int_{\Omega} ((\mathbf{u} \cdot \nabla)(T + \bar{T}_0))(\mathbf{x}) S(\mathbf{x}) d\mathbf{x} - \langle \mathbf{f}, \mathbf{v} \rangle_{\Omega} - \langle g, S \rangle_{\Omega}. \end{aligned}$$

It follows from (2.1) and the imbedding of  $H^1(\Omega)$  into  $L^6(\Omega)$  that  $\Phi$  is continuous on  $\mathbb{V} \times H_0^1(\Omega)$ . Moreover, it follows from (2.1), (2.5) and (2.7) and the antisymmetry property

$$\int_{\Omega} ((\mathbf{u} \cdot \nabla) \bar{T}_0)(\mathbf{x}) S(\mathbf{x}) d\mathbf{x} = - \int_{\Omega} ((\mathbf{u} \cdot \nabla) S)(\mathbf{x}) \bar{T}_0(\mathbf{x}) d\mathbf{x}, \quad (2.8)$$

that

$$\begin{aligned} \langle \Phi(U), U \rangle \geq & \nu_1 | \mathbf{u} |_{H^1(\Omega)^d}^2 + \alpha | T |_{H^1(\Omega)}^2 - \alpha | \bar{T}_0 |_{H^1(\Omega)} | T |_{H^1(\Omega)} \\ & - \frac{c\varepsilon}{2} \| T_0 \|_{H^{\frac{1}{2}}(\partial\Omega)} ( | \mathbf{u} |_{H^1(\Omega)^d}^2 + | T |_{H^1(\Omega)}^2 ) \\ & - \| \mathbf{f} \|_{H^{-1}(\Omega)^d} | \mathbf{u} |_{H^1(\Omega)^d} - \| g \|_{H^{-1}(\Omega)} | T |_{H^1(\Omega)}. \end{aligned}$$

We now take  $\varepsilon$  such that

$$c\varepsilon \| T_0 \|_{H^{\frac{1}{2}}(\partial\Omega)} \leq \min \{ \nu_1, \alpha \}.$$

Thus, we deduce from the previous inequality that

$$\begin{aligned} \langle \Phi(U), U \rangle \geq & \frac{\min\{\nu_1, \alpha\}}{2} (|\mathbf{u}|_{H^1(\Omega)^d}^2 + |T|_{H^1(\Omega)}^2) \\ & - \left( \alpha c \|T_0\|_{H^{\frac{1}{2}}(\partial\Omega)} + (\|\mathbf{f}\|_{H^{-1}(\Omega)^d}^2 + \|g\|_{H^{-1}(\Omega)}^2)^{\frac{1}{2}} \right) \\ & (|\mathbf{u}|_{H^1(\Omega)^d}^2 + |T|_{H^1(\Omega)}^2)^{\frac{1}{2}}. \end{aligned}$$

All this yields that  $\langle \Phi(U), U \rangle$  is nonnegative on the sphere of  $\mathbb{V} \times H_0^1(\Omega)$  with radius

$$\mu = \frac{2}{\min\{\nu_1, \alpha\}} \left( \alpha c \|T_0\|_{H^{\frac{1}{2}}(\partial\Omega)} + (\|\mathbf{f}\|_{H^{-1}(\Omega)^d}^2 + \|g\|_{H^{-1}(\Omega)}^2)^{\frac{1}{2}} \right). \quad (2.9)$$

3) We recall from [9, Chap. I, Cor 2.5] that  $\mathcal{D}(\Omega)^d \cap \mathbb{V}$  is dense in  $\mathbb{V}$ . Thus, there exist an increasing sequence  $(\mathbb{V}_n)_n$  of finite-dimensional subspaces of  $\mathbb{V}$  and an increasing sequence  $(\mathbb{W}_n)_n$  of finite-dimensional subspaces of  $H_0^1(\Omega)$  such that  $\bigcup_{n \in \mathbb{N}} (\mathbb{V}_n \times \mathbb{W}_n)$  is dense in  $\mathbb{V} \times H_0^1(\Omega)$ . Moreover, the properties of the function  $\Phi$  established above still hold with  $\mathbb{V} \times H_0^1(\Omega)$  replaced by  $\mathbb{V}_n \times \mathbb{W}_n$ . Thus, applying Brouwer's fixed-point theorem (see [9, Chap. IV, Cor. 1.1] for instance) yields that, for each  $n$ , there exists a  $U_n = (\mathbf{u}_n, T_n)$  satisfying

$$\begin{aligned} \forall V_n \in \mathbb{V}_n \times \mathbb{W}_n, \quad \langle \Phi(U_n), V_n \rangle &= 0 \\ \text{and} \quad (|\mathbf{u}_n|_{H^1(\Omega)^d}^2 + |T_n|_{H^1(\Omega)}^2)^{\frac{1}{2}} &\leq \mu. \end{aligned} \quad (2.10)$$

4) Since the norm of  $\mathbf{u}_n$  in  $H^1(\Omega)^d$  and of  $T_n$  in  $H^1(\Omega)$  are bounded by a constant  $c$  (due to the Poincaré–Friedrichs inequality on  $\Omega$ ) and owing to the compactness of the imbedding of  $H^1(\Omega)$  into  $L^4(\Omega)$ , there exists a subsequence, still denoted by  $(\mathbf{u}_n, T_n)_n$  for simplicity, which converges to a pair  $(\mathbf{u}, \tilde{T})$  of  $H_0^1(\Omega)^d \times H_0^1(\Omega)$  weakly in  $H^1(\Omega)^d \times H^1(\Omega)$  and strongly in  $L^4(\Omega)^d \times L^4(\Omega)$ . Next, we observe that, for  $m \leq n$ , these  $(\mathbf{u}_n, T_n)$  satisfy

$$\forall V_m \in \mathbb{V}_m \times \mathbb{W}_m, \quad \langle \Phi(U_n), V_m \rangle = 0.$$

Passing to the limit on  $n$  is obvious for the linear terms and follows from the strong convergence in  $L^4(\Omega)^d \times L^4(\Omega)$  for the terms  $(\mathbf{u}_n \cdot \nabla)\mathbf{u}_n$  and  $(\mathbf{u}_n \cdot \nabla)T_n$ . On the other hand, due to this strong convergence, the sequence  $(\nu(T_n + \bar{T}_0) \mathbf{grad} \mathbf{v}_m)_n$  converges to  $(\nu(\tilde{T} + \bar{T}_0) \mathbf{grad} \mathbf{v}_m)$  a.e. in  $\Omega$  and its norm is bounded by  $\nu_2 \|\mathbf{grad} \mathbf{v}_m\|_{L^2(\Omega)^{d \times d}}$ , so that using the Lebesgue dominated convergence theorem yields the convergence of  $(\nu(T_n + \bar{T}_0) \mathbf{grad} \mathbf{v}_m)_n$  to  $\nu(\tilde{T} + \bar{T}_0) \mathbf{grad} \mathbf{v}_m$  in  $L^2(\Omega)^{d \times d}$ . All this leads to

$$\forall V_m \in \mathbb{V}_m \times \mathbb{W}_m, \quad \langle \Phi(\mathbf{u}, \tilde{T}), V_m \rangle = 0,$$

and passing to the limit on  $m$  is now easy. Thus, we derive the pair  $(\mathbf{u}, T = \tilde{T} + \bar{T}_0)$  satisfies the second and third equation in (2.4) and also

$$\begin{aligned} \forall \mathbf{v} \in \mathbb{V}, \quad \int_{\Omega} \nu(T) (\mathbf{grad} \mathbf{u})(\mathbf{x}) : (\mathbf{grad} \mathbf{v})(\mathbf{x}) \, d\mathbf{x} \\ + \int_{\Omega} ((\mathbf{u} \cdot \nabla)\mathbf{u})(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \, d\mathbf{x} = \langle \mathbf{f}, \mathbf{v} \rangle_{\Omega}. \end{aligned} \quad (2.11)$$

5) We recall from [9, Chap. I, Cor. 2.4] the following inf-sup condition for a positive constant  $\beta$

$$\forall q \in L_0^2(\Omega), \quad \sup_{\mathbf{v} \in H_0^1(\Omega)^d} \frac{\int_{\Omega} (\operatorname{div} \mathbf{v})(\mathbf{x}) q(\mathbf{x}) d\mathbf{x}}{\|\mathbf{v}\|_{H^1(\Omega)^d}} \geq \beta \|q\|_{L^2(\Omega)}. \quad (2.12)$$

Thus, owing to equation (2.11), there exists a  $p$  in  $L_0^2(\Omega)$ , see [9, Chap. I, Lemma 4.1] such that

$$\begin{aligned} \forall \mathbf{v} \in H_0^1(\Omega)^d, \quad & \int_{\Omega} \nu(T)(\mathbf{grad} \mathbf{u})(\mathbf{x}) : (\mathbf{grad} \mathbf{v})(\mathbf{x}) d\mathbf{x} \\ & + \int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{u})(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) d\mathbf{x} - \langle \mathbf{f}, \mathbf{v} \rangle_{\Omega} = \int_{\Omega} (\operatorname{div} \mathbf{v})(\mathbf{x}) p(\mathbf{x}) d\mathbf{x}. \end{aligned}$$

Then the triple  $(\mathbf{u}, p, T)$  is a solution of problem (2.3) – (2.4), and estimate (2.6) is easily derived from (2.7) and (2.10), see (2.9).

