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# Branching Law for the Finite Subgroups of $\mathbf{SL}_4\mathbb{C}$

Frédéric BUTIN<sup>1</sup>

## Abstract

In the framework of McKay correspondence we determine, for every finite subgroup  $\Gamma$  of  $\mathbf{SL}_4\mathbb{C}$ , how the finite dimensional irreducible representations of  $\mathbf{SL}_4\mathbb{C}$  decompose under the action of  $\Gamma$ .

Let  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{sl}_4\mathbb{C}$  and let  $\varpi_1, \varpi_2, \varpi_3$  be the corresponding fundamental weights. For  $(p, q, r) \in \mathbb{N}^3$ , the restriction  $\pi_{p,q,r}|_\Gamma$  of the irreducible representation  $\pi_{p,q,r}$  of highest weight  $p\varpi_1 + q\varpi_2 + r\varpi_3$  of  $\mathbf{SL}_4\mathbb{C}$  decomposes as  $\pi_{p,q,r}|_\Gamma = \bigoplus_{i=0}^l m_i(p, q, r)\gamma_i$ . We determine the multiplicities  $m_i(p, q, r)$  and prove that the series  $P_\Gamma(t, u, w)_i = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} m_i(p, q, r)t^p u^q w^r$  are rational functions.

This generalizes results from Kostant for  $\mathbf{SL}_2\mathbb{C}$  and our preceding works about  $\mathbf{SL}_3\mathbb{C}$ .

**Keywords:** McKay correspondence; branching law; representations; finite subgroups of  $\mathbf{SL}_4\mathbb{C}$ .

**Mathematics Subject Classifications (2000):** 20C15; 17B10; 15A09; 17B67.

## 1 Introduction and results

• Let  $\Gamma$  be a finite subgroup of  $\mathbf{SL}_4\mathbb{C}$  and  $\{\gamma_0, \dots, \gamma_l\}$  the set of equivalence classes of irreducible finite dimensional complex representations of  $\Gamma$ , where  $\gamma_0$  is the trivial representation. The character associated to  $\gamma_j$  is denoted by  $\chi_j$ .

Consider  $\gamma : \Gamma \rightarrow \mathbf{SL}_4\mathbb{C}$  the natural 4-dimensional representation, and  $\gamma^*$  its contragredient representation. The character of  $\gamma$  is denoted by  $\chi$ . By complete reducibility we get the decompositions

$$\forall j \in \llbracket 0, l \rrbracket, \quad \gamma_j \otimes \gamma = \bigoplus_{i=0}^l a_{ij}^{(1)} \gamma_i, \quad \gamma_j \otimes (\gamma \wedge \gamma) = \bigoplus_{i=0}^l a_{ij}^{(2)} \gamma_i \quad \text{and} \quad \gamma_j \otimes \gamma^* = \bigoplus_{i=0}^l a_{ij}^{(3)} \gamma_i.$$

This defines the three following square matrices of  $\mathbf{M}_{l+1}\mathbb{N}$ :

$$A^{(1)} := \left( a_{ij}^{(1)} \right)_{(i,j) \in \llbracket 0, l \rrbracket^2}, \quad A^{(2)} := \left( a_{ij}^{(2)} \right)_{(i,j) \in \llbracket 0, l \rrbracket^2} \quad \text{and} \quad A^{(3)} := \left( a_{ij}^{(3)} \right)_{(i,j) \in \llbracket 0, l \rrbracket^2}.$$

• Let  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{sl}_4\mathbb{C}$  and let  $\varpi_1, \varpi_2, \varpi_3$  be the corresponding fundamental weights, and  $V(p\varpi_1 + q\varpi_2 + r\varpi_3)$  the simple  $\mathfrak{sl}_4\mathbb{C}$ -module of highest weight  $p\varpi_1 + q\varpi_2 + r\varpi_3$  with  $(p, q, r) \in \mathbb{N}^3$ . Then we get an irreducible representation  $\pi_{p,q,r} : \mathbf{SL}_4\mathbb{C} \rightarrow \mathbf{GL}(V(p\varpi_1 + q\varpi_2 + r\varpi_3))$ . The restriction of  $\pi_{p,q,r}$  to the subgroup  $\Gamma$  is a representation of  $\Gamma$ , and by complete reducibility, we get the decomposition

