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# Homogenization of random parabolic operators. Diffusion approximation.

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## Abstract

The paper deals with homogenization of divergence form second order parabolic operators whose coefficients are periodic in spatial variables and random stationary in time. Under proper mixing assumptions, we study the limit behaviour of the normalized difference between solutions of the original and the homogenized problems. The asymptotic behaviour of this difference depends crucially on the ratio between spatial and temporal scaling factors. Here we study the case of self-similar parabolic diffusion scaling.

## 1 Introduction

The goal of this paper is to characterize the rate of convergence in the homogenization problem for a second order divergence form parabolic operator with random stationary in time and periodic in spatial variables coefficients. We are also aimed at describing the limit behaviour of a normalized difference between solutions of the original and homogenized problems

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To avoid boundary effects we study a Cauchy problem that takes the form

$$(1) \quad \begin{aligned} \partial_t u^\varepsilon &= \operatorname{div} \left( a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla u^\varepsilon \right), & x \in \mathbb{R}^n, t > 0, \\ u^\varepsilon(x, 0) &= g(x). \end{aligned}$$

with  $\alpha > 0$ . In this paper we consider the case  $\alpha = 2$ . We assume that the matrix  $a(z, s) = \{a^{ij}(z, s)\}$  is uniformly elliptic,  $(0, 1)^n$ -periodic in  $z$  variable, and random stationary ergodic in  $s$ . We denote  $Y = (0, 1)^n$  and in what follows identify  $Y$ -periodic function with functions define on the torus  $\mathbb{T}^n$ .

It is known (see [14], [8]) that under these assumptions problem (1) admits homogenization. More precisely, for any  $g \in L^2(\mathbb{R}^n)$ , almost surely (a.s.) solutions  $u^\varepsilon$  of problem (1) converge, as  $\varepsilon \rightarrow 0$ , to a solution of the homogenized problem

$$(2) \quad \begin{aligned} \partial_t u^0 &= \operatorname{div} (a^{\text{eff}} \nabla u^0) \\ u^0(x, 0) &= g(x) \end{aligned}$$

with a constant non-random coefficients. The convergence is in  $L^2(\mathbb{R}^n \times (0, T))$ . More detailed description of the existing homogenization results is given in Sections 3 and 3.1.

The paper focuses on the rate of this convergence and on higher order terms of the asymptotics of  $u^\varepsilon$ . Our goal is to describe the limit behaviour of the normalized difference  $\varepsilon^{-1}(u^\varepsilon - u^0)$ .

Clearly, the main oscillating term of the asymptotics of this normalized difference should be expressed in terms of the corrector. We recall (see [8], [3]) that the equation

$$\partial_s \chi(z, s) = \operatorname{div}_z (a(z, s) (\nabla_z \chi(z, s) + \mathbf{I}))$$

has a unique up to an additive (random) constant periodic in  $z$  and stationary in  $s$  solution. Thus, the gradient  $\nabla_z \chi$  is uniquely defined. The principal corrector takes the form  $\varepsilon \chi(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}) \nabla u^0(x, t)$ . We study the limit behaviour of the expression

$$U^\varepsilon(x, t) := \frac{u^\varepsilon(x, t) - u^0(x, t)}{\varepsilon} - \chi \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u^0(x, t).$$

For generic stationary ergodic coefficients  $a(z, s)$  the family  $\{U^\varepsilon\}$  need not be compact or tight in  $L^2(\mathbb{R}^n \times (0, T))$ .

For this reason we assume that (see Section 2 for further details)

- Coefficients  $a(z, s)$  have good mixing properties.
- Initial function  $g$  is sufficiently smooth.

Under these conditions we show (see Theorem 3, Section 6) that  $U^\varepsilon$  converges in law in  $L^2(\mathbb{R}^n \times (0, T))$  equipped with the strong topology to a solution of a SPDE with constant coefficients and an additive noise. This SPDE reads

$$dU^0 = \operatorname{div} \left( a^{\text{eff}} \nabla U^0 + \mu \frac{\partial^3}{\partial x^3} u^0 \right) dt + \Lambda^{1/2} \frac{\partial^2}{\partial x^2} u^0 dW_t,$$

$$U^0(x, 0) = 0;$$

here  $a^{\text{eff}}$  is the homogenized coefficients matrix,  $u^0$  is a solution of (2),  $W_t$  is a standard  $n^2$ -dimensional Wiener process, and  $\mu$  and  $\Lambda$  are constant tensors which are defined in Section 6. We show that this SPDE is well-posed and, thus, defines the limit law of  $U^\varepsilon$  uniquely.

Notice that under proper choice of an additive constant the mean value of  $\chi(z, s)$  is equal to zero. Therefore, the function  $\chi(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}) \nabla u^0(x, t)$  converges a.s. to zero weakly in  $L^2(\mathbb{R}^n \times (0, T))$ , as  $\varepsilon \rightarrow 0$ . Therefore, in the weak topology of  $L^2(\mathbb{R}^n \times (0, T))$ , the limit in law of the normalized difference  $\varepsilon^{-1}(u^\varepsilon(x, t) - u^0(x, t))$  coincides with that of  $U^\varepsilon$ .