**Proposition 2.3** *Assume that the function  $\nu$  is Lipschitz-continuous, with Lipschitz constant  $\nu^*$ . There exist two positive constants  $c_{\sharp}$  and  $c_{\flat}$  such that*

(i) *if the data  $(\mathbf{f}, g)$  in  $H^{-1}(\Omega)^d \times H^{-1}(\Omega)$  and  $T_0$  in  $H^{\frac{1}{2}}(\partial\Omega)$  satisfy*

$$c_{\sharp} (\|\mathbf{f}\|_{H^{-1}(\Omega)^d} + \|g\|_{H^{-1}(\Omega)} + \|T_0\|_{H^{\frac{1}{2}}(\partial\Omega)}) < 1, \quad (2.13)$$

(ii) *if problem (2.3) – (2.4) admits a solution  $(\mathbf{u}, p, T)$  such that  $\mathbf{u}$  belongs to  $W^{1,q}(\Omega)^d$  with  $q > 2$  in dimension  $d = 2$  and  $q \geq 3$  in dimension  $d = 3$ , and satisfies*

$$c_{\flat} \nu^* |\mathbf{u}|_{W^{1,q}(\Omega)^d} < 1, \quad (2.14)$$

*then this solution is unique.*

**Proof:** For brevity, we set:

$$c_1 = c (\|\mathbf{f}\|_{H^{-1}(\Omega)^d} + \|g\|_{H^{-1}(\Omega)} + \|T_0\|_{H^{\frac{1}{2}}(\partial\Omega)}).$$

where  $c$  is the constant in (2.6). Let  $(\mathbf{u}_1, p_1, T_1)$  and  $(\mathbf{u}_2, p_2, T_2)$  be two solutions of problem (2.3) – (2.4), with  $\mathbf{u}_1$  in  $W^{1,q}(\Omega)^d$  satisfying (2.14).

Setting for a while  $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2, p = p_1 - p_2$  and  $T = T_1 - T_2$ , we proceed in three steps.

1) It follows from the third equation in (2.4) that, since  $T$  belongs to  $H_0^1(\Omega)$ ,

$$\begin{aligned} \alpha |T|_{H^1(\Omega)}^2 &= - \int_{\Omega} ((\mathbf{u}_1 \cdot \nabla) T_1 - (\mathbf{u}_2 \cdot \nabla) T_2)(\mathbf{x}) T(\mathbf{x}) d\mathbf{x} \\ &= - \int_{\Omega} ((\mathbf{u} \cdot \nabla) T_1)(\mathbf{x}) T(\mathbf{x}) d\mathbf{x} \end{aligned}$$

whence

$$\alpha |T|_{H^1(\Omega)} \leq c_1 c_2 |\mathbf{u}|_{H^1(\Omega)^d}, \quad (2.15)$$

where  $c_2$  is the square of the norm of the imbedding of  $H_0^1(\Omega)$  into  $L^4(\Omega)$ .

2) Similarly, we derive from the first equation in (2.4) that

$$\begin{aligned} \int_{\Omega} \nu(T_2)(\mathbf{x}) |\mathbf{grad} \mathbf{u}|^2(\mathbf{x}) \, d\mathbf{x} &= - \int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{u}_1)(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \, d\mathbf{x} \\ &\quad - \int_{\Omega} (\nu(T_1) - \nu(T_2))(\mathbf{x}) (\mathbf{grad} \mathbf{u}_1)(\mathbf{x}) : (\mathbf{grad} \mathbf{u})(\mathbf{x}) \, d\mathbf{x}. \end{aligned}$$

Using appropriate Hölder's inequalities thus yields

$$\nu_1 |\mathbf{u}|_{H^1(\Omega)^d}^2 \leq c_1 c_2 |\mathbf{u}|_{H^1(\Omega)^d}^2 + \nu^* c_3 |\mathbf{u}_1|_{W^{1,q}(\Omega)^d} |T|_{H^1(\Omega)} |\mathbf{u}|_{H^1(\Omega)^d},$$

where  $c_3$  stands for the norm of the imbedding of  $H_0^1(\Omega)$  into  $L^{q^*}(\Omega)$ , with  $\frac{1}{q} + \frac{1}{q^*} = \frac{1}{2}$ .

By combining this with (2.15) and choosing  $c_{\sharp}$  and  $c_{\flat}$  such that

$$c_1 c_2 \nu_1^{-1} (1 + \nu^* c_3 \alpha^{-1} |\mathbf{u}_1|_{W^{1,q}(\Omega)^d}) < 1,$$

we obtain that  $\mathbf{u}$  is zero, so that  $\mathbf{u}_1$  and  $\mathbf{u}_2$  are equal.

3) It then follows from (2.15) that  $T_1$  and  $T_2$  are equal. Finally, the function  $p$  satisfies

$$\forall \mathbf{v} \in H_0^1(\Omega)^d, \quad - \int_{\Omega} (\operatorname{div} \mathbf{v})(\mathbf{x}) p(\mathbf{x}) \, d\mathbf{x} = 0,$$

so that it is zero (see [9, Chap. I, §2] for instance). Thus,  $p_1$  and  $p_2$  coincide.

This concludes the proof.

Assumptions (2.13) and (2.14) are clearly very restrictive and will not be used in what follows. We conclude with a regularity result.

**Proposition 2.4** *There exist a real number  $q_0 > 2$  only depending on the geometry of  $\Omega$  and on the ratio  $\nu_2/\nu_1$  and a real number  $q_1 > 1$  only depending on the geometry of  $\Omega$  such that, for any  $q$ ,  $2 \leq q \leq q_0$ , and  $q'$ ,  $1 \leq q' \leq q_1$ , and for any data  $(\mathbf{f}, g)$  in the space  $W^{-1,q}(\Omega)^d \times L^{q'}(\Omega)$  and  $T_0$  in  $W^{2-\frac{1}{q'}, q'}(\partial\Omega)$ , any solution  $(\mathbf{u}, p, T)$  of problem (2.3) – (2.4) belongs to  $W^{1,q}(\Omega)^d \times L^q(\Omega) \times W^{2,q'}(\Omega)$ . Moreover,  $q_1$  is  $\geq \frac{4}{3}$  for a general domain  $\Omega$  and  $\geq 2$  when  $\Omega$  is convex.*

**Proof:** Proving the regularity of the velocity follows from the approach in [12]. The regularity of the pressure is a direct consequence of this. Finally, the regularity of the temperature is deduced from the standard properties of the Laplace operator (see [10, Thm 4.3.2.4], [7, Thm. 2] or [8, Cor. 3.10]).

### 3 The discrete problem and its a priori analysis.

We are now interested in the discretization of problem (2.3) – (2.4) in the case where  $\Omega = ]-1, 1[^d$ ,  $d = 2$  or  $3$ . Let  $N$  be an integer  $\geq 2$ , we introduce the space  $\mathbb{P}_N(\Omega)$  of polynomials with  $d$  variables and degree  $\leq N$  with respect to each variable and the space  $\mathbb{P}_N^0(\Omega)$  of polynomials in  $\mathbb{P}_N(\Omega)$  vanishing on the boundary of  $\Omega$ . Relying on these definitions, we introduce the discrete spaces

$$\begin{aligned} \mathbb{X}_N &= \mathbb{P}_N^0(\Omega)^d, \quad \mathbb{M}_N = \mathbb{P}_{N-2}(\Omega) \cap L_0^2(\Omega), \\ \mathbb{Y}_N &= \mathbb{P}_N(\Omega), \quad \mathbb{Y}_N^0 = \mathbb{Y}_N \cap H_0^1(\Omega). \end{aligned}$$

The reason for the choice of the space  $\mathbb{M}_N$  is that it does not contain spurious modes, see [2, Chap V].

We introduce the space  $\mathbb{P}_N(-1, 1)$  of restrictions to  $[-1, 1]$  of polynomials with degree  $\leq N$ . Setting  $\xi_0 = -1$  and  $\xi_N = 1$ , we consider the  $N - 1$  nodes  $\xi_j$ ,  $1 \leq j \leq N - 1$ , and the  $N + 1$  weights  $\rho_j$ ,  $0 \leq j \leq N$ , of the Gauss-Lobatto quadrature formula. We recall that the following equality holds

$$\forall \phi \in \mathbb{P}_{2N-1}(-1, 1), \quad \int_{-1}^1 \phi(\zeta) d\zeta = \sum_{i=0}^N \phi(\xi_i) \rho_i. \quad (3.1)$$

We also recall [3, Chap IV, Cor. 1.10] the following property, which is useful in what follows

$$\forall \phi_N \in \mathbb{P}_N(-1, 1), \quad \|\phi_N\|_{L^2(-1,1)}^2 \leq \sum_{i=0}^N \phi_N^2(\xi_i) \rho_i \leq 3 \|\phi_N\|_{L^2(-1,1)}^2. \quad (3.2)$$

Relying on this formula, we introduce the discrete product, defined on continuous functions  $u$  and  $v$  by

$$(u, v)_N = \begin{cases} \sum_{i=0}^N \sum_{j=0}^N u(\xi_i, \xi_j) v(\xi_i, \xi_j) \rho_i \rho_j & \text{if } d = 2, \\ \sum_{i=0}^N \sum_{j=0}^N \sum_{k=0}^N u(\xi_i, \xi_j, \xi_k) v(\xi_i, \xi_j, \xi_k) \rho_i \rho_j \rho_k & \text{if } d = 3. \end{cases}$$

It follows from (3.2) that it is a scalar product on  $\mathbb{P}_N(\Omega)$ . Let  $\mathcal{I}_N$  denote the Lagrange interpolation operator at the nodes of the grid

$$\Sigma_N = \begin{cases} \{\mathbf{x} = (\xi_i, \xi_j); \quad 0 \leq i, j \leq N\} & \text{if } d = 2, \\ \{\mathbf{x} = (\xi_i, \xi_j, \xi_k); \quad 0 \leq i, j, k \leq N\} & \text{if } d = 3. \end{cases}$$

with values in the space  $\mathbb{P}_N(\Omega)$ . Finally, let  $i_N^{\partial\Omega}$  stand for the Lagrange interpolation operator at the nodes of  $\Sigma_N \cap \partial\Omega$  with values in the space of traces of  $\mathbb{P}_N(\Omega)$ .