$$\pi_{p,q,r}|_\Gamma = \bigoplus_{i=0}^l m_i(p, q, r)\gamma_i,$$

where the  $m_i(p, q, r)$ 's are non negative integers. Let  $\mathcal{E} := (e_0, \dots, e_l)$  be the canonical basis of  $\mathbb{C}^{l+1}$ , and

$$v_{p,q,r} := \sum_{i=0}^l m_i(p, q, r)e_i \in \mathbb{C}^{l+1}.$$

We have in particular  $v_{0,0,0} = e_0$  as  $\gamma_0$  is the trivial representation. Let us consider the vector

$$P_\Gamma(t, u, w) := \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} v_{p,q,r} t^p u^q w^r \in (\mathbb{C}[[t, u, w]])^{l+1},$$

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and denote by  $P_\Gamma(t, u, w)_j$  its  $j$ -th coordinate in the basis  $\mathcal{E}$ , which is an element of  $\mathbb{C}[[t, u, w]]$ . Note that  $P_\Gamma(t, u, w)$  can also be seen as a formal power series with coefficients in  $\mathbb{C}^{l+1}$ . The aim of this article is to prove the following theorem.

**Theorem 1**

*The coefficients of  $P_\Gamma(t, u, w)$  are rational fractions in  $t, u, w$ , i.e. the formal power series  $P_\Gamma(t, u, w)_i$  are rational functions*

$$P_\Gamma(t, u, w)_i = \frac{N_\Gamma(t, u, w)_i}{D_\Gamma(t, u, w)}, \quad i \in \llbracket 0, l \rrbracket,$$

where the  $N_\Gamma(t, u, w)_i$ 's and  $D_\Gamma(t, u, w)$  are elements of  $\mathbb{Q}[[t, u, w]]$ .

- The proof of this theorem uses a key-relation satisfied by  $P_\Gamma(t, u, w)$  as well as a so-called inversion formula. Two essential ingredients are the decomposition of the tensor product of  $\pi_{p,q,r}$  with the natural representation of  $\mathbf{SL}_4\mathbb{C}$  and the simultaneous diagonalizability of certain matrices. The effective calculation of  $P_\Gamma(t, u, w)$  then reduces to matrix multiplication.

In [BP09] we applied a similar method for  $\mathbf{SL}_2\mathbb{C}$  — recovering thereby in a quite easy way the results obtained by Kostant in [Kos85], [Kos06], and by Gonzalez-Sprinberg and Verdier in [GSV83] — and for  $\mathbf{SL}_3\mathbb{C}$  in order to get explicit computations of the series for every finite subgroup of  $\mathbf{SL}_3\mathbb{C}$ .

The general framework of that study is the construction of a minimal resolution of singularities of the orbifold  $\mathbb{C}^n/\Gamma$ . It is related to the McKay correspondence (see [BKR01], [GSV83] and [GNS04]). For example, Gonzalez-Sprinberg and Verdier use in [GSV83] a Poincaré series to construct explicitly minimal resolutions for singularities of  $V = \mathbb{C}^2/\Gamma$  when  $\Gamma$  is a finite subgroup of  $\mathbf{SL}_2\mathbb{C}$ . To go further in this approach, our results for  $\mathbf{SL}_4\mathbb{C}$  could be used to construct an explicit synthetic minimal resolution of singularities for orbifolds of the form  $\mathbb{C}^4/\Gamma$  where  $\Gamma$  is a finite subgroup of  $\mathbf{SL}_4\mathbb{C}$ .

## 2 Properties of the matrices $A^{(1)}, A^{(2)}, A^{(3)}$

In order to compute the series  $P_\Gamma(t, u, w)$ , we first establish here some properties of the matrices  $A^{(1)}, A^{(2)}, A^{(3)}$ . The first proposition essentially follows from the uniqueness of the decomposition of a representation as sum of irreducible representations.

**Proposition 2**

- $A^{(3)} = {}^t A^{(1)}$ .
- $A^{(2)}$  is a symmetric matrix.
- $A^{(1)}, A^{(2)}$  and  $A^{(3)}$  commute. In particular,  $A^{(1)}$  is a normal matrix.

Proof:

Since  $a_{ij}^{(1)} = (\chi_i | \chi_{\gamma_j}) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \overline{\chi_i(g)} \chi_j(g)$ , we have  $\gamma \otimes \gamma_j = \bigoplus_{i=0}^l a_{ij}^{(1)} \gamma_i$ . In the same way,  $(\gamma \wedge \gamma) \otimes \gamma_j = \bigoplus_{i=0}^l a_{ij}^{(2)} \gamma_i$  and  $\gamma^* \otimes \gamma_j = \bigoplus_{i=0}^l a_{ij}^{(3)} \gamma_i$ .