The first results on homogenization of elliptic operators with random statistically homogeneous coefficients were obtained in [9], [11]. At present there is an extensive literature on this topic. However, optimal estimates for the rate of convergence is an open issue. In [13] some power estimates for the rate of convergence were obtained in dimension three and more. In the recent work [5] the further important progress has been made in this problem.

Parabolic operators with random coefficients depending both on spatial and temporal variables have been considered in [14]. In the case of a diffusive scaling, the a.s. homogenization theorem has been proved.

The case of non-diffusive scaling has been studied in [7] under the assumption that the coefficients are periodic in spatial variables and random stationary in time.

It turns out that the structure of the higher order terms of the asymptotics of  $u^\varepsilon$  depends crucially on whether the scaling is diffusive or not. Here we study the diffusive scaling. The case of non-diffusive scaling will be addressed elsewhere.

## 2 The setup

Let  $(\Omega, \mathcal{F}, \mathbf{P})$  be a standard probability space equipped with a measure preserving ergodic dynamical system  $\mathcal{T}_s$ ,  $s \in \mathbb{R}$ .

Given a measurable matrix function  $\tilde{a}(z, \omega) = \{\tilde{a}^{ij}(z, \omega)\}_{i,j=1}^d$  which is periodic in  $z$  variable with a period one in each coordinate direction, we define a random field  $a(z, s)$  by

$$a(z, s) = \tilde{a}(z, \mathcal{T}_s \omega).$$

Then  $a(z, s)$  is periodic in  $z$  and stationary ergodic in  $s$ .

We consider the following Cauchy problem in  $\mathbb{R}^d \times (0, T]$ ,  $T > 0$ :

$$(3) \quad \begin{cases} \frac{\partial u^\varepsilon}{\partial t} &= \operatorname{div} \left( a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u^\varepsilon \right) \\ u^\varepsilon(0, x) &= g(x) \end{cases}$$

with a small positive parameter  $\varepsilon$ .

We assume that the coefficients in (3) possess the following properties.

**H1** The matrix  $a(z, s)$  is symmetric and satisfies uniform ellipticity conditions that is there is  $\lambda > 0$  such that for all  $(z, \omega)$  the following inequality holds:

$$\lambda |\xi|^2 \leq \tilde{a}(z, \omega) \xi \cdot \xi \leq \lambda^{-1} |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d.$$

**H2** The initial condition  $g$  is sufficiently smooth and decays fast enough at infinity.

In order to formulate one more condition we introduce the so-called maximum correlation coefficient. Setting  $\mathcal{F}_{\leq r} = \sigma\{a(z, s) : s \leq r\}$  and  $\mathcal{F}_{\geq r} = \sigma\{a(z, s) : s \geq r\}$ , we define

$$\rho(r) = \sup_{\xi_1, \xi_2} \mathbf{E}(\xi_1 \xi_2)$$

where the supremum is taken over all  $\mathcal{F}_{\leq 0}$ -measurable  $\xi_1$  and  $\mathcal{F}_{\geq r}$ -measurable  $\xi_2$  such that  $\mathbf{E}\xi_1 = \mathbf{E}\xi_2 = 0$ , and  $\mathbf{E}\{(\xi_1)^2\} = \mathbf{E}\{(\xi_2)^2\} = 1$ . We then assume that

**H3** The function  $\rho$  satisfies the estimate  $\int_0^\infty \rho(r) dr < +\infty$ .

**Remark 1** Condition **H3** is somehow implicit. In applications various sufficient conditions are often used. In particular, **H3** is fulfilled if  $\rho(r) \leq cr^{-(1+\delta)}$  for some  $\delta > 0$ .

### 3 Homogenization results

In this section we remind of the existing homogenization results for problem (1). Although we only deal in this paper with the case  $\alpha = 2$ , for convenience of the reader we formulate the homogenization results for all  $\alpha > 0$ . To this end we first introduce the so-called cell problem. For  $\alpha = 2$  it reads

$$(4) \quad \partial_s \chi(z, s) = \operatorname{div}(a(z, s)(\mathbf{I} + \nabla \chi(z, s))), \quad (z, s) \in (-\infty, +\infty) \times \mathbb{T}^n$$

with  $\mathbf{I}$  being the unit matrix; here  $\chi = \{\chi^j\}_{j=1}^n$  is a vector function. In what follows for the sake of brevity we denote  $\operatorname{div} a = \operatorname{div}(a\mathbf{I}) = \frac{\partial}{\partial z^i} a^{ij}(z)$ .

According to Lemma 4.1, under assumption **H1** this equation has a stationary periodic in  $y$  vector-valued solution. This solution is unique up to an additive constant. We define

$$(5) \quad a^{\text{eff}} = \mathbb{E} \int_{\mathbb{T}^n} a(z, s)(\mathbf{I} + \nabla \chi(z, s)) dz$$

Notice that due to the stationarity the expression on the right-hand side does not depend on  $s$ .

