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On some partial differential equation for non coercive functional and critical Sobolev exponent

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1 Introduction and results

Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $n > 2$. We assume that q and f are some continuous functions on $\overline{\Omega}$ which change sign in Ω.

We are interested in the following partial differential equation

$$\begin{cases} -\Delta u + (q-\lambda)u = f(x)u^{2^\star-1}, & u > 0 \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where λ is a real parameter and $2^\star = \frac{2n}{n-2}$ is the critical exponent for the Sobolev embedding.

Let λ_1 be the principal eigenvalue of $-\Delta + q$ in Ω, or equivalently

$$\lambda_1 = \inf_{\{v \in H_0^1(\Omega),\ |v|_2 = 1\}} \{\int_\Omega |\nabla v|^2 + \int_\Omega q|v|^2\}. \tag{2}$$

It is well known that there exists $\phi \geq 0$ which realizes the infimum in (2); it satisfies the Euler-Lagrange equation

$$\begin{cases} -\Delta\phi + (q-\lambda_1)\phi = 0, & \phi \geq 0 \text{ in } \Omega \\ \phi = 0 \text{ on } \partial\Omega \end{cases} \tag{3}$$

The strict maximum principle implies that $\phi > 0$ in Ω. In the sequel we shall choose ϕ normalized in $\mathrm{L}^2(\Omega)$.

In this paper, we will study the existence of classical solutions of (1) when $\lambda > \lambda_1$.

In the sub-critical case (i.e. the exponent of u on the right hand side of (1) is less than $2^\star - 1$) , the problem has been studied by Berestycki, Capuzzo-Dolcetta, and Nirenberg ,[4], [5] and Alama and Tarantello[1] . The same problem (with sub-critical exponent) and the Heisenberg Laplacian has been studied in[6]. On another hand the case where $\lambda < \lambda_1$ and the right hand side is critical has been studied by Hebey and Vaugon in [12], see also Demengel and Hebey [11] for the p-Laplacian case.

In [1], [5] [6], the authors establish the following necessary condition

Theorem 1 *Assume that (1) has a solution. Then the following conditions are satisfied:*
(i) $\int_\Omega f\phi^{2^\star} < 0$ *if* $\lambda > \lambda_1$
(ii) $\Omega^+ := \{x \in \Omega,\ f(x) > 0\} \neq \emptyset$ *if* $\lambda < \lambda_1$
(iii) $\Omega^+ \neq \emptyset,\ \Omega^- := \{x \in \Omega,\ f(x) < 0\} \neq \emptyset$ *if* $\lambda = \lambda_1$.

The main result of the present article is the following:

Theorem 2 *Let* $K(n,2)$ *be the best constant for the Sobolev embedding of* $H^1(\mathbb{R}^n)$ *into* $L^{2^\star}(\mathbb{R}^n)$. *We assume that* $\int_\Omega f\phi^{2^\star} < 0$ *and that*

$$M_1 = \sup_{\{\int_\Omega |\nabla v|^2 + \int_\Omega (q-\lambda_1)|v|^2 = 1,\}} \{\int_\Omega f|v|^{2^\star}\} > K(n,2)^{2^\star} \sup f \tag{4}$$

then there exists a real $\lambda^\star > \lambda_1$, *such that for every* $\lambda \in [\lambda_1, \lambda^\star[$, *(1) possesses a solution, and there is no solution for* $\lambda > \lambda^\star$.

Remark 1 *1. The same problem with* $q = 0$ *was considered in [14] and [1]. In particular in [14] Ouyang, through a bifurcation method, proved the existence of a branch of solutions of*

$$\begin{cases} -\Delta u - \lambda u = f u^p, & u > 0 \text{ in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5}$$

bifurcating on the right from λ_1 *under conditions similar to i), ii), iii) in Theorem 1,* $\lambda \neq \lambda_1$, *and for* p *large (i.e.* $p \leq \frac{n-6}{n-10}$ *if* $n \geq 10$ *and for any* p *if* $n \leq 10$*). These solutions* $u(\lambda)$ *don't coincide with those considered here,*

since they are "small" and $\int_\Omega fu(\lambda)^{p+1} < 0$. *On the other hand, for* $p+1$ *sub-critical, through variational methods, he proved the existence of another solution, with* λ *sufficiently close to* λ_1.

In [1], Alama and Tarantello completed these results by perturbing the bifurcation solution and applying the mountain pass lemma, obtaining in that fashion the second solution, even in the critical case. This requires a condition similar to our condition (4), adapted to the case $q = 0$.

Our proof is direct. It uses the geometry of the functionals involved and the concentration compactness Lemma, which permits to overcome the lack of compactness.

2. *The value* $K(n,2)$ *has been computed independently by Aubin in [2, 3] and Talenti [15]. It is achieved on functions of the form*

$$u_\epsilon(x) = \left(\epsilon^2 + |x|^2\right)^{1-\frac{n}{2}}$$

and then

$$K(n,2)^2 = \frac{4}{n(n-2)\omega_{n-1}^{\frac{2}{n}}}.$$

The condition (4) is comparable with the conditions in [12], [2],[11].