Then

$$\begin{aligned} a_{ij}^{(3)} &= (\chi_i | \chi_{\gamma_j \otimes \gamma^*}) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \overline{\chi_i(g)} \chi_j(g) \chi_{\gamma^*}(g) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \overline{\chi_i(g)} \chi_j(g) \chi(g^{-1}) \\ &= \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \overline{\chi_i(g^{-1})} \chi_j(g^{-1}) \chi(g) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_i(g) \chi_j(g) \chi(g) = a_{ji}^{(1)}, \end{aligned}$$

hence  $A^{(3)} = {}^t A^{(1)}$ .

We also have  $(\gamma_j \otimes \gamma) \otimes \gamma^* = \left( \bigoplus_{i=0}^l a_{ij}^{(1)} \gamma_i \right) \otimes \gamma^* = \bigoplus_{i=0}^l a_{ij}^{(1)} \left( \bigoplus_{k=0}^l a_{ki}^{(3)} \gamma_k \right) = \bigoplus_{k=0}^l \left( \sum_{i=0}^l a_{ki}^{(3)} a_{ij}^{(1)} \right) \gamma_k$  and  $\gamma \otimes (\gamma_j \otimes \gamma^*) = \gamma \otimes \left( \bigoplus_{i=0}^l a_{ij}^{(3)} \gamma_i \right) = \bigoplus_{i=0}^l a_{ij}^{(3)} \left( \bigoplus_{k=0}^l a_{ki}^{(1)} \gamma_k \right) = \bigoplus_{k=0}^l \left( \sum_{i=0}^l a_{ki}^{(1)} a_{ij}^{(3)} \right) \gamma_k$ , hence  $A^{(3)} A^{(1)} = A^{(1)} A^{(3)}$ . The proofs of the other statements are the same. ■

Since  $A^{(1)}, A^{(2)}, A^{(3)}$  are normal, we know that they are diagonalizable with eigenvectors forming an orthogonal basis. Now we will diagonalize these matrices by using the character table of the group  $\Gamma$ . Let

us denote by  $\{C_0, \dots, C_l\}$  the set of conjugacy classes of  $\Gamma$ , and for any  $j \in \llbracket 0, l \rrbracket$ , let  $g_j$  be an element of  $C_j$ . So the character table of  $\Gamma$  is the matrix  $T_\Gamma \in \mathbf{M}_{l+1} \mathbb{C}$  defined by  $(T_\Gamma)_{i,j} := \chi_i(g_j)$ .

**Proposition 3**

- For  $k \in \llbracket 0, l \rrbracket$ , set  $w_k := (\chi_0(g_k), \dots, \chi_l(g_k)) \in \mathbb{C}^{l+1}$ . Then  $w_k$  is an eigenvector of  $A^{(3)}$  associated to the eigenvalue  $\chi(g_k)$ . Similarly,  $w_k$  is an eigenvector of  $A^{(1)}$  associated to the eigenvalue  $\overline{\chi(g_k)}$ .
- For  $k \in \llbracket 0, l \rrbracket$ ,  $w_k$  is an eigenvector of  $A^{(2)}$  associated to the eigenvalue  $\frac{1}{2}(\chi(g_k)^2 + \chi(g_k^2))$ .

Proof:

From the relation  $\gamma_i \otimes \gamma = \sum_{j=0}^l a_{ji}^{(1)} \gamma_j$ , we get  $\chi_i \chi = \chi_{\gamma_i \otimes \gamma} = \sum_{j=0}^l a_{ji}^{(1)} \chi_j$ . By evaluating this on  $g_k$ , we obtain  $\chi_i(g_k) \chi(g_k) = \sum_{j=0}^l a_{ji}^{(1)} \chi_j(g_k) = \sum_{j=0}^l a_{ij}^{(3)} \chi_j(g_k)$  according to Proposition 2. So  $w_k$  is an eigenvector of  $A^{(3)}$  associated to the eigenvalue  $\chi(g_k)$ . The method is similar for the other results. ■