If  $\alpha < 2$ , the cell problem reads

$$(6) \quad \operatorname{div}(a(z, s)(\mathbf{I} + \nabla \chi_-(z, s))) = 0, \quad z \in \mathbb{T}^n;$$

here  $s$  is a parameter. This equation has a unique up to a multiplicative constant solution. We then set

$$(7) \quad a_-^{\text{eff}} = \mathbb{E} \int_{\mathbb{T}^n} a(z, s)(\mathbf{I} + \nabla \chi_-(z, s)) dz.$$

For  $\alpha > 2$  we first define  $\bar{a}(z) = \mathbb{E}a(z, s)$ , then introduce a deterministic function  $\chi_+(z)$  as a periodic solution to the problem

$$(8) \quad \operatorname{div}(\bar{a}(z)(\mathbf{I} + \nabla \chi_+(z))) = 0, \quad z \in \mathbb{T}^n,$$

and finally define

$$(9) \quad a_+^{\text{eff}} = \int_{\mathbb{T}^n} \bar{a}(z)(\mathbf{I} + \nabla \chi_+(z)) dz.$$

The following statement has been obtained in [14] and [3].

**Theorem 1** *Let  $g \in L^2(\mathbb{R}^n)$ . If  $\alpha = 2$ , then a solution  $u^\varepsilon$  of problem (1) converges a.s. in  $L^2(\mathbb{R}^n \times (0, T))$  to a solution of the limit problem (2) with  $a^{\text{eff}}$  given by (5).*

*If  $\alpha < 2$ , then a solution  $u^\varepsilon$  of problem (1) converges in probability in  $L^2(\mathbb{R}^n \times (0, T))$  to a solution of the limit problem (2) with  $a^{\text{eff}} = a_-^{\text{eff}}$  defined in (7).*

*If  $\alpha > 2$ , then a solution  $u^\varepsilon$  of problem (1) converges in probability in  $L^2(\mathbb{R}^n \times (0, T))$  to a solution of the limit problem (2) with  $a^{\text{eff}} = a_+^{\text{eff}}$  defined in (9).*

### 3.1 Diffusive dependence of time

In this section as a particular case of (3) we introduce the following problem

$$(10) \quad \begin{cases} \frac{\partial u^\varepsilon}{\partial t} &= \operatorname{div} \left( \tilde{a} \left( \frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) u^\varepsilon \right) \\ u^\varepsilon(0, x) &= g(x) \end{cases}$$

with a diffusion process  $\xi_s$ ,  $s \in (-\infty, +\infty)$ , with values in  $\mathbb{R}^N$  or on a compact manifold. This process is defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . For the sake of definiteness we consider here the case of a diffusion in  $\mathbb{R}^N$ . The corresponding Itô equation reads

$$d\xi_t = b(\xi_t)dt + \sigma(\xi_t)dW_t,$$

here  $W$  stands for a standard  $N$ -dimensional Wiener process. The infinitesimal generator of  $\xi$  is denoted by  $\mathcal{L}$ :

$$\mathcal{L}f(y) = q^{ij}(y) \frac{\partial^2}{\partial y^i \partial y^j} f(y) + b(y) \cdot \nabla f(y), \quad y \in \mathbb{R}^N,$$

with a  $N \times N$  matrix  $q(y) = \frac{1}{2} \sigma(y) \sigma^*(y)$ . We also introduce an operator

$$\mathcal{A}^\varepsilon f(x) = \operatorname{div}_x \left( a \left( \frac{x}{\varepsilon}, y \right) \nabla_x f \right);$$

here  $y$  is a parameter.  $\mathcal{A}$  denotes  $\mathcal{A}^\varepsilon$  for  $\varepsilon = 1$ .

Applied to a function  $f(z, y)$ ,  $\mathcal{L}$  acts on the function  $y \mapsto f(z, y)$  for  $z$  fixed, and  $\mathcal{A}$  acts on the function  $z \mapsto f(z, y)$  for  $y$  fixed.

In the diffusive case condition **H3** can be replaced with certain assumptions on the generator  $\mathcal{L}$ . More precisely, we suppose that the following conditions hold true.

**A1.** The coefficients  $a$  and  $q$  are uniformly bounded as well as their first order derivatives in all variables:

$$|a(z, y)| + |\nabla_z a(z, y)| + |\nabla_y a(z, y)| \leq C_1,$$

$$|q(y)| + |\nabla q(y)| \leq C_1.$$

The function  $b$  as well as its derivatives satisfy polynomial growth condition:

$$|b(y)| + |\nabla b(y)| \leq C_1(1 + |y|)^{N_1}.$$

**A2.** Both  $\mathcal{A}$  and  $\mathcal{L}$  are uniformly elliptic:

$$C_2 \mathbf{I} \leq a(z, y), \quad C_2 \mathbf{I} \leq q(y), \quad \text{with } C_2 > 0,$$

where  $\mathbf{I}$  stands for a unit matrix of the corresponding dimension.

**A3.** There exist  $N_2 > -1$ ,  $R > 0$  and  $C_3 > 0$  such that

$$b(y) \frac{y}{|y|} \leq -C_3 |y|^{N_2}$$

for all  $y$ ,  $|y| > R$ .

Under above assumptions the process  $\xi$  has a unique invariant probability measure (see [12]). This measure possesses a smooth density  $\pi$  that forms the kernel of the formal adjoint operator  $\mathcal{L}^*$  of  $\mathcal{L}$ .