Furthermore it is not very restrictive since, in section 3, we prove that one always has

$$M_1 \geq K(n,2)^{2\star} \sup f.$$

3. *Let us remark that* I_λ *defined as* $I_\lambda(u) := \int_\Omega |\nabla u|^2 + \int_\Omega (g-\lambda)u^2$ *is not coercive on the space* H_0^1. *But it is coercive on the hyperplan* $\phi^\perp$ *as soon as* λ *is close to* λ_1. *Indeed, let* $v = u - \left(\int_\Omega u\phi\right)\phi$, *where* ϕ *is supposed to be normalized. Let us consider the set of eigenfunctions,* $\{\phi_k\}_{k\in\mathbb{R}}$ $(\phi_1 = \phi)$ *for* $-\Delta + q$, *which is complete in* L^2. *Using the fact that* ϕ *is simple, one can write* $v = \sum_{k\geq 2} t_k\phi_k$ *where* $t_k = \langle v, \phi_k\rangle$, *hence*

$$I_\lambda(v) = \sum_k (\lambda_k - \lambda)t_k^2 \geq (\lambda_2 - \lambda)|v|_2^2 \tag{6}$$

taking λ *less than* λ_2, *one gets the result.*

4. *It would be interesting to have a better estimate for* $\lambda^\star$.

While working on this case, we also considered similar equations for the p-Laplacian. The results we obtained will be published in a forthcoming paper [7].

2 Proof of Theorem 2

The proof of Theorem 2 is based on some variational methods similar to those used in [5].

One of the key ingredient in the proof of Theorem 2 will be the famous concentration compactness principle of P.L. Lions [13], that we enounce here:

Lemma 1 *Let Ω be some bounded open set in $\mathbb{R}^n$, and (u_k) be some sequence in $H_0^1(\Omega)$, which is bounded in $H^1(\Omega)$. Then there exist a subsequence of (u_k), still denoted (u_k) for simplicity, two nonnegative measures μ and ν on $\overline{\Omega}$, a sequence of points x_i in $\overline{\Omega}$, two sequences of nonnegative reals (μ_i) and (ν_i) and a function u in H_0^1, such that*

$$|\nabla u_k|^2 \rightharpoonup \mu \geq |\nabla u|^2 + \sum_i \mu_i \delta_{x_i}$$

(the convergence being tight i.e. $\int_\Omega |\nabla u_k|^2$ towards $\int_{\overline{\Omega}} \mu$,),

$$|u_k|^{2^\star} \rightharpoonup \nu = |u|^{2^\star} + \sum_i \nu_i \delta_{x_i}$$

(the convergence being tight on $\overline{\Omega}$ i.e. $\int_\Omega |u_k|^{2^\star}$ towards $\int_{\overline{\Omega}} \nu$), with the inequality

$$\nu_i^{\frac{2}{2^\star}} \leq K(n,2)^2 \mu_i. \tag{7}$$

Remark 2 *In the sequel we shall denote the space of bounded measures on Ω by $M^1(\Omega)$. This space will be endowed with the weak topology or the tight one. Recall that the tight convergence is equivalent to weak convergence and convergence of the total variations.*

The proof of Theorem 2 is divided in several steps.

Step 1 For large $\lambda > \lambda_1$ (1) has no solution.

Step 2 If there exists a solution for a certain $\lambda' > \lambda_1$, then (1) has a solution for any $\lambda_1 < \lambda \leq \lambda'$.

Step 3 For $\lambda = \lambda_1$ and for $\lambda > \lambda_1$ sufficiently close to λ_1 there exists a solution to (1) .

In the first two steps, we follow the proofs given in [5][6].

Step 1. Let $x_0 \in \Omega^+$ and $R > 0$ such that $B(x_0, R) \subset \Omega^+$. Let $\mu^\star$ and ψ be respectively the principal eigenvalue and eigenfunction for $-\Delta$ with Dirichlet conditions.

$$\begin{cases} -\Delta\psi - \mu^\star\psi = 0 & \text{in } B(x_0, R) \\ \psi = 0 & \text{on } \partial B(x_0, R). \end{cases}$$

Suppose that there exists a solution of (1) for $\lambda > ||q||_\infty + \mu^\star$. Then since $f > 0$ in $B(x_0, R)$, one has

$$\begin{cases} -\Delta u + (q - \lambda)u & \geq 0 \text{ in } B(x_0, R) \\ u > 0 & \text{on } \partial B(x_0, R). \end{cases}$$

On the other hand, $\frac{\partial\psi}{\partial n} \leq 0$ on $\partial B(x_0, R)$ implies

$$0 \leq \int_{\partial B(x_0,R)} -u\frac{\partial\psi}{\partial n} = \int_{B(x_0,R)} (-u\Delta\psi + \psi\Delta u) \leq \int_{B(x_0,R)} (\mu^\star + q - \lambda)u\psi < 0$$

from which one derives a contradiction.

Step 2 One uses a sub and super solution argument. Namely, let $\lambda' > \lambda_1$, and denote by w a solution for $\lambda = \lambda'$. For $\lambda \in]\lambda_1, \lambda'[$, w satisfies

$$\begin{cases} -\Delta w + (q - \lambda)w \geq -\Delta w + (q - \lambda')w = fw^{2^\star - 1} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

and w is a super solution of (1) for λ. On the other hand, if ϕ is defined by (3), $\epsilon\phi$ is a sub solution of (1) as soon as ϵ is sufficiently small. Indeed

$$-\Delta(\epsilon\phi) + (q - \lambda)(\epsilon\phi) = (\lambda_1 - \lambda)\epsilon\phi \leq f(x)\epsilon^{2^\star}\phi^{2^\star}$$

since f is bounded from below. Since by Hopff's maximum principle, $\epsilon\phi \leq w$ for ϵ small enough one obtains that there exists a solution for λ.

Step 3 One defines as in [5] [6] the following

$$S_\lambda = \{u \in H^1(\Omega),\ I_\lambda(u) = 1\}$$

and for ϵ small and positive,

$$M_{\epsilon,\lambda} = \sup_{u \in S_\lambda} \{\int_\Omega f|u|^{2^\star - \epsilon}\}$$

$$M_\lambda = \sup_{u\in S_\lambda}\{\int_\Omega f|u|^{2^\star}\}.$$

Let us observe that according to [5] [6], $S_\lambda \neq \emptyset$, $M_{\epsilon,\lambda} > 0$ and $M_{\epsilon,\lambda}$ is achieved by some maximizer u_ϵ . u_ϵ can be chosen positive since $|u_\epsilon|$ is a maximizer as well. The proof that $M_\lambda > 0$ is identical to the sub-critical case. Furthermore, it is not difficult to see that for $\lambda \geq \lambda_1$, $M_\lambda \geq M_{\lambda_1} = M_1$.

Claim 1: *The sequences* $(M_{\epsilon,\lambda})$, (M_λ) *and* $||u_\epsilon||_{H^1(\Omega)}$ *are bounded uniformly with respect to* ϵ *and* λ *for* ϵ *small enough and* λ *close to* λ_1.

Let u be such that $I_\lambda(u) = 1$. Remarking that $I_\lambda(|u|) = I_\lambda(u)$ and that $||u_\epsilon||_{H^1}$ is bounded if and only if $||(|u_\epsilon|)||_{H^1}$ is, one can assume in what follows that $u \geq 0$. Let then $v \in H^1_o(\Omega)$, $v \in \phi^\perp$, and $t \in \mathbb{R}$ such that $u = v + t\phi$, and $\int_\Omega fu^{2^\star} > 0$, one gets

$$I_\lambda(v) = 1 + (\lambda - \lambda_1)t^2$$

and noticing that $t = \int_\Omega u\phi > 0$,

$$\begin{aligned}\int_\Omega f(v+t\phi)^{2^\star-\epsilon} &\leq t^{2^\star-\epsilon}\int_\Omega f\phi^{2^\star-\epsilon} + |f|_\infty 2^\star 2^{2^\star-\epsilon-1}(\int_\Omega |v|^{2^\star-\epsilon} + \\ &+ \int_\Omega |v|(t\phi)^{2^\star-\epsilon-1}) \qquad (8)\end{aligned}$$

using the mean-value Theorem. Let $\alpha = -\int_\Omega f|\phi|^{2^\star}$, and ϵ_0 be such that $\int_\Omega f\phi^{2^\star-\epsilon} \leq -\frac{3\alpha}{4}$ for $\epsilon < \epsilon_0$. By Young's inequality, there exists some constant C_1 such that

$$|f|_\infty 2^\star 2^{2^\star-\epsilon-1}\int_\Omega v(t\phi)^{2^\star-\epsilon-1} \leq C_1|v|^{2^\star-\epsilon}_{L^{2^\star-\epsilon}} + \frac{\alpha}{4}t^{2^\star-\epsilon}$$

hence for some C_2 positive

$$\int_\Omega f(v+t\phi)^{2^\star-\epsilon} \leq \frac{-\alpha}{2}t^{2^\star-\epsilon} + C_2|v|^{2^\star-\epsilon}_{2^\star-\epsilon}. \qquad (9)$$

Using remark 1 on the coerciveness of I_λ on $\phi^\perp$, and choosing $\lambda \in [\lambda_1, \frac{\lambda_1+\lambda_2}{2}]$, one has

$$\begin{aligned}|\nabla v|_2^2 &\leq I_\lambda(v) + |q-\lambda|_\infty |v|_2^2 \\ &\leq I_\lambda(v)\left(1 + 2\frac{|q|_\infty + |\lambda_2|}{|\lambda_1 - \lambda_2|}\right). \qquad (10)\end{aligned}$$

Using Poincaré's inequality and (10), there exists some constant C_3 which does not depend on λ, such that

$$\begin{aligned} |v|_{2^\star}^2 &\leq C_3(I_\lambda(v)) \\ &= C_3(1+(\lambda-\lambda_1)t^2). \end{aligned} \tag{11}$$

Finally, through the convexity inequality:

$$\begin{aligned} (1+(\lambda-\lambda_1)t^2)^{\frac{2^\star-\epsilon}{2}} &\leq 2^{\frac{2^\star-\epsilon}{2}-1}(1+(\lambda-\lambda_1)^{\frac{2^\star-\epsilon}{2}}t^{2^\star-\epsilon}) \\ &\leq 2^{\frac{2^\star}{2}}+2^{\frac{2^\star}{2}}(\lambda-\lambda_1)^{\frac{2^\star-\epsilon}{2}}t^{2^\star-\epsilon}. \end{aligned}$$

Then, we choose λ sufficiently close to λ_1 in order to have

$$2^{\frac{2^\star}{2}}C_2C_3^{\frac{2^\star}{2}}|\lambda-\lambda_1|^{\frac{2^\star-1}{2}} \leq \frac{\alpha}{4},$$

and (9) becomes

$$\int_\Omega fu^{2^\star-\epsilon}+t^{2^\star-\epsilon}\frac{\alpha}{2} \leq C_2|v|_{2^\star-\epsilon}^{2^\star-\epsilon} \leq \frac{\alpha}{4}t^{2^\star-\epsilon}+C_2C_3^{\frac{2^\star}{2}}2^{\frac{2^\star}{2}}.$$

Finally there exists C_5 independent of ϵ and λ such that

$$\int_\Omega fu^{2^\star-\epsilon}+t^{2^\star-\epsilon}\frac{\alpha}{4} \leq C_5. \tag{12}$$

We may choose in (12) $u=u_\epsilon$, with $u_\epsilon=v_\epsilon+t_\epsilon\phi$, $v_\epsilon\in\phi^\perp$, one obtains that both $\int_\Omega fu_\epsilon^{2^\star-\epsilon}$ and t_ϵ are bounded for $\lambda-\lambda_1$ sufficiently small. More precisely, we have proved that for λ sufficiently close to λ_1, one has

$$M_{\epsilon,\lambda}+t_\epsilon^{2^\star-\epsilon}\frac{\alpha}{4} \leq C_5. \tag{13}$$

This implies that $(M_{\epsilon,\lambda})$ and (t_ϵ) are bounded. Using $I_\lambda(v_\epsilon)=1+(\lambda-\lambda_1)t_\epsilon^2$, one obtains that (v_ϵ) is bounded as well. Finally (u_ϵ) is bounded independently of ϵ.

This ends the proof of claim 1.

Our purpose now is to let ϵ go to zero. Since the sequences (u_ϵ), (t_ϵ), $(M_{\epsilon,\lambda})$ are bounded, one can extract from them some subsequences, denoted in the same manner, such that

$$
\begin{aligned}
u_\epsilon &\rightharpoonup u \text{ in } H^1(\Omega) \text{ weakly} && (14)\\
t_\epsilon &\to t && (15)\\
M_{\epsilon,\lambda} &\to \overline{M}_\lambda. && (16)
\end{aligned}
$$

Let us note that $\underline{\lim}_{\epsilon\to 0} M_{\epsilon,\lambda} = \overline{M}_\lambda \geq M_\lambda$. Indeed, let $\delta > 0$ be given and $u \in S_\lambda$, such that $\int_\Omega f u^{2^\star} \geq M_\lambda - \delta$. Then $\int_\Omega f u^{2^\star-\epsilon} \geq M_\lambda - 2\delta$ for ϵ small enough. We then deduce that

$$\overline{M_\lambda} = \underline{\lim} M_{\epsilon,\lambda} \geq M_\lambda.$$

Since u_ϵ satisfies the following partial differential equation

$$
\begin{cases}
-\Delta u_\epsilon + (q-\lambda)u_\epsilon &= (M_{\epsilon,\lambda})^{-1} f u_\epsilon^{2^\star-\epsilon-1} \text{ in } \Omega\\
u_\epsilon = 0 \text{ on } \partial\Omega
\end{cases}
\tag{17}
$$

by passing to the limit when $\epsilon \to 0$, one has

$$
\begin{cases}
-\Delta u + (q-\lambda)u &= (\overline{M}_\lambda)^{-1} f u^{2^\star-1} \text{ in } \Omega\\
u = 0 \text{ on } \partial\Omega.
\end{cases}
\tag{18}
$$

The point here is to prove that

i) u is not identically zero.

ii) $M_\lambda = \overline{M}_\lambda$ and u is a maximizer for M_λ.

In order to prove i) let us multiply (17) by $u_\epsilon\varphi$, where $\varphi \in \mathcal{D}(\bar{\Omega})$ and integrate by parts. One obtains

$$\int_\Omega |\nabla u_\epsilon|^2\varphi + \int_\Omega u_\epsilon \nabla u_\epsilon.\nabla\varphi + \int_\Omega (q-\lambda)u_\epsilon^2\varphi = M_{\epsilon,\lambda}^{-1}\int_\Omega f u_\epsilon^{2^\star-\epsilon}\varphi. \tag{19}$$

Similarly, multiplying (18) by $u\varphi$ one gets

$$\int_\Omega |\nabla u|^2\varphi + \int_\Omega u\nabla u.\nabla\varphi + \int_\Omega (q-\lambda)u^2\varphi = \overline{M}_\lambda^{-1}\int_\Omega f u^{2^\star}\varphi. \tag{20}$$

Using the concentration compactness principle as given in Lemma 1 we get that there exist two nonnegative measures, μ and ν, compactly supported in $\overline{\Omega}$, a numerable set of points x_i in $\overline{\Omega}$, some sequences of nonnegative numbers ν_i and μ_i, such that

$$|\nabla u_\epsilon|^2 \rightharpoonup \mu \geq |\nabla u|^2 + \sum_i \mu_i\delta_{x_i} \text{ in } M^1(\overline{\Omega}) \text{ tightly}$$

$$|u_\epsilon|^{2^\star} \rightharpoonup \nu = |u|^{2^\star} + \sum_i \nu_i \delta_{x_i} \text{ in } M^1(\overline{\Omega}) \text{ tightly}$$

with

$$\nu_i^{\frac{2}{2^*}} \leq K(n,2)^2 \mu_i.$$

Let us observe that $\int_\Omega u_\epsilon^{2^\star-\epsilon}$ is bounded and hence, $u_\epsilon^{2^\star-\epsilon}$ converges to some measure $\tilde{\nu}$ in $M^1(\bar{\Omega})$ tightly, up to a subsequence. By Hölder's inequality, one has $\tilde{\nu} \leq \nu$. Since one can extract once more a subsequence from u_ϵ, one can also assume that u_ϵ converges almost everywhere towards u. By Fatou's lemma, for all $\varphi \in \mathcal{D}(\bar{\Omega})$, $\varphi \geq 0$, one has $\int_\Omega |u|^{2^\star}\varphi \leq \langle \tilde{\nu}, \varphi \rangle$, and then $\tilde{\nu} \geq |u|^{2^\star}$. Finally there exists some $0 \leq \tilde{\nu}_i \leq \nu_i$ such that

$$\tilde{\nu} = |u|^{2^\star} + \sum_i \tilde{\nu}_i \delta_{x_i}.$$

One then obtains, by passing to the limit in (19)

$$\langle \mu, \varphi \rangle + \int_\Omega u \nabla u . \nabla \varphi + \int_\Omega (q-\lambda)|u|^2 \varphi = \frac{1}{\overline{M}_\lambda} \left(\int_\Omega f |u|^{2^\star} \varphi + \sum_i \tilde{\nu}_i f(x_i) \varphi(x_i) \right). \quad (21)$$

Subtracting (20) from (21) one gets

$$\langle (\mu - |\nabla u|^2, \varphi \rangle = \overline{M}_\lambda^{-1} \sum_i \tilde{\nu}_i f(x_i) \varphi(x_i) \quad (22)$$

which implies that, if $\mu = \mu^{ac} + \mu^S$ is the Lebesgue decomposition of μ in its absolutely continuous part and a singular one, $\mu^{ac} = |\nabla u|^2$ and $\mu^S = \sum_i \tilde{\mu}_i \delta_{x_i} = \overline{M}_\lambda^{-1} \sum_i \tilde{\nu}_i f(x_i) \delta_{x_i}$, hence

$$\mu_i \leq \tilde{\mu}_i = \overline{M}_\lambda^{-1} \tilde{\nu}_i f(x_i). \quad (23)$$

Clearly if $f(x_i) \leq 0$ then $\mu_i = 0$ and therefore $\tilde{\nu}_i = 0$.

Observe that

$$\frac{1}{\overline{M}_\lambda} \int_\Omega f u^{2^\star} = I_\lambda(u) = I_\lambda(v) + (\lambda_1 - \lambda) t^2 \geq (\lambda_1 - \lambda) t^2.$$

Let us define $B = \dfrac{M_1 + K(n,2)^{2^\star} \sup f(x)}{2}$. By assumption one has $M_1 > B > K(n,2)^{2^\star} \sup f(x)$. From claim 1, we know that $|t| \leq K$ a constant independent

of λ. Suppose that $|\lambda-\lambda_1|$ is sufficiently small in order to have $M_1^{\frac{2}{2^\star}}(1+(\lambda-\lambda_1)K^2)^{\frac{2}{2^\star}-1} > B^{\frac{2}{2^\star}}$. Passing to the limit in the definition of $M_{\epsilon,\lambda}$, one has

$$\int_\Omega fu^{2^\star} + \sum_i \nu_i f(x_i) = \overline{M}_\lambda$$

and then

$$\sum_i \nu_i f(x_i) \leq \overline{M}_\lambda(1+(\lambda-\lambda_1)t^2) \tag{24}$$

then, using (4), lemma 1, (7) and the definition of B,

$$\tilde{\mu}_i \leq \left(\frac{\nu_i f(x_i)}{\overline{M}_\lambda(1+(\lambda-\lambda_1)t^2)}\right)^{1-\frac{2}{2^\star}} \left(\frac{\nu_i f(x_i)}{\overline{M}_\lambda(1+(\lambda-\lambda_1)t^2)}\right)^{\frac{2}{2^\star}} (1+(\lambda-\lambda_1)t^2)$$

$$\leq \frac{K(n,2)^2 \sup f^{\frac{2}{2^\star}}}{B^{\frac{2}{2^\star}}} \tilde{\mu}_i \leq \beta\tilde{\mu}_i$$

for some $\beta<1$. This implies that $\tilde{\mu}_i=0$ and then $\nu_i=0$. Finally, $\bar{M}_\lambda = \int_\Omega fu^{2^\star}$, u is a minimizer and $\overline{M}_\lambda = M_\lambda$. This ends the proof of Theorem 2. ■

Corollary

Under the assumptions $\int_\Omega f\phi^{2^\star} < 0$ and $M_{\lambda_1} > K(n,2)^{2^\star}\sup f$, then for $\lambda > \lambda_1$, $\lambda \to \lambda_1$,

$$M_\lambda \to M_{\lambda_1} = M_1.$$

Moreover, from any sequence $(u_\lambda)_\lambda$ of maximizers, one can extract a sequence which converges strongly towards a maximizer for λ_1.

Proof

Clearly, $M_{\lambda_1} \leq M_\lambda$ for every $\lambda > \lambda_1$. Let (u_λ) be a sequence of maximizers obtained by Theorem 2, for $\lambda > \lambda_1$ sufficiently close to λ_1. By the inequality (12) one gets that (u_λ) is uniformly bounded in H^1 with respect to λ. Extracting a subsequence from (M_λ) and (u_λ), one obtains that $u_\lambda \rightharpoonup w$ which satisfies

$$-\Delta w + (q-\lambda_1)w = \frac{1}{M_1'} f w^{2^\star-1}$$

where $M_1' = \lim M_\lambda \geq M_{\lambda_1}$. Arguing as in the previous proof, one gets that w cannot be zero, and by the positiveness of I_{λ_1}, one has $\int_\Omega fw^{2^\star} \geq 0$. Since

$\int_\Omega f u_\lambda^{2^\star} = M_\lambda \to M_1' = \int_\Omega f w^{2^\star} + \sum_i \nu_i f(x_i)$, then for every i, $\nu_i f(x_i) \leq M_1'$. Proceeding as we already did in the proof of Theorem 2

$$\mu_i = \frac{\nu_i f(x_i)}{M_1'} \leq \left(\frac{\nu_i f(x_i)}{M_1'}\right)^{1-\frac{2}{2^\star}} \left(\frac{\nu_i f(x_i)}{M_1'}\right)^{\frac{2}{2^\star}} \leq \beta \mu_i$$

where $\beta < 1$. Finally one gets that $\mu_i = \nu_i = 0$, $\int_\Omega f w^{2^\star} = 1$, w is a maximizer for M_{λ_1}, and $M_{\lambda_1} = M_1'$. Moreover, there is strong convergence of u_λ towards w.

3 About the condition (4). Test functions.

In this section, we follow the ideas developped in [11]. However, we give some details of the proof for completeness sake.

We assume from now on that $n > 4$. We first prove that

$$M_{\lambda_1} \geq K(n,2)^{2^\star} \sup f.$$

Let $\delta > 0$ be given, suppose that f achieves its supremum on a point x_0 inside Ω and let u_ϵ be defined as

$$u_\epsilon(x) = (\epsilon^2 + |x - x_0|^2)^{1-\frac{n}{2}}.$$

Suppose that R is small enough to have $f > \sup f - \delta$ on $B(x_0, 2R)$ and choose $\varphi \in \mathcal{D}(B(x_0, 2R))$, $\varphi = 1$ on $B(x_0, R)$, $0 \leq \varphi \leq 1$. Finally define $v_\epsilon = u_\epsilon \varphi$. Then

$$\begin{aligned}
I_{\lambda_1}(v_\epsilon) &= \int_{\mathbb{R}^n} |\nabla(u_\epsilon\varphi)|^2 + \int_{\mathbb{R}^n} (q - \lambda_1)(u_\epsilon\varphi)^2 \\
&\leq \int_{\mathbb{R}^n} |\nabla u_\epsilon|^2 + \int_{B(x_0,2R)\setminus B(x_0,R)} \left(u_\epsilon^2 |\nabla\varphi|^2 + 2|\nabla u_\epsilon||\nabla\varphi|\right) \\
&+ \int_\Omega (q - \lambda_1)(u_\epsilon\varphi)^2 \\
&\leq \frac{(n-2)^2\omega_{n-1}}{2} \int_0^\infty \frac{r^{n+1}dr}{(\epsilon^2 + r^2)^n} + \int_{B(x_0,2R)\setminus B(x_0,R)} u_\epsilon^2 |\nabla\varphi|^2 + \\
&+ 2\int_{B(x_0,2R)\setminus B(x_0,R)} |\nabla u_\epsilon||\nabla\varphi| + \int_\Omega (|q|_\infty + |\lambda_1|) C |u_\epsilon|_2^2 \\
&\leq C_1 \epsilon^{2-n} + o(\epsilon^{2-n}) \qquad (25)
\end{aligned}$$

where

$$C_1 = \omega_{n-1}\frac{(n-2)^2}{2}\int_0^{+\infty}\frac{r^{n+1}dr}{(1+r^2)^n} = \omega_{n-1}\frac{(n-2)^2}{2}\frac{\Gamma(\frac{n+3}{2})\Gamma(\frac{n-1}{2})}{n!}.$$

On the other hand

$$\begin{aligned}\int_\Omega f|v_\epsilon|^{2^\star} &\geq (f(x_0)-\delta)\int_{B(x_0,R)}|u_\epsilon\varphi|^{2^\star}\\ &\geq (f(x_0)-\delta)C_2\epsilon^{-n}+O(\epsilon^{-1})\end{aligned} \tag{26}$$

where

$$C_2 = \frac{\omega_{n-1}}{2}\int_0^{\infty}\frac{r^{n-1}dr}{(1+r^2)^n} = \frac{\omega_{n-1}}{2}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}+1)}{n!}.$$

Since $\frac{C_1}{C_2^{\frac{2}{2^\star}}} = K(n,2)^{-2}$ one gets that

$$\frac{I_{\lambda_1}(v_\epsilon)}{(\int_\Omega f|v_\epsilon|^{2^\star})^{\frac{2}{2^\star}}} \leq \frac{K(n,2)^{-2}}{(\sup f-\delta)^{\frac{2}{2^\star}}}$$

which implies the desired result, since δ is arbitrary.

Let us introduce, for $n > 4+2j$,

$$\alpha_{n,j} = \omega_{n-1}\int_0^{+\infty}\frac{r^{2j+n-1}dr}{(1+r^2)^{n-2}} = \frac{\omega_{n-1}}{2}\frac{\Gamma(\frac{n}{2}+j)\Gamma(\frac{n}{2}-j-1)}{(n-2)!}$$

and for $n > 2k$

$$\beta_{n,k} = \omega_{n-1}\int_0^{+\infty}\frac{r^{2k+n-1}dr}{(1+r^2)^n} = \frac{\omega_{n-1}}{2}\frac{\Gamma(k+\frac{n}{2})\Gamma(n+1-\frac{k}{2})}{n!}.$$

We have the following result

Theorem 3 *Suppose that $n > 4$, that q and f are smooth functions, and suppose that f achieves its maximum on an interior point x_0. Let also k_f and k_q be the integers defined by*

$$k_q = \inf\{j \geq 0,\ \Delta^j(q-\lambda_1)(x_0) \neq 0\}$$

$$k_f = \inf\{k \geq 1,\ \Delta^k(f)(x_0) \neq 0\}.$$

Then one has $M_{\lambda_1} > K(n,2)^{2^\star}\sup f$ in each one of the following situations

1. *If* $n > 2k_q + 4$, $k_f = k_q + 1$ *and*

$$\frac{\alpha_{n,k_q}}{(2k_q)!}\Delta^{k_q}(q-\lambda_1)(x_0) - \left(\frac{n-2}{n}\right)\frac{\beta_{n,k_f}}{f(x_0)(2k_f)!}\Delta^{k_f}f(x_0) < 0$$

2. *If* $k_f > k_q + 1$ *and* $\Delta^{k_q}(q-\lambda_1)(x_0) < 0$.

3. *If* $n > 2k_f$ *and* $k_q > k_f$, *and* $\Delta^{k_f}f(x_0) > 0$.

Remark 3 *When* $q = 0$ *the condition in 1) becomes if* $\lambda_1 > 0$, *and if* $k_f = 1$

$$-\alpha_{n,0}\lambda_1 - \left(\frac{n-2}{2n}\right)\frac{\beta_{n,1}\Delta f(x_0)}{f(x_0)} < 0$$

and no condition when $k_f \geq 2$, *which is exactly the condition given in [1].*

Proof of Theorem 3. Suppose that $\delta > 0$ is such that $B(x_0, 2\delta) \subset\subset \Omega$ and let φ be some smooth function, compactly supported in $B(x_0, 2\delta)$ whose value is 1 on a neighborhood of x_0. Let

$$u_\epsilon(r) = (\epsilon^2 + r^2)^{1-\frac{n}{2}}$$

and let v_ϵ be defined as

$$v_\epsilon = u_\epsilon\varphi$$

Suppose that J is the biggest integer such that $n > 2J + 4$ and let $\eta = 2J + 2$, one can write

$$|\int_{\mathbb{R}^n}|\nabla(u_\epsilon\varphi)|^2 - \int_{\mathbb{R}^n}|\nabla u_\epsilon|^2| \leq c\epsilon^{-n+2}\int_{\frac{\delta}{\epsilon}}^{\infty}\frac{r^{n+1}dr}{(1+r^2)^n} \leq C\epsilon^{-n+2+\eta}\int_0^{+\infty}\frac{r^{\eta+n+1}dr}{(1+r^2)^n}$$

this last integral being convergent. Using the definition of η, one gets that

$$|\int_{\mathbb{R}^n}|\nabla(u_\epsilon\varphi)|^2 - \int_{\mathbb{R}^n}|\nabla u_\epsilon|^2| = O(\epsilon^{-n+4+2J}).$$

Let us now treat the term $\int_{\mathbb{R}^n}(q-\lambda_1)v_\epsilon^2 = \int_{\mathbb{R}^n}(q-\lambda_1)(u_\epsilon\varphi)^2$.

We remark that for all j such that $j \leq J$ one has

$$|\int_{\mathbb{R}^n} r^{2j}(u_\epsilon\varphi)^2 - \int_{\mathbb{R}^n} r^{2j}u_\epsilon^2| = o(\epsilon^{n+2J-4}).$$

Indeed, this difference can be majorized (up to a constant) by

$$\epsilon^{2j-4+n}\int_{\frac{\delta}{\epsilon}}^{\infty}\frac{u^{2j+n-1}}{(1+u^2)^{n-2}}du.$$

Let η be such that $n-4-2j>\eta>2(J-j)$, then one may write

$$\epsilon^{2j-4+n}\int_{\frac{\delta}{\epsilon}}^{\infty}\frac{u^{2j+n-1}}{(1+u^2)^{n-2}}=\epsilon^{2j-4+n}\int_{\frac{\delta}{\epsilon}}^{\infty}\frac{u^{2j+n+\eta-1}}{u^{\eta}(1+u^2)^{n-2}}du\leq\epsilon^{2j-4+n+\eta}\delta^{-\eta}C$$

where the constant C is given by the rest of some convergent integral.

Then, using a Taylor expansion around x_0, one gets that

$$\int_{\mathbb{R}^n}(q-\lambda_1)(u_\epsilon\varphi)^2=(q(x_0)-\lambda_1)\int_{\mathbb{R}^n}u_\epsilon^2+\sum_{1\leq j\leq[\frac{n-4}{2}]=J}\frac{\Delta^j q(x_0)\alpha_{n,j}}{(2j)!}\epsilon^{2j}+O(\epsilon^{4-n+2J})$$

and then

$$\int_{\mathbb{R}^n}(q-\lambda_1)v_\epsilon^2=(q(x_0)-\lambda_1)\int_{\mathbb{R}^n}u_\epsilon^2+\sum_{1\leq j\leq[\frac{n-4}{2}]=J}\frac{\Delta^j q(x_0)\alpha_{n,j}}{(2j)!}\epsilon^{2j}+O(\epsilon^{4-n+2J}),$$

where $\alpha_{n,j}=\omega_{n-1}\int_0^{\infty}\frac{u^{2j+n-1}du}{(1+u^2)^{n-2}}$. Here we have used the technical Lemma 6 in [11], which basically says that the integrals over odd order terms in the Taylor expansion are zero.

Let us now treat the denominator. Let K be the greatest integer such that $n>2K$, let us see that for all $k\leq K$

$$\int_{\mathbb{R}^n}r^{2k}u_\epsilon^{2^\star}-\int_{\mathbb{R}^n}r^{2k}(u_\epsilon\varphi)^{2^\star}=o(\epsilon^{-n+2K}).$$

Indeed the difference above can be majorized by

$$C\epsilon^{-n+2k}\int_{\frac{\delta}{\epsilon}}^{\infty}\frac{u^{2k+n-1}du}{(1+u^2)^n}\leq C\epsilon^{-n+2k}\int_{\frac{\delta}{\epsilon}}^{\infty}\frac{u^{2k+n+\eta-1}du}{(1+u^2)^n u^{\eta}},$$

hence by choosing η such that $n-2k>\eta>2(K-k)$, one gets

$$C\epsilon^{-n+2k}\int_{\frac{\delta}{\epsilon}}\frac{u^{2k+n+\eta-1}du}{u^{\eta}(1+u^2)^n}\leq C\epsilon^{-n+2k+\eta}\delta^{-\eta}=o(\epsilon^{2K-n}).$$

Using a Taylor expansion around x_0 for f and Lemma 6 in [11], one gets

$$\int_{\mathbb{R}^n} f(u_\epsilon\varphi)^{2^\star} = f(x_0)\int_{\mathbb{R}^n}(u_\epsilon\varphi)^{2^\star} + \sum_{k\leq[\frac{n}{2}]}\beta_{k,n}\Delta^k f(x_0)\int_{\mathbb{R}^n} r^{2k}(u_\epsilon\varphi)^{2^\star} + O(\epsilon^{n-2[\frac{n}{2}]})$$

$$= f(x_0)C_2\epsilon^{-n} + \sum_{k\leq[\frac{n}{2}]}\Delta^{(k)}f(x_0)\beta_{k,n}\epsilon^{2k-n} + O(\epsilon^{-n+2[\frac{n}{2}]})$$

$$(\int_{\mathbb{R}^n} f(u_\epsilon\varphi)^{2^\star})^{\frac{2}{2^\star}} = (f(x_0)C_2\epsilon^{-n})^{\frac{2}{2^\star}}\left(1+\sum_{k\leq[\frac{n}{2}]}\beta_{k,n}\frac{\Delta^k f(x_0)}{f(x_0)C_2}\epsilon^{2k} + o(\epsilon^{2[\frac{n}{2}]})\right).$$

Suppose that $k_f \leq [\frac{n}{2}]$, then the sum above may be reduced, say

$$(\int_{\mathbb{R}^n} tf(u_\epsilon\varphi)^{2^\star})^{\frac{2}{2^\star}} = (f(x_0)c_2\epsilon^{-n})^{\frac{2}{2^\star}}(1+\beta_{k_f,n}\frac{\Delta^{k_f}f(x_0)}{f(x_0)c_2}\epsilon^{2k_f} + o(\epsilon^{2k_f}))$$

where $\beta_{k,n} = \omega_{n-1}\int_0^\infty \frac{u^{2k+n-1}}{(1+u^2)^n}du$.

Suppose that $n > 2k_q+4$, and $2k_f > 2k_q+2$, then one may write

$$\frac{I_{\lambda_1}(v_\epsilon)}{(\int f v_\epsilon^{2^\star})^{\frac{2}{2^\star}}} = \frac{c_1}{(f(x_0)c_2)^{\frac{2}{2^\star}}}(1+\epsilon^{2k_q+4}\alpha_{k_q,n}\Delta^{k_q}q(x_0) + o(\epsilon^{2k_q+4}))(1+o(\epsilon^{2k_q+4})).$$

As a consequence, the first nonzero term which appears in this development is the term with $\Delta^{k_q}q(x_0)$. Since in this calculation $\frac{c_1}{(f(x_0)c_2)^{\frac{2}{2^\star}}}$ is nothing else that $K(n,2)^{-2}(\sup f)^{\frac{2}{2^\star}}$, the result follows as soon as $\Delta^{k_q}q(x_0) < 0$.

Suppose that $n > 2k_f$ and $2k_q+4 > 2k_f$ then

$$\begin{aligned}\frac{I_{\lambda_1}(v_\epsilon)}{(\int f v_\epsilon^{2^\star})^{\frac{2}{2^\star}}} &= \frac{c_1}{(f(x_0)c_2)^{\frac{2}{2^\star}}}\left(1+o(\epsilon^{2k_f})(1-\frac{2}{2^\star}\frac{\beta_{k,n}\Delta^{k_f}f(x_0)\epsilon^{2k_f}}{f(x_0)C_2} + o(\epsilon^{2k_f}))\right)\\ &< \frac{K(n,2)^{-2}}{f(x_0)^{\frac{2}{2^\star}}}\end{aligned}$$

as soon as $\Delta^{k_f}f(x_0) > 0$.

Finally suppose that $n > 2k_q+4 = 2k_f+2$, then the first nonzero term in the expansion above are of the same order at the numerator and the denominator. Then the result follows as soon as

$$\Delta^{k_q}(q-\lambda_1)(x_0)\alpha_{k_q,n} - \frac{2}{2^\star}\frac{\Delta^{k_f}f(x_0)\beta_{k_f,n}}{f(x_0)C_2} < 0.$$

■

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