As the  $w_j$ 's are the column of  $T_\Gamma$ , which are always orthogonal, the matrix  $T_\Gamma$  is invertible and the family  $\mathcal{W} := (w_0, \dots, w_l)$  is a common basis of eigenvectors of  $A^{(1)}$ ,  $A^{(2)}$  and  $A^{(3)}$ . Then  $\Lambda^{(1)} := T_\Gamma^{-1} A^{(1)} T_\Gamma$ ,  $\Lambda^{(2)} := T_\Gamma^{-1} A^{(2)} T_\Gamma$  and  $\Lambda^{(3)} := T_\Gamma^{-1} A^{(3)} T_\Gamma$  are diagonal matrices, with  $\Lambda_{jj}^{(1)} = \overline{\chi(g_j)}$ ,  $\Lambda_{jj}^{(2)} = \frac{1}{2}(\chi(g_j)^2 + \chi(g_j^2))$  and  $\Lambda_{jj}^{(3)} = \chi(g_j)$ .

Now, we make use of the Clebsch-Gordan formula

$$\begin{aligned} \pi_{1,0,0} \otimes \pi_{p,q,r} &= \pi_{p+1,q,r} \oplus \pi_{p,q,r-1} \oplus \pi_{p-1,q+1,r} \oplus \pi_{p,q-1,r+1}, \\ \pi_{0,1,0} \otimes \pi_{p,q,r} &= \pi_{p,q+1,r} \oplus \pi_{p,q-1,r} \oplus \pi_{p+1,q-1,r+1} \oplus \pi_{p-1,q+1,r-1} \oplus \pi_{p-1,q,r+1} \oplus \pi_{p+1,q,r-1}, \\ \pi_{0,0,1} \otimes \pi_{p,q,r} &= \pi_{p,q,r+1} \oplus \pi_{p-1,q,r} \oplus \pi_{p,q+1,r-1} \oplus \pi_{p+1,q-1,r}. \end{aligned} \quad (1)$$

**Proposition 4**

The vectors  $v_{m,n}$  satisfy the following recurrence relations

$$\begin{aligned} A^{(1)} v_{p,q,r} &= v_{p+1,q,r} + v_{p,q,r-1} + v_{p-1,q+1,r} + v_{p,q-1,r+1}, \\ A^{(2)} v_{p,q,r} &= v_{p,q+1,r} + v_{p,q-1,r} + v_{p+1,q-1,r+1} + v_{p-1,q+1,r-1} + v_{p-1,q,r+1} + v_{p+1,q,r-1}, \\ A^{(3)} v_{p,q,r} &= v_{p,q,r+1} + v_{p-1,q,r} + v_{p,q+1,r-1} + v_{p+1,q-1,r}. \end{aligned}$$

Proof:

The definition of  $v_{p,q,r}$  reads  $v_{p,q,r} = \sum_{i=0}^l m_i(p, q, r) e_i$ , thus  $A^{(1)} v_{p,q,r} = \sum_{i=0}^l \left( \sum_{j=0}^l m_j(p, q, r) a_{ij}^{(1)} \right) e_i$ . Now

$$(\pi_{1,0,0} \otimes \pi_{p,q,r})|_\Gamma = \pi_{p,q,r}|_\Gamma \otimes \gamma = \sum_{j=0}^l m_j(p, q, r) \gamma_j \otimes \gamma = \sum_{i=0}^l \left( \sum_{j=0}^l m_j(p, q, r) a_{ij}^{(1)} \right) \gamma_i,$$

and

$$\begin{aligned} &\pi_{p+1,q,r}|_\Gamma + \pi_{p,q,r-1}|_\Gamma + \pi_{p-1,q+1,r}|_\Gamma + \pi_{p,q-1,r+1}|_\Gamma \\ &= \sum_{i=0}^l (m_i(p+1, q, r) + m_i(p, q, r-1) + m_i(p-1, q+1, r) + m_i(p, q-1, r+1)) \gamma_i. \end{aligned}$$

By uniqueness,

$$\sum_{j=0}^l m_j(p, q, r) a_{ij}^{(1)} = m_i(p+1, q, r) + m_i(p, q, r-1) + m_i(p-1, q+1, r) + m_i(p, q-1, r+1). \quad \blacksquare$$

### 3 The series $P_\Gamma(t, u, w)$ is a rational function

This section is mainly devoted to the proof of Theorem 1.

### 3.1 A key-relation satisfied by the series $P_\Gamma(t, u, w)$

#### Proposition 5

Set

$$J(t, u, w) := (1 - u^2)((1 + ut^2)(1 + uw^2) - