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► **To cite this version:**

Nadine Guillotin-Plantard, Yueyun Hu. The quenched limiting distributions of a one-dimensional random walk in random scenery.. 2013. hal-00842602v1

**HAL Id: hal-00842602**

**<https://hal.science/hal-00842602v1>**

Preprint submitted on 8 Jul 2013 (v1), last revised 17 Sep 2013 (v2)

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# The quenched limiting distributions of a one-dimensional random walk in random scenery

Nadine Guillotin-Plantard\* and Yueyun Hu†

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**Summary.** For a one-dimensional random walk in random scenery (RWRS) on  $\mathbb{Z}$ , we determine its quenched weak limits by applying Strassen [14]’s functional law of iterated logarithm. As a consequence, conditioned on the random scenery, the one-dimensional RWRS does not converge in law, in contrast with the multi-dimensional case.

**Keywords:** Random walk in random scenery; Weak limit theorem; Law of the iterated logarithm; Brownian motion in Brownian Scenery; Strong approximation.

**AMS Subject Classification:** 60F05, 60G52.

## 1. INTRODUCTION

Random walks in random sceneries were introduced independently by Kesten and Spitzer [9] and by Borodin [3, 4]. Let  $S = (S_n)_{n \geq 0}$  be a random walk in  $\mathbb{Z}^d$  starting at 0, i.e.,  $S_0 = 0$  and  $(S_n - S_{n-1})_{n \geq 1}$  is a sequence of i.i.d.  $\mathbb{Z}^d$ -valued random variables. Let  $\xi = (\xi_x)_{x \in \mathbb{Z}^d}$  be a field of i.i.d. real random variables independent of  $S$ . The field  $\xi$  is called the random scenery. The random walk in random scenery (RWRS)  $K := (K_n)_{n \geq 0}$  is defined by setting  $K_0 := 0$  and, for  $n \in \mathbb{N}^*$ ,

$$K_n := \sum_{i=1}^n \xi_{S_i}. \quad (1.1)$$

We will denote by  $\mathbb{P}$  the joint law of  $S$  and  $\xi$ . The law  $\mathbb{P}$  is called the *annealed* law, while the conditional law  $\mathbb{P}(\cdot|\xi)$  is called the *quenched* law.

Limit theorems for RWRS have a long history, we refer to [6] or [7] for a complete review. Distributional limit theorems for *quenched* sceneries (i.e. under the quenched law) are however quite recent. The first result in this direction that we are aware of was obtained by Ben Arous and Černý [1], in the case of a heavy-tailed scenery and planar random walk. In [6], quenched central limit theorems (with the usual  $\sqrt{n}$ -scaling and Gaussian law in the limit) were proved for a large class of transient random walks. More recently, in [7], the case of the planar random walk was studied, the authors proved a quenched version of the annealed central limit theorem obtained by Bolthausen in [2].

In this note we consider the case of the simple symmetric random walk  $(S_n)_{n \geq 0}$  on  $\mathbb{Z}$ , the random scenery  $(\xi_x)_{x \in \mathbb{Z}}$  is assumed to be centered with finite variance equal to one and there

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exists some  $\delta > 0$  such that  $\mathbb{E}(|\xi_0|^{2+\delta}) < \infty$ . We prove that under these assumptions, there is no quenched distributional limit theorem for  $K$ .

**Theorem 1.** For  $\mathbb{P}$ -a.e.  $\xi$ , under the quenched probability  $\mathbb{P}(\cdot | \xi)$ , the process

$$\tilde{K}_n := \frac{K_n}{(2n^{3/2} \log \log n)^{1/2}}, \quad n > e^e,$$

does not converge in law. More precisely, for  $\mathbb{P}$ -a.e.  $\xi$ , under the quenched probability  $\mathbb{P}(\cdot | \xi)$ , the limit points of the law of  $\tilde{K}_n$ , as  $n \rightarrow \infty$ , under the topology of weak convergence of measures, are equal to the closure of the laws of random variables in  $\mathcal{K}_B$ , with

$$\mathcal{K}_B := \left\{ \int_{m_1}^{M_1} f(x) dL_1(x) : f \in \mathcal{C}([m_1, M_1] \rightarrow \mathbb{R}) \text{ such that } f(0) = 0, \int_{m_1}^{M_1} (\dot{f}(x))^2 dx \leq 1 \right\}, \quad (1.2)$$

where  $(L_1(x), x \in \mathbb{R})$  denotes the family of local times at time 1 of a one-dimensional Brownian motion  $B$  starting from 0 and  $m_1 := \inf_{0 \leq s \leq 1} B_s$ ,  $M_1 := \sup_{0 \leq s \leq 1} B_s$ .

Let us mention that the random set  $\mathcal{K}_B$  directly comes from Strassen [14]'s limiting set. Moreover,  $\dot{f}$  is the derivative of the (random) function  $f$  and the precise meaning of  $\int_{m_1}^{M_1} f(x) dL_1(x)$  can be given by the integration by parts and the occupation times formulas:  $\int_{m_1}^{M_1} f(x) dL_1(x) = - \int_{-\infty}^{\infty} L_1(x) \dot{f}(x) dx = - \int_0^1 \dot{f}(B_s) ds$ .

Instead of Theorem 1, we shall prove that there is no quenched limit theorem for the continuous analogue of  $K$  introduced by Kesten and Spitzer [9] and deduce Theorem 1 by using a strong approximation for the one-dimensional RWRS. Let us define this continuous analogue: Assume that  $B := (B(t))_{t \geq 0}$ ,  $W := (W(t))_{t \geq 0}$ ,  $\tilde{W} := (\tilde{W}(t))_{t \geq 0}$  are three real Brownian motions starting from 0, defined on the same probability space and independent of each other. For brevity, we shall write  $W(x) := W(x)$  if  $x \geq 0$  and  $\tilde{W}(-x)$  if  $x < 0$  and say that  $W$  is a two-sided Brownian motion. We denote by  $\mathbb{P}_B, \mathbb{P}_W$  the law of these processes. We will also denote by  $(L_t(x))_{t \geq 0, x \in \mathbb{R}}$  a continuous version with compact support of the local time of the process  $B$ . We define the continuous version of the RWRS, also called *Brownian motion in Brownian scenery*, as

$$Z_t := \int_0^{+\infty} L_t(x) dW(x) + \int_0^{+\infty} L_t(-x) d\tilde{W}(x) \equiv \int_{-\infty}^{+\infty} L_t(x) dW(x).$$

It was proved by Kesten and Spitzer [9] that the process  $(n^{-3/4} K([nt]))_{t \geq 0}$  weakly converges in the space of continuous functions to the continuous process  $Z = (Z(t))_{t \geq 0}$ . Zhang [15] (see also [5, 10]) gave a stronger version of this result in the special case when the scenery has a finite moment of order  $2 + \delta$  for some  $\delta > 0$ , more precisely, there is a coupling of  $\xi$ ,  $S$ ,  $B$  and  $W$  such that  $(\xi, W)$  is independent of  $(S, B)$  and for any  $\varepsilon > 0$ , almost surely,

$$\max_{0 \leq m \leq n} |K(m) - Z(m)| = o(n^{\frac{1}{2} + \frac{1}{2(2+\delta)} + \varepsilon}), \quad n \rightarrow +\infty. \quad (1.3)$$

Theorem 1 will follow from this strong approximation and the following result.

**Theorem 2.**  $\mathbb{P}_W$ -almost surely, under the quenched probability  $\mathbb{P}(\cdot | W)$ , the limit points of the law of

$$\tilde{Z}_t := \frac{Z_t}{(2t^{3/2} \log \log t)^{1/2}}, \quad t \rightarrow \infty,$$

under the topology of weak convergence of measures, are equal to the closure of the laws of random variables in  $\mathcal{K}_B$  defined in Theorem 1. Consequently under  $\mathbb{P}(\cdot | W)$ , as  $t \rightarrow \infty$ ,  $\tilde{Z}_t$  does not converge in law.

To prove Theorem 2, we shall apply Strassen [14]'s functional law of iterated logarithm applied to the two-sided Brownian motion  $W$ ; we shall also need to estimate the stochastic integral  $\int g_{m_1, M_1}(x) dL_1(x)$  for a bounded Borel function  $g$ , this will be done by using the theory of enlargement of filtrations, see Section 2 for the details.

## 2. PROOFS

At first, we reformule Strassen's theorem as follows: For a two-sided one-dimensional Brownian motion  $(W(t), t \in \mathbb{R})$  starting from 0, define for any  $\lambda > e^e$ ,

$$W_\lambda(t) := \frac{W(\lambda t)}{(2\lambda \log \log \lambda)^{1/2}}, \quad t \in \mathbb{R}.$$

**Theorem 3** (Strassen). *Almost surely, for any  $s < 0 < r$ ,  $(W_\lambda(t), s \leq t \leq r)$  is relatively compact in the uniform topology and the set of its limit points is  $\mathcal{K}_{s,r}$ , with*

$$\mathcal{K}_{s,r} := \left\{ f \in \mathcal{C}([s, r] \rightarrow \mathbb{R}) : f(0) = 0, \int_s^r (\dot{f}(x))^2 dx \leq 1 \right\}.$$

**Proof:** For any fixed  $s < 0 < r$ , by applying Strassen's theorem ([14]) to the two-dimensional rescaled Brownian motion:  $(\frac{W(\lambda r u)}{\sqrt{2\lambda r \log \log \lambda}}, \frac{W(\lambda |s| u)}{\sqrt{2\lambda |s| \log \log \lambda}})_{0 \leq u \leq 1}$ , we get that  $(W_\lambda(t), s \leq t \leq r)$  is relatively compact in the uniform topology with  $\mathcal{K}_{s,r}$  as the set of limit points. By inverting a.s. and  $s, r$ , we see that there exists an event  $\Omega_0$  with full probability such that for any  $\omega \in \Omega_0$ , for all  $s < 0 < r$  integers,  $(W_\lambda(t)(\omega), s \leq t \leq r)$  is relatively compact in the uniform topology and the set of its limit points is  $\mathcal{K}_{s,r}$ .

Now, let  $\omega \in \Omega_0$ . We are going to check that for any  $s < 0 < r$ ,  $(W_\lambda(t)(\omega), s \leq t \leq r)$  is relatively compact in the uniform topology and the set of its limit points is  $\mathcal{K}_{s,r}$ .

To this end, for any  $s < 0 < r$ , let  $k < 0 < l$  be integers such that  $k \leq s < k+1 \leq 0 \leq l-1 < r \leq l$ . Let  $d(\cdot, \cdot)$  be the distance in the uniform topology:  $d(f, g) := \sup_{a \leq t \leq b} |f(t) - g(t)|$  for  $f, g \in \mathcal{C}([a, b] \rightarrow \mathbb{R})$ , which is consistent on  $a < 0 < b$ . For any  $\varepsilon > 0$ , for all large  $\lambda \geq \lambda_0(\varepsilon, \omega)$ ,  $d(W_\lambda(\cdot)(\omega), \mathcal{K}_{k,l}) < \varepsilon$ , hence there exists some  $f \equiv f_{\lambda, \omega, k, l} \in \mathcal{K}_{k,l}$  such that  $d(W_\lambda(\cdot)(\omega), f) < \varepsilon$ . Notice that if we denote by  $g := f1_{[s,r]}(\cdot)$  the restriction of  $f$  on  $[s, r]$ , then  $g \in \mathcal{K}_{s,r}$  and  $d(W_\lambda(\cdot)(\omega), g) < \varepsilon$ , this proves that for any  $\omega \in \Omega_0$ , all possible limits of  $(W_\lambda(t)(\omega), s \leq t \leq r)$  as  $\lambda \rightarrow \infty$  are in  $\mathcal{K}_{s,r}$ .

It remains to check that for any  $f \in \mathcal{K}_{s,r}$ , infinitely often  $\lambda \rightarrow \infty$ ,  $(W_\lambda(t)(\omega), s \leq t \leq r)$  are in the ball centered at  $f$  of radius  $\varepsilon$ . It is enough to check this for all  $f \in \mathcal{K}_{s,r}$  such that  $\int_s^r |\dot{f}(t)|^2 dt < 1$ . We may find some function  $h \in \mathcal{K}_{k,l}$  such that  $f$  is the restriction of  $h$  on  $[s, r]$ . Since infinitely often as  $\lambda \rightarrow \infty$ ,  $(W_\lambda(t)(\omega), k \leq t \leq l)$  are in the ball centered at  $h$  of radius  $\varepsilon$ , the desired conclusion follows.  $\square$

Next, we recall some properties on Brownian local times: The process  $x \rightarrow L_1(x)$  is a (continuous) semimartingale (by Perkins [12]), moreover, the quadratic variation of  $x \rightarrow L_1(x)$  equals  $4 \int_{-\infty}^x L_1(z) dz$ . By Revuz and Yor ([13], Exercice VI (1.28)), for any  $f$  locally bounded Borel function,

$$\frac{1}{2} \int f(x) dL_1(x) = - \int_0^{B_1} f(u) du + \int_0^1 f(B_u) dB_u. \quad (2.4)$$

Recall that  $m_1 = \inf_{0 \leq t \leq 1} B_t$  and  $M_1 = \sup_{0 \leq t \leq 1} B_t$ . When  $f$  is random but measurable with respect to  $\sigma\{m_1, M_1\}$ , we define  $\int f(x) dL_1(x)$  through (2.4): the integral  $\int_0^1 f(B_u) dB_u$  is well defined since by the theory of (initial) enlargement of filtration (see e.g. Jeulin and Yor [8], Mansuy and Yor [11]),  $B$  is still a continuous semimartingale with respect to the filtration

$(\sigma\{B_s, s \leq t, m_1, M_1\})_{0 \leq t \leq 1}$  (see the forthcoming (2.11) for the semimartingale decomposition). Denote by  $L^1(B) := L^1(\mathbb{P}_B, \sigma\{B_s, 0 \leq s \leq 1\})$  and  $\|\cdot\|_1$  the  $L^1$ -norm.

**Lemma 4.** *There exists some constant  $c_1 > 0$  such that for any bounded Borel function  $g : (a, b, x) \in \mathbb{R}^3 \rightarrow g_{a,b}(x) \in \mathbb{R}$ , we have*

$$\left\| \int g_{m_1, M_1}(x) dL_1(x) \right\|_1 \leq c_1 \sup_{a,b,x} |g_{a,b}(x)|.$$

**Proof:** Let  $\|g\|_\infty := \sup_{a,b,x} |g_{a,b}(x)|$ . Using (2.4) and observing that  $\|\int_0^{B_1} g_{m_1, M_1}(u) du\|_1 \leq \|g\|_\infty \mathbb{E}[|B_1|]$ , it is enough to prove that for some positive constant  $c_2$ ,

$$\left\| \int_0^1 g_{m_1, M_1}(B_u) dB_u \right\|_1 \leq c_2 \|g\|_\infty. \quad (2.5)$$

The difficulty here is the dependence of  $g$  on  $(m_1, M_1)$ , in particular, there is no  $L^2$ -isometry for  $\int_0^1 g_{m_1, M_1}(B_u) dB_u$ . We need here an (initial) enlargement of filtration (see e.g. Jeulin and Yor [8], Mansuy and Yor [11]): Let  $(\mathcal{F}_t)$  be the natural filtration of  $B$  and  $\tilde{\mathcal{F}}_t := \sigma\{M_1, m_1\} \vee \mathcal{F}_t$  for  $0 \leq t \leq 1$ . We compute at first the law of  $(m_1, M_1)$  conditioned on  $\mathcal{F}_t$ : for any measurable bounded function  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $0 < t < 1$ ,

$$\mathbb{E}\left(h(m_1, M_1) | \mathcal{F}_t\right) = \mathbb{E}\left(h(m_t \wedge (B_t + \widehat{m}_{1-t}), M_t \vee (B_t + \widehat{M}_{1-t})) | \mathcal{F}_t\right),$$

where  $(\widehat{m}_{1-t}, \widehat{M}_{1-t})$  are independent of  $\mathcal{F}_t$  and have the same law as  $(m_{1-t}, M_{1-t})$ , and  $x \wedge y := \min(x, y)$ ,  $x \vee y := \max(x, y)$  for any  $x, y \in \mathbb{R}$ . For any fixed  $r > 0$ , the law of  $(m_r, B_r, M_r)$  is given as follows (cf. Revuz and Yor [13], Exercice III (3.15)): For any  $a < 0 < b$ ,

$$\mathbb{P}\left(a \leq m_r < M_r \leq b, B_r \in dx\right) / dx = \sum_{k=-\infty}^{\infty} \left(p_r(x+2k(b-a)) - p_r(x-2b+2k(b-a))\right) =: \Upsilon_{r,a,b}(x), \quad (2.6)$$

with  $p_r(x) := (2\pi r)^{-1/2} e^{-x^2/(2r)}$ . Integrating on  $x \in [a, b]$  gives that

$$\phi_r(a, b) := \mathbb{P}\left(a \leq m_r < M_r \leq b\right) = \sum_{k=-\infty}^{\infty} (-1)^k \int_{a+k(b-a)}^{b+k(b-a)} p_r(x) dx. \quad (2.7)$$

Therefore for any fixed  $r > 0$ ,

$$\mathbb{P}\left(m_r \in da, M_r \in db\right) = \psi_r(a, b) da db, \quad a < 0 < b,$$

with

$$\begin{aligned} \psi_r(a, b) &:= \sum_{k=-\infty}^{\infty} (-1)^k \left(k(k+1)p'_r(b+k(b-a)) - k(k-1)p'_r(a+k(b-a))\right) \\ &= 2 \sum_{k=-\infty}^{\infty} (-1)^k k(k+1)p'_r(b+k(b-a)), \end{aligned} \quad (2.8)$$

where  $p'_r(x) = -\frac{x}{r} p_r(x)$  denotes the derivative of  $p_r(x)$  with respect to  $x$ . Plainly,  $\psi_r(a, b)$  is continuous on  $(a, b, r)$  for  $a < 0 < b$  and  $r > 0$ . We can show that  $\psi_r(a, b) > 0$  for any  $a < 0 < b$  and  $r > 0$ , see Remark 5.

It follows that for any  $0 < t < 1$ ,

$$\begin{aligned}\mathbb{E}\left(h(m_1, M_1)|\mathcal{F}_t\right) &= \int_{a < 0 < b} dadb \psi_{1-t}(a, b) h(m_t \wedge (B_t + a), M_t \vee (B_t + b)) \\ &= I_1 + I_2 + I_3 + I_4,\end{aligned}$$

with

$$\begin{aligned}I_1 &:= \int_{-\infty}^{m_t} dx \int_{M_t}^{\infty} dy \psi_{1-t}(x - B_t, y - B_t) h(x, y), \\ I_2 &:= \int_{-\infty}^{m_t} dx \int_0^{M_t - B_t} db \psi_{1-t}(x - B_t, b) h(x, M_t), \\ I_3 &:= \int_{m_t - B_t}^0 da \int_0^{M_t - B_t} db \psi_{1-t}(a, b) h(m_t, M_t) = \phi_{1-t}(m_t - B_t, M_t - B_t) h(m_t, M_t), \\ I_4 &:= \int_{m_t - B_t}^0 da \int_{M_t}^{\infty} dy \psi_{1-t}(a, y - B_t) h(m_t, y).\end{aligned}$$

Let us adopt the following notation: for any  $f \in C^1((-\infty, 0) \times (0, \infty) \rightarrow \mathbb{R})$ ,

$$\nabla f(a, b) := \frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}.$$

Define for  $a < 0 < b$ ,

$$\psi_r^{(1)}(a, b) := \int_a^0 dx \psi_r(x, b), \quad \psi_r^{(2)}(a, b) := \int_0^b \psi_r(a, y) dy. \quad (2.9)$$

Then  $I_2 = \int_{-\infty}^{m_t} dx \psi_{1-t}^{(2)}(x - B_t, M_t - B_t) h(x, M_t)$  and by Ito's formula,

$$dI_2 = - \left[ \int_{-\infty}^{m_t} dx \nabla \psi_{1-t}^{(2)}(x - B_t, M_t - B_t) h(x, M_t) \right] dB_t + \text{f.v.},$$

where here and in the sequel, f.v. denotes some finite variation process. Similarly,

$$\begin{aligned}dI_1 &= - \left[ \int_{-\infty}^{m_t} dx \int_{M_t}^{\infty} dy \nabla \psi_{1-t}(x - B_t, y - B_t) h(x, y) \right] dB_t + \text{f.v.}, \\ dI_3 &= - \left[ \nabla \phi_{1-t}(m_t - B_t, M_t - B_t) h(m_t, M_t) \right] dB_t + \text{f.v.}, \\ dI_4 &= - \left[ \int_{M_t}^{\infty} dy \nabla \psi_{1-t}^{(1)}(m_t - B_t, y - B_t) h(m_t, y) \right] dB_t + \text{f.v.}.\end{aligned}$$

It follows that if we denote by  $X_t := \mathbb{E}\left(h(m_1, M_1)|\mathcal{F}_t\right)$  for  $0 \leq t < 1$ , then  $X$  is a martingale and

$$dX_t = \dot{Q}_t(h) dB_t,$$

where  $\dot{Q}_t(h) := \int_{\mathbb{R}^2} \dot{Q}_t(dxdy) h(x, y)$  with

$$\begin{aligned}\dot{Q}_t(dx, dy) &:= - \left[ 1_{(x < m_t)} 1_{(M_t < y)} \nabla \psi_{1-t}(x - B_t, y - B_t) \right] dx dy - \\ &\quad - \left[ 1_{(x < m_t)} \nabla \psi_{1-t}^{(2)}(x - B_t, M_t - B_t) \right] dx \otimes \delta_{M_t}(dy) - \\ &\quad - \left[ \nabla \phi_{1-t}(m_t - B_t, M_t - B_t) \right] \delta_{m_t}(dx) \otimes \delta_{M_t}(dy) - \\ &\quad - \left[ 1_{(y > M_t)} \nabla \psi_{1-t}^{(1)}(m_t - B_t, y - B_t) \right] \delta_{m_t}(dx) \otimes dy.\end{aligned}$$

On the other hand, we may re-write  $X_t = \int_{\mathbb{R}^2} Q_t(dxdy)h(x, y)$  with

$$\begin{aligned} Q_t(dx, dy) &:= \left[ \mathbf{1}_{(x < m_t)} \mathbf{1}_{(M_t < y)} \psi_{1-t}(x - B_t, y - B_t) \right] dx dy + \\ &+ \left[ \mathbf{1}_{(x < m_t)} \psi_{1-t}^{(2)}(x - B_t, M_t - B_t) \right] dx \otimes \delta_{M_t}(dy) + \\ &+ \left[ \phi_{1-t}(m_t - B_t, M_t - B_t) \right] \delta_{m_t}(dx) \otimes \delta_{M_t}(dy) + \\ &+ \left[ \mathbf{1}_{(y > M_t)} \psi_{1-t}^{(1)}(m_t - B_t, y - B_t) \right] \delta_{m_t}(dx) \otimes dy. \end{aligned}$$

Then we have  $\dot{Q}_t(dxdy) = \varrho_t(x, y) Q_t(dxdy)$  with the density  $\varrho_t(x, y)$  given by

$$\begin{aligned} \varrho_t(x, y) &:= -\mathbf{1}_{(x < m_t)} \mathbf{1}_{(M_t < y)} \nabla \log \psi_{1-t}(x - B_t, y - B_t) \\ &\quad -\mathbf{1}_{(x < m_t, y = M_t)} \nabla \log \psi_{1-t}^{(2)}(x - B_t, M_t - B_t) \\ &\quad -\mathbf{1}_{(x = m_t, y = M_t)} \nabla \log \phi_{1-t}(m_t - B_t, M_t - B_t) \\ &\quad -\mathbf{1}_{(x = m_t, y > M_t)} \nabla \log \psi_{1-t}^{(1)}(m_t - B_t, y - B_t). \end{aligned} \quad (2.10)$$

The theory of enlargement of filtrations says that

$$B_t = \tilde{B}_t + \int_0^t \varrho_s(m_1, M_1) ds, \quad 0 \leq t < 1, \quad (2.11)$$

with  $\tilde{B}$  a Brownian motion with respect to  $(\tilde{\mathcal{F}}_t)$ .

By Cauchy-Schwarz' inequality and the  $L^2$ -isometry for  $\tilde{B}$ , we get that

$$\mathbb{E} \left| \int_0^1 g_{m_1, M_1}(B_s) d\tilde{B}_s \right| \leq \sqrt{\mathbb{E} \int_0^1 g_{m_1, M_1}^2(B_s) ds} \leq \|g\|_\infty,$$

which in view of (2.11) imply that

$$\mathbb{E} \left| \int_0^1 g_{m_1, M_1}(B_s) dB_s \right| \leq \|g\|_\infty \left( 1 + \mathbb{E} \left( \int_0^1 |\varrho_s(m_1, M_1)| ds \right) \right).$$

Then (2.5) follows once we have checked that

$$\mathbb{E} \left( \int_0^1 |\varrho_s(m_1, M_1)| ds \right) < \infty. \quad (2.12)$$

To prove (2.12), we use (2.10) and write

$$\varrho_t(m_1, M_1) = J_1(t) + J_2(t) + J_3(t) + J_4(t),$$

with obvious definitions of  $J_i, 1 \leq i \leq 4$ . Notice that  $\{m_1 < m_t\} = \{\hat{m}_{1-t} < m_t - B_t\}$  and  $\{M_t < M_1\} = \{\widehat{M}_{1-t} > M_t - B_t\}$  where  $\hat{m}, \widehat{M}$  are the minimum and maximum processes of  $\hat{B}_s := B_{s+t} - B_s, s \geq 0$ . Recalling that  $\psi_{1-t}(a, b)$  are the densities of  $(\hat{m}_{1-t}, \widehat{M}_{1-t})$ , we have that

$$\begin{aligned} \mathbb{E} \left[ |J_1(t)| \right] &= \mathbb{E} \left[ \mathbf{1}_{(m_1 < m_t)} \mathbf{1}_{(M_t < M_1)} \left| \nabla \log \psi_{1-t}(m_1 - B_t, M_1 - B_t) \right| \right] \\ &= \mathbb{E} \int_{-\infty}^{m_t - B_t} da \int_{M_t - B_t} db \left| \nabla \psi_{1-t}(a, b) \right|. \end{aligned}$$

Elementary computations show that for any  $r > 0$ ,

$$\nabla \psi_r(a, b) = 2 \sum_{k \in \mathbb{Z}} (-1)^k k(k+1) p_r''(b + k(b-a)),$$

where  $p_r''(x) = (-\frac{1}{r} + \frac{x^2}{r^2})p_r(x)$  is the second order derivative of  $p_r(\cdot)$ . Bounding all terms by its absolute values, we get that

$$\mathbb{E}\left[|J_1(t)|\right] \leq 2\mathbb{E}\int_{-\infty}^{m_t-B_t} da \int_{M_t-B_t}^{\infty} db \sum_{k \in \mathbb{Z}} |k(k+1)| \frac{1+(b+k(b-a))^2}{(1-t)^2} p_{1-t}(b+k(b-a)).$$