In the following,

- $\langle f(\cdot, y) \rangle = \int_0^1 f(z, y) dz$  denotes the mean value over the period;
- $\overline{f(z, \cdot)} = \int_{\mathbf{R}} f(z, y) \pi(y) dy$ ;
- $\overline{\langle f \rangle} = \langle \overline{f} \rangle = \int_0^1 \int_{\mathbf{R}} f(z, y) \pi(y) dy dz$ .

We denote by  $L_\pi^2 = L_\pi^2([0, 1] \times \mathbf{R})$  the weighted space with norm

$$\|f\|_\pi^2 = \int_0^1 \int_{\mathbf{R}} f^2(z, y) \pi(y) dz dy.$$

We also define  $\overline{L}_\pi^2$  the subspace

$$\overline{L}_\pi^2 = \left\{ f \in L_\pi^2 \mid \overline{\langle f \rangle} = 0 \right\}.$$



$V$  will denote the space

$$V = L_w^2([0, T]; H^1(\mathbb{R})) \cap C([0, T]; L_w^2(\mathbb{R})),$$

where the symbol  $w$  means the corresponding space is endowed with its weak topology; the space  $C([0, T]; L_w^2(\mathbb{R}))$  is endowed with the topology of uniform convergence.

Let us recall the result of [7] (see also [2]).

**Theorem 2** *Under Assumptions A1–A3, the solution  $u^\varepsilon$  of (3) converges in probability in the space  $V$  to the solution of the problem*

$$(11) \quad \begin{cases} \frac{\partial u_0}{\partial t} &= \widehat{a}^{ij} \frac{\partial^2 u_0}{\partial x^i \partial x^j} \\ u_0(0, x) &= g(x) \end{cases}$$

with

- for  $\alpha = 2$ ,  $\widehat{a}$  is given by

$$(12) \quad \widehat{a} = \overline{\langle a(1 + \nabla_x \chi^0) \rangle}$$

with  $\chi^0$  being the solution of the following equation

$$(13) \quad (\mathcal{A} + \mathcal{L})\chi^0 = -\operatorname{div}_z a(z, y).$$

- for  $0 < \alpha < 2$ , the formula for  $\widehat{a}$  remains unchanged, and  $\chi^0$  satisfies:

$$(14) \quad \mathcal{A}\chi^0 = -\operatorname{div}_z a(z, y);$$

- for  $\alpha > 2$ ,

$$(15) \quad \widehat{a} = \langle \bar{a}(1 + \nabla_z \chi^0) \rangle,$$

and  $\chi^0$  is a solution of

$$(16) \quad \overline{\mathcal{A}}\chi^0 = \operatorname{div}(\bar{a}(z) \nabla \chi^0) = -\operatorname{div} \bar{a}(z).$$

## 4 Technical statements

In this section we provide a number of technical statements required for formulating and proving the main results.

Consider an equation

$$(17) \quad \partial_s \psi(z, s) - \operatorname{div}(a(z, s) \nabla \psi(z, s)) = g(z, s)$$

with a stationary in  $s$  and periodic in  $z$  random function  $g$ . Denote by  $\rho_{(a,g)}(r)$  the maximum correlation coefficient of the stationary field  $\{a(z, s), g(z, s)\}$ , and by  $\rho_{a,g,\psi}(r)$  the maximum correlation coefficient of  $\{a(\cdot, s), g(\cdot, s), \psi(\cdot, s)\}$ .

**Lemma 4.1** *Let  $g \in L^2_{\text{loc}}(\mathbb{R}; H^{-1}(Y))$ , and assume that  $\mathbb{E}\|g\|_{L^2((0,1); H^{-1}(Y))} < \infty$ , and*

$$\int_Y g(z, s) dz = 0 \quad a.s.$$

*Then equation (17) has a stationary solution  $\psi \in L^\infty_{\text{loc}}((-\infty, +\infty); L^2(Y)) \cap L^2_{\text{loc}}((-\infty, +\infty); H^1(Y))$ . It is unique up to an additive (random) constant. If, in addition,*

$$\int_0^\infty (\rho_{a,g}(r))^\beta dr < +\infty$$

*for some  $\beta > 0$ , then*

$$\int_0^\infty (\rho_{a,g,\psi}(r))^\beta dr < +\infty.$$

*If  $\rho_{a,g}(r) \leq C(1+r)^{-\beta}$ , then  $\rho_{a,g,\psi}(r) \leq C_1(1+r)^{-\beta}$ .*

A proof of this statement is similar to that of Lemmata 2 and 4 in [8].

**Corollary 1** *If the function  $g$  in (17) belongs to  $L^\infty(\mathbb{R}; W^{-1,\infty}(Y))$ , then  $\psi \in L^\infty(\mathbb{R} \times Y)$  and*

$$\|\psi\|_{L^\infty(\mathbb{R} \times Y)} \leq C \|g\|_{L^\infty(\mathbb{R}; W^{-1,\infty}(Y))}$$

*with a deterministic constant  $C$ .*

## 5 Formal asymptotic expansion

In this section we deal with the formal asymptotic expansion of a solution of problem (1). Although, in contrast with the periodic case, this method fails to work in full generality in the case under consideration, we can use it in order to understand the structure of the leading terms of the difference  $u^\varepsilon - u^0$ . As usually in the multi-scale asymptotic expansion method we consider  $z = x/\varepsilon$  and  $s = t/\varepsilon^2$  as independent variables and use repeatedly the formulae

$$\begin{aligned}\frac{\partial}{\partial x_j} f\left(x, \frac{x}{\varepsilon}\right) &= \left(\frac{\partial}{\partial x_j} f(x, z) + \frac{1}{\varepsilon} \frac{\partial}{\partial z_j} f(x, z)\right)_{z=\frac{x}{\varepsilon}}, \\ \frac{\partial}{\partial t} f\left(t, \frac{t}{\varepsilon^2}\right) &= \left(\frac{\partial}{\partial t} f(t, s) + \frac{1}{\varepsilon^2} \frac{\partial}{\partial s} f(t, s)\right)_{s=\frac{t}{\varepsilon^2}}.\end{aligned}$$

We represent a solution  $u^\varepsilon$  as the following asymptotic series in integer powers of  $\varepsilon$ :

$$(18) \quad u^\varepsilon(x, t) = u^0(x, t) + \varepsilon u^1\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + \varepsilon^2 u^2\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + \dots;$$

here all the functions  $u^j(x, t, z, s)$  are periodic in  $z$ . The dependence of  $s$  is not always stationary.

Substituting the expression on the right-hand side of (18) for  $u^\varepsilon$  in (3) and collecting power-like terms in (3) yields

$$\begin{aligned}\varepsilon^{-1} : \quad & \partial_s u^1 - \operatorname{div}_z(a(z, s)\nabla_z u^1) = -\operatorname{div}_z(a(z, s)\nabla_x u^0). \\ \varepsilon^0 : \quad & \partial_s u^2 - \operatorname{div}_z(a(z, s)\nabla_z u^2) = -\partial_t u^0 + \operatorname{div}_x(a(z, s)\nabla_x u^0) \\ & \quad + \operatorname{div}_z(a(z, s)\nabla_x u^1) + \operatorname{div}_x(a(z, s)\nabla_z u^1). \\ \varepsilon^1 : \quad & \partial_s u^3 - \operatorname{div}_z(a(z, s)\nabla_z u^3) = -\partial_t u^1 + \operatorname{div}_x(a(z, s)\nabla_x u^1) \\ & \quad + \operatorname{div}_z(a(z, s)\nabla_x u^2) + \operatorname{div}_x(a(z, s)\nabla_z u^2).\end{aligned}$$

We will see later on that dealing with the first three equations is sufficient.

The structure of the first equation suggests that

$$u^1(x, t, z, s) = \chi(z, s)\nabla u^0(x, t).$$

with  $\chi$  solving the equation (4)

$$\partial_s \chi - \operatorname{div}_z(a(z, s)\nabla_z \chi) = \operatorname{div}_z(a(z, s)).$$

By Corollary 1 there is a stationary solution  $\chi \in L^\infty(\mathbb{R} \times Y) \cap L^2_{\text{loc}}(\mathbb{R}; H^1(Y))$ . For the sake of definiteness we assume from now on that

$$\int_Y \chi(z, s) dz = 0.$$

One can easily check that this integral does not depend on  $s$  so that the normalization condition makes sense. Notice also that

$$(19) \quad \|\chi\|_{L^\infty(\mathbb{R} \times Y)} \leq C, \quad \|\chi\|_{L^2([0,1]; H^1(Y))} \leq C$$

with a deterministic constant  $C$ . The first inequality follows from the Nash elliptic estimates, see [4], to obtain the second one it suffice to multiply the equation related to  $\varepsilon^{-1}$  by  $\chi$  and integrate the resulting relation over  $Y \times [0, 1]$ .

We turn to the equation related to  $\varepsilon^0$ . We do not reprove here the homogenization results (see [14]) and assume that  $u^0$  satisfies problem (2) with  $a^{\text{eff}}$  given by (5). Then the right-hand side of the equation related to  $\varepsilon^0$  takes the form

$$\begin{aligned} & -\partial_t u^0 + \text{div}_x(a(z, s)\nabla_x u^0) + \text{div}_z(a(z, s)\nabla_x u^1) + \text{div}_x(a(z, s)\nabla_z u^1) = \\ & = \text{div}_x(\{a(z, s)(\mathbf{I} + \nabla_z \chi(z, s)) - a^{\text{eff}}\}\nabla_x u^0) + \text{div}_z(a(z, s)\nabla_x u^1) \end{aligned}$$

By the definition of  $a^{\text{eff}}$  we have

$$\mathbb{E} \int_Y \{a(z, s)(\mathbf{I} + \nabla_z \chi(z, s)) - a^{\text{eff}}\} dz = 0.$$

Letting

$$\Psi_{0,1}(s) = \int_Y \{a(z, s)(\mathbf{I} + \nabla_z \chi(z, s)) - a^{\text{eff}}\} dz$$

and

$$\Psi_{0,2}(z, s) = \{a(z, s)(\mathbf{I} + \nabla_z \chi(z, s)) - a^{\text{eff}}\} - \Psi_{0,1}(s) + \text{div}_z(a(z, s) \otimes \chi(z, s)),$$

we rewrite the equation related to  $\varepsilon^0$  as follows

$$(20) \quad \partial_s u^2 - \text{div}_z(a(z, s)\nabla_z u^2) = (\Psi_{0,1}^{ij}(s) + \Psi_{0,2}^{ij}(z, s)) \frac{\partial^2}{\partial x^i \partial x^j} u^0.$$

Since the process  $\int_0^s \Psi_{0,1}(r)dr$  need not be stationary, we cannot follow any more the same strategy as in the periodic case. Instead, we consider the equation

$$(21) \quad \begin{cases} \frac{\partial V^{\varepsilon,1}}{\partial t} &= \operatorname{div} \left( a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla V^{\varepsilon,1} \right) + \Psi_{0,1}^{ij} \left( \frac{t}{\varepsilon^2} \right) \frac{\partial^2}{\partial x^i \partial x^j} u^0(x, t) \\ V^{\varepsilon,1}(0, x) &= 0. \end{cases}$$

This suggests the representation

$$(22) \quad \begin{aligned} u^\varepsilon(x, t) &= u^0(x, t) + \varepsilon \chi \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u^0(x, t) \\ &\quad + \left\{ \varepsilon^2 u^2 \left( x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) + V^{\varepsilon,1} \right\} + \dots \end{aligned}$$

with

$$u^2 = \chi_{0,2}^{ij}(z, s) \frac{\partial^2}{\partial x^i \partial x^j} u^0(x, t),$$

where  $\chi_{0,2}^{ij}(z, s)$  is a stationary zero mean solution of the equation

$$\partial_s \chi_{0,2}^{ij}(z, s) - \operatorname{div}_z (a(z, s) \nabla_z \chi_{0,2}^{ij}(z, s)) = \Psi_{0,2}^{ij}(z, s).$$

It is straightforward to check that

$$(23) \quad \|\chi_{0,2}\|_{L^2([0,1]; H^1(Y))} \leq C$$

with a deterministic constant  $C$ .

By its definition,  $\Psi_{0,1}(s)$  is a stationary zero average process. Denote

$$\chi_{0,1}^{ij}(s) = \int_0^s \Psi_{0,1}^{ij}(r) dr.$$

Estimates (19) imply that

$$\|\Psi_{0,1}\|_{L^2(0,1)} \leq C$$

with a deterministic constant  $C$ . It follows from Lemma 4.1 that under condition **H3** the maximum correlation coefficient  $\rho_{\Psi_{0,1}}$  is such that

$$\int_0^\infty \rho_{\Psi_{0,1}}(r) dy < \infty.$$

Therefore, the invariance principle holds for this process (see [6, Theorem VIII.3.79]), that is for any  $T > 0$

$$(24) \quad \varepsilon \chi_{0,1} \left( \frac{\cdot}{\varepsilon^2} \right) \xrightarrow{\varepsilon \rightarrow 0} \Lambda^{1/2} W.$$

in law in the space  $(C[0, T])^{n^2}$  with

$$\Lambda^{ijkl} = \int_0^\infty \mathbb{E}(\Psi_{0,1}^{ij}(0)\Psi_{0,1}^{kl}(s) + \Psi_{0,1}^{kl}(0)\Psi_{0,1}^{ij}(s)) ds,$$

here  $W$  is a standard  $n^2$ -dimensional Wiener process. Since the  $n^2 \times n^2$  matrix  $\Lambda$  is symmetric and positive (but not necessary positive definite), its square root is well defined.

**Lemma 5.1** *The functions  $\varepsilon^{-1}V^{\varepsilon,1}$  converges in law, as  $\varepsilon \rightarrow 0$ , in the space  $C((0, T); L^2(\mathbb{R}^n))$  to a unique solution of the following SPDE with a finite dimensional additive noise:*

$$(25) \quad \begin{cases} dV^{0,1} &= \operatorname{div}(a^{\text{eff}} \nabla V^{0,1}) dt + \Lambda^{1/2,ijkl} \frac{\partial^2}{\partial x^i \partial x^j} u^0(x, t) dW_{t,kl} \\ V^{0,1}(0, x) &= 0. \end{cases}$$

**Proof.** The proof is an immediate consequence of (24) and the fact that  $u^0(x, t)$  is a smooth deterministic function vanishing with all its derivatives at infinity.  $\square$

We proceed with the equation related to  $\varepsilon^1$ . Its right-hand side can be rearranged as follows:

$$\begin{aligned} & -\partial_t u^1 + \operatorname{div}_x(a(z, s) \nabla_x u^1) + \operatorname{div}_z(a(z, s) \nabla_x u^2) + \operatorname{div}_x(a(z, s) \nabla_z u^2) \\ &= \left\{ -a^{\text{eff}} \otimes \chi(z, s) + a(z, s) \otimes \chi(z, s) + \operatorname{div}_z[a(z, s) \otimes \chi_{0,2}(z, s)] \right. \\ & \quad \left. + a(z, s) \nabla_z \chi_{0,2}(z, s) \right\} \frac{\partial^3}{\partial x^3} u^0(x, t) := \Psi_{1,2}(z, s) \frac{\partial^3}{\partial x^3} u^0(x, t), \end{aligned}$$

here and in what follows the symbol  $\frac{\partial^3}{\partial x^3} u^0(x, t)$  stands for the tensor of third order partial derivatives of  $u^0$ , that is  $\frac{\partial^3}{\partial x^3} = \left\{ \frac{\partial^3}{\partial x^i \partial x^j \partial x^k} \right\}_{i,j,k=1}^n$ . We introduce the following constant tensor:

$$\mu = \mathbb{E} \int_Y \left\{ -a^{\text{eff}} \otimes \chi(z, s) + a(z, s) \otimes \chi(z, s) + a(z, s) \nabla_z \chi_{0,2}(z, s) \right\} dz$$

and consider the following problems:

$$(26) \quad \begin{cases} \frac{\partial \Xi_{\varepsilon,1}}{\partial t} &= \operatorname{div} \left( a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla \Xi_{\varepsilon,1} \right) + \left( \Psi_{1,2} \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) - \mu \right) \frac{\partial^3}{\partial x^3} u^0(x, t) \\ \Xi_{\varepsilon,1}(x, 0) &= 0, \end{cases}$$

and

$$(27) \quad \begin{cases} \frac{\partial \Xi_{\varepsilon,2}}{\partial t} &= \operatorname{div} \left( a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla \Xi_{\varepsilon,2} \right) + \mu \frac{\partial^3}{\partial x^3} u^0(x, t) \\ \Xi_{\varepsilon,2}(0, x) &= 0. \end{cases}$$

**Lemma 5.2** *A solution of problem (26) tends to zero a.s., as  $\varepsilon \rightarrow 0$ , in  $L^2(\mathbb{R}^n \times [0, T])$ . Moreover,*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left( \|\Xi_{\varepsilon,1}\|_{L^2(\mathbb{R}^n \times [0, T])}^2 \right) = 0.$$

**Proof.** Splitting further the right-hand side  $\Psi_{1,2}$  in (26) into two parts

$$\begin{aligned} \Psi_{1,2}(z, s) &= \operatorname{div}_z [a(z, s) \otimes \chi_{0,2}(z, s)] \\ &\quad + \left\{ (a(z, s) - a^{\text{eff}}) \otimes \chi(z, s) + a(z, s) \nabla_z \chi^{0,2}(z, s) \right\} \\ &= \Psi_{1,2}^1(z, s) + \Psi_{1,2}^2(z, s), \end{aligned}$$

we represent the solution  $\Xi_{\varepsilon,1}$  as the sum  $\Xi_{\varepsilon,1}^1$  and  $\Xi_{\varepsilon,1}^2$ .

Since the right-hand side  $g$  in (1) is smooth and has a finite support, the function  $\frac{\partial^3}{\partial x^3} u^0$  is smooth and decays exponentially as  $|x| \rightarrow \infty$  uniformly in  $t \in [0, T]$ . Combining this with (23) we conclude that

$$\left\| \Psi_{1,2}^1 \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \frac{\partial^3}{\partial x^3} u_0(x, t) \right\|_{L^2([0, T]; H^{-1}(\mathbb{R}^n))} \leq C\varepsilon.$$

Therefore,

$$(28) \quad \|\Xi_{\varepsilon,1}^1\|_{L^2([0, T]; H^1(\mathbb{R}^n))} \leq C\varepsilon.$$

Due to (23) and the properties of  $u_0$ , we have

$$\left\| \left( \Psi_{1,2}^2 \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) - \mu \right) \frac{\partial^3}{\partial x^3} u_0 \right\|_{L^2(\mathbb{R}^n \times (0, T))} \leq C$$

with a deterministic  $C$ . Using Theorem 1.5.1 in [10] we derive from this estimate that a.s. the family  $\Xi_{\varepsilon,1}^2$  is compact in  $L^2((0, T); L_{\text{loc}}^2(\mathbb{R}^n))$ . Considering condition **H2** and Aronson's estimate (see [1]), we then conclude that the family  $\Xi_{\varepsilon,1}^2$  is compact in  $L^2(\mathbb{R}^n \times (0, T))$ .

By the Birkhoff ergodic theorem, the function  $(\Psi_{1,2}^2(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}) - \mu) \frac{\partial^3}{\partial x^3} u_0$  converges a.s. to zero weakly in  $L^2(\mathbb{R}^n \times (0, T))$ . Combining this with the above compactness arguments, we conclude that a.s.  $\Xi_2^{\varepsilon,1}$  converges to zero in  $L^2(\mathbb{R}^n \times (0, T))$ . Then in view of (28),  $\Xi^{\varepsilon,1}$  tends to zero in  $L^2(\mathbb{R}^n \times (0, T))$  a.s. This yields the first statement of the lemma. The second statement follows from the first one by the Lebesgue dominated convergence theorem.  $\square$

According to [14], problem (27) admits homogenization. In particular,  $\Xi^{\varepsilon,2}$  converges a.s. in  $L^2(\mathbb{R}^n \times (0, T))$  to a solution of the following problem:

$$(29) \quad \begin{cases} \frac{\partial \Xi^{0,2}}{\partial t} &= \operatorname{div}(a^{\text{eff}} \nabla \Xi^{0,2}) + \mu \frac{\partial^3}{\partial x^3} u^0(x, t) \\ \Xi^{0,2}(0, x) &= 0 \end{cases}$$

This is not the end of the story with the asymptotic expansion because the initial condition is not satisfied at the level  $\varepsilon^1$ . In order to fix this problem we introduce one more term of order  $\varepsilon^1$  so that the expansion takes the form

$$(30) \quad \begin{aligned} u^\varepsilon(x, t) &= u^0(x, t) + \varepsilon \left( \chi\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + \chi_{\text{il}}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \right) \nabla u^0(x, t) \\ &\quad + \varepsilon^2 \left\{ u^2\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + V^{\varepsilon,1} \right\} + \dots \end{aligned}$$

The initial layer type function  $\chi_{\text{il}}$  has been added in order to compensate the discrepancy in the initial condition. This function solves the following problem:

$$(31) \quad \begin{cases} \frac{\partial \chi_{\text{il}}}{\partial s} &= \operatorname{div}(a(y, s) \nabla \chi_{\text{il}}) \\ \chi_{\text{il}}(0, y) &= -\chi(0, y). \end{cases}$$

**Lemma 5.3** *A solution of problem (31) decays exponentially as  $s \rightarrow \infty$ .*

**Proof.** The desired statement is an immediate consequence of the parabolic Harnack inequality.  $\square$



## 6 Main results

In this section we present the main result. Consider the expression

$$U^\varepsilon(x, t) = \frac{u^\varepsilon(x, t) - u^0(x, t)}{\varepsilon} - \chi\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \nabla_x u^0$$

In fact,  $U^\varepsilon$  is equal to the normalized difference between  $u^\varepsilon$  and the first two terms of the asymptotic expansion. The limit behaviour of  $U^\varepsilon$  is described by the following statement.

**Theorem 3** *Under the assumptions **H1–H3** the function  $U^\varepsilon$  converges in law, as  $\varepsilon \rightarrow 0$ , in the space  $L^2(\mathbb{R}^n \times (0, T))$  to a solution of the following SPDE*

$$(32) \quad \begin{aligned} dU^0 &= \operatorname{div}\left(a^{\text{eff}} \nabla U^0 + \mu \frac{\partial^3}{\partial x^3} u^0\right) dt + \Lambda^{1/2} \frac{\partial^2}{\partial x^2} u^0 dW_t, \\ U^0(x, 0) &= 0. \end{aligned}$$

**Proof.** We set

$$\begin{aligned} \mathcal{V}^\varepsilon(x, t) &= U^\varepsilon(x, t) - \varepsilon^{-1} V^{\varepsilon,1}(x, t) - \Xi^{\varepsilon,2}(x, t) \\ &\quad - \chi_{\text{il}}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \nabla u^0(x, t) - \varepsilon \chi^{0,2}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \frac{\partial^2}{\partial x^2} u^0(x, t) - \Xi^{\varepsilon,1}(x, t). \end{aligned}$$

Substituting this expression in (1) for  $u^\varepsilon$  and combining the above equations, we obtain after straightforward computations that  $\mathcal{V}^\varepsilon$  satisfies the problem

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{V}^\varepsilon - \operatorname{div}\left(a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \nabla \mathcal{V}^\varepsilon\right) &= R^\varepsilon, \\ \mathcal{V}^\varepsilon(x, 0) &= R_1^\varepsilon \end{aligned}$$

where  $R^\varepsilon$  a.s. tends to zero in  $L^2(\mathbb{R}^n \times (0, T))$ , as  $\varepsilon \rightarrow 0$ , and  $R^\varepsilon$  a.s. tends to zero in  $L^2(\mathbb{R}^n)$ . Moreover,

$$\mathbb{E} \|R^\varepsilon\|_{L^2(\mathbb{R}^n \times (0, T))}^2 \rightarrow 0, \quad \mathbb{E} \|R_1^\varepsilon\|_{L^2(\mathbb{R}^n)}^2 \rightarrow 0.$$

Considering Lemmata 5.2 and 5.3, we conclude that  $(U^\varepsilon - \varepsilon^{-1} V^{\varepsilon,1} - \Xi^{\varepsilon,2})$  tends a.s. to zero in  $L^2(\mathbb{R}^n \times (0, T))$ , and

$$\mathbb{E} \|U^\varepsilon - \varepsilon^{-1} V^{\varepsilon,1}(x, t) - \Xi^{\varepsilon,2}(x, t)\|_{L^2(\mathbb{R}^n \times (0, T))}^2 \rightarrow 0.$$

By Lemma 5.1 the function  $\varepsilon^{-1}V^{\varepsilon,1}$  converges in law to a solution of (25). Also,  $\Xi^{\varepsilon,2}$  converges a.s. to  $\Xi^{0,2}$  in  $L^2(\mathbb{R}^n \times (0, T))$ . This yields the convergence

$$U^\varepsilon \longrightarrow V^{0,1} + \Xi^{0,2}$$

in law in the space  $L^2(\mathbb{R}^n \times (0, T))$ . It remains to note that due to (25) and (29) the random function  $U^0 := (V^{0,1} + \Xi^{0,2})$  satisfies the stochastic PDE (32) as required.  $\square$

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