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The quenched limiting distributions of a one-dimensional random walk in random scenery

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Summary. For a one-dimensional random walk in random scenery (RWRS) on \mathbb{Z} , we determine its quenched weak limits by applying Strassen [13]'s functional law of the iterated logarithm. As a consequence, conditioned on the random scenery, the one-dimensional RWRS does not converge in law, in contrast with the multi-dimensional case.

Keywords: Random walk in random scenery; Weak limit theorem; Law of the iterated logarithm; Brownian motion in Brownian Scenery; Strong approximation.

AMS Subject Classification: 60F05, 60G52.

1. INTRODUCTION

Random walks in random sceneries were introduced independently by Kesten and Spitzer [9] and by Borodin [3, 4]. Let $S = (S_n)_{n \geq 0}$ be a random walk in \mathbb{Z}^d starting at 0, i.e., $S_0 = 0$ and $(S_n - S_{n-1})_{n \geq 1}$ is a sequence of i.i.d. \mathbb{Z}^d -valued random variables. Let $\xi = (\xi_x)_{x \in \mathbb{Z}^d}$ be a field of i.i.d. real random variables independent of S . The field ξ is called the random scenery. The random walk in random scenery (RWRS) $K := (K_n)_{n \geq 0}$ is defined by setting $K_0 := 0$ and, for $n \in \mathbb{N}^*$,

$$K_n := \sum_{i=1}^n \xi_{S_i}. \quad (1)$$

We will denote by \mathbb{P} the joint law of S and ξ . The law \mathbb{P} is called the *annealed* law, while the conditional law $\mathbb{P}(\cdot | \xi)$ is called the *quenched* law.

Limit theorems for RWRS have a long history, we refer to [7] or [8] for a complete review. Distributional limit theorems for *quenched* sceneries (i.e. under the quenched law) are however quite recent. The first result in this direction that we are aware of was obtained by Ben Arous and Černý [1], in the case of a heavy-tailed scenery and planar random walk. In [7], quenched central limit theorems (with the usual \sqrt{n} -scaling and Gaussian law in the limit) were proved for a large class of transient random walks. More recently, in [8], the case of the planar random walk was studied, the authors proved a quenched version of the annealed central limit theorem obtained by Bolthausen in [2].

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In this note we consider the case of the simple symmetric random walk $(S_n)_{n \geq 0}$ on \mathbb{Z} , the random scenery $(\xi_x)_{x \in \mathbb{Z}}$ is assumed to be centered with finite variance equal to one and there exists some $\delta > 0$ such that $\mathbb{E}(|\xi_0|^{2+\delta}) < \infty$. We prove that under these assumptions, there is no quenched distributional limit theorem for K . In the sequel, for $-\infty \leq a < b \leq \infty$, we will denote by $\mathcal{AC}([a, b] \rightarrow \mathbb{R})$ the set of absolutely continuous functions defined on the interval $[a, b]$ with values in \mathbb{R} . Recall that if $f \in \mathcal{AC}([a, b] \rightarrow \mathbb{R})$, then the derivative of f (denoted by \dot{f}) exists almost everywhere and is Lebesgue integrable on $[a, b]$. Define

$$\mathcal{K}^* := \left\{ f \in \mathcal{AC}(\mathbb{R} \rightarrow \mathbb{R}) : f(0) = 0, \int_{-\infty}^{\infty} (\dot{f}(x))^2 dx \leq 1 \right\}. \quad (2)$$

Theorem 1. *For \mathbb{P} -a.e. ξ , under the quenched probability $\mathbb{P}(\cdot | \xi)$, the process*

$$\tilde{K}_n := \frac{K_n}{(2n^{3/2} \log \log n)^{1/2}}, \quad n > e^e,$$

does not converge in law. More precisely, for \mathbb{P} -a.e. ξ , under the quenched probability $\mathbb{P}(\cdot | \xi)$, the limit points of the law of \tilde{K}_n , as $n \rightarrow \infty$, under the topology of weak convergence of measures, are equal to the set of the laws of random variables in Θ_B , with

$$\Theta_B := \left\{ \int_{-\infty}^{\infty} f(x) dL_1(x) : f \in \mathcal{K}^* \right\}, \quad (3)$$

where $(L_1(x), x \in \mathbb{R})$ denotes the family of local times at time 1 of a one-dimensional Brownian motion B starting from 0.

The set Θ_B is closed for the topology of weak convergence of measures, and is a compact subset of $L^2((B_t)_{t \in [0,1]})$.

Let us mention that the set \mathcal{K}^* directly comes from Strassen [13]'s limiting set. The precise meaning of $\int_{-\infty}^{\infty} f(x) dL_1(x)$ can be given by the integration by parts and the occupation times formula:

$$\int_{-\infty}^{\infty} f(x) dL_1(x) = - \int_{-\infty}^{\infty} L_1(x) \dot{f}(x) dx = - \int_0^1 \dot{f}(B_s) ds, \quad (4)$$

where as before, \dot{f} denotes the almost everywhere derivative of f .

Instead of Theorem 1, we shall prove that there is no quenched limit theorem for the continuous analogue of K introduced by Kesten and Spitzer [9] and deduce Theorem 1 by using a strong approximation for the one-dimensional RWRS. Let us define this continuous analogue: Assume that $B := (B(t))_{t \geq 0}$, $W := (W(t))_{t \geq 0}$, $\tilde{W} := (\tilde{W}(t))_{t \geq 0}$ are three real Brownian motions starting from 0, defined on the same probability space and independent of each other. For brevity, we shall write $W(x) := W(x)$ if $x \geq 0$ and $\tilde{W}(-x)$ if $x < 0$ and say that W is a two-sided Brownian motion. We denote by $\mathbb{P}_B, \mathbb{P}_W$ the law of these processes. We will also denote by $(L_t(x))_{t \geq 0, x \in \mathbb{R}}$ a continuous version with compact support of the local time of the process B . We define the continuous version of the RWRS, also called *Brownian motion in Brownian scenery*, as

$$Z_t := \int_0^{+\infty} L_t(x) dW(x) + \int_0^{+\infty} L_t(-x) d\tilde{W}(x) \equiv \int_{-\infty}^{+\infty} L_t(x) dW(x).$$

In dimension one, under the annealed measure, Kesten and Spitzer [9] proved that the process $(n^{-3/4} K([nt]))_{t \geq 0}$ weakly converges in the space of continuous functions to the continuous process $Z = (Z(t))_{t \geq 0}$. Zhang [14] (see also [6, 10]) gave a stronger version of this result in the special case when the scenery has a finite moment of order $2 + \delta$ for some $\delta > 0$, more precisely, there is

a coupling of ξ , S , B and W such that (ξ, W) is independent of (S, B) and for any $\varepsilon > 0$, almost surely,

$$\max_{0 \leq m \leq n} |K(m) - Z(m)| = o(n^{\frac{1}{2} + \frac{1}{2(2+\delta)} + \varepsilon}), \quad n \rightarrow +\infty. \quad (5)$$

Theorem 1 will follow from this strong approximation and the following result.

Theorem 2. \mathbb{P}_W -almost surely, under the quenched probability $\mathbb{P}(\cdot|W)$, the limit points of the law of

$$\tilde{Z}_t := \frac{Z_t}{(2t^{3/2} \log \log t)^{1/2}}, \quad t \rightarrow \infty,$$

under the topology of weak convergence of measures, are equal to the set of the laws of random variables in Θ_B defined in Theorem 1. Consequently under $\mathbb{P}(\cdot|W)$, as $t \rightarrow \infty$, \tilde{Z}_t does not converge in law.

To prove Theorem 2, we shall apply Strassen [13]'s functional law of the iterated logarithm applied to the two-sided Brownian motion W ; we shall also need to estimate the stochastic integral $\int g(x)dL_1(x)$ for a Borel function g , see Section 2 for the details.

2. PROOFS

For a two-sided one-dimensional Brownian motion $(W(t), t \in \mathbb{R})$ starting from 0, let us define for any $\lambda > e^e$,

$$W_\lambda(t) := \frac{W(\lambda t)}{(2\lambda \log \log \lambda)^{1/2}}, \quad t \in \mathbb{R}.$$

Lemma 3. (i) Almost surely, for any $s < 0 < r$ rational numbers, $(W_\lambda(t), s \leq t \leq r)$ is relatively compact in the uniform topology and the set of its limit points is $\mathcal{K}_{s,r}$, with

$$\mathcal{K}_{s,r} := \left\{ f \in \mathcal{AC}([s, r] \rightarrow \mathbb{R}) : f(0) = 0, \int_s^r (\dot{f}(x))^2 dx \leq 1 \right\}.$$

(ii) There exists some finite random variable \mathcal{A}_W only depending on $(W(x), x \in \mathbb{R})$ such that for all $\lambda \geq e^{36}$,

$$\sup_{t \in \mathbb{R}, t \neq 0} \frac{|W_\lambda(t)|}{\sqrt{|t| \log \log(|t| + \frac{1}{|t|}) + 36}} \leq \mathcal{A}_W < \infty.$$

Remark 4. The statement (i) is a reformulation of Strassen's theorem and holds in fact for all real numbers s and r . Moreover, using the notation \mathcal{K}^* in (2), we remark that $\mathcal{K}_{s,r}$ coincides with the restriction of \mathcal{K}^* on $[s, r]$: for any $s < 0 < r$,

$$\mathcal{K}_{s,r} = \left\{ f|_{[s,r]} : f \in \mathcal{K}^* \right\}.$$

Proof: (i) For any fixed $s < 0 < r$, by applying Strassen's theorem ([13]) to the two-dimensional rescaled Brownian motion: $(\frac{W(\lambda r u)}{\sqrt{2\lambda r \log \log \lambda}}, \frac{W(\lambda s u)}{\sqrt{2\lambda |s| \log \log \lambda}})_{0 \leq u \leq 1}$, we get that a.s., $(W_\lambda(t), s \leq t \leq r)$ is relatively compact in the uniform topology with $\mathcal{K}_{s,r}$ as the set of limit points. By inverting a.s. and s, r , we obtain (i).

(ii) By the classical law of the iterated logarithm for the Brownian motion W (both at 0 and at ∞), we get that

$$\tilde{\mathcal{A}}_W := \sup_{x \in \mathbb{R}, x \neq 0} \frac{|W(x)|}{\sqrt{|x| \log \log(|x| + \frac{1}{|x|}) + 36}}$$

is a finite variable. Observe that for any $t > 0$ and $\lambda > e^{36}$, $\log \log(\lambda t + \frac{1}{\lambda t} + 36) \leq \log \log \lambda + \log \log(t + \frac{1}{t} + 36)$. The Lemma follows if we take for e.g. $\mathcal{A}_W := 2\tilde{\mathcal{A}}_W$. \square

Next, we recall some properties of Brownian local times: The process $x \mapsto L_1(x)$ is a (continuous) semimartingale (by Perkins [11]), moreover, the quadratic variation of $x \mapsto L_1(x)$ equals $4 \int_{-\infty}^x L_1(z) dz$. By Revuz and Yor ([12], Exercise VI (1.28)), for any locally bounded Borel function f ,

$$\frac{1}{2} \int_{-\infty}^{\infty} f(x) dL_1(x) = - \int_0^{B_1} f(u) du + \int_0^1 f(B_u) dB_u. \quad (6)$$

Let us define for all $\lambda > e^e$ and $n \geq 0$,

$$H_\lambda := \int_{-\infty}^{\infty} W_\lambda(x) dL_1(x), \quad H_\lambda^{(n)} := \int_{-n}^n W_\lambda(x) dL_1(x),$$

with $H_\lambda^{(0)} = 0$. Denote by \mathbb{E}_B the expectation with respect to the law of B .

Lemma 5. *There exists some positive constant c_1 such that for any $\lambda > e^{36}$ and $n \geq 0$, we have*

$$\mathbb{E}_B \left| H_\lambda - H_\lambda^{(n)} \right| \leq c_1 e^{-\frac{n^2}{4}} \mathcal{A}_W, \quad (7)$$

$$\mathbb{E}_B \left(\int_{-\infty}^{\infty} f(x) dL_1(x) \right)^2 \leq 16 s(f), \quad (8)$$

$$\mathbb{E}_B \left| \int_{-\infty}^{\infty} f(x) dL_1(x) - \int_{-n}^n f(x) dL_1(x) \right| \leq 4 \sqrt{2s(f)} e^{-\frac{n^2}{4}}, \quad (9)$$

for any Borel function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $s(f) := \sup_{0 \leq u \leq 1} \mathbb{E}_B [f^2(B_u)] < \infty$.

Remark that if f is bounded, then $s(f) \leq \sup_{x \in \mathbb{R}} f^2(x)$.

Proof: We first prove that there exists some positive constant c_2 such that for all $n \geq 0$ and $\lambda > e^{36}$,

$$\mathbb{E}_B \left[(H_\lambda - H_\lambda^{(n)})^2 \right] \leq c_2 \mathcal{A}_W^2. \quad (10)$$

In fact, by applying (6) and using the Brownian isometry for $f(x) = W_\lambda(x) 1_{(|x| > n)}$, we get that

$$\mathbb{E}_B \left[(H_\lambda - H_\lambda^{(n)})^2 \right] \leq 8 \mathbb{E}_B \left[F_{n,\lambda}(B_1)^2 \right] + 8 \mathbb{E}_B \left[\int_0^1 (W_\lambda(B_u))^2 1_{(|B_u| > n)} du \right],$$

with $F_{n,\lambda}(x) := \int_0^x W_\lambda(y) 1_{(|y| > n)} dy$ for any $x \in \mathbb{R}$. By Lemma 3 (ii),

$$|F_{n,\lambda}(x)| \leq \mathcal{A}_W \left| \int_0^x (|y| \log \log(|y| + \frac{1}{|y|} + 36))^{1/2} dy \right| \leq c_3 \mathcal{A}_W (1 + x^2), \quad \forall x \in \mathbb{R},$$

with some constant $c_3 > 0$. Hence $\mathbb{E}_B \left[F_{n,\lambda}(B_1)^2 \right] \leq 6 c_3^2 \mathcal{A}_W^2$. In the same way, $\mathbb{E}_B \left[(W_\lambda(B_u))^2 \right] \leq \mathcal{A}_W^2 \mathbb{E}_B \left[|B_u| \log \log(|B_u| + \frac{1}{|B_u|} + 36) \right]$ which is integrable for $u \in (0, 1]$. Then (10) follows.

To check (7), we remark that $H_\lambda - H_\lambda^{(n)} = 0$ if $\sup_{0 \leq u \leq 1} |B_u| \leq n$. Then by Cauchy-Schwarz' inequality and (10), we have that

$$\begin{aligned} \mathbb{E}_B \left| H_\lambda - H_\lambda^{(n)} \right| &= \mathbb{E}_B \left[\left| H_\lambda - H_\lambda^{(n)} \right| 1_{(\sup_{0 \leq u \leq 1} |B_u| > n)} \right] \\ &\leq \sqrt{\mathbb{E}_B \left[(H_\lambda - H_\lambda^{(n)})^2 \right]} \sqrt{\mathbb{P}_B \left(\sup_{0 \leq u \leq 1} |B_u| > n \right)} \\ &\leq \sqrt{c_2} \mathcal{A}_W \sqrt{2} e^{-\frac{n^2}{4}}, \end{aligned}$$

by the standard Gaussian tail: $\mathbb{P}_B(\sup_{0 \leq u \leq 1} |B_u| > x) \leq 2e^{-x^2/2}$ for any $x > 0$. Then we get (7).

To prove (8), we use again (6) and the Brownian isometry to arrive at

$$\mathbb{E}_B \left(\int_{-\infty}^{\infty} f(x) dL_1(x) \right)^2 \leq 8\mathbb{E}_B \left[G^2(B_1) \right] + 8 \int_0^1 \mathbb{E}_B \left[f^2(B_u) \right] du \leq 8\mathbb{E}_B \left[G^2(B_1) \right] + 8s(f),$$

with $G(x) := \int_0^x f(y) dy$ for any $x \in \mathbb{R}$. By Cauchy-Schwarz' inequality, $(G(x))^2 \leq \left| x \int_0^x f^2(y) dy \right|$ for any $x \in \mathbb{R}$, from which we use the integration by parts for the density of B_1 and deduce that $\mathbb{E}_B \left[G^2(B_1) \right] \leq \mathbb{E}_B \left[f^2(B_1) \right]$. Then (8) follows.

Finally for (9), we use (8) to see that

$$\mathbb{E}_B \left(\int_{-\infty}^{\infty} f(x) dL_1(x) - \int_{-n}^n f(x) dL_1(x) \right)^2 = \mathbb{E}_B \left(\int_{-\infty}^{\infty} f(x) 1_{(|x| > n)} dL_1(x) \right)^2 \leq 16s(f),$$

for any n . Then (9) follows from the Cauchy-Schwarz inequality and the Gaussian tail, exactly in the same way as (7). \square

Recalling (3) for the definition of Θ_B . For any $p > 0$, it is easy to see that $\Theta_B \subset L^p(B)$, since from Cauchy-Schwarz' inequality, using the relation (4), we deduce that

$$\left(\int_{-\infty}^{\infty} f(x) dL_1(x) \right)^2 \leq \left(\int_{-\infty}^{\infty} (L_1(x))^2 dx \right) \left(\int_{-\infty}^{\infty} (f(x))^2 dx \right) \leq \sup_x L_1(x) \in L^p(B),$$

see Csáki [5], Lemma 1 for the tail of $\sup_x L_1(x)$. Write $d_{L^1(B)}(\xi, \eta)$ for the distance in $L^1(B)$ for any $\xi, \eta \in L^1(B)$.

Lemma 6. \mathbb{P}_W -almost surely,

$$d_{L^1(B)}(H_\lambda, \Theta_B) \rightarrow 0, \quad \text{as } \lambda \rightarrow \infty,$$

where Θ_B is defined in (3). Moreover, \mathbb{P}_W -almost surely for any $\xi \in \Theta_B$, $\liminf_{\lambda \rightarrow \infty} d_{L^1(B)}(H_\lambda, \xi) = 0$.

Proof: Let $\varepsilon > 0$. Choose a large $n = n(\varepsilon)$ such that $c_1 e^{-n^2/4} \leq \varepsilon$. By Lemma 3 (i), for all large $\lambda \geq \lambda_0(W, \varepsilon, n)$, there exists some function $g = g_{\lambda, W, \varepsilon, n} \in \mathcal{K}_{-n, n}$ such that $\sup_{|x| \leq n} |W_\lambda(x) - g(x)| \leq \varepsilon$. Applying (8) to $f(x) = (W_\lambda(x) - g(x)) 1_{(|x| \leq n)}$ which is a bounded by ε , we get that

$$\mathbb{E}_B \left| H_\lambda^{(n)} - \int_{-n}^n g(x) dL_1(x) \right| \leq 4\sqrt{s(f)} \leq 4\varepsilon.$$

We extend g to \mathbb{R} by letting $g(x) = g(n)$ if $x \geq n$ and $g(x) = g(-n)$ if $x \leq -n$, then $g \in \mathcal{K}^*$ and $\int_{-\infty}^{\infty} g(x)dL_1(x) = \int_{-n}^n g(x)dL_1(x)$. By the triangular inequality and (7),

$$\mathbb{E}_B \left| H_\lambda - \int_{-\infty}^{\infty} g(x)dL_1(x) \right| \leq 4\varepsilon + \mathbb{E}_B \left| H_\lambda - H_\lambda^{(n)} \right| \leq (4 + c_1 \mathcal{A}_W) \varepsilon.$$

It follows that $d_{L^1(B)}(H_\lambda, \Theta_B) \leq (4 + c_1 \mathcal{A}_W) \varepsilon$. Hence \mathbb{P}_W -a.s., $\limsup_{\lambda \rightarrow \infty} d_{L^1(B)}(H_\lambda, \Theta_B) \leq (4 + c_1 \mathcal{A}_W) \varepsilon$, showing the first part in the Lemma.

For the other part of the Lemma, let $h \in \mathcal{K}^*$ such that $\xi = \int_{-\infty}^{\infty} h(x)dL_1(x)$. Observe that $|h(x)| \leq \sqrt{\left| x \int_0^x (\dot{h}(y))^2 dy \right|} \leq \sqrt{|x|}$ for all $x \in \mathbb{R}$, $s(h) = \sup_{0 \leq u \leq 1} \mathbb{E}_B[h^2(B_u)] \leq \mathbb{E}_B[|B_1|]$, then for any $\varepsilon > 0$, we may use (9) and choose an integer $n = n(\varepsilon)$ such that $(c_1 + 4\sqrt{2})e^{-n^2/4} \leq \varepsilon$ and

$$d_{L^1(B)}(\xi, \xi_n) \leq \varepsilon,$$

where $\xi_n := \int_{-n}^n h(x)dL_1(x)$. Applying Lemma 3 (i) to the restriction of h on $[-n, n]$, we may find a sequence $\lambda_j = \lambda_j(\varepsilon, W, n) \rightarrow \infty$ such that $\sup_{|x| \leq n} |W_{\lambda_j}(x) - h(x)| \leq \varepsilon$. By applying (8) to $f(x) = (W_{\lambda_j}(x) - h(x))1_{(|x| \leq n)}$, we have that

$$d_{L^1(B)}(H_{\lambda_j}^{(n)}, \xi_n) \leq 4\varepsilon.$$

By (7) and the choice of n , $d_{L^1(B)}(H_{\lambda_j}^{(n)}, H_{\lambda_j}) \leq \varepsilon \mathcal{A}_W$ for all large λ_j , it follows from the triangular inequality that

$$d_{L^1(B)}(\xi, H_{\lambda_j}) \leq (5 + \mathcal{A}_W) \varepsilon,$$

implying that \mathbb{P}_W -a.s., $\liminf_{\lambda \rightarrow \infty} d_{L^1(B)}(H_\lambda, \xi) \leq (5 + \mathcal{A}_W) \varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. \square

We now are ready to give the proof of Theorems 2 and 1.

Proof of Theorem 2. Firstly, we remark that by Brownian scaling, \mathbb{P}_W -a.s.,

$$\frac{Z_t}{t^{3/4}} \stackrel{(d)}{=} - \int_{m_1}^{M_1} \frac{1}{t^{1/4}} W(\sqrt{t}y) dL_1(y). \quad (11)$$

In fact, by the change of variables $x = y\sqrt{t}$, we get

$$\int_{-\infty}^{+\infty} L_t(x) dW(x) = \sqrt{t} \int_{-\infty}^{+\infty} \left(\frac{L_t(y\sqrt{t})}{\sqrt{t}} \right) dW(y\sqrt{t})$$

which has the same distribution as

$$\sqrt{t} \int_{-\infty}^{+\infty} L_1(y) dW(y\sqrt{t})$$

from the scaling property of the local time of the Brownian motion. Since $(L_1(x))_{x \in \mathbb{R}}$ is a continuous semi-martingale, independent from the process W , from the formula of integration by parts, we get that \mathbb{P}_W -a.s.,

$$\sqrt{t} \int_{-\infty}^{+\infty} L_1(y) dW(y\sqrt{t}) = -t^{3/4} \int_{m_1}^{M_1} \left(\frac{W(\sqrt{t}y)}{t^{1/4}} \right) dL_1(y),$$

yielding (11). The first part of Theorem 2 follows from Lemma 6.

Let $(\zeta_n)_n$ be a sequence of random variables in Θ_B , each ζ_n being associated to a function $f_n \in \mathcal{K}^*$. The sequence of the (almost everywhere) derivatives of f_n is then a bounded sequence in the Hilbert space $L^2(\mathbb{R})$, so we can extract a subsequence which weakly converges. Using the definition of the weak convergence and the relation (4), $(\zeta_n)_n$ converges almost surely and

the closure of Θ_B follows. Since the sequence $(\zeta_n)_n$ is bounded in $L^p(B)$ for any $p \geq 1$, the convergence also holds in $L^2(B)$. Therefore Θ_B is a compact set of $L^2(B)$ as closed and bounded subset. \square

Proof of Theorem 1. We use the strong approximation of Zhang [14] : there exists on a suitably enlarged probability space, a coupling of ξ , S , B and W such that (ξ, W) is independent of (S, B) and for any $\varepsilon > 0$, almost surely,

$$\max_{0 \leq m \leq n} |K(m) - Z(m)| = o(n^{\frac{1}{2} + \frac{1}{2(2+\delta)} + \varepsilon}), \quad n \rightarrow +\infty.$$

From the independence of (ξ, W) and (S, B) , we deduce that for \mathbb{P} -a.e. (ξ, W) , under the quenched probability $\mathbb{P}(\cdot | \xi, W)$, the limit points of the laws of \tilde{K}_n and \tilde{Z}_n are the same ones. Now, by adapting the proof of Theorem 2, we have that for \mathbb{P} -a.e. (ξ, W) , under the quenched probability $\mathbb{P}(\cdot | \xi, W)$, the limit points of the laws of \tilde{Z}_n , as $n \rightarrow \infty$, under the topology of weak convergence of measures, are equal to the set of the laws of random variables in Θ_B . It gives that for \mathbb{P} -a.e. (ξ, W) , under the quenched probability $\mathbb{P}(\cdot | \xi, W)$, the limit points of the laws of \tilde{K}_n , as $n \rightarrow \infty$, under the topology of weak convergence of measures, are equal to the set of the laws of random variables in Θ_B and Theorem 1 follows. \square

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