We now assume that the function  $T_0$  is continuous on  $\partial\Omega$  and  $\mathbf{f}$ ,  $g$  are continuous on  $\bar{\Omega}$ . Thus the discrete problem is constructed from (2.3) – (2.4) by using the Galerkin method combined with numerical integration. It reads

Find  $(\mathbf{u}_N, p_N, T_N)$  in  $\mathbb{X}_N \times \mathbb{M}_N \times \mathbb{Y}_N$  such that

$$T_N = i_N^{\partial\Omega} T_0 \quad \text{on } \partial\Omega, \quad (3.3)$$

and that

$$\begin{aligned}
\forall \mathbf{v}_N \in \mathbb{X}_N, \quad & (\nu(T_N) \mathbf{grad} \mathbf{u}_N, \mathbf{grad} \mathbf{v}_N)_N + ((\mathbf{u}_N \cdot \nabla) \mathbf{u}_N, \mathbf{v}_N)_N \\
& - (\operatorname{div} \mathbf{v}_N, p_N)_N = (\mathbf{f}, \mathbf{v}_N)_N, \\
\forall q_N \in \mathbb{M}_N, \quad & -(\operatorname{div} \mathbf{u}_N, q_N)_N = 0, \\
\forall S_N \in \mathbb{Y}_N^0, \quad & \alpha(\mathbf{grad} T_N, \mathbf{grad} S_N)_N + ((\mathbf{u}_N \cdot \nabla) T_N, S_N)_N = (g, S_N)_N.
\end{aligned} \tag{3.4}$$

The existence of a solution can be derived by the same arguments in Section 2, however we prefer to follow the approach of [5] to obtain directly more precise results.

We recall the existence of a discrete inf-sup condition between the spaces  $\mathbb{X}_N$  and  $\mathbb{M}_N$ , see [2, Chap. V, Thm 25.7]

$$\forall q_N \in \mathbb{M}_N, \quad \sup_{\mathbf{v}_N \in \mathbb{X}_N} \frac{\int_{\Omega} (\operatorname{div} \mathbf{v}_N)(\mathbf{x}) q_N(\mathbf{x}) \, d\mathbf{x}}{\|\mathbf{v}_N\|_{H^1(\Omega)^d}} \geq c N^{-(d-1)/2} \|q_N\|_{L^2(\Omega)}. \tag{3.5}$$

For any real-valued measurable function  $\tau$  on  $\Omega$ , we introduce the modified Stokes operator  $\mathcal{S}(\tau)$ , which associates with any datum  $\mathbf{F}$  in  $H^{-1}(\Omega)^d$  the part  $\mathbf{u}$  of the solution  $(\mathbf{u}, p)$  of the generalized Stokes problem

$$\begin{cases} -\operatorname{div}(\nu(\tau) \nabla \mathbf{u}) + \mathbf{grad} p = \mathbf{F} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

We also consider the operator  $\tilde{\mathcal{S}}(\tau)$  which associates with any datum  $\mathbf{F}$  in  $H^{-1}(\Omega)^d$  the part  $p$  of the solution  $(\mathbf{u}, p)$  of this same problem.

We introduce the inverse  $\mathcal{L}$  of the Laplace operator which associates with any datum  $(g, T_0)$  in  $H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$  the solution  $T$  in  $H^1(\Omega)$  of the problem

$$\begin{cases} -\alpha \Delta T = g & \text{in } \Omega, \\ T = T_0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Thus it is readily checked that, when setting  $U = (\mathbf{u}, T)$ , problem (2.3)–(2.4) can be written equivalently as

$$\begin{aligned}
\mathcal{F}(U) = U + \begin{pmatrix} \mathcal{S}(T) & 0 \\ 0 & \mathcal{L} \end{pmatrix} \mathcal{G}(U) = 0, \\
\text{with } \mathcal{G}(U) = \begin{pmatrix} (\mathbf{u} \cdot \nabla) \mathbf{u} - \mathbf{f} \\ ((\mathbf{u} \cdot \nabla) T - g, T_0) \end{pmatrix}.
\end{aligned} \tag{3.8}$$

Similarly, let  $\mathcal{S}_N(\tau)$  denote the discrete Stokes operator, i.e., the operator which associates with any data  $\mathbf{F}$  in  $H^{-1}(\Omega)^d$ , the part  $\mathbf{u}_N$  of the solution  $(\mathbf{u}_N, p_N)$  in  $\mathbb{X}_N \times \mathbb{M}_N$  of the Stokes

problem

$$\begin{aligned} \forall \mathbf{v}_N \in \mathbb{X}_N, \quad & (\nu(\tau) \nabla \mathbf{u}_N, \nabla \mathbf{v}_N)_N - (\operatorname{div} \mathbf{v}_N, p_N)_N = \langle \mathbf{F}, \mathbf{v}_N \rangle_\Omega, \\ \forall q_N \in \mathbb{M}_N, \quad & -(\operatorname{div} \mathbf{u}_N, q_N)_N = 0. \end{aligned} \quad (3.9)$$

Let finally  $\mathcal{L}_N$  denote the operator which associates with any datum  $G$  in  $H^{-1}(\Omega)$  and any continuous function  $R_0$  in  $H^{\frac{1}{2}}(\partial\Omega)$ , the function  $R_N$  in  $\mathbb{Y}_N$ , equal to  $i_N^{\partial\Omega} R_0$  on  $\partial\Omega$  and which satisfies

$$\forall S_N \in \mathbb{Y}_N^0, \quad \alpha (\mathbf{grad} R_N, \mathbf{grad} S_N)_N = \langle G, S_N \rangle_\Omega \quad (3.10)$$

With the notation  $U_N = (\mathbf{u}_N, T_N)$ , problem (3.3) – (3.4) can equivalently be written as

$$\begin{aligned} \mathcal{F}_N(U_N) = U_N + \begin{pmatrix} \mathcal{S}_N(T_N) & 0 \\ 0 & \mathcal{L}_N \end{pmatrix} \mathcal{G}_N(U_N) = 0, \\ \text{with } \mathcal{G}_N(U_N) = \begin{pmatrix} \mathcal{G}_{N1} \\ (\mathcal{G}_{N2}, T_0) \end{pmatrix}. \end{aligned} \quad (3.11)$$

The two components  $\mathcal{G}_{N1}$  and  $\mathcal{G}_{N2}$  are defined in the dual spaces of  $\mathbb{X}_N$  and  $\mathbb{Y}_N^0$ , respectively, by

$$\begin{aligned} \forall \mathbf{v}_N \in \mathbb{X}_N, \quad & \int_\Omega \mathcal{G}_{N1}(\mathbf{x}) \cdot \mathbf{v}_N(\mathbf{x}) d\mathbf{x} = ((\mathbf{u}_N \cdot \nabla) \mathbf{u}_N - \mathbf{f}, \mathbf{v}_N)_N \\ \forall S_N \in \mathbb{Y}_N^0, \quad & \int_\Omega \mathcal{G}_{N2}(\mathbf{x}) S_N(\mathbf{x}) d\mathbf{x} = ((\mathbf{u}_N \cdot \nabla) T_N - g, S_N)_N. \end{aligned}$$

**Lemma 3.1** *There exists a constant  $c > 0$  such that the following continuity property holds*

$$\langle \mathcal{G}_{N1}(\mathbf{u}_N), \mathbf{v}_N \rangle \leq c (\| \mathbf{u}_N \|_{H^1(\Omega)^d} + \| \mathcal{I}_N \mathbf{f} \|_{L^2(\Omega)^d}) \| \mathbf{v}_N \|_{H^1(\Omega)^d} \quad (3.12)$$

**Proof:** By definition, we have

$$\begin{aligned} \langle \mathcal{G}_{N1}(\mathbf{u}_N), \mathbf{v}_N \rangle &= \sum_{i,j=1}^d (\mathbf{u}_{Nj} \frac{\partial \mathbf{u}_{Ni}}{\partial x_j} - \mathbf{f}, \mathbf{v}_{Ni})_N \\ &= \sum_{i,j=1}^d \left( (\mathbf{u}_{Nj} \mathbf{v}_{Ni}, \frac{\partial \mathbf{u}_{Ni}}{\partial x_j})_N - (\mathbf{f}, \mathbf{v}_{Ni})_N \right) \\ &= \sum_{i,j=1}^d \left( (\mathcal{I}_N(\mathbf{u}_{Nj} \mathbf{v}_{Ni}), \frac{\partial \mathbf{u}_{Ni}}{\partial x_j})_N - (\mathcal{I}_N \mathbf{f}, \mathbf{v}_{Ni})_N \right) \end{aligned}$$

whence, by using (3.2), we obtain

$$\begin{aligned} \langle \mathcal{G}_{N1}(\mathbf{u}_N), \mathbf{v}_N \rangle &\leq 3^d \sum_{i,j=1}^d \left( \| \mathcal{I}_N(\mathbf{u}_{Nj} \mathbf{v}_{Ni}) \|_{L^2(\Omega)^d} \| \frac{\partial \mathbf{u}_{Ni}}{\partial x_j} \|_{L^2(\Omega)^d} \right. \\ &\quad \left. + \| \mathcal{I}_N \mathbf{f} \|_{L^2(\Omega)^d} \| \mathbf{v}_{Ni} \|_{L^2(\Omega)^d} \right). \end{aligned}$$

We recall from [2, Rem. 13.5] that,

$$\forall \varphi_M \in \mathbb{P}_M(\Omega), \quad \|\mathcal{I}_N \varphi_M\|_{L^2(\Omega)} \leq c \left(1 + \frac{M}{N}\right)^d \|\varphi_M\|_{L^2(\Omega)}. \quad (3.13)$$

By taking  $M = 2N$ , we derive

$$\begin{aligned} \langle \mathcal{G}_{N1}(\mathbf{u}_N), \mathbf{v}_N \rangle &\leq c \sum_{i,j=1}^d \left( \|\mathbf{u}_{Nj}\|_{L^4(\Omega)^d} \|\mathbf{v}_{Ni}\|_{L^4(\Omega)^d} \|\mathbf{u}_{Ni}\|_{H^1(\Omega)^d} \right. \\ &\quad \left. + \|\mathcal{I}_N \mathbf{f}\|_{L^2(\Omega)^d} \|\mathbf{v}_{Ni}\|_{L^2(\Omega)^d} \right) \end{aligned}$$

We conclude by the Hölder's inequality and by noting that  $H^1(\Omega)$  is embedded in  $L^4(\Omega)$ .

We recall the basic properties of the discrete operators  $\mathcal{S}_N(\tau)$  and  $\mathcal{L}_N$ . The operator  $\mathcal{S}_N(\tau)$  satisfies the following properties: For any  $\mathbf{F}$  in  $H^{-1}(\Omega)^d$ ,

$$\|\mathcal{S}_N(\tau) \mathbf{F}\|_{H^1(\Omega)^d} \leq c \|\mathbf{F}\|_{H^{-1}(\Omega)^d}, \quad (3.14)$$

and, if moreover  $\tilde{\mathcal{S}}(\tau) \mathbf{F}$  belongs to  $H^{s-1}(\Omega)$  and  $\mathcal{S}(\tau) \mathbf{F}$  to  $H^s(\Omega)^d$  for a real number  $s$ ,  $s \geq 1$ ,

$$\|(\mathcal{S}(\tau) - \mathcal{S}_N(\tau)) \mathbf{F}\|_{H^1(\Omega)^d} \leq c N^{1-s} \left( \|\mathcal{S}(\tau) \mathbf{F}\|_{H^s(\Omega)^d} + \|\tilde{\mathcal{S}}(\tau) \mathbf{F}\|_{H^{s-1}(\Omega)} \right). \quad (3.15)$$

The analogous properties concerning the operator  $\mathcal{L}_N$  read: For any  $G$  in  $H^{-1}(\Omega)$ ,

$$\|\mathcal{L}_N(G, 0)\|_{H^1(\Omega)} \leq c \|G\|_{H^{-1}(\Omega)}. \quad (3.16)$$

and, if moreover  $\mathcal{L}G$  belongs to  $H^s(\Omega)^d$ ,  $s \geq 1$  and  $R_0$  belongs to  $H^\sigma(\partial\Omega)$ , for a real number  $\sigma$ ,  $\sigma > \frac{d-1}{2}$ ,

$$\|(\mathcal{L} - \mathcal{L}_N)(G, R_0)\|_{H^1(\Omega)} \leq c N^{1-s} \|\mathcal{L}G\|_{H^s(\Omega)} + N^{\frac{1}{2}-\sigma} \|R_0\|_{H^\sigma(\partial\Omega)}. \quad (3.17)$$

Note that these properties yield the following convergence result, for any  $\mathbf{F}$  in  $H^{-1}(\Omega)^d$  and any  $G$  in  $H^{-1}(\Omega)$ ,

$$\lim_{N \rightarrow +\infty} \|(\mathcal{S}(\tau) - \mathcal{S}_N(\tau)) \mathbf{F}\|_{H^1(\Omega)^d} = 0, \quad \lim_{N \rightarrow +\infty} \|(\mathcal{L} - \mathcal{L}_N)(G, 0)\|_{H^1(\Omega)} = 0. \quad (3.18)$$

From now on, we denote by

$$\mathcal{X}(\Omega) = H_0^1(\Omega)^d \times H^1(\Omega), \quad \mathcal{X}_N = \mathbb{X}_N \times \mathbb{Y}_N.$$

### Assmption 3.2

The solution  $(\mathbf{u}, p, T)$  of problem (2.3)–(2.4) is such that  $D\mathcal{F}(U)$  is an isomorphism of  $\mathcal{X}(\Omega)$ .

Note that these assumptions are not restrictive, compared with the hypotheses of Proposition 2.3 for these uniqueness of the solution.

We are thus in a position to prove the preliminary results which we need for applying the theorem of Brezzi, Rappaz and Raviart [5]. This requires an approximation  $U_N^\diamond = (u_N^\diamond, T_N^\diamond)$  of  $U$  in  $\mathbb{X}_N \times \mathbb{Y}_N$  which satisfies (see [3, Chap. III, Th. 2.4]) for  $0 \leq t \leq 1$ ,

$$\begin{aligned} \|u - u_N^\diamond\|_{H^t(\Omega)^d} &\leq c N^{t-1} \|u\|_{H^1(\Omega)^d}, \\ \|T - T_N^\diamond\|_{H^t(\Omega)} &\leq c N^{t-1} \|T\|_{H^1(\Omega)^d}. \end{aligned} \quad (3.19)$$

**Lemma 3.3** *If the data  $\mathbf{f}$  belong to  $H^\sigma(\Omega)^d$ ,  $\sigma > \frac{d}{2}$ , the following result holds*

$$\begin{aligned} \langle \mathcal{G}_1(\mathbf{u}_N) - \mathcal{G}_{N1}(\mathbf{u}_N), \mathbf{v}_N \rangle_\Omega &\leq c \left( N^{-\frac{1}{2}} \|\mathbf{u}_N\|_{H^1(\Omega)^d} \right. \\ &\quad \left. + N^{-\sigma} \|\mathbf{f}\|_{H^\sigma(\Omega)^d} \right) \|\mathbf{v}_N\|_{H^1(\Omega)^d} \end{aligned} \quad (3.20)$$

with

$$\langle \mathcal{G}_1(\mathbf{u}_N), \mathbf{v}_N \rangle_\Omega = \int_\Omega ((\mathbf{u}_N \cdot \nabla) \mathbf{u}_N)(\mathbf{x}) \cdot \mathbf{v}_N(\mathbf{x}) \, d\mathbf{x}$$

**Proof:** Denoting for brevity the scalar product of  $L^2(\Omega)$  by  $(\cdot, \cdot)$ , we have,

$$\langle \mathcal{G}_1(\mathbf{u}_N) - \mathcal{G}_{N1}(\mathbf{u}_N), \mathbf{v}_N \rangle_\Omega = ((\mathbf{u}_N \cdot \nabla) \mathbf{u}_N - \mathbf{f}, \mathbf{v}_N) - ((\mathbf{u}_N \cdot \nabla) \mathbf{u}_N - \mathbf{f}, \mathbf{v}_N)_N$$

If  $N'$  stands for the integer part of  $\frac{N-1}{2}$ , we introduce an approximation  $\mathbf{u}_{N'}$  of  $\mathbf{u}_N$  in  $\mathbb{P}_{N'}(\Omega)$  and we note the identity

$$((\mathbf{u}_{N'} \cdot \nabla) \mathbf{u}_{N'}, \mathbf{v}_N) = ((\mathbf{u}_{N'} \cdot \nabla) \mathbf{u}_{N'}, \mathbf{v}_N)_N$$

Inserting it, we obtain

$$\begin{aligned} \langle \mathcal{G}_1(\mathbf{u}_N) - \mathcal{G}_{N1}(\mathbf{u}_N), \mathbf{v}_N \rangle_\Omega &= ((\mathbf{u}_N \cdot \nabla) \mathbf{u}_N - (\mathbf{u}_{N'} \cdot \nabla) \mathbf{u}_{N'}, \mathbf{v}_N) \\ &\quad + ((\mathbf{u}_{N'} \cdot \nabla) \mathbf{u}_{N'} - (\mathbf{u}_N \cdot \nabla) \mathbf{u}_N, \mathbf{v}_N)_N \\ &\quad - ((\mathbf{f}, \mathbf{v}_N) - (\mathcal{I}_N \mathbf{f}, \mathbf{v}_N)_N) \end{aligned}$$

The arguments for evaluating the first two quantities are the same, so we only consider the first one. We have

$$((\mathbf{u}_N \cdot \nabla) \mathbf{u}_N - (\mathbf{u}_{N'} \cdot \nabla) \mathbf{u}_{N'}, \mathbf{v}_N) = ((\mathbf{u}_N - \mathbf{u}_{N'} \cdot \nabla) \mathbf{u}_N, \mathbf{v}_N) + ((\mathbf{u}_{N'} \cdot \nabla) (\mathbf{u}_N - \mathbf{u}_{N'}), \mathbf{v}_N)$$

Writing

$$((\mathbf{u}_N - \mathbf{u}_{N'} \cdot \nabla) \mathbf{u}_N, \mathbf{v}_N) = \int_\Omega \sum_{i,j=1}^d (\mathbf{u}_{N'i} - \mathbf{u}_{N'i})(\mathbf{x}) \mathbf{v}_{Nj}(\mathbf{x}) \nabla \mathbf{u}_{N'i}(\mathbf{x}) \, d\mathbf{x},$$

we obtain

$$\begin{aligned} ((\mathbf{u}_N - \mathbf{u}_{N'} \cdot \nabla) \mathbf{u}_N, \mathbf{v}_N) &\leq \|\mathbf{u}_N - \mathbf{u}_{N'}\|_{L^3(\Omega)^d} \|\mathbf{v}_N\|_{L^6(\Omega)^d} \|\nabla \mathbf{u}_N\|_{L^2(\Omega)^d} \\ &\leq \|\mathbf{u}_N - \mathbf{u}_{N'}\|_{H^{\frac{1}{2}}(\Omega)^d} \|\mathbf{v}_N\|_{H^1(\Omega)^d} \|\mathbf{u}_N\|_{H^1(\Omega)^d} \end{aligned}$$

and we use similar arguments for the second part. We conclude thanks to (3.19) for  $t = \frac{1}{2}$ . To evaluating the third term, we have for any  $\mathbf{f}_{N-1}$  in  $\mathbb{P}_{N-1}(\Omega)^d$

$$\begin{aligned} (\mathbf{f}, \mathbf{v}_N) - (\mathbf{f}, \mathbf{v}_N)_N &= \int_{\Omega} \mathbf{f}(\mathbf{x}) \cdot \mathbf{v}_N(\mathbf{x}) d\mathbf{x} - (\mathbf{f}, \mathbf{v}_N)_N \\ &= \int_{\Omega} (\mathbf{f} - \mathbf{f}_{N-1})(\mathbf{x}) \cdot \mathbf{v}_N(\mathbf{x}) d\mathbf{x} - (\mathbf{f} - \mathbf{f}_{N-1}, \mathbf{v}_N)_N, \end{aligned}$$

whence

$$(\mathbf{f}, \mathbf{v}_N) - (\mathbf{f}, \mathbf{v}_N)_N \leq (\|\mathbf{f} - \mathbf{f}_{N-1}\|_{L^2(\Omega)^d} + 3^d \|\mathcal{I}_N \mathbf{f} - \mathbf{f}_{N-1}\|_{L^2(\Omega)^d}) \|\mathbf{v}_N\|_{L^2(\Omega)^d}$$

$$(\mathbf{f}, \mathbf{v}_N) - (\mathbf{f}, \mathbf{v}_N)_N \leq c(\|\mathbf{f} - \mathcal{I}_N \mathbf{f}\|_{L^2(\Omega)^d} + \inf_{\mathbf{f}_{N-1} \in \mathbb{P}_{N-1}(\Omega)} \|\mathbf{f} - \mathbf{f}_{N-1}\|_{L^2(\Omega)^d}) \|\mathbf{v}_N\|_{H^1(\Omega)^d}$$

By taking  $\mathbf{f}_{N-1}$  equal to the image of  $\mathbf{f}$  by the  $L^2(\Omega)$  orthogonal projection operator (see [3, Chap. III]), using [3, Chap. IV, Thm 2.6] and [3, Chap. III, Thm 2.4], we obtain

$$(\mathbf{f}, \mathbf{v}_N) - (\mathbf{f}, \mathbf{v}_N)_N \leq cN^{-\sigma} \|\mathbf{f}\|_{H^\sigma(\Omega)^d} \|\mathbf{v}_N\|_{H^1(\Omega)^d}.$$

We are now in a position to prove the following lemmas, we denote by  $\mathcal{E}$  the space of endomorphisms of  $\mathcal{X}(\Omega)$ . Here,  $D$  stand for the differential operator.

**Lemma 3.4** *Assume that  $\nu$  is of class  $\mathcal{C}^2$  on  $\mathbb{R}$ , with bounded derivatives, and Assumption 3.2 holds. There exists a positive integer  $N_0$  such that, for all then  $N \geq N_0$ , the operator  $D\mathcal{F}_N(U_N^\diamond)$  is an isomorphism of  $\mathbb{X}_N \times \mathbb{Y}_N$  with the norm of its inverse bounded independently of  $N$ .*

**Proof:** We write the expansion

$$\begin{aligned} D\mathcal{F}_N(U_N^\diamond) &= D\mathcal{F}(U) - \begin{pmatrix} (\mathcal{S} - \mathcal{S}_N)(T) & 0 \\ 0 & \mathcal{L} - \mathcal{L}_N \end{pmatrix} D\mathcal{G}(U) - \begin{pmatrix} \mathcal{S}_N & 0 \\ 0 & \mathcal{L}_N \end{pmatrix} (D\mathcal{G}(U) - D\mathcal{G}(U_N^\diamond)) \\ &\quad - \begin{pmatrix} \mathcal{S}_N & 0 \\ 0 & \mathcal{L}_N \end{pmatrix} (D\mathcal{G}(U_N^\diamond) - D\mathcal{G}_N(U_N^\diamond)) - \begin{pmatrix} D(\mathcal{S} - \mathcal{S}_N)(T) & 0 \\ 0 & 0 \end{pmatrix} \mathcal{G}(U) \\ &\quad - \begin{pmatrix} D\mathcal{S}_N(T) & 0 \\ 0 & 0 \end{pmatrix} (\mathcal{G}(U) - \mathcal{G}(U_N^\diamond)) - \begin{pmatrix} D\mathcal{S}_N(T) & 0 \\ 0 & 0 \end{pmatrix} (\mathcal{G}(U_N^\diamond) - \mathcal{G}_N(U_N^\diamond)). \end{aligned}$$

Due to part (ii) of Assumption 3.2, it suffices to check that the last six terms in the right-hand side tend to zero when  $N$  tends to  $+\infty$  in the norm of the space  $\mathcal{E}$ . Let  $W_N = (\mathbf{w}_N, R_N)$  be any element in the unit sphere of  $\mathcal{X}_N$ .

1) We observe that

$$D\mathcal{G}(U) \cdot W_N = \begin{pmatrix} (\mathbf{u} \cdot \nabla) \mathbf{w}_N + (\mathbf{w}_N \cdot \nabla) \mathbf{u} \\ ((\mathbf{u} \cdot \nabla) R_N + (\mathbf{w}_N \cdot \nabla) T, 0) \end{pmatrix}.$$

Thus, the compactness of the imbedding of  $H^1(\Omega)$  into  $L^q(\Omega)$ , with  $q < \infty$  in dimension  $d = 2$  and  $q < 6$  in dimension  $d = 3$ , combined with the regularity of  $\mathbf{u}$  yields, that both terms  $(\mathbf{u} \cdot \nabla)\mathbf{w}_N + (\mathbf{w}_N \cdot \nabla)\mathbf{u}$  and  $(\mathbf{u} \cdot \nabla)R_N + (\mathbf{w}_N \cdot \nabla)T$  belongs to a compact subset of  $H^{-1}(\Omega)^d$  and  $H^{-1}(\Omega)$  respectively. Combining all this with (3.18), leads to

$$\lim_{N \rightarrow +\infty} \left\| \begin{pmatrix} (\mathcal{S} - \mathcal{S}_N)(T) & 0 \\ 0 & \mathcal{L} - \mathcal{L}_N \end{pmatrix} D\mathcal{G}(U) \right\|_{\mathcal{E}} = 0. \quad (3.21)$$

2) Due to the definition of  $D\mathcal{G}$ , we must now investigate the convergence of the two terms

$$((\mathbf{u} - \mathbf{u}_N^\diamond) \cdot \nabla)\mathbf{w}_N + (\mathbf{w}_N \cdot \nabla)(\mathbf{u} - \mathbf{u}_N^\diamond), \quad ((\mathbf{u} - \mathbf{u}_N^\diamond) \cdot \nabla)R_N + (\mathbf{w}_N \cdot \nabla)(T - T_N^\diamond)$$

We applied (3.19) with a  $t < 1$  such that  $H^t(\Omega)$  is imbedded in  $L^r(\Omega)$ , with  $r > 2$  in dimension  $d = 2$  and  $r = 3$  in dimension  $d = 3$ . Combining with (3.14) and (3.16) yields

$$\lim_{N \rightarrow +\infty} \left\| \begin{pmatrix} \mathcal{S}_N & 0 \\ 0 & \mathcal{L}_N \end{pmatrix} (D\mathcal{G}(U) - D\mathcal{G}(U_N^\diamond)) \right\|_{\mathcal{E}} = 0. \quad (3.22)$$

3) Similarly, using an extension of Lemma 3.3 obviously yields

$$\lim_{N \rightarrow +\infty} \left\| \begin{pmatrix} \mathcal{S}_N & 0 \\ 0 & \mathcal{L}_N \end{pmatrix} (D\mathcal{G}(U_N^\diamond) - D\mathcal{G}_N(U_N^\diamond)) \right\|_{\mathcal{E}} = 0. \quad (3.23)$$

4) On the other hand, we note that, for any  $\mathbf{F}$  in  $H^{-1}(\Omega)^d$ ,

$$\begin{aligned} (D\mathcal{S}(T)R_N)\mathbf{F} &= \mathcal{S}(T)(-\operatorname{div}(\partial_\tau \nu(T) R_N \nabla \mathcal{S}(T)\mathbf{F})), \\ (D\mathcal{S}_N(T)R_N)\mathbf{F} &= \mathcal{S}_N(T)(-\operatorname{div}(\partial_\tau \nu(T) R_N \nabla \mathcal{S}_N(T)\mathbf{F})). \end{aligned} \quad (3.24)$$

By subtracting the second line from the first one, we derive

$$\begin{aligned} (D(\mathcal{S} - \mathcal{S}_N)(T)R_N)\mathbf{F} &= (\mathcal{S} - \mathcal{S}_N)(T)(-\operatorname{div}(\partial_\tau \nu(T) R_N \nabla \mathcal{S}(T)\mathbf{F})) \\ &\quad + \mathcal{S}_N(T)(-\operatorname{div}(\partial_\tau \nu(T) R_N \nabla (\mathcal{S} - \mathcal{S}_N)(T)\mathbf{F})) \end{aligned}$$

Denoting by  $\mathbf{F}$  the first component of  $\mathcal{G}(U)$ , we see that  $\mathcal{S}(T)\mathbf{F}$  is equal to  $-\mathbf{u}$ , see (3.8). First, using the compactness of the imbedding of  $H^1(\Omega)$  into  $L^r(\Omega)$  for any  $r < \infty$  in dimension  $d = 2$  and  $r < 6$  in dimension  $d = 3$ , we deduce from the regularity assumption on  $\mathbf{u}$  that, when  $W_N$  runs through the unit sphere of  $\mathcal{X}_N$ , the quantity  $-\operatorname{div}(\partial_\tau \nu(T) R_N \nabla \mathcal{S}(T)\mathbf{F})$  belongs to a compact subset of  $H^{-1}(\Omega)^d$ . Thus, the convergence of the first term to zero follows from (3.18).

To handle the second term, we observe from (3.14) that it suffices to prove the convergence of  $\|\nabla(\mathcal{S} - \mathcal{S}_N)(T)\mathbf{F}\|_{L^{q^*}(\Omega)^{d \times d}}$ , with  $\frac{1}{q} + \frac{1}{q^*} = \frac{1}{2}$  for the  $q$  introduced in the beginning of the proof. Since  $\mathcal{S}(T)\mathbf{F}$  coincides with  $-\mathbf{u}$ , by using the injection of  $H^1(\Omega)$  into  $L^{q^*}(\Omega)$ , we obtain

$$\|\nabla(\mathcal{S} - \mathcal{S}_N)(T)\mathbf{F}\|_{L^{q^*}(\Omega)^{d \times d}} \leq c \left( N^{1-s} \left( \|\mathbf{u}\|_{H^s(\Omega)^d} + \|p\|_{H^{s-1}(\Omega)} \right) \right).$$

We derive

$$\lim_{N \rightarrow +\infty} \left\| \begin{pmatrix} D(\mathcal{S} - \mathcal{S}_N)(T) & 0 \\ 0 & 0 \end{pmatrix} \mathcal{G}(U) \right\|_{\mathcal{E}} = 0. \quad (3.25)$$

5) The convergence of the fifth term is deduced from (3.19), (3.12) and the stability of  $D\mathcal{S}_N(\tau)$  and the convergence of the last term is obtained with the same arguments as Lemma 3.3.

This concludes the proof.

**Lemma 3.5** *If the function  $\nu$  belongs to  $W^{2,\infty}(\mathbb{R})$ , with Lipschitz-continuous derivatives, there exist a neighbourhood of  $U_N^\diamond$  in  $\mathcal{X}_N$  and a constant  $c > 0$  such that the operator  $D\mathcal{F}_N$  satisfies the following Lipschitz property, for all  $U_N^*$  in this neighbourhood,*

$$\| D\mathcal{F}_N(U_N^\diamond) - D\mathcal{F}_N(U_N^*) \|_{\mathcal{E}} \leq c \mu(N) \| U_N^\diamond - U_N^* \|_{\mathcal{X}(\Omega)}. \quad (3.26)$$

with  $\mu(N)$  equal to  $|\log N|^{\frac{1}{2}}$  in dimension  $d = 2$  and to  $N$  in dimension  $d = 3$ .

**Proof:** Let us introduce the matrix operators

$$\mathcal{M}(\xi) = \begin{pmatrix} \mathcal{S}(\xi) & 0 \\ 0 & \mathcal{L} \end{pmatrix}, \quad \mathcal{M}_N(\xi) = \begin{pmatrix} \mathcal{S}_N(\xi) & 0 \\ 0 & \mathcal{L}_N \end{pmatrix}.$$

Setting  $U_N^* = (\mathbf{u}_N^*, T_N^*)$ , we have

$$\begin{aligned} D\mathcal{F}_N(U_N^\diamond) - D\mathcal{F}_N(U_N^*) &= (\mathcal{M}_N(T_N^\diamond) - \mathcal{M}_N(T_N^*)) D\mathcal{G}_N(U_N^\diamond) + (D\mathcal{M}_N(T_N^\diamond) - D\mathcal{M}_N(T_N^*)) \mathcal{G}_N(U_N^\diamond) \\ &\quad + \mathcal{M}_N(T_N^*) (D\mathcal{G}_N(U_N^\diamond) - D\mathcal{G}_N(U_N^*)) + D\mathcal{M}_N(T_N^*) (\mathcal{G}_N(U_N^\diamond) - \mathcal{G}_N(U_N^*)). \end{aligned}$$

We have to evaluate these quantities, for any  $W_N = (\mathbf{w}_N, R_N)$  in the unit sphere of  $\mathcal{X}_N$  and  $\psi_N$  in the unit sphere of  $\mathbb{X}_N$ . Since evaluating the last two terms follow from Lemma 3.3 and an extension of it, we only consider the first two terms. All constants  $c$  in what follows only depend on the norms  $\| U_N^\diamond \|_{\mathcal{X}(\Omega)}$ ,  $\| U_N \|_{\mathcal{X}(\Omega)}$  and  $\| \nu \|_{W^{2,\infty}(\mathbb{R})}$ .

1) We have

$$(\mathcal{M}_N(T_N^\diamond) - \mathcal{M}_N(T_N^*)) D\mathcal{G}(U_N^\diamond) W_N = \mathcal{M}_N(l_1) \begin{pmatrix} A \\ 0 \end{pmatrix},$$

with

$$A = \operatorname{div}((\nu(T_N^\diamond) - \nu(T_N^*)) \nabla \mathcal{S}_N(T_N^*) ((\mathbf{u}_N^\diamond \cdot \nabla) \mathbf{w}_N + (\mathbf{w}_N \cdot \nabla) \mathbf{u}_N^\diamond, \psi_N)_N)$$

There exists a constant  $c$  only depending on the Lipschitz property of  $\nu$  such that,

$$\| (\mathcal{M}_N(T_N^\diamond) - \mathcal{M}_N(T_N^*)) D\mathcal{G}(V_1) W_N \|_{\mathcal{X}(\Omega)} \leq c \| T_N^\diamond - T_N^* \|_{L^2(\Omega)} \| \mathbf{w}_N \|_{L^\infty(\Omega)}$$

We conclude by applying the inverse inequality [13], valid for any polynomial  $\varphi_N$  in  $\mathbb{P}_N(\Omega)$ ,

$$\|\varphi_N\|_{L^\infty(\Omega)} \leq cN^{\frac{2d}{\delta}} \|\varphi_N\|_{L^\delta(\Omega)},$$

and noting that

-in dimension  $d = 3$ ,  $H^1(\Omega)$  is embedded in  $L^6(\Omega)$

-in dimension  $d = 2$ ,  $H^1(\Omega)$  is embedded in any  $L^\delta(\Omega)$  with the norm of the imbedding smaller than  $c\sqrt{\delta}$  (see [14]), (we thus take  $\delta$  equal to  $\log N$ ).

2) On the other hand, combining the second part of (3.24) with a further triangle inequality

$$\|((D\mathcal{M}_N(T_N^\diamond) - D\mathcal{M}_N(T_N^*)R_N)\mathcal{G}(U_N^\diamond) \|_{\mathcal{X}(\Omega)} \leq c \|T_N^\diamond - T_N^*\|_{L^2(\Omega)} \|R_N\|_{L^\infty(\Omega)}$$

The same arguments as in part 1) yields the desired result.

**Lemma 3.6** *Assume that  $\nu$  is of class  $\mathcal{C}^2$  on  $\mathbb{R}$  and that the solution  $(\mathbf{u}, p, T)$  of problem (2.3)-(2.4) belongs to  $H^s(\Omega)^d \times H^{s-1}(\Omega) \times H^s(\Omega)$  for a real number  $s$ ,  $s \geq 1$ , and the data  $\mathbf{f}$  belong to  $H^\sigma(\Omega)^d$  for a real number  $\sigma$ ,  $\sigma > \frac{d}{2}$ . Then, the following estimate is satisfied*

$$\|\mathcal{F}_N(U_N^\diamond) \|_{\mathcal{X}(\Omega)} \leq cN^{1-s} (\|\mathbf{u}\|_{H^s(\Omega)^d} + \|p\|_{H^{s-1}(\Omega)} + \|T\|_{H^s(\Omega)}) + c'N^{-\sigma} \|\mathbf{f}\|_{H^\sigma(\Omega)^d}.$$

**Proof:** Since  $\mathcal{F}(U)$  is zero, we have

$$\begin{aligned} \|\mathcal{F}_N(U_N^\diamond) \|_{\mathcal{X}(\Omega)} &\leq \|U - U_N^\diamond\|_{\mathcal{X}(\Omega)} + \left\| \begin{pmatrix} (\mathcal{S} - \mathcal{S}_N)(T) & 0 \\ 0 & \mathcal{L} - \mathcal{L}_N \end{pmatrix} \mathcal{G}(U) \right\|_{\mathcal{X}(\Omega)} \\ &+ \left\| \begin{pmatrix} \mathcal{S}_N(T) & 0 \\ 0 & \mathcal{L}_N \end{pmatrix} (\mathcal{G}(U) - \mathcal{G}(U_N^\diamond)) \right\|_{\mathcal{X}(\Omega)} + \left\| \begin{pmatrix} \mathcal{S}_N(T) & 0 \\ 0 & \mathcal{L}_N \end{pmatrix} (\mathcal{G}(U_N^\diamond) - \mathcal{G}_N(U_N^\diamond)) \right\|_{\mathcal{X}(\Omega)}. \end{aligned}$$

The first term is bounded in (3.19). Evaluating the second term follows from (3.15) and (3.17) by noting that: If  $\mathbf{F}$  denotes the first component of  $\mathcal{G}(U)$ ,  $\mathcal{S}\mathbf{F}$  is equal to  $-\mathbf{u}$  and  $\mathbf{F}$  is equal to  $\mathbf{f} + \text{div}(\nu(T)\nabla\mathbf{u}) - \mathbf{grad} p$ . To bound the third one, we apply (3.14) and by using triangle inequalities and estimate (3.19). Finally, proving the estimate for the fourth term is obtained by using the standard arguments for the error issued from numerical integration combined with the same arguments as in the proof of Lemma 3.3.

Thanks to Lemmas 3.4 to 3.6, we are now in a position to prove the main result of this section.

**Theorem 3.7** *Let  $(\mathbf{u}, p, T)$  be a solution of problem (2.3) – (2.4) which satisfies Assumption 3.2 and belongs to  $H^s(\Omega)^d \times H^{s-1}(\Omega) \times H^s(\Omega)$ ,  $s > 1$ . We moreover assume that the function  $\nu$  is of class  $\mathcal{C}^2$  on  $\mathbb{R}$  with Lipschitz-continuous derivatives and that the data  $\mathbf{f}$  belong to  $H^\sigma(\Omega)^d$  for a real number  $\sigma$ ,  $\sigma > \frac{d}{2}$ . Then, there exist a positive number  $N_\diamond$  and a constant  $c_\diamond$  such that, for all  $N \geq N_\diamond$ , problem (3.3) – (3.4) has a unique solution  $(\mathbf{u}_N, p_N, T_N)$  such*

that  $(\mathbf{u}_N, T_N)$  belongs to the ball of  $\mathcal{X}(\Omega)$  with center  $(\mathbf{u}, T)$  and radius  $c_\diamond \mu_N^{-1}$  for the constant  $\mu_N$  introduced in Lemma 3.5. Moreover, this solution satisfies

$$\begin{aligned} & \| \mathbf{u} - \mathbf{u}_N \|_{H^1(\Omega)^d} + \| T - T_N \|_{H^1(\Omega)} + N^{-(d-1)/2} \| p - p_N \|_{L^2(\Omega)} \\ & \leq c N^{1-s} (\| \mathbf{u} \|_{H^s(\Omega)^d} + \| p \|_{H^{s-1}(\Omega)} + \| T \|_{H^s(\Omega)}) + c' N^{-\sigma} \| \mathbf{f} \|_{H^\sigma(\Omega)^d}. \end{aligned} \quad (3.22)$$

**Proof:** Combining Lemmas 3.4 to 3.6 with the Brezzi-Rappaz-Raviart Theorem [5] ( see also [9, Chap. IV, Thm 3.1]) yields for  $N$  large enough, the existence of a solution  $(u_N, T_N)$ , its local uniqueness and the desired estimate for  $\| \mathbf{u} - \mathbf{u}_N \|_{H^1(\Omega)^d}$  and  $\| T - T_N \|_{H^1(\Omega)}$ . Moreover, thanks to the discrete inf-sup condition (3.5), there exists a unique  $p_N$  in  $\mathbb{M}_N$  such that:

$$\begin{aligned} \int_{\Omega} (\operatorname{div} v_N)(\mathbf{x}) p_N(\mathbf{x}) \, d\mathbf{x} &= \int_{\Omega} \nu(T_N)(\mathbf{grad} \mathbf{u}_N)(\mathbf{x}) : (\mathbf{grad} v_N)(\mathbf{x}) \, d\mathbf{x} \\ &+ \int_{\Omega} ((\mathbf{u}_N \cdot \nabla) \mathbf{u}_N)(\mathbf{x}) \cdot v_N(\mathbf{x}) \, d\mathbf{x} - (\mathbf{f}, v_N). \end{aligned}$$

whence, for any  $q_N$  in  $\mathbb{M}_N$ ,

$$\begin{aligned} & \int_{\Omega} (\operatorname{div} v_N)(\mathbf{x}) (p_N - q_N)(\mathbf{x}) \, d\mathbf{x} \\ &= \int_{\Omega} (\nu(T_N)(\mathbf{grad} \mathbf{u}_N)(\mathbf{x}) - \nu(T)(\mathbf{grad} \mathbf{u})(\mathbf{x})) : (\mathbf{grad} v_N)(\mathbf{x}) \, d\mathbf{x} \\ &+ \int_{\Omega} ((\mathbf{u}_N \cdot \nabla) \mathbf{u}_N - (\mathbf{u} \cdot \nabla) \mathbf{u})(\mathbf{x}) \cdot v_N(\mathbf{x}) \, d\mathbf{x} + \int_{\Omega} (\operatorname{div} v_N)(\mathbf{x}) (p - q_N)(\mathbf{x}) \, d\mathbf{x}. \end{aligned}$$

So by using successively (3.5), triangle inequalities and the error estimates on  $\mathbf{u}$  and  $T$ , we derive the estimate for  $\| p - p_N \|_{L^2(\Omega)}$ .

Estimate (3.22) is fully optimal. Moreover the regularity assumptions on the solution  $(\mathbf{u}, p, T)$  are not at all restrictive in dimension  $d = 2$ .

## 4 Numerical experiments.

The numerical experiments have been performed in the two-dimensional case, on the square  $\Omega = ]-1, 1[^2$ . Problem (3.3)-(3.4) is solved via the following iterative algorithm. We set:

$$\mathcal{H}(U_N) = U_N + \mathcal{M}_N(T_N)\mathcal{G}_N(U_N)$$

Applying Newton's method consists in solving iteratively the equation

$$U_N^m = U_N^{m-1} - D\mathcal{H}(U_N^{m-1})^{-1}\mathcal{H}(U_N^{m-1}),$$

which can equivalently be written as:  $U_N^m = U_N^{m-1} - W_N^{m-1}$ , where  $W_N^{m-1} = (\mathbf{z}_N^{m-1}, \chi_N^{m-1})$  is a solution of this problem:

$$D\mathcal{H}(U_N^{m-1})W_N^{m-1} = \mathcal{H}(U_N^{m-1})$$

We iteratively solve this problem, for  $m \geq 1$ . It reads

Find  $(\mathbf{z}_N^{m-1}, \varphi_N^{m-1}, \chi_N^{m-1})$  in  $\mathbb{X}_N \times \mathbb{M}_N \times \mathbb{Y}_N$  such that

$$\begin{aligned} \forall \mathbf{v}_N \in \mathbb{X}_N, \quad & (\nu'(T_N^{m-1})\chi_N^{m-1}\nabla\mathbf{u}_N^{m-1}, \nabla\mathbf{v}_N)_N + (\nu(T_N^{m-1})\nabla\mathbf{z}_N^{m-1}, \nabla\mathbf{v}_N)_N \\ & + ((\mathbf{u}_N^{m-1} \cdot \nabla)\mathbf{z}_N^{m-1} + (\mathbf{z}_N^{m-1} \cdot \nabla)\mathbf{u}_N^{m-1}, \mathbf{v}_N)_N - (\operatorname{div} v_N, \varphi_N^{m-1})_N \\ & = (\nu(T_N^{m-1})\nabla\mathbf{u}_N^{m-1}, \nabla\mathbf{v}_N)_N + ((\mathbf{u}_N^{m-1} \cdot \nabla)\mathbf{u}_N^{m-1}, \mathbf{v}_N)_N \\ & \quad - (\operatorname{div} \mathbf{v}_N, p_N^{m-1})_N - (\mathbf{f}, \mathbf{v}_N)_N \end{aligned}$$

$$\forall q_N \in \mathbb{M}_N, \quad -(\operatorname{div} \mathbf{z}_N^{m-1}, q_N)_N = 0$$

$$\begin{aligned} \forall S_N \in \mathbb{Y}_N^0, \quad & \alpha(\nabla\chi_N^{m-1}, \nabla S_N)_N + ((\mathbf{z}_N^{m-1} \cdot \nabla)T_N^{m-1} + (\mathbf{u}_N^{m-1} \cdot \nabla)\chi_N^{m-1}, S_N)_N \\ & = \alpha(\nabla T_N^{m-1}, \nabla S_N)_N + ((\mathbf{u}_N^{m-1} \cdot \nabla)T_N^{m-1}, S_N)_N - (g, S_N)_N. \end{aligned}$$

The convergence of this method can easily be derived from [5] (see [9, Chap. IV, Thm 6.5]) owing to Lemmas 3.4 to 3.6.

The numerical experiments that we present in the next lines are performed on the code MATLAB software and the global system is solved by the GMRES method.

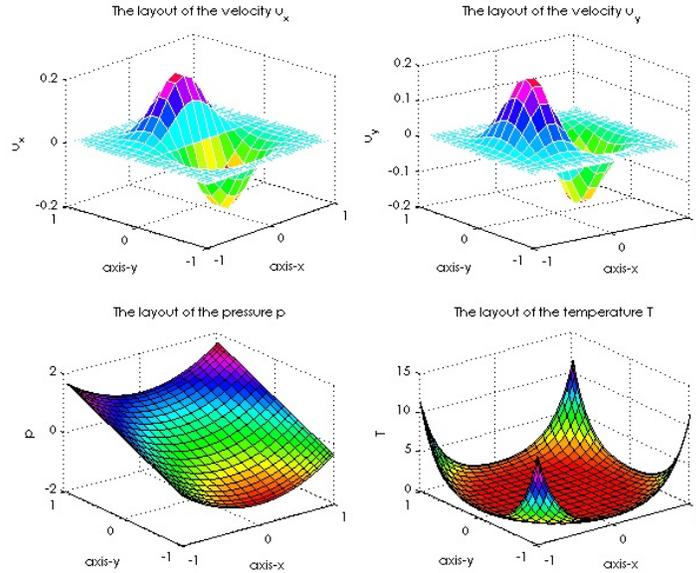
To start, we take in all the calculations  $\mathbf{u}_N^0, T_N^0, p_N^0$  zeros on internal nodes, knowing that whatever the choice of these solutions the algorithm converges. In all tests the number of iterations needed for better convergence of the Newton algorithm, vary between 5 and 10. We work with  $\nu$  as a constant, as a function dependent of the space variable and finally as a function which depends on the temperature  $T$

- case where  $\nu(T)$  is constant equal to  $10^{-2}$

In the first experiment, the exact solution is given by

$$\mathbf{u}(x, y) = \begin{pmatrix} y(1-x^2)^{\frac{11}{2}}(1-y^2)^{\frac{9}{2}} \\ -x(1-x^2)^{\frac{9}{2}}(1-y^2)^{\frac{11}{2}} \end{pmatrix}, \quad p(x, y) = x^2 + y - \frac{1}{3}, \quad T(x, y) = (x^2 + y^2)^{\frac{7}{2}} \quad (4.1)$$

The discrete solution computed with  $N = 24$  is presented in Figure 1, the two components of the velocity on the top, the pressure and the temperature on the bottom.

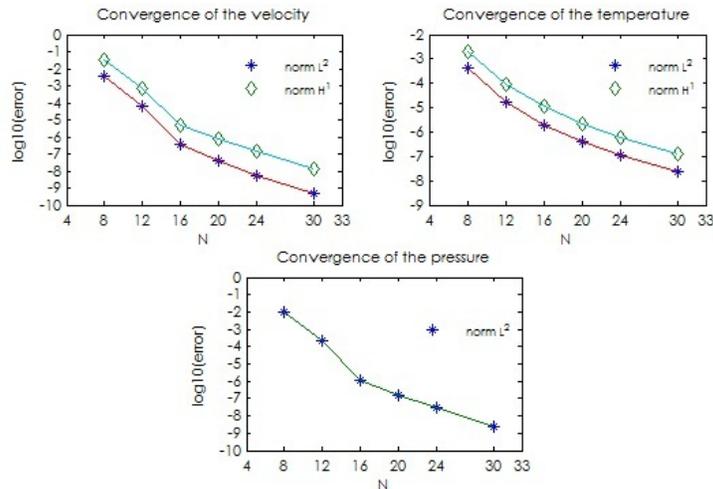


**Figure 1:** The discrete solution issued from (4.1)

- case where  $\nu(T)$  is a function equal to  $x + y + 1$

i) In the first case, we work with regular functions where the boundary conditions are not homogeneous, we attained the convergence of  $10^{-13}$  from  $N = 6$ .

ii) In the second experiment, we work with the solution given in (4.1). In Figure 2, we present the convergence of the relative error on  $\mathbf{u}$ ,  $p$  and  $T$  in the  $L^2(\Omega)^2$ ,  $H^1(\Omega)^2$ ,  $L^2(\Omega)$  or  $H^1(\Omega)$  norm in logarithmic scales, as a function of  $N$ , for  $N$  varying from 8 to 30.



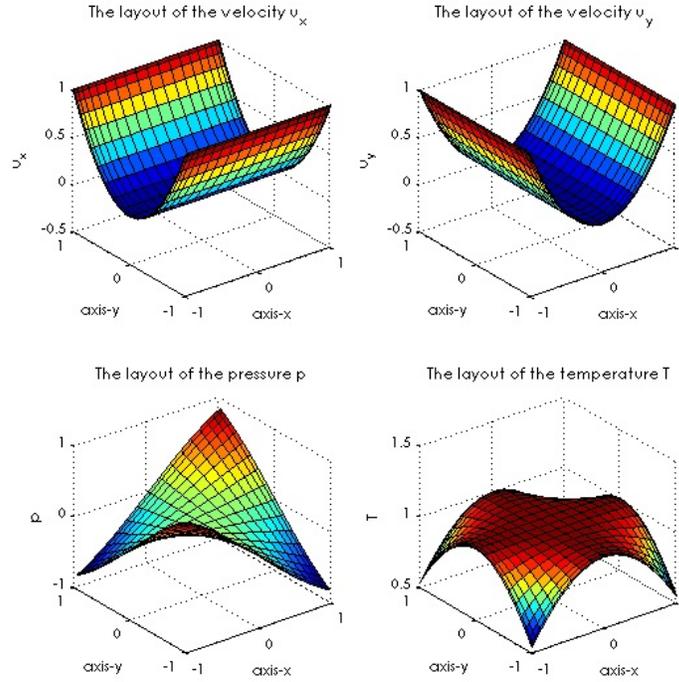
**Figure 2:** The estimations of error of the solution issued from (4.1)

- case where  $\nu(T)$  depend on  $T$

i) We first consider a smooth solution in the case where  $\nu(T)$  equal to  $3\sqrt{T^2 + 1} + 2$

$$\mathbf{u}(x, y) = \begin{pmatrix} y^2 \\ x^2 \end{pmatrix}, \quad p(x, y) = \sin(x + y), \quad T(x, y) = \cos(xy) \quad (4.2)$$

In Figure 3 we present the discrete solution issued from (4.2) computed with  $N = 24$ .



**Figure 3:** The values of the discrete solution issued from (4.2)

ii) Second we consider the solution in the case where  $\nu(T)$  is given by  $T + 1$

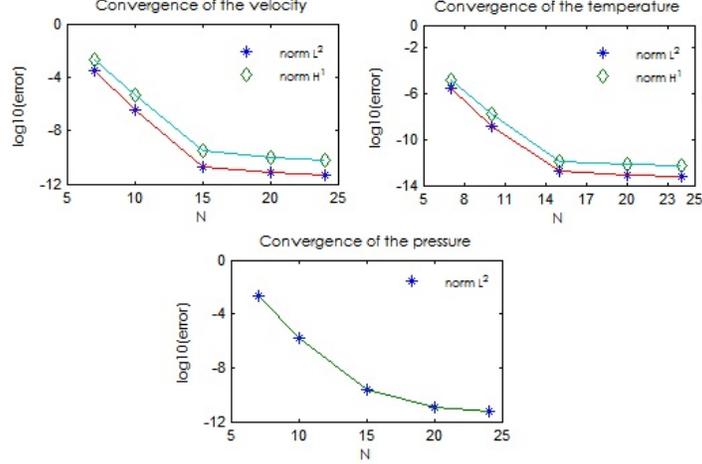
$$\mathbf{u}(x, y) = \begin{pmatrix} x \sin(\pi xy) \\ -y \sin(\pi xy) \end{pmatrix}, \quad p(x, y) = x + y, \quad T(x, y) = xy \quad (4.3)$$

In Figure 4 we present the quantities:

$$\log_{10} \|u - u_N\|_{L^2(\Omega)^2}, \log_{10} \|u - u_N\|_{H^1(\Omega)^2}, \log_{10} \|p - p_N\|_{L^2(\Omega)},$$

$$\log_{10} \|T - T_N\|_{L^2(\Omega)} \quad \text{and} \quad \log_{10} \|T - T_N\|_{H^1(\Omega)}$$

as functions of  $N$ , for  $N$  varying from 8 to 24. We observe the perfect convergence for  $N = 24$ .



**Figure 4:** The estimations of error of the solution obtained from (4.3)

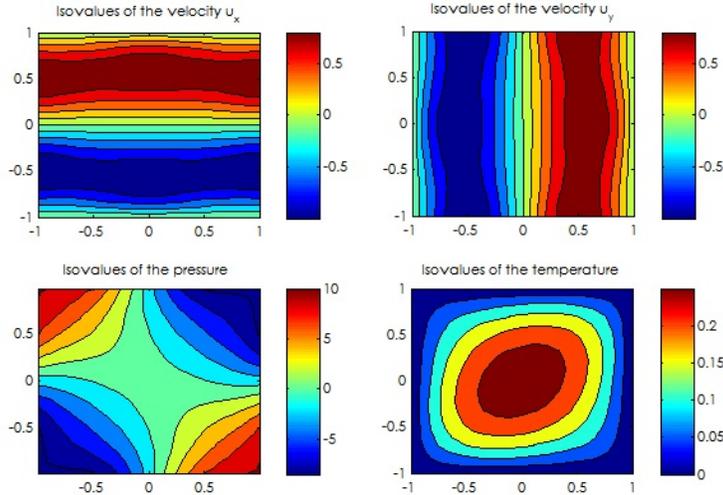
iii) We present now numerical experiments where  $\nu(T)$  is taken equal to  $T^2 + T$ , the datum  $\mathbf{f}$  is equal to zero, and the datum  $g$  is given by 1, but the boundary condition is replaced by

$$\mathbf{u} = \mathbf{h} \quad \text{on } \partial\Omega, \quad T_0 = 0,$$

the boundary velocity  $\mathbf{h} = (h_x, h_y)$  being given by

$$h_x(x, y) = \begin{cases} 0 & \text{si } y = \pm 1, \\ \sin(\pi y) & \text{otherwise,} \end{cases} \quad h_y(x, y) = \begin{cases} 0 & \text{si } x = \pm 1, \\ \sin(\pi x) & \text{otherwise.} \end{cases} \quad (4.4)$$

Note that the data satisfy the usual compatibility condition  $\int_{\partial\Omega} \mathbf{h}(\tau) \cdot \mathbf{n}(\tau) d\tau = 0$ . We present in Figure 5 the isovalues of the two components of the velocity, the pressure and the temperature issued from (4.4) computed with  $N = 26$ .



**Figure 5:** Isovalues of the discrete solution issued from (4.4)

iv) For the last numerical experiment we work with the solution given by

$$\mathbf{u}(x, y) = \begin{pmatrix} 2 \sin(\pi x)^2 \sin(\pi y) \cos(\pi y) \\ -2 \sin(\pi y)^2 \sin(\pi x) \cos(\pi x) \end{pmatrix}, \quad p(x, y) = x^2 - y^2, \quad T(x, y) = xy. \quad (4.5)$$

In Table 1 we present different values of  $\nu(T)$ . We see the stability of the algorithm and the variation of the errors of the components : velocity, pressure and temperature computed with  $N = 16$ .

$\nu(T)$	$T+1$	$\frac{1}{25}(T+1)$	$\frac{1}{50}(T+1)$	$\frac{1}{75}(T+1)$	$\frac{1}{100}(T+1)$
$\ \mathbf{u} - \mathbf{u}_{app}\ _{L^2(\Omega)^2}$	7.64e-009	1.48e-008	4.16e-008	5.50e-008	9.62e-008
$\ \mathbf{u} - \mathbf{u}_{app}\ _{H^1(\Omega)^2}$	8.76e-008	2.76e-007	4.23e-007	7.67e-007	9.33e-007
$\ p - p_{app}\ _{L^2(\Omega)}$	1.35e-008	4.16e-008	6.16e-008	9.02e-008	4.73e-007
$\ T - T_{app}\ _{L^2(\Omega)}$	4.23e-0011	3.03e-0010	4.06e-0010	8.39e-0010	1.58e-009
$\ T - T_{app}\ _{H^1(\Omega)}$	5.68e-0010	9.18e-0010	2.28e-009	5.21e-009	6.93e-009

**Table 1:** Convergence of the solution (4.5) as a function of  $\nu(T)$ .

We note that the precision of convergence decreases if  $\nu(T)$  decreases but it remains stable. So to overcome this handicap we should increase the discretization parameter  $N$ : For example by taking  $N = 18$  and  $\nu(T) = \frac{1}{100}(T+1)$  we attained the precision of  $N = 16$  and  $\nu(T) = T+1$ .

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