In the above sum  $\sum_{k \in \mathbb{Z}}$ , we decompose as  $\sum_{k \geq 1}$  and  $\sum_{k \leq -2}$ . By changing  $B$  to  $-B$ , it is easy to see that the two sums  $\sum_{k \geq 1}$  and  $\sum_{k \leq -2}$  give the same contribution in  $\mathbb{E}\left[\int da \int db\right]$ . It follows that

$$\begin{aligned} \mathbb{E}\left[|J_1(t)|\right] &\leq 4\mathbb{E}\int_{-\infty}^{m_t-B_t} da \int_{M_t-B_t}^{\infty} db \sum_{k \geq 1} k(k+1) \frac{1+(b+k(b-a))^2}{(1-t)^2} p_{1-t}(b+k(b-a)) \\ &= \frac{4}{(1-t)^2} \int_0^{\infty} dx (1+x^2) p_{1-t}(x) \mathbb{E}\left[\sum_{k=1}^{\infty} k \int_{-\infty}^{m_t-B_t} da 1_{(ka \geq (k+1)(M_t-B_t)-x)}\right] \\ &\leq \frac{4}{(1-t)^2} \int_0^{\infty} dx (1+x^2) p_{1-t}(x) \mathbb{E}\left[1_{(x \geq M_t-m_t)} \frac{x^2}{M_t-m_t}\right], \end{aligned}$$

where the above equality follows from a change of variable  $x = b + k(b - a)$ . If we denote by  $\gamma$  a Brownian motion independent of  $B$ , then for any  $0 < t < 1$ ,

$$\mathbb{E}\left[|J_1(t)|\right] \leq \frac{4}{(1-t)^2} \mathbb{E}\left[1_{(\gamma_{1-t} \geq M_t-m_t)} \frac{(1+\gamma_{1-t}^2)\gamma_{1-t}^2}{M_t-m_t}\right],$$

which is continuous for  $t \in (0, 1)$ , behaves as  $\frac{c_3}{\sqrt{t}}$  as  $t \rightarrow 0$  (with  $c_3 := 4\mathbb{E}[1_{(\gamma_1 > 0)}(1+\gamma_1^2)\gamma_1^2] \mathbb{E}[\frac{1}{M_1-m_1}]$ ) and converges to 0 as  $t \uparrow 1$  (since  $\frac{1}{M_1-m_1}$  has finite moments of any order and ). Consequently,

$$\int_0^1 \mathbb{E}\left[|J_1(t)|\right] dt < \infty.$$

The other terms  $J_2(t), J_3(t), J_4(t)$  can be estimated in the same way: By symmetry,  $\int_0^1 \mathbb{E}\left[|J_4(t)|\right] dt = \int_0^1 \mathbb{E}\left[|J_2(t)|\right] dt$ . For  $J_2(t)$ , we observe that for any  $0 < t < 1$  and  $a < 0 < b$ ,

$$\psi_{1-t}^{(2)}(a, b) da = \mathbb{P}(m_{1-t} \in da, M_{1-t} \leq b).$$

It follows that

$$\begin{aligned} \mathbb{E}\left[|J_2(t)|\right] &= \mathbb{E}\left[1_{(m_1 < m_t)} 1_{(M_t = M_1)} |\nabla \log \psi_{1-t}^{(2)}(m_1 - B_t, M_t - B_t)|\right] \\ &= \mathbb{E}\left[1_{(\widehat{m}_{1-t} < m_t - B_t)} 1_{(\widehat{M}_{1-t} \leq M_t - B_t)} |\nabla \log \psi_{1-t}^{(2)}(\widehat{m}_{1-t}, M_t - B_t)|\right] \\ &= \mathbb{E}\int_{-\infty}^{m_t-B_t} da |\nabla \psi_{1-t}^{(2)}(a, M_t - B_t)|. \end{aligned} \tag{2.13}$$

By (2.9) and (2.8),  $\frac{\partial \psi_{1-t}^{(2)}}{\partial a}(a, b) = \int_0^b \frac{\partial \psi_{1-t}}{\partial a}(a, y) dy = 2 \sum_{k \in \mathbb{Z}} (-1)^{k+1} k^2 (p'_{1-t}(b+k(b-a)) - p'_{1-t}(-ka)) = 2 \sum_{k \in \mathbb{Z}} (-1)^{k+1} k^2 p'_{1-t}(b+k(b-a))$ , by the asymmetry of  $p'_{1-t}(\cdot)$ . Hence,

$$\nabla \psi_{1-t}^{(2)}(a, b) = \frac{\partial \psi_{1-t}^{(2)}}{\partial a}(a, b) + \frac{\partial \psi_{1-t}^{(2)}}{\partial b}(a, b) = 2 \sum_{k \in \mathbb{Z}} (-1)^k k p'_{1-t}(b+k(b-a)).$$



It follows that

$$\begin{aligned} \int_{-\infty}^{m_t - B_t} da |\nabla \psi_{1-t}^{(2)}(a, b)| &\leq 2 \sum_{k \in \mathbb{Z}} \int_{-\infty}^{m_t - B_t} da |k p'_{1-t}(b + k(b - a))| \\ &= 2 \sum_{k \in \mathbb{Z}, k \neq 0} p_{1-t}(b + k(b - (m_t - B_t))). \end{aligned}$$

Using the elementary fact: for any  $x > 0$  and  $r > 0$ ,

$$\sum_{k \in \mathbb{Z}, k \neq 0} p_r(kx) \leq 2 \int_0^\infty p_r(tx) dt = \frac{1}{x}, \quad (2.14)$$

we deduce from (2.13) that for any  $0 < t < 1$ ,

$$\mathbb{E} \left[ |J_2(t)| \right] \leq \mathbb{E} \left( \frac{1}{M_t - m_t} \right) = \frac{1}{\sqrt{t}} \mathbb{E} \left( \frac{1}{M_1 - m_1} \right).$$

Hence

$$\int_0^1 \mathbb{E} \left[ |J_2(t)| \right] dt < \infty.$$

Finally for  $J_3(t)$ , we have that

$$\begin{aligned} \mathbb{E} \left[ |J_3(t)| \right] &= \mathbb{E} \left[ \mathbf{1}_{(m_1 = m_t)} \mathbf{1}_{(M_t = M_1)} |\nabla \log \phi_{1-t}(m_t - B_t, M_t - B_t)| \right] \\ &= \mathbb{E} \left[ \mathbf{1}_{(\widehat{m}_{1-t} \geq m_t - B_t)} \mathbf{1}_{(\widehat{M}_{1-t} \leq M_t - B_t)} |\nabla \log \phi_{1-t}(m_t - B_t, M_t - B_t)| \right] \\ &= \mathbb{E} \left[ |\nabla \phi_{1-t}(m_t - B_t, M_t - B_t)| \right]. \end{aligned} \quad (2.15)$$

Recalling (2.7). Elementary computations yield that

$$\nabla \phi_{1-t}(a, b) = \sum_{k \in \mathbb{Z}} (-1)^k \left[ p_{1-t}(b + k(b - a)) - p_{1-t}(a + k(b - a)) \right] = 2 \sum_{k \in \mathbb{Z}} (-1)^k p_{1-t}(b + k(b - a)).$$

Hence  $|\nabla \phi_{1-t}(a, b)| \leq 2p_{1-t}(a) + 2p_{1-t}(b) + 2 \sum_{j \in \mathbb{Z}, j \neq 0} p_{1-t}(j(b - a)) \leq \sqrt{\frac{8}{\pi(1-t)}} + \frac{2}{b-a}$ . It follows that

$$\mathbb{E} \left[ |J_3(t)| \right] \leq \sqrt{\frac{8}{\pi(1-t)}} + 2 \mathbb{E} \left( \frac{1}{M_t - m_t} \right) \leq \sqrt{\frac{8}{\pi(1-t)}} + \frac{2}{\sqrt{t}} \mathbb{E} \left( \frac{1}{M_1 - m_1} \right).$$

Therefore  $\int_0^1 \mathbb{E} \left[ |J_3(t)| \right] dt < \infty$ , which completes the proof of (2.12). Then Lemma 4 follows.  $\square$

**Remark 5.** Recall that  $\psi_r(a, b)$  is the density of  $(m_r, M_r)$  (cf. (2.8)). To check that  $\psi_r(a, b) > 0$ , M. Lifshits suggested to use the following integration equation: For any  $a < 0 < b$  and any  $r > s > 0$ , by the Markov property at  $s$ ,

$$\begin{aligned} \psi_r(a, b) &= \int_a^b \mathbb{P} \left( B_s \in dx, a < m_s < M_s < b \right) \psi_{r-s}(a - x, b - x) \\ &= \int_a^b \Upsilon_{s,a,b}(x) \psi_{r-s}(a - x, b - x) dx, \end{aligned} \quad (2.16)$$

by using the notation in (2.6). Elementary computations show that for any  $s > 0$  and  $a < 0 < b$ ,

$$\Upsilon_{s,a,b}(x) \geq p_s(x) - p_s(2b - x) - p_s(x - 2a), \quad \forall a < x < b,$$

which implies that for any  $a < 0 < b$  and  $0 < s < a^2 \wedge b^2$ ,

$$\Upsilon_{s,a,b}(0) \geq (1 - 2e^{-2})p_s(0) > 0. \quad (2.17)$$

If  $\psi_r(a, b) = 0$ , then by the continuity of  $\Upsilon_{s,a,b}(\cdot)$  and  $\psi_{r-s}(\cdot, \cdot)$ , we deduce from (2.16) that  $\psi_{r-s}(a, b) = 0$  for any  $0 < s < a^2 \wedge b^2 \wedge r$ . By iterating this procedure, we would get that  $\psi_v(a, b) = 0$  for all  $0 < v < r$ ; but this contradicts with the fact that for any fixed  $a < 0 < b$ ,

$$\psi_v(a, b) \sim 4\frac{2b-a}{v}p_v(2b-a) + 4\frac{b-2a}{v}p_v(b-2a), \quad v \rightarrow 0,$$

which is positive for all small  $v$ . Hence  $\psi_r(a, b) > 0$  for any  $r > 0$  and  $a < 0 < b$ .

**Remark 6.** If we enlarge the filtration only with  $M_1$ , it was known that (see Jeulin and Yor [8])

$$B_t = \widehat{B}_t + \int_0^t \frac{ds}{\sqrt{1-s}} \left( 1_{(M_s < M_1)} \frac{M_1 - B_s}{\sqrt{1-s}} - 1_{(M_s = M_1)} k\left(\frac{M_1 - B_s}{\sqrt{1-s}}\right) \right), \quad 0 \leq t < 1,$$

with  $\widehat{B}$  a Brownian motion in the filtration  $(\mathcal{F}_t \vee \sigma\{M_1\})_{0 \leq t \leq 1}$  and  $k(x) := e^{-\frac{x^2}{2}} \frac{1}{\int_0^x dr e^{-r^2/2}}$ .

Let

$$H_\lambda := \int W_\lambda(y) dL_1(y), \quad \text{with } W_\lambda(y) = \frac{W(\lambda y)}{(2\lambda \log \log \lambda)^{1/2}}, \quad y \in \mathbb{R}.$$

By using (2.4), it is easy to check that  $\mathbb{P}_W$ -almost surely,  $H_\lambda \in L^1(B) \equiv L^1(\mathbb{P}_B, \sigma\{B_s, 0 \leq s \leq 1\})$ . Write  $d_{L^1}(\xi, \eta)$  for the distance in  $L^1(B)$  for any  $\xi, \eta \in L^1(B)$ .

**Lemma 7.**  $\mathbb{P}(dW)$ -almost surely,

$$d_{L^1}(H_\lambda, \mathcal{K}_B) \rightarrow 0, \quad \text{as } \lambda \rightarrow \infty,$$

where  $\mathcal{K}_B$  is defined in (1.2). Moreover,  $\mathbb{P}_W$ -almost surely for any  $\xi \in \mathcal{K}_B$ ,  $\liminf_{\lambda \rightarrow \infty} d(H_\lambda, \xi) = 0$ .

It is easy to see that  $\mathcal{K}_B \subset L^1(B)$ , since by Cauchy-Schwarz' inequality,  $(\int_{m_1}^{M_1} f(x) dL_1(x))^2 \leq (\int L_1(x)^2 dx) (\int_{m_1}^{M_1} f(x)^2 dx) \leq \sup_x L_1(x) \in L^p(B)$  for any  $p > 0$ .

**Proof:** Let  $\varepsilon > 0$ . Applying Theorem 3 to  $s = m_1$  and  $r = M_1$ , we get that  $\mathbb{P}_W$ -a.s. for all large  $\lambda \geq \lambda_0(W)$ , there exists a (random) function  $f \equiv f_\lambda \in \mathcal{K}_{m_1, M_1}$  such that  $\sup_{m_1 \leq x \leq M_1} |W_\lambda(x) - f(x)| \leq \varepsilon$ . Let  $g(x) := W_\lambda(x) - f(x)$ . Notice that  $\sup_{m_1 \leq x \leq M_1} |g(x)| \leq \varepsilon$ , and conditioned on  $W$ ,  $g$  is measurable with respect to  $\sigma\{m_1, M_1\}$ . By Lemma 4,  $\left\| \int g(x) dL_1(x) \right\|_1 \leq c_1 \varepsilon$ . Hence  $d_{L^1}(H_\lambda, \mathcal{K}_B) \rightarrow 0$ . The another part of the Lemma is again a consequence of Strassen's law of iterated logarithm (Theorem 3) and Lemma 4.  $\square$

We now are ready to give the proof of Theorems 2 and 1.

**Proof of Theorem 2.** Firstly, we remark that by Brownian scaling,  $\mathbb{P}_W$ -a.s.,

$$\frac{Z_t}{t^{3/4}} \stackrel{(d)}{=} - \int_{m_1}^{M_1} \frac{1}{t^{1/4}} W(\sqrt{t}y) dL_1(y). \quad (2.18)$$

In fact, by the change of variables  $x = y\sqrt{t}$ , we get

$$\int_{-\infty}^{+\infty} L_t(x) dW(x) = \sqrt{t} \int_{-\infty}^{+\infty} \left( \frac{L_t(y\sqrt{t})}{\sqrt{t}} \right) dW(y\sqrt{t})$$

which has the same distribution as

$$\sqrt{t} \int_{-\infty}^{+\infty} L_1(y) dW(y\sqrt{t})$$

from the scaling property of the local time of the Brownian motion. Since  $(L_1(x))_{x \in \mathbb{R}}$  is a continuous semi-martingale, independent from the process  $W$ , from the formula of integration by parts, we get that  $\mathbb{P}_W$  -a.s.,

$$\sqrt{t} \int_{-\infty}^{+\infty} L_1(y) dW(y\sqrt{t}) = -t^{3/4} \int_{m_1}^{M_1} \left( \frac{W(\sqrt{t}y)}{t^{1/4}} \right) dL_1(y),$$

yielding (2.18). Theorem 2 follows from Lemma 7.  $\square$

**Proof of Theorem 1.** We use the strong approximation of Zhang [15] : there exists on a suitably enlarged probability space, a coupling of  $\xi$ ,  $S$ ,  $B$  and  $W$  such that  $(\xi, W)$  is independent of  $(S, B)$  and for any  $\varepsilon > 0$ , almost surely,

$$\max_{0 \leq m \leq n} |K(m) - Z(m)| = o(n^{\frac{1}{2} + \frac{1}{2(2+\delta)} + \varepsilon}), \quad n \rightarrow +\infty. \quad (2.19)$$

From the independence of  $(\xi, W)$  and  $(S, B)$ , we deduce that for  $\mathbb{P}$ -a.e.  $(\xi, W)$ , under the quenched probability  $\mathbb{P}(\cdot | \xi, W)$ , the limit points of the laws of  $\tilde{K}_n$  and  $\tilde{Z}_n$  are the same ones. Now, by adapting the proof of Theorem 2, we have that for  $\mathbb{P}$ -a.e.  $(\xi, W)$ , under the quenched probability  $\mathbb{P}(\cdot | \xi, W)$ , the limit points of the laws of  $\tilde{Z}_n$ , as  $n \rightarrow \infty$ , under the topology of weak convergence of measures, are equal to the closure of the laws of random variables in  $\mathcal{K}_B$ . It gives that for  $\mathbb{P}$ -a.e.  $(\xi, W)$ , under the quenched probability  $\mathbb{P}(\cdot | \xi, W)$ , the limit points of the laws of  $\tilde{K}_n$ , as  $n \rightarrow \infty$ , under the topology of weak convergence of measures, are equal to the closure of the laws of random variables in  $\mathcal{K}_B$  and Theorem 1 follows.  $\square$

**Acknowledgments.** We are grateful to Mikhail Lifshits for Remark 5.

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