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Existence of solutions for semi-linear equations involving the p -Laplacian : the non coercive case

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1 Introduction

In this paper we give necessary and sufficient conditions for the existence of solutions of the following equation

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + (g - \lambda)u^{p-1} = fu^{q-1}, & u \geq 0 \text{ in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded smooth domain of \mathbb{R}^N , $1 < p < N$, $p < q \leq \frac{pN}{N-p} := p^*$, f and g belong to L^∞ , and $\lambda \in \mathbb{R}$. By solution of (1.1), we mean a function $u \in W_0^{1,p}(\Omega)$ satisfying (1.1) in the weak usual sense.

In particular we shall study (1.1) considering the position of λ with respect to the principal eigenvalue. Precisely, it is well known that the concept of ”eigenvalue” and ”eigenfunction” has been generalized by many authors to the quasi-linear setting of the p -Laplacian $\Delta_p := \operatorname{div}(|\nabla \cdot|^{p-2}\nabla \cdot)$, in particular let us recall the works of Allegretto and Huang in [2], Anane in [3] and Lindqvist in [19]. We shall now state their definitions and the principal properties obtained in the works cited above.

Definition 1.1 λ_1 the first "eigenvalue" of $-\operatorname{div}(|\nabla \cdot|^{p-2} \nabla \cdot) + g$ in $W_0^{1,p}(\Omega)$ is defined by

$$\lambda_1 := \inf_{\{\psi \in W_0^{1,p}(\Omega), |\psi|_p=1\}} \left\{ \int_{\Omega} |\nabla \psi|^p + \int_{\Omega} g |\psi|^p \right\}.$$

It is by now a classical result that there exists ϕ , positive in Ω for which this infimum is achieved. ϕ is called the "eigenfunction" corresponding to λ_1 .

In particular ϕ satisfies

$$\begin{cases} -\operatorname{div}(|\nabla \phi|^{p-2} \nabla \phi) + (g - \lambda_1) \phi^{p-1} = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

Furthermore ϕ is simple, i.e. any solution of (1.2) satisfies $v = k\phi$ for some $k \in \mathbb{R}$. In the sequel we will normalize ϕ in the $L^p(\Omega)$ norm.

Clearly for any $\lambda < \lambda_1$ the only nonnegative solution of

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2} \nabla u) + (g - \lambda) u^{p-1} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.3)$$

is $u \equiv 0$.

On the other hand λ_1 is isolated, i.e. there exists $\delta > 0$ such that for any λ in $(\lambda_1, \lambda_1 + \delta)$ the only solution of (1.3) is $u \equiv 0$ as well.

Our first results concern some necessary conditions for the existence of solutions.

Theorem 1.2 *Suppose that there exists a nonnegative solution $u \not\equiv 0$ of equation (1.1). Then*

1) For $\lambda < \lambda_1$, the set Ω^+ defined as

$$\Omega^+ := \{x \in \Omega, f(x) > 0\}$$

is nonempty.

2) For $\lambda > \lambda_1$, $\Omega^- := \{x \in \Omega, f(x) < 0\} \neq \emptyset$ and $\int_{\Omega} f \phi^q < 0$.

3) For $\lambda = \lambda_1$, $\Omega^+ \neq \emptyset$, $\Omega^- \neq \emptyset$ and $\int_{\Omega} f \phi^q < 0$.

Theorem 1.3 *There exists $\lambda' > \lambda_1$ such that there are no non trivial non negative solutions of equation (1.1) for $\lambda > \lambda'$.*

Theorem 1.4 *Suppose that there exists $\bar{\lambda} > \lambda_1$ for which (1.1) possesses a solution. Then, (1.1) has a solution for $\lambda \in]\lambda_1, \bar{\lambda}]$.*

Our next result concerns the existence of solutions of equation (1.1) in the sub-critical case:

Theorem 1.5 *Suppose that Ω^+ and Ω^- are nonempty, that $p < q < p^*$, and $\int_{\Omega} f\phi^q < 0$. Then there exists $\delta > 0$ such that for $\lambda \in (\lambda_1, \lambda_1 + \delta)$ equation (1.1) has at least two non zero and nonnegative solutions of equation (1.1). For $\lambda = \lambda_1$ there exists at least one solution of (1.1) nonnegative and not identically zero.*

Remark 1: The solutions are obtained as minima of the two variational problems:

$$\alpha_{\lambda,q} = \inf_{\{u \in W_o^{1,p}(\Omega), \int_{\Omega} f|u|^q = -1\}} \left\{ \int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda)|u|^p \right\}$$

and

$$\mu_{\lambda,q} = \inf_{\{u \in W_o^{1,p}(\Omega), \int_{\Omega} f|u|^q = 1\}} \left\{ \int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda)|u|^p \right\}.$$

Indeed, if $u \in W_o^{1,p}(\Omega)$ realizes $\alpha_{\lambda,q}$ (respectively $\mu_{\lambda,q}$), so does $|u|$, and it is easy to see that u satisfies:

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + (g - \lambda)u^{p-1} = -\alpha_{\lambda,q}fu^{q-1}$$

(respectively

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + (g - \lambda)u^{p-1} = \mu_{\lambda,q}fu^{q-1}).$$

By a standard scaling argument one obtains two nonnegative solutions of equation (1.1), one being such that $\int_{\Omega} fu^q > 0$ and the other such that $\int_{\Omega} fu^q < 0$.

For simplicity of notation let $\alpha_{\lambda} := \alpha_{\lambda,p^*}$ and $\mu_{\lambda} := \mu_{\lambda,p^*}$.

Theorem 1.6 *Suppose that $q = p^*$ and that $\Omega^+, \Omega^- \neq \emptyset$, that $\lambda > \lambda_1$ and that $\int_{\Omega} f\phi^{p^*} < 0$. Then there exists $\delta > 0$ such that if $\lambda \in (\lambda_1, \lambda_1 + \delta)$ there exists at least one solution of equation (1.1). If moreover,*

$$\mu_{\lambda} < K(N, p)^{-p} \sup |f|^{\frac{-p}{p^*}},$$

then, there exist at least two non zero solutions of equation (1.1).

Remark 2: As in the subcritical case, the solutions are obtained as minima of α_λ and μ_λ .

Remark 3: According to Theorems 1.4 and 1.5 the solutions of equation (1.1) exist for an interval, $(\lambda_1, \bar{\lambda})$. On the other hand for some $\lambda \in]\lambda_1, \bar{\lambda}[$, there may be only one solution, because for λ not close to λ_1 nothing can be said about the sign of $\int_\Omega f u_\lambda^q$ when u_λ is a solution obtained by Theorem 1.4.

For $p = 2$ i.e. the classical Laplacian and $2 < q < \frac{2n}{n-2}$ problem (1.1) has been extensively studied when $f > 0$. Since we are concerned with the case where f changes sign, let us recall the main results in that case. Necessary and sufficient conditions for the existence of solutions for (1.1) have been given by Alama and Tarantello [1], Berestycki, Capuzzo Dolcetta and Nirenberg [5] and Ouyang [20] in the non coercive case.

Alama and Tarantello in [1] and the authors of the present paper in [6] have studied the critical case i.e. $q = \frac{2n}{n-2}$. Let us also mention the very interesting work of Chen and Li in [7].

It is well known that the p -Laplacian appears in many contexts : Non-Newtonian fluids, nonlinear elasticity and reaction diffusion problems just to name a few. Indeed equation (1.1) has been extensively studied for general p and q ; in particular for q critical, existence of solutions of problem (1.1) was studied by Guedda and Veron in [14] for $f \equiv 1$, $g(x) \equiv \lambda = 0$. Demengel and Hebey in [10] gave existence of variational solutions when f changes sign and the functional $\int_\Omega |\nabla u|^p + \int_\Omega (g - \lambda)|u|^p$ is coercive i.e. $\lambda < \lambda_1$.

In [12], the authors study a similar problem with $(g - \lambda)u^{p-1}$ replaced by $cte u^{k-1}$ with $k \neq p$.

Always for general p but q subcritical the non coercive case was also studied by Drabek and Pohozaev in [11]; they use the fibering method to obtain some existence results for λ close to λ_1 . See also Pohozaev and Veron [21] for the Neumann problem.

Finally for q critical, Drabek and Huang studied the problem in \mathbb{R}^N [10], while Arioli and Gazzola in [4] proved existence for solutions changing sign through a linking method.

The above Theorems are the natural extension to the p -Laplacian of the results obtained in [6]. Nonetheless the proofs differ from the case $p = 2$. In particular the proofs of Theorems 1.5 and 1.6 follow the approach taken by Ouyang in [20]. Although we should mention that Ouyang treats the

sub-critical case and he uses bifurcation technic that don't hold for $p \neq 2$.

The outline of the paper is the following. In the next section we prove the necessary conditions (i.e. Theorem 1.2 and 1.3) using among other things Picone's identity for the p -Laplacian (cf Allegretto and Huang [2]). In the third section we prove the existence results first for the sub-critical case and then for the critical case. Finally in the last section we construct some test functions to show that the condition on μ_λ of Theorem 1.6 can be satisfied and easily verified.

2 Proofs of Theorem 1.2, 1.3, 1.4.

Let us recall Picone's identity for the p -Laplacian as formulated by Allegretto and Huang in [2]. Suppose that v and w belong to $W^{1,p}(\Omega)$ with $v \geq 0$ and $w > 0$, then

$$|\nabla v|^p - \nabla \left(\frac{v^p}{w^{p-1}} \right) \cdot \sigma(w) \geq 0$$

everywhere in Ω , for $\sigma(w) := |\nabla w|^{p-2} \nabla w$.

Moreover if equality holds then $w = kv$ for some constant $k \in \mathbb{R}$.

Proof of Theorem 1.2

Since in the case $\lambda < \lambda_1$ the functional

$$I_\lambda(u) := \int_\Omega |\nabla u|^p + \int_\Omega (g - \lambda)|u|^p$$

is coercive the first assertion is obvious.

Let us prove 2. Suppose that $\lambda > \lambda_1$, and let u be a nonnegative solution of (1.1). Adapting the strict maximum principle of Vasquez, one has $u > 0$ inside Ω . In addition, from regularity results of [13], [23], [17], [9], u is $\mathcal{C}^{1,\alpha}(\bar{\Omega})$, for every $\alpha \in [0, 1[$. Using once more the strict maximum principle inspired from Hopf's lemma, as given in [24], one has the existence of some real $\epsilon > 0$ such that $\phi \geq \epsilon u$ on $\bar{\Omega}$. As a consequence, one is allowed to multiply the equation (1.1) by $(u)^{1-q} \phi^q$. Integrating by parts on Ω , one obtains

$$\int_\Omega f \phi^q = \int_\Omega \sigma(u) \cdot \nabla (u^{1-q} \phi^q) + \int_\Omega (g - \lambda) u^{p-1} u^{1-q} \phi^q$$

$$\begin{aligned}
&= (1-q) \int_{\Omega} |\nabla u|^p \left(\frac{\phi}{u}\right)^q + q \int_{\Omega} (\sigma(u) \cdot \nabla \phi) \left(\frac{\phi}{u}\right)^{q-1} \\
&+ \int_{\Omega} (g - \lambda) u^{p-q} \phi^q. \tag{2.4}
\end{aligned}$$

Now we multiply equation (1.2) by $\phi^{q-p+1} u^{p-q}$ and integrate over Ω ;

$$\int_{\Omega} \sigma(\phi) \cdot \nabla (\phi^{q-p+1} u^{p-q}) + \int_{\Omega} (g - \lambda_1) \phi^q u^{p-q} = 0$$

and then

$$\begin{aligned}
(q-p+1) \int_{\Omega} |\nabla \phi|^p \left(\frac{\phi}{u}\right)^{q-p} &+ (p-q) \int_{\Omega} \sigma(\phi) \cdot \nabla u \left(\frac{\phi}{u}\right)^{q-p+1} + \\
&+ \int_{\Omega} (g - \lambda_1) \phi^q u^{p-q} = 0. \tag{2.5}
\end{aligned}$$

Subtracting (2.4) to (2.5), one gets

$$\begin{aligned}
(q-p+1) \int_{\Omega} |\nabla \phi|^p \left(\frac{\phi}{u}\right)^{q-p} &+ (p-q) \int_{\Omega} \sigma(\phi) \cdot \nabla u \left(\frac{\phi}{u}\right)^{q-p+1} \\
-q \int_{\Omega} \left(\frac{\phi}{u}\right)^{q-1} \nabla \phi \cdot \sigma(u) &+ (q-1) \int_{\Omega} \left(\frac{\phi}{u}\right)^q |\nabla u|^p + \\
(\lambda - \lambda_1) \int_{\Omega} \phi^q u^{p-q} &= - \int_{\Omega} f \phi^q. \tag{2.6}
\end{aligned}$$

Now apply Picone's identity as follows

$$|\nabla u|^p - \nabla \left(\frac{u^p}{\phi^{p-1}} \right) \cdot \sigma(\phi) \geq 0.$$

Multiplying it by $\left(\frac{\phi}{u}\right)^q$ and integrating over Ω it becomes

$$\begin{aligned}
\int_{\Omega} |\nabla u|^p \left(\frac{\phi}{u}\right)^q &- p \int_{\Omega} \nabla u \cdot \sigma(\phi) u^{p-q-1} \phi^{q-p+1} + \\
&+ (p-1) \int_{\Omega} |\nabla \phi|^p u^{p-q} \phi^{q-p} \geq 0. \tag{2.7}
\end{aligned}$$

Similarly, exchanging the role of u and ϕ i.e. considering

$$|\nabla\phi|^p - \nabla\left(\frac{\phi^p}{u^{p-1}}\right) \cdot \sigma(u) \geq 0$$

and multiplying by $\left(\frac{\phi}{u}\right)^{q-p}$ one gets

$$\begin{aligned} \int_{\Omega} |\nabla\phi|^p \left(\frac{\phi}{u}\right)^{q-p} - p \int_{\Omega} \left(\frac{\phi}{u}\right)^{q-1} \nabla\phi \cdot \sigma(u) \\ + (p-1) \int_{\Omega} \left(\frac{\phi}{u}\right)^q |\nabla u|^p \geq 0. \end{aligned} \quad (2.8)$$

Multiply (2.8) by $\frac{q}{p}$ and (2.7) by $\frac{q}{p} - 1$ their sum gives

$$\begin{aligned} (q-p+1) \int_{\Omega} |\nabla\phi|^p \left(\frac{\phi}{u}\right)^{q-p} + (p-q) \int_{\Omega} \nabla u \cdot \sigma(\phi) \left(\frac{\phi}{u}\right)^{q-p+1} + \\ -q \int_{\Omega} \left(\frac{\phi}{u}\right)^{q-1} \nabla\phi \cdot \sigma(u) + (q-1) \int_{\Omega} |\nabla u|^p \left(\frac{\phi}{u}\right)^q \geq 0. \end{aligned} \quad (2.9)$$

Subtracting (2.9) from (2.6) we obtain

$$\int_{\Omega} f\phi^q + (\lambda - \lambda_1) \int_{\Omega} \phi^q u^{p-q} \leq 0. \quad (2.10)$$

When $\lambda > \lambda_1$, this implies that $\int_{\Omega} f\phi^q < 0$ and 2) is proved.

For the proof of 3), let $\lambda = \lambda_1$ and let u be a nonnegative solution of equation (1.1). Multiplying it by u one obtains

$$\int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda_1)u^p = \int_{\Omega} fu^q.$$

Since the functional I_{λ_1} is non negative, one has $\int_{\Omega} fu^q \geq 0$. Suppose that it is zero. Then u would be an eigenfunction for the eigenvalue λ_1 , which would imply that $fu^{q-1} = 0$. Then u must be zero on a set of positive measure, which contradicts the fact that u is parallel to $\phi > 0$ in Ω . We have proved that $\int_{\Omega} fu^q > 0$, this implies that $\Omega^+ \neq \emptyset$.

We shall now prove that $\int_{\Omega} f\phi^q < 0$, this of course implies also that $\Omega^- \neq \emptyset$.

From the previous computations in the proof of 2), and precisely from (2.6) with $\lambda = \lambda_1$ and from (2.9), we obtain that

$$\begin{aligned}
& (q-p+1) \int_{\Omega} |\nabla \phi|^p \left(\frac{\phi}{u}\right)^{q-p} + (p-q) \int_{\Omega} \nabla u \cdot \sigma(\phi) \left(\frac{\phi}{u}\right)^{q-p+1} \\
& -q \int_{\Omega} \left(\frac{\phi}{u}\right)^{q-1} \nabla \phi \cdot \sigma(u) + (q-1) \int_{\Omega} |\nabla u|^p \left(\frac{\phi}{u}\right)^q + \\
& = - \int_{\Omega} f \phi^q. \tag{2.11}
\end{aligned}$$

As a consequence $\int_{\Omega} f \phi^q \leq 0$. Suppose by contradiction that $\int_{\Omega} f \phi^q = 0$, then the left hand side of the previous identity is zero. Recalling (2.8) and (2.9) the left hand side is a sum of two nonnegative quantities, hence they must be both null. Therefore we have obtained that

$$\begin{aligned}
& \int_{\Omega} |\nabla \phi|^p \left(\frac{\phi}{u}\right)^{q-p} - p \int_{\Omega} \left(\frac{\phi}{u}\right)^{q-1} \nabla \phi \cdot \sigma(u) \\
& + (p-1) \int_{\Omega} \left(\frac{\phi}{u}\right)^q |\nabla u|^p = 0 \tag{2.12}
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\Omega} |\nabla u|^p \left(\frac{\phi}{u}\right)^q - p \int_{\Omega} \nabla u \cdot \sigma(\phi) u^{p-q-1} \phi^{q-p+1} + \\
& + (p-1) \int_{\Omega} |\nabla \phi|^p u^{p-q} \phi^{q-p} = 0. \tag{2.13}
\end{aligned}$$

Clearly (2.12) and (2.13) imply that

$$|\nabla u|^p - \nabla \left(\frac{u^p}{\phi^{p-1}} \right) \cdot \sigma(\phi) = 0$$

and

$$|\nabla \phi|^p - \nabla \left(\frac{\phi^p}{u^{p-1}} \right) \cdot \sigma(u) = 0.$$

Each of these identities implies that ϕ is parallel to u . Then u is an eigenfunction. This implies that $f u^{q-1}$ is identically zero which is a contradiction. ■

Proof of Theorem 1.3.

Let B be a ball on which $f > 0$, $B \subset\subset \Omega^+$. Let then (ψ, μ^*) be the non zero and non negative normalized solution, of

$$\begin{cases} -\Delta_p \psi + (-\mu^*)\psi^{p-1} = 0 & \text{in } B \\ \psi = 0 & \text{on } \partial B. \end{cases}$$

Suppose that a solution of equation (1.1) exists for λ such that $|g|_\infty + \mu^* < \lambda$, $u \geq 0$ and non identically zero. On B , by the strict maximum principle of Vasquez, $u > 0$. Using Picone's identity, one has

$$|\nabla \psi|^p - \nabla \left(\frac{\psi^p}{u^{p-1}} \right) \cdot \sigma(u) \geq 0$$

in B , hence, integrating over B

$$0 \leq \int_B (\mu^*)\psi^p + \int_B (g - \lambda)\psi^p \quad (2.14)$$

here, we have used the fact that $\psi = 0$ on ∂B and the equation verified by u , since

$$-\Delta_p u + (g - \lambda)u^{p-1} = fu^{q-1} \geq 0$$

on B . (2.14) of course contradicts the choice of λ . ■

Proof of Theorem 1.4.

Let $\bar{\lambda}$ be such that $\lambda_1 < \bar{\lambda}$ and take $\lambda \in]\lambda_1, \bar{\lambda}[$. Let \bar{u} be a solution of (1.1) for $\bar{\lambda}$. Then \bar{u} is a supersolution of (1.1) for λ . Indeed

$$-\Delta_p \bar{u} + (g - \lambda)\bar{u}^{p-1} = f\bar{u}^{q-1} + (\bar{\lambda} - \lambda)\bar{u}^{p-1} \geq f\bar{u}^{q-1}$$

and $\bar{u} = 0$ on the boundary. On another hand, taking ϵ small enough, $\epsilon\phi$ is a subsolution, since

$$-\Delta_p(\epsilon\phi) + (g - \lambda)(\epsilon\phi)^{p-1} = (\lambda_1 - \lambda)\epsilon^{p-1}\phi^{p-1} \leq f\epsilon^{q-1}\phi^{q-1},$$

(using $p < q$ and $(\lambda_1 - \lambda)\epsilon^{p-1}\phi^{p-1} < 0$). Moreover, using strong maximum principle of Vasquez and regularity results, one can choose ϵ small enough in order to have $\bar{u} \geq \epsilon\phi$. Finally we use the following Proposition, whose proof can be found in the appendix and is a mere adaptation of the classical sub and super solution for $p = 2$. (see e.g. [15], see also [22]):

Proposition 2.1 *Suppose that $f(x, t) = a(x)|t|^{q-2}t + b(x)|t|^{p-2}t$ with $1 < p < q$ with a and b two continuous and bounded functions on Ω . Suppose that \bar{u} is a weak supersolution for $-\Delta_p u + f(x, u)$, $\bar{u} = 0$ on $\partial\Omega$, and that \underline{u} is a weak subsolution with $\underline{u} = 0$ on $\partial\Omega$. Suppose that there exists some constant c and C such that*

$$-\infty < c \leq \underline{u} \leq \bar{u} \leq C < +\infty$$

Then, there exists a solution u between \underline{u} and \bar{u}

Using this Proposition with $f(x, u) = (g - \lambda)u^{p-1} - fu^{q-1}$, and $\underline{u} = \epsilon\phi$, one obtains that there exists a solution which is such that

$$\epsilon\phi \leq u \leq \bar{u}.$$

3 Existence of solutions

Proof of Theorem 1.5

This proof is inspired by the arguments used in [20]. We begin with the subcritical case. Suppose that $q < p^*$. Let us recall the following notations:

$$\begin{aligned} \lambda_q^* &= \inf_{\{u \in W_0^{1,p}(\Omega), |u|_p^p=1, \int_{\Omega} fu^q=0\}} \left\{ \int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda_1)|u|^p \right\} \\ \alpha_{\lambda,q} &= \inf_{\{u, \int_{\Omega} f|u|^q=-1\}} \left\{ \int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda)|u|^p \right\} \end{aligned} \quad (3.15)$$

and

$$\mu_{\lambda,q} = \inf_{\{u, \int_{\Omega} f|u|^q=1\}} \left\{ \int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda)|u|^p \right\}. \quad (3.16)$$

Let $I_{\lambda}(u) := \int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda)|u|^p$.

We will prove the following facts

1. $\lambda_q^* > 0$.
2. For $\lambda \in]\lambda_1, \lambda_1 + \lambda_q^*[$, $\alpha_{\lambda,q} < 0$ and it is achieved; $\alpha_{\lambda_1,q} = 0$.
3. For $\lambda \in]\lambda_1, \lambda_1 + \lambda_q^*[$, $\mu_{\lambda,q} > 0$ and it is achieved. Moreover $\mu_{\lambda_1,q} > 0$.

Proof of 1.

By the definition of λ_1 , $\lambda_q^* \geq 0$. Suppose by contradiction that $\lambda_q^* = 0$. Let (u_n) be a minimizing sequence. Since $|\nabla|u_n|| = |\nabla u_n|$, one can assume that $u_n \geq 0$. Since $|u_n|_p = 1$ and $\int_{\Omega} |\nabla u_n|^p + \int_{\Omega} (g - \lambda_1)u_n^p \rightarrow 0$, then $\int_{\Omega} |\nabla u_n|^p$ is bounded; hence (u_n) is bounded in $W_0^{1,p}$. Extracting from it a subsequence and passing to the limit, one gets that there exists some $u \geq 0$, weak limit of (u_n) in $W^{1,p}(\Omega)$, such that

$$\int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda_1)u^p \leq 0. \quad (3.17)$$

Clearly (3.17) implies that

$$\int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda_1)u^p = 0.$$

and then u is an eigenfunction for λ_1 and then it is parallel to ϕ . Moreover $u \in W_0^{1,p}$, $\int_{\Omega} |u|^p = 1$ and $\int_{\Omega} f u^q = 0$, which contradicts the assumption $\int_{\Omega} f \phi^q < 0$. Finally $\lambda_q^* > 0$.

Proof of 2.

In order to prove that $\alpha_{\lambda,q} < 0$ for $\lambda > \lambda_1$, let us take, as an admissible function, $v = \frac{\phi}{(-\int_{\Omega} f \phi^q)^{\frac{1}{q}}}$. We then have

$$\alpha_{\lambda,q} \leq I_{\lambda}(v) = \frac{1}{(-\int_{\Omega} f \phi^q)^{\frac{p}{q}}} I_{\lambda}(\phi) = \frac{1}{(-\int_{\Omega} f \phi^q)^{\frac{p}{q}}} (\lambda_1 - \lambda) < 0.$$

Now we will check that

$$\alpha_{\lambda,q} > -\infty.$$

If not, there would exist a subsequence (u_i) , $u_i \geq 0$ for all i , such that $\int_{\Omega} f u_i^q = -1$ and $I_{\lambda}(u_i) \rightarrow -\infty$. Clearly $|u_i|_p \rightarrow +\infty$ since

$$\overline{\lim} \int_{\Omega} (g - \lambda)u_i^p \leq \alpha_{\lambda,q}.$$

Let $w_i = \frac{u_i}{|u_i|_p}$. One has $\int_{\Omega} f w_i^q \rightarrow 0$, and (w_i) is bounded in $W_0^{1,p}(\Omega)$, since

$|w_i|_p = 1$ and $\int_{\Omega} |\nabla w_i|^p + \int_{\Omega} (g - \lambda)w_i^p = \frac{I_{\lambda}(u_i)}{|u_i|_p^p} \leq 0$ implies

$$\int_{\Omega} |\nabla w_i|^p \leq |g - \lambda|_{\infty}.$$

Then, there exists a subsequence still denoted (w_i) , such that $w_i \rightharpoonup w$ weakly in $W^{1,p}(\Omega)$. Observe that

$$\int_{\Omega} |w|^p = 1 \text{ and } I_{\lambda}(w) \leq 0.$$

This contradicts the definition of λ , since $\int_{\Omega} fw^q = 0$ and $\lambda \in]\lambda_1, \lambda_1 + \lambda_q^*[$. We have proved that $\alpha_{\lambda,q} > -\infty$.

We shall now see that $\alpha_{\lambda,q}$ is achieved. Let (u_n) , $u_n \geq 0$ be a minimizing sequence for $\alpha_{\lambda,q}$ i.e.

$$\int_{\Omega} |\nabla u_n|^p + \int_{\Omega} (g - \lambda)u_n^p \rightarrow \alpha_{\lambda,q},$$

$$\int_{\Omega} fu_n^q = -1.$$

Let us prove first that $|u_n|_p$ is bounded. If not, one can argue as previously by considering $w_n = \frac{u_n}{|u_n|_p}$. It is easy to see that (w_n) converges weakly in $W^{1,p}(\Omega)$, up to a subsequence, towards some function $w \geq 0$ which satisfies $\int_{\Omega} fw^q = 0$, $|w|_p = 1$ and

$$\int_{\Omega} |\nabla w|^p + \int_{\Omega} (g - \lambda)w^p = 0.$$

This contradicts the definition of λ . Hence $\int_{\Omega} |u_n|^p$ is bounded, and so is $\int_{\Omega} |\nabla u_n|^p$. By extracting from (u_n) a subsequence, one obtains that there exists $u \in W_0^{1,p}$, $u \geq 0$, such that $\int_{\Omega} fu^q = -1$ and by lower semi-continuity of the semi-norm $|\nabla u|_p$ with respect to the weak topology,

$$\int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda)u^p \leq \alpha_{\lambda,q}.$$

Finally using the definition of $\alpha_{\lambda,q}$, u is a minimizer for $\alpha_{\lambda,q}$, hence it is a nonzero solution of

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + (g - \lambda)u^{p-1} = -\alpha_{\lambda,q}fu^{q-1}.$$

Proof of 3.

Acting as we did for $\alpha_{\lambda,q}$ one can prove that $\mu_{\lambda,q} > -\infty$. We are now going to check that $\mu_{\lambda,q}$ is achieved.

Indeed, let u_n be a sequence such that $u_n \geq 0$,

$$\int_{\Omega} |\nabla u_n|^p + \int_{\Omega} (g - \lambda) u_n^p \rightarrow \mu_{\lambda, q},$$

$$\int_{\Omega} f u_n^q = 1.$$

Suppose that $|u_n|_p \rightarrow \infty$. Then considering $w_n = \frac{u_n}{|u_n|_p}$ one gets, by passing to the limit that there exists $w \geq 0$, a weak limit of (w_n) in $W^{1,p}(\Omega)$, such that

$$\int_{\Omega} |\nabla w|^p + \int_{\Omega} (g - \lambda) w^p \leq 0$$

and $\int_{\Omega} f w^q = 0$, which contradicts the assumption $\lambda \in]\lambda_1, \lambda_1 + \lambda_q^*[$. Then (u_n) is bounded and we pass to the limit to obtain

$$\int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda) u^p = \mu_{\lambda, q}$$

and $\int_{\Omega} f u^q = 1$. Hence $\mu_{\lambda, q}$ is achieved.

For $\lambda = \lambda_1$, $\mu_{\lambda_1, q} \geq 0$, but since it is achieved, if $\mu_{\lambda_1, q} = 0$, we would have an eigenfunction u such that $\int_{\Omega} f u^q = 1$, which contradicts the assumptions. Then $\mu_{\lambda_1, q} > 0$.

For $\lambda > \lambda_1$ let $u_q \geq 0$ which realizes the minimum in $\mu_{\lambda, q}$. Then :

$$-\Delta_p u_q + (g - \lambda) u_q^{p-1} = \mu_{\lambda, q} f u_q^{q-1}.$$

Using the procedure of the proof of Theorem 1.2 for u_q , inequality (2.10) becomes

$$\mu_{\lambda, q} \int_{\Omega} f \phi^q + (\lambda - \lambda_1) \int_{\Omega} \phi^q u_q^{p-q} \leq 0.$$

Using $\int_{\Omega} f \phi^q < 0$ and $\lambda - \lambda_1 > 0$, one gets $\mu_{\lambda, q} > 0$. ■

Let us now state and prove some results concerning $\alpha_{\lambda, q}$ and $\mu_{\lambda, q}$.

Lemma 3.1 *The following convergences hold:*

$$\lim_{\lambda \rightarrow \lambda_1} \alpha_{\lambda, q} = \alpha_{\lambda_1, q} = 0, \tag{3.18}$$

$$\lim_{\lambda \rightarrow \lambda_1} \mu_{\lambda, q} = \mu_{\lambda_1, q} \tag{3.19}$$

Lemma 3.2 1. $\lambda_{p^*}^* \geq \overline{\lim}_{q \rightarrow p^*} \lambda_q^* \geq \underline{\lim}_{q \rightarrow p^*} \lambda_q^* := \lambda^* > 0$.

2. For $\lambda_1 \leq \lambda < \lambda_1 + \lambda^*$, then $0 \leq \underline{\lim}_{q \rightarrow p^*} \mu_{\lambda,q} \leq \overline{\lim}_{q \rightarrow p^*} \mu_{\lambda,q} \leq \mu_\lambda (= \mu_{\lambda,p^*})$.

3. For λ close to λ_1 , $\alpha_\lambda (= \alpha_{\lambda,p^*}) > -\infty$ and $\overline{\lim}_{q \rightarrow p^*} \alpha_{\lambda,q} \leq \alpha_\lambda$.

Proof of Lemma 3.1

Suppose by contradiction that (3.18) does not hold, then there exist some number $\alpha < 0$ and a sequence of $\lambda \in \mathbb{R}$, $\lambda \rightarrow \lambda_1$, and $(u_\lambda) \subset W_0^{1,p}(\Omega)$ such that

$$\int_{\Omega} |\nabla u_\lambda|^p + \int_{\Omega} (g - \lambda) |u_\lambda|^p \leq \alpha.$$

Moreover one can assume that $u_\lambda \geq 0$. If (u_λ) is bounded, we may extract from it a subsequence weakly convergent to some $u \in W_0^{1,p}$, such that

$$\int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda_1) u^p \leq \alpha < 0,$$

which is absurd.

On the other hand if (u_λ) diverges we can normalize it and then we obtain a sequence (w_λ) such that $\int_{\Omega} |w_\lambda|^p = 1$. By extracting a subsequence, there exists $w \geq 0$, such that $\int_{\Omega} |w|^p = 1$, $\int_{\Omega} f w^q = 0$ and

$$\int_{\Omega} |\nabla w|^p + \int_{\Omega} (g - \lambda_1) w^p \leq 0.$$

This would imply that w is parallel to ϕ which is absurd since $\int_{\Omega} f \phi^q < 0$.

Let us now prove (3.19). Let us define $\bar{\mu}_q := \overline{\lim}_{\lambda \rightarrow \lambda_1} \mu_{\lambda,q}$. One already has $\bar{\mu}_q \leq \mu_{\lambda_1,q}$. Let u_λ which satisfies $u_\lambda \geq 0$ and

$$-\Delta_p u_\lambda + (g - \lambda) u_\lambda^{p-1} = \mu_{\lambda,q} f u_\lambda^{q-1} \quad (3.20)$$

$$\int_{\Omega} f u_\lambda^q = 1.$$

As we did above, one can prove that (u_λ) is bounded in the $W^{1,p}$ norm. By extracting a subsequence, one gets by passing to the limit when $\lambda \rightarrow \lambda_1$

$$\int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda_1) u^p \leq \bar{\mu}_q$$

and $u \geq 0$, $\int_{\Omega} f u^q = 1$. This clearly implies that $\bar{\mu}_q \geq \mu_{\lambda_1,q}$ and gives the required result. \blacksquare

Proof of Lemma 3.2

Let us prove 1, and first that $\underline{\lim}_{q \rightarrow p^*} \lambda_q^* > 0$. Since λ_q^* is achieved, let $u_q \geq 0$ be a solution of

$$\int_{\Omega} |\nabla u_q|^p + \int_{\Omega} (g - \lambda_1) u_q^p = \lambda_q^*$$

$|u_q|_p = 1$ and $\int_{\Omega} f u_q^q = 0$. Suppose by contradiction that $\underline{\lim}_{q \rightarrow p^*} \lambda_q^* = 0$. Then, by extracting from (u_q) a subsequence, one gets by passing to the limit when q tends to p^* :

$$\int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda_1) u^p \leq 0$$

and $|u|_p = 1$. Since I_{λ_1} is coercive, $\int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda_1) u^p = 0$, and the sequence $\int_{\Omega} |\nabla u_q|^p$ tends to $\int_{\Omega} |\nabla u|^p$. Hence u_q tends to u strongly in $W^{1,p}(\Omega)$, and finally $\int_{\Omega} f u^{p^*} = \lim_{q \rightarrow p^*} \int_{\Omega} f u_q^q = 0$. This is a contradiction since ϕ is simple and $\int_{\Omega} f \phi^{p^*} < 0$. As a consequence $\lambda^* > 0$.

We now prove that $\lambda^* \leq \lambda_{p^*}^*$. Indeed, let $u \geq 0$ be a \mathcal{C}^1 function, such that $\int_{\Omega} f u^{p^*} = 0$, $|u|_p = 1$, and

$$I_{\lambda_1}(u) \leq \lambda_{p^*}^* + \epsilon.$$

If there exists an infinite sequence $q \rightarrow p^*$, such that $\int_{\Omega} f u^q = 0$, one has the desired result. If not, there exists an infinite sequence $q \rightarrow p^*$ such that either $\int_{\Omega} f u^q > 0$ for all q , or $\int_{\Omega} f u^q < 0$ for all q . Suppose that we are in the first case and define $\alpha(q) = \frac{\int_{\Omega} f u^q}{\int_{\Omega} f u^q - \int_{\Omega} f \phi^q}$. Then $\alpha(q) \in [0, 1]$, and $\alpha(q) \rightarrow 0$ when $q \rightarrow p^*$. Let us define

$$v_q = (\alpha(q)\phi^q + (1 - \alpha(q))u^q)^{\frac{1}{q}}.$$

By the regularity properties of ϕ and u , v_q belongs to $W_0^{1,p}(\Omega)$, $v_q \geq 0$ and $\int_{\Omega} f v_q^q = 0$ by the choice of $\alpha(q)$. Moreover it is easy to check that v_q tends to u in $W^{1,p}(\Omega)$ strongly. As a consequence

$$\lambda_q^*(1 + o(1)) \leq \lambda_q^*(\int_{\Omega} v_q^p) \leq \int_{\Omega} |\nabla v_q|^p + \int_{\Omega} (g - \lambda_1) v_q^p \leq \lambda_{p^*}^* + \epsilon + o(1)$$

when $q \rightarrow p^*$. This implies that $\lambda^* \leq \lambda_{p^*}^*$. Suppose now that there exists a sequence $q \rightarrow p^*$ such that $\int_{\Omega} f u^q < 0$. Let u_0 be nonnegative in $\mathcal{C}^1(\bar{\Omega})$, such that $\int_{\Omega} f u_0^q > 0$ and define

$$v_{\alpha} = (\alpha(q)u_0^q + (1 - \alpha(q))u^q)^{\frac{1}{q}},$$

where $\alpha(q) = \frac{\int_{\Omega} f u^q}{\int_{\Omega} f u^q - \int_{\Omega} f u_0^q}$. One concludes as in the case $\int_{\Omega} f u^q > 0$.

To prove 2., let $\varepsilon > 0$ be given and let u be such that $u \geq 0$, $\int_{\Omega} f u^{p^*} = 1$ and

$$I_{\lambda}(u) \leq \mu_{\lambda} + \varepsilon.$$

Then for q close to p^* , $\int_{\Omega} f u^q > \frac{1}{2}$ and taking $v_q = \frac{u}{(\int_{\Omega} f u^q)^{\frac{1}{q}}}$, one gets, for q sufficiently close to p^* ,

$$\mu_{\lambda, q} \leq I_{\lambda}(v_q) \leq \mu_{\lambda} + 2\varepsilon.$$

We will prove 3. by contradiction. Hence suppose that there exists a sequence $\lambda_n \rightarrow \lambda_1$ and a sequence (u_n) , $u_n \geq 0$ such that $\int_{\Omega} f u_n^{p^*} = -1$ and $I_{\lambda_n}(u_n) \leq -n$. Clearly $|u_n|_p \rightarrow +\infty$. Then defining $w_n = \frac{u_n}{|u_n|_p}$, and extracting a subsequence from it, one gets that there exists $w \geq 0$ such that

$$I_{\lambda_1}(w) \leq 0.$$

This in fact implies that strong convergence holds and then $\int_{\Omega} f w^{p^*} = 0$, which contradicts $|w|_p = 1$ and ϕ is simple. ■

Before giving the proof of Theorem 1.6 let us recall one of the key ingredients employed herein i.e. the famous concentration compactness principle of P.L. Lions[18]:

Lemma 3.3 *Let Ω be some bounded open set in \mathbb{R}^n , and (u_k) be some sequence in $W_o^{1,p}(\Omega)$, which is bounded in $W^{1,p}(\Omega)$. Then there exist a subsequence of (u_k) , still denoted (u_k) for simplicity, two nonnegative measures μ*

and ν on $\bar{\Omega}$, a sequence of points x_i in $\bar{\Omega}$, two sequences of nonnegative real numbers μ_i and ν_i and a function u in $W_o^{1,p}(\Omega)$, such that

$$|\nabla u_k|^p \rightharpoonup \mu \geq |\nabla u|^p + \sum_i \mu_i \delta_{x_i}$$

(the convergence being tight on $\bar{\Omega}$ i.e. $\int_{\Omega} |\nabla u_k|^p \rightarrow \int_{\bar{\Omega}} \mu$),

$$|u_k|^{p^*} \rightharpoonup \nu = |u|^{p^*} + \sum_i \nu_i \delta_{x_i}$$

(the convergence being tight on $\bar{\Omega}$ i.e. $\int_{\Omega} |u_k|^{p^*} \rightarrow \int_{\bar{\Omega}} \nu$), with the inequality

$$\nu_i \leq K(n,p)^{\frac{p^*}{p}} \mu_i. \quad (3.21)$$

Proof of Theorem 1.6

First part.

We prove the existence of solutions for α_λ and for λ sufficiently close to λ_1 . According to Lemma 3.1 above, $\lim_{\lambda \rightarrow \lambda_1} \alpha_\lambda = 0$. One takes λ sufficiently close to λ_1 in order to have $-\alpha_\lambda < K(N,p)^{-p} (\sup |f|)^{\frac{-p}{p^*}}$, and $\lambda < \lambda_1 + \lambda^*$. Let (u_q) , $u_q \geq 0$ be a solution for the problem defining $\alpha_{\lambda,q}$.

Claim: $(u_q)_q$ is bounded in L^p .

Suppose that it is not true. Then, proceeding as in the proof of Theorem 1.5, there would exist a sequence (w_q) such that $w_q \geq 0$, $|w_q|_p = 1$, and

$$\int_{\Omega} |\nabla w_q|^p + \int_{\Omega} (g - \lambda) w_q^p \leq 0. \quad (3.22)$$

Extracting from (w_q) a subsequence one obtains that there exists w , weak limit of w_q in $W^{1,p}$ such that $w \geq 0$, $|w|_p = 1$, and

$$\int_{\Omega} |\nabla w|^p + \int_{\Omega} (g - \lambda) w^p \leq 0.$$

If $\int_{\Omega} f w^{p^*} = 0$, this contradicts the assumption $\lambda < \lambda_1 + \lambda^* \leq \lambda_1 + \lambda_p^*$. If $\int_{\Omega} f w^{p^*} > 0$, $I_\lambda(w) \geq \mu_\lambda (\int_{\Omega} f w^{p^*})^{\frac{p}{p^*}} > 0$, and since $\mu_\lambda \geq 0$ one would obtain that $\mu_\lambda = 0 = I_\lambda(w)$, and using lower semi-continuity for the weak topology

$$I_\lambda(w) \leq \liminf_{q \rightarrow p^*} I_\lambda(w_q) \leq 0.$$

Finally $I_\lambda(w) = \lim_{q \rightarrow p^*} I_\lambda(w_q)$ and then $\int_\Omega |\nabla w_q|^p \rightarrow \int_\Omega |\nabla w|^p$, strong convergence holds in fact, hence $\int_\Omega f w^{p^*} = \lim_{q \rightarrow p^*} \int_\Omega f w_q^q = 0$, which is a contradiction of the assumption $\int_\Omega f w^{p^*} > 0$.

Finally suppose that $\int_\Omega f w^{p^*} < 0$. Then, applying P.L. Lions' concentration compactness lemma recalled above, one gets that there exists two bounded and nonnegative measures μ and ν on $\bar{\Omega}$, some countable set of points (x_i) in $\bar{\Omega}$, and some sequence of non-negative numbers (μ_i) and (ν_i) , which satisfy, up to a subsequence

$$|\nabla w_q|_p^p \rightharpoonup \mu \geq |\nabla w|_p^p + \sum_i \mu_i \delta_{x_i} \quad (3.23)$$

$$|w_q|^q \rightharpoonup \nu = |w|^{p^*} + \sum_i \nu_i \delta_{x_i}. \quad (3.24)$$

Passing to the limit in (3.22), in the equality $\int_\Omega f w_q^q = \frac{-1}{|u_q|_p^q}$, and using (3.23) and (3.24), one obtains

$$I_\lambda(w) \leq - \sum_i \mu_i,$$

$$\int_\Omega f w^{p^*} + \sum_i \nu_i f(x_i) = 0.$$

On the other hand, using $\int_\Omega f w^{p^*} < 0$, one has

$$\alpha_\lambda \left(- \int_\Omega f w^{p^*} \right)^{\frac{p}{p^*}} \leq I_\lambda(w) \leq - \sum_i \mu_i.$$

Hence,

$$\sum_i \mu_i \leq - \alpha_\lambda \left(\sum_i \nu_i f(x_i) \right)^{\frac{p}{p^*}}$$

Finally

$$\sum_i \mu_i \leq - \alpha_\lambda \sum_i (\nu_i f(x_i))^{\frac{p}{p^*}} \leq - \alpha_\lambda \sup |f|^{\frac{p}{p^*}} K(N, p)^p \sum_i \mu_i \leq \delta \sum_i \mu_i$$

for some $\delta < 1$. One obtains that $\mu_i = 0$ and then $\nu_i = 0$, as well as $\int_\Omega f w^{p^*} = 0$, which contradicts the assumption.

As a consequence the claim is proved i.e. (u_q) is bounded in L^p .

Furthermore, since

$$\alpha_{\lambda, q} \geq (\lambda_1 - \lambda) \int_\Omega |u_q|^p$$

the sequence $\alpha_{\lambda,q}$ is bounded too. Let us denote by $\bar{\alpha}$ the limit of a subsequence. Clearly $\bar{\alpha} \leq \alpha_\lambda$. Since $(u_q), (u_q \geq 0)$ is bounded, one may extract a subsequence such that $u_q \rightharpoonup u$ in $W^{1,p}$. Let us recall that u_q satisfies:

$$\begin{cases} -\Delta_p u_q + (g - \lambda)u_q^{p-1} = -\alpha_{\lambda,q} f u_q^{q-1}, \\ \int_{\Omega} f u_q^q = -1 \end{cases} \quad (3.25)$$

Let us denote by σ the weak limit of a subsequence in $L^{\frac{p}{p-1}}(\Omega)$ of $\sigma_q := |\nabla u_q|^{p-1} \nabla u_q$. Then, passing to the limit in equation (3.25) one gets $u \geq 0$ and

$$-\operatorname{div}(\sigma) + (g - \lambda)u^{p-1} = -\bar{\alpha} f u^{p^*-1}. \quad (3.26)$$

Using again P.L. Lions' concentration lemma, there exist two bounded and nonnegative measures μ and ν on $\bar{\Omega}$, some countable sets of points (x_i) in $\bar{\Omega}$, and some sequence of nonnegative numbers (μ_i) and (ν_i) , which satisfy, up to a subsequence

$$|\nabla u_q|_p^p \rightharpoonup \mu \geq |\nabla u|_p^p + \sum_i \mu_i \delta_{x_i} \text{ tightly on } \bar{\Omega},$$

$$|u_q|^q \rightharpoonup \nu = |u|^{p^*} + \sum_i \nu_i \delta_{x_i}, \text{ tightly on } \bar{\Omega}.$$

Let us multiply equation (3.25) (resp. equation (3.26)) by $u_q \varphi$ (resp. $u \varphi$), for a function φ in $\mathcal{D}(\bar{\Omega})$. One obtains

$$\int_{\Omega} |\nabla u_q|^p \varphi + \int_{\Omega} \sigma_q \cdot \nabla \varphi u_q + \int_{\Omega} (g - \lambda) u_q^p \varphi = -\alpha_{\lambda,q} \int_{\Omega} f u_q^q \varphi \quad (3.27)$$

and

$$\int_{\Omega} (\sigma \cdot \nabla u) \varphi + \int_{\Omega} (\sigma \cdot \nabla \varphi) u + \int_{\Omega} (g - \lambda) u^p \varphi = -\bar{\alpha} \int_{\Omega} f u^{p^*} \varphi. \quad (3.28)$$

By passing to the limit in (3.27), one gets

$$\int_{\Omega} \mu \varphi + \int_{\Omega} (\sigma \cdot \nabla \varphi) u + \int_{\Omega} (g - \lambda) u^p \varphi = -\bar{\alpha} \left(\int_{\Omega} f u^{p^*} \varphi + \sum_i \nu_i f(x_i) \varphi(x_i) \right). \quad (3.29)$$

Subtracting (3.28) from (3.29) one obtains

$$\int_{\Omega} (\mu - \sigma \cdot \nabla u) \varphi = -\bar{\alpha} \left(\sum_i \nu_i f(x_i) \varphi(x_i) \right). \quad (3.30)$$

Using Lebesgue decomposition of $\mu := \mu^{ac} + \mu^s$, where μ^{ac} is the absolutely continuous part of μ , one derives

$$|\nabla u|^p \leq \mu^{ac} = \sigma \cdot \nabla u, \quad (3.31)$$

$$\sum_i \mu_i \delta_{x_i} \leq \mu^s = -\bar{\alpha} \nu_i f(x_i) \delta_{x_i}. \quad (3.32)$$

Suppose first that x_i is such that $f(x_i) \leq 0$, then $\mu_i = \nu_i = 0$.

On the other hand, passing to the limit in equation (3.27) and using lower semi-continuity one has

$$I_\lambda(u) \leq \bar{\alpha} < 0.$$

If $\int_\Omega f u^{p^*} = 0$ this contradicts the assumption $\lambda \in]\lambda_1, \lambda_1 + \lambda_{p^*}^*[$. If $\int_\Omega f u^{p^*} > 0$ one also gets a contradiction, since

$$0 \leq \mu_\lambda \left(\int_\Omega f u^{p^*} \right)^{\frac{p}{p^*}} \leq I_\lambda(u).$$

Suppose that $\int_\Omega f u^{p^*} < 0$, then using (3.31) and (3.28) one has

$$\alpha_\lambda \left(- \int_\Omega f u^{p^*} \right)^{\frac{p}{p^*}} \leq I_\lambda(u) \leq -\bar{\alpha} \int_\Omega f u^{p^*} \leq -\alpha_\lambda \int_\Omega f u^{p^*}.$$

From this, one obtains that $-\int_\Omega f u^{p^*} \leq 1$.

On the other hand the identity

$$\int_\Omega f u^{p^*} + \sum_i \nu_i f(x_i) = -1$$

yields to $\sum_i \nu_i f(x_i) \leq 0$, and since we are in the case $f(x_i) \geq 0$ we get $\nu_i f(x_i) = 0$ for all i . Using (3.32) one obtains that $\int_\Omega f u^{p^*} = -1$ and $\mu_i = 0$. We have then

$$\alpha_\lambda \leq \int_\Omega |\nabla u|^p + \int_\Omega (g - \lambda) u^p \leq \int_\Omega \sigma \cdot \nabla u + \int_\Omega (g - \lambda) u^p = \bar{\alpha} \leq \alpha_\lambda$$

which implies that $\bar{\alpha} = \alpha_\lambda$, $\sigma \cdot \nabla u = |\nabla u|^p$, the convergence of ∇u_q is strong in $W^{1,p}(\Omega)$ and α_λ is achieved.

Second part. Since $\lim_{\lambda \rightarrow \lambda_1} \alpha_\lambda = \alpha_{\lambda_1} = 0$, one can choose λ sufficiently close to λ_1 in order to have

$$\alpha_\lambda > - \left(\sup |f|^{\frac{p}{p^*}} K(N, p)^p \right).$$

Now let u_q be a function for which $\mu_{\lambda, q}$ is achieved, $u_q \geq 0$.

Claim: (u_q) is bounded in L^p when q goes to p^* .

Suppose on the contrary that $|u_q|_p$ tends to infinity. Then, defining $w_q = \frac{u_q}{|u_q|_p}$, one obtains that w_q tends, up to a subsequence, to a function $w \in W_0^{1,p}(\Omega)$, $w \geq 0$ which satisfies $|w|_p = 1$, and

$$\begin{aligned} \int_{\Omega} |\nabla w|^p + \sum_i \mu_i + \int_{\Omega} (g - \lambda) w^p &\leq 0, \\ \int_{\Omega} f w^{p^*} + \sum_i \nu_i f(x_i) &= 0 \end{aligned}$$

where (μ_i) and (ν_i) are as in the first part.

Suppose first that $\int_{\Omega} f w^{p^*} = 0$. Then one gets a contradiction with the conditions on λ since

$$\int_{\Omega} |\nabla w|^p + \int_{\Omega} (g - \lambda) w^p \leq 0.$$

Suppose that $\int_{\Omega} f w^{p^*} > 0$. Then by the definition of μ_λ one would obtain that

$$\mu_\lambda \left(\int_{\Omega} f w^{p^*} \right)^{\frac{p}{p^*}} \leq \int_{\Omega} |\nabla w|^p + \int_{\Omega} (g - \lambda) w^p \leq 0$$

Since $\mu_\lambda \geq 0$, this may happen only if $\mu_\lambda = 0$, and in the same time $I_\lambda(w) = 0$. Then, coming back to the previous inequalities, one has

$$I_\lambda(w) = 0 \leq \underline{\lim}_{q \rightarrow p^*} I_\lambda(w_q) \leq 0$$

hence $I_\lambda(w_q) \rightarrow I_\lambda(w)$, and strong convergence holds. This implies that $\int_{\Omega} f w^{p^*} = \lim_{q \rightarrow p^*} \int_{\Omega} f w_q^q = 0$, which contradicts the assumption $\int_{\Omega} f w^{p^*} > 0$.

Suppose finally that $\int_{\Omega} f w^{p^*} < 0$, then one can write

$$\alpha_\lambda \left(- \int_{\Omega} f w^{p^*} \right)^{\frac{p}{p^*}} \leq \int_{\Omega} |\nabla w|^p + \int_{\Omega} (g - \lambda) w^p \leq - \sum_i \mu_i$$

and then

$$\begin{aligned}
\sum_i \mu_i &\leq (-\alpha_\lambda) \left(\sum_i \nu_i |f(x_i)| \right)^{\frac{p}{p^*}} \\
&\leq (-\alpha_\lambda) \left(\sum_i \nu_i^{\frac{p}{p^*}} |f(x_i)|^{\frac{p}{p^*}} \right) \\
&\leq (-\alpha_\lambda) \sup |f|^{\frac{p}{p^*}} \sum_i \mu_i K(N, p)^p \\
&\leq \delta \sum_i \mu_i
\end{aligned}$$

for some $\delta < 1$. Finally one has $\mu_i = 0$ for all i and then $\nu_i = 0$. Then $\int_\Omega f w^{p^*} = 0$ which is absurd, as we remarked before. We have obtained that (u_q) is bounded. This proves the claim.

Let $\beta = \frac{1}{2} \left(K(N, p)^{-p} \sup |f|^{\frac{-p}{p^*}} - \mu_{\lambda_1} \right)$ and suppose that λ is sufficiently close to λ_1 in order to ensure that

$$|\alpha_\lambda| < \beta.$$

Let (u_q) be a sequence of nonnegative minimizers for $\mu_{\lambda, q}$, $u_q \geq 0$. Then

$$-\Delta_p u_q + (g - \lambda) u_q^{p-1} = \mu_{\lambda, q} f u_q^{q-1} \quad (3.33)$$

$$\int_\Omega f u_q^q = 1.$$

By the previous computations, the sequence (u_q) is bounded in L^p , and since $(\mu_{\lambda, q})$ is bounded too, (u_q) is in fact bounded in $W^{1, p}$. Let us extract from it a subsequence such that

$$u_q \rightharpoonup u$$

in $W^{1, p}$ weakly. Let us denote by γ the limit of some subsequence of $\mu_{\lambda, q}$. One has $\gamma \leq \mu_\lambda \leq \mu_{\lambda_1}$.

Acting as we did in the first part, one gets

$$-\operatorname{div}(\sigma) + (g - \lambda) u^{p-1} = \gamma f u^{p^*-1}, \quad (3.34)$$

denoting by σ a weak limit of $|\nabla u_q|^{p-1} \nabla u_q$ in $L^{\frac{p}{p-1}}(\Omega)$.

Multiplying equation (3.33) (respectively (3.34)) by $u_q\varphi$ (respectively by $u\varphi$) with $\varphi \in \mathcal{D}(\bar{\Omega})$ and integrating over Ω , introducing measures μ and ν as in the concentration compactness lemma one gets

$$\begin{aligned}\mu^{ac} - \sigma \cdot \nabla u &= 0 \\ \sum_i \mu_i \delta_i &\leq \mu^s = \gamma \sum_i \nu_i f(x_i) \delta_i.\end{aligned}\tag{3.35}$$

This last identity yields that γ cannot be zero: if it was, one would have $\mu_i = 0$, hence $\nu_i = 0$, and in the same time,

$$\int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda)u^p = 0$$

and

$$\int_{\Omega} f u^{p^*} = 1.$$

This is impossible, since for example, one has supposed that λ is not an eigenvalue. Then $\gamma > 0$. Moreover, if x_i is such that $f(x_i) < 0$, then $\mu_i = 0$, and so is ν_i . Since one has

$$|\nabla u|^p \leq \mu^{ac} = \sigma \cdot \nabla u,$$

coming back to (3.34), one gets

$$\int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda)u^p \leq \int_{\Omega} \sigma \cdot \nabla u + \int_{\Omega} (g - \lambda)u^p = \gamma \int_{\Omega} f u^{p^*}.$$

On another hand the identity

$$\int_{\Omega} f u^{p^*} + \sum_i \nu_i f(x_i) = 1$$

implies that $\sum_i \nu_i f(x_i) \leq 1$ if $\int_{\Omega} f u^{p^*} \geq 0$. Suppose now that $\int_{\Omega} f u^{p^*} < 0$. Then $\nu_f = \sum_i \nu_i f(x_i) > 1$. In the same time one has

$$\alpha_{\lambda} \left(- \int_{\Omega} f u^{p^*} \right)^{\frac{p}{p^*}} \leq \int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda)u^p \leq \gamma \int_{\Omega} f u^{p^*}$$

and then

$$\nu_f \leq 1 + \left(\frac{-\alpha}{\gamma} \right)^{\frac{1}{1-\frac{p}{p^*}}}.$$

As seen before if $f(x_i) < 0$, $\mu_i = 0$, hence $\nu_i = 0$. If $f(x_i) \geq 0$, the previous calculations imply that for all i , $\nu_i f(x_i) \leq 1 + \left(\frac{-\alpha}{\gamma}\right)^{\frac{1}{1-\frac{p}{p^*}}}$. Finally

$$\begin{aligned}
\mu_i &\leq \gamma \left(\frac{\nu_i f(x_i)}{1 + \left(\frac{-\alpha}{\gamma}\right)^{\frac{1}{1-\frac{p}{p^*}}}} \right) \left(1 + \left(\frac{-\alpha}{\gamma}\right)^{\frac{1}{1-\frac{p}{p^*}}} \right) \\
&\leq \gamma \left(\frac{\nu_i f(x_i)}{1 + \left(\frac{-\alpha}{\gamma}\right)^{\frac{1}{1-\frac{p}{p^*}}}} \right)^{1-\frac{p}{p^*}} \left(\frac{\nu_i f(x_i)}{1 + \left(\frac{-\alpha}{\gamma}\right)^{\frac{1}{1-\frac{p}{p^*}}}} \right)^{\frac{p}{p^*}} \left(1 + \left(\frac{-\alpha}{\gamma}\right)^{\frac{1}{1-\frac{p}{p^*}}} \right) \\
&\leq \gamma \left(1 + \left(\frac{-\alpha}{\gamma}\right)^{\frac{1}{1-\frac{p}{p^*}}} \right)^{1-\frac{p}{p^*}} K(N, p)^p \sup |f|^{\frac{p}{p^*}} \mu_i \\
&\leq \gamma \left(1 + \frac{-\alpha}{\gamma} \right) K(N, p)^p \sup |f|^{\frac{p}{p^*}} \mu_i \\
&\leq K(N, p)^p \sup |f|^{\frac{p}{p^*}} \mu_i (\gamma - \alpha) \\
&\leq \delta \mu_i
\end{aligned} \tag{3.36}$$

for some $\delta < 1$. As a consequence $\mu_i = 0$ and then $\nu_i = 0$. Finally

$$\int_{\Omega} f u^{p^*} = 1,$$

$$\mu_{\lambda} \leq \int_{\Omega} |\nabla u|^p + \int_{\Omega} (g - \lambda) |u|^p \leq \int_{\Omega} \sigma \cdot \nabla u + \int_{\Omega} (g - \lambda) |u|^p \leq \gamma$$

hence $\mu_{\lambda} = \gamma$, $|\nabla u|^p = \sigma \cdot \nabla u = \mu$, the convergence is strong, and u is a minimizer for μ_{λ} . \blacksquare

Remark 3.4 *We have also obtained that $\mu_{\lambda} > 0$.*

Corollary 3.5 *Suppose that $\int_{\Omega} f \phi^{p^*} < 0$ and that there exists a minimizer for $\lambda = \lambda_1$, then there exist at least two minimizers for $\lambda > \lambda_1$, and λ sufficiently close to λ_1 .*

Proof

Suppose that there exists a minimizer u_1 for the problem with $\lambda = \lambda_1$. Then

$$\begin{aligned} \inf_{\{u \in W_0^{1,p}(\Omega), \int_{\Omega} f|u|^{p^*} = 1\}} \left\{ \int_{\Omega} |\nabla u|_p^p + \int_{\Omega} (g - \lambda)u^p \right\} &\leq I_{\lambda}(u_1) < I_{\lambda_1}(u_1) \\ &= \inf I_{\lambda_1}(u) \\ &\leq \frac{1}{K(N, p)^p \sup f(x)^{\frac{p}{p^*}}}. \end{aligned}$$

As a consequence, using Theorem 1.6 one obtains that I_{λ} has a minimizer. ■

4 Estimates and test functions

Let $x_0 \in \mathbf{R}^N$ and $r = |x - x_0|$ the euclidean distance from x_0 to x . For $p > 1$ given, p real such that $p < N$, we define the function u_{ϵ} by

$$u_{\epsilon}(x) = (\epsilon + r^{p/p-1})^{1-N/p}$$

and the function v_{ϵ} by

$$v_{\epsilon}(x) = (\epsilon + r^{p/p-1})^{1-N/p} \phi(r)$$

where $\phi : \mathbf{R} \rightarrow \mathbf{R}$, nonnegative and smooth, is such that $\phi(r) = 1$ for $r \leq \delta/4$ and $\phi(r) = 0$ for $r \geq \delta$, $\delta > 0$ small. Recall here that

$$u_1(x) = (1 + r^{p/p-1})^{1-N/p}$$

realizes the best constant for the embedding of $W^{1,p}(\mathbf{R}^N)$ in $L^{p^*}(\mathbf{R}^N)$. Let also a and f be smooth functions defined in a neighborhood Ω of x_0 . We assume in what follows that $f > 0$ in $B_{x_0}(\delta)$, and that $B_{x_0}(\delta) \subset \Omega$. For $u \in W_0^{1,p}(\Omega)$, we set

$$I(u) = \frac{\int_{\Omega} |\nabla u|^p dx + \int_{\Omega} (g(x) - \lambda_1)|u|^p dx}{\left(\int_{\Omega} f(x)|u|^{p^*} dx\right)^{\frac{p}{p^*}}}.$$

We also introduce

$$k_g = 0 \text{ if } g(x_0) < \lambda_1$$

$$k_g = \inf\{j \in \mathbf{N}, / j \geq 1 \text{ and } \Delta^j g(x_0) < 0\} \text{ if not}$$

$$k_f = \inf\{j \in \mathbf{N}^*, / \Delta^j f(x_0) < 0\}$$

with the convention that $k_g = +\infty$ (resp. $k_f = +\infty$) if the corresponding set above is empty. Here $\Delta^j = \Delta^{j-1} \circ \Delta$, $j \geq 1$, where Δ is the usual Laplacian. When $N > p^2$, we define as in [10], [6]

$$k = \sup\{m \in \mathbf{N} / N > p^2 + 2m(p-1)\}$$

and for j integer, we set

$$\alpha_{N,j} = \frac{\Gamma(j + \frac{1}{2})\Gamma(\frac{1}{2})^{N-1}(2j + N)}{\Gamma(j + \frac{N}{2} + 1)}$$

and

$$\tilde{\alpha}_j^{p,N} = \frac{\alpha_{N,j}}{(2j)!} \int_0^\infty \frac{r^{N+2j-1} dr}{\left(1 + r^{\frac{p}{p-1}}\right)^{N-p}}$$

$$\tilde{\beta}_j^{p,N} = \frac{\alpha_{N,j}}{(2j)!} \frac{(N-p)^p}{(p-1)^{p-1}} \int_0^\infty \frac{r^{N+2j-1} dr}{\left(1 + r^{\frac{p}{p-1}}\right)^N}.$$

Note that $\tilde{\alpha}_j^{p,N}$ exists as soon as $N > p^2 + 2j(p-1)$, that $\tilde{\beta}_j^{p,N}$ exists as soon as $N > 2j(p-1)$. One can find the explicit values of $\tilde{\alpha}_j^{p,N}$, $\tilde{\beta}_j^{p,N}$ in [10], Lemma 7.

Proposition 4.1 *Suppose that $1 < p^2 < N$ and that f and g are $C^\infty(\bar{\Omega})$. For $\epsilon > 0$ sufficiently small,*

$$I(v_\epsilon) < \frac{1}{K(N,p)^p f(x_0)^{\frac{p}{p^*}}}$$

in each of the following cases

- 1. $k \geq k_g$, $k_f > k_g + \frac{p}{2}$, and $\Delta^{k_g}(g(x_0) - \lambda_1) < 0$.
- 2. $k \geq k_g$, $k_f < k_g + \frac{p}{2}$, and $\Delta^{k_f} f(x_0) > 0$.
- 3. $k \geq k_g$, $k_f = k_g + \frac{p}{2}$, and $\tilde{\alpha}_{k_g}^{p,n}(\Delta^{k_g}(g(x_0) - \lambda_1)f(x_0) - \tilde{\beta}_{k_f}^{p,n} \Delta^{k_f} f(x_0)) < 0$

- 4. $k \leq k_g$, $k_f \leq k + \frac{p}{2}$, and $\Delta^{k_f} f(x_0) > 0$.

For example, the following corollary presents particular situations which enclose the results in the case where $p = 2$ obtained in [6], see also [1] in the case $p = 2$ and $g = 0$:

Corollary 4.2 *Suppose that $1 < p^2 < n$. For $\epsilon > 0$ small, one has that*

$$I(v_\epsilon) < \frac{1}{K(N, p)^p f(x_0)^{1 - \frac{p}{N}}}$$

in each of the following situations

- 1. $1 < p < 2$ and $g(x_0) < \lambda_1$.
- 2. $p = 2$ and $\frac{8(N-1)}{(N-2)(N-4)}(g(x_0) - \lambda_1)f(x_0) - \Delta f(x_0) < 0$.
- 3. $p > 2$ and $g(x_0) = \lambda_1$, $\Delta g(x_0) = \Delta f(x_0) = 0$ and $\Delta^2 f(x_0) > 0$.

As a consequence of Proposition 4.1 One obtains that if f achieves its supremum on an interior point x_0 such that one of the situations described in 1. 2. 3. 4. occurs, then, there exists a solution to equation 1.1 for $\lambda = \lambda_1$ and for λ close to λ_1 .

We do not give the proofs of Proposition 4.1 and Corollary 4.2, because they are very technical and are already written in [10], in the coercive case. One must just replace in [10] the function a by the function $g - \lambda_1$.

5 Appendix

As mentioned in the introduction, in this appendix we want to prove the following

Proposition 2.1 *Suppose that $f(x, t) = a(x)|t|^{q-2}t + b(x)|t|^{p-2}t$ with $1 < p < q$, and a and b two continuous and bounded functions on Ω . Suppose that \bar{u} is a weak supersolution for $-\Delta_p u + f(x, u) \bar{u} = 0$ on $\partial\Omega$, and that \underline{u} is a weak subsolution with $\underline{u} = 0$ on $\partial\Omega$. Suppose that there exists some constant c and C such that*

$$-\infty < c \leq \underline{u} \leq \bar{u} \leq C < +\infty$$

Then, there exists a solution u between \underline{u} and \bar{u}

Proof We follow the method of E. Hebey in [?].

Let k be chosen in order that the function

$$H(x, t) = f(x, t) + k|t|^{p-2}t$$

be increasing on $[\inf_{x \in \bar{\Omega}} \underline{u}, \sup_{x \in \bar{\Omega}} \bar{u}]$. Let u_1 be the solution of the variational problem

$$\inf_{u \in W_0^{1,p}(\Omega)} \frac{1}{p} \int_{\Omega} |\nabla u|^p + \frac{k}{p} \int_{\Omega} |u|^p - \int_{\Omega} H(x, \bar{u})u.$$

The solution u_1 is unique and satisfies the following partial differential equation

$$-\Delta_p u_1 + k|u_1|^{p-2}u_1 = H(x, \bar{u})$$

and in particular

$$-\Delta_p u_1 + k|u_1|^{p-2}u_1 \leq -\Delta_p \bar{u} + k|\bar{u}|^{p-2}\bar{u}$$

and by the comparison principle one gets that $u_1 \leq \bar{u}$. On the other hand by the monotonicity properties of H

$$-\Delta_p u_1 + k|u_1|^{p-2}u_1 = H(x, \bar{u}) \geq H(x, \underline{u}) \geq -\Delta_p \underline{u} + k|\underline{u}|^{p-2}\underline{u}$$

and then

$$u_1 \geq \underline{u}.$$

Finally u_1 is a supersolution since

$$-\Delta_p u_1 + k|u_1|^{p-2}u_1 = H(x, \bar{u}) \geq H(x, u_1),$$

hence

$$\underline{u} \leq u_1 \leq \bar{u}.$$

Iterating this process, one obtains the existence of a decreasing sequence u_n of supersolutions and

$$\underline{u} \leq u_n \leq \bar{u},$$

with

$$-\Delta_p u_n + k|u_n|^{p-2}u_n = H(x, u_{n-1}).$$

The sequence is, then, simply convergent and furthermore u_n is bounded in $W^{1,p}$ since it is bounded in L^∞ and

$$\int_{\Omega} |\nabla u_n|^p + k \int_{\Omega} |u_n|^p - \int_{\Omega} H(x, u_{n-1}) u_n \leq \int_{\Omega} |\nabla \bar{u}|^p + k \int_{\Omega} |\bar{u}|^p - \int_{\Omega} H(x, \bar{u}) u.$$

Extracting from it a subsequence one gets that there exists u such that $u_n \rightharpoonup u$ in $W^{1,p}$ weakly. Let σ be a weak limit of $|\nabla u_n|^{p-2} \nabla u_n$ in $L^{p'}$. It satisfies

$$-\operatorname{div} \sigma + k|u|^{p-2}u = H(x, u).$$

Multiplying this by u and integrating by parts one gets

$$\int_{\Omega} \nabla u \cdot \sigma + k \int_{\Omega} |u|^p = \int_{\Omega} H(x, u) u.$$

and on another hand passing to the limit in the equation satisfied by u_n , multiplied by u_n , one has

$$\lim \int_{\Omega} |\nabla u_n|^p + k \int_{\Omega} |u|^p = \int_{\Omega} H(x, u) u.$$

We have obtained that

$$\int_{\Omega} \sigma \cdot \nabla u = \lim \int_{\Omega} |\nabla u_n|^p.$$

By using lower semicontinuity for the weak topology,

$$|\int_{\Omega} \sigma \cdot \nabla u| \leq \lim (\int_{\Omega} |\nabla u_n|^p)^{\frac{p}{p-1}} (\int_{\Omega} |\nabla u|^p)^{\frac{1}{p}}$$

and then

$$\lim \int_{\Omega} |\nabla u_n|^p \leq \lim (\int_{\Omega} |\nabla u_n|^p)^{\frac{p}{p-1}} (\int_{\Omega} |\nabla u|^p)^{\frac{1}{p}}$$

hence

$$\lim (\int_{\Omega} |\nabla u_n|^p)^{\frac{1}{p}} \leq (\int_{\Omega} |\nabla u|^p)^{\frac{1}{p}}.$$

Since the other inequality is always true, one obtains that the convergence is strong, $\sigma = |\nabla u|^{p-2} \nabla u$, and u is a solution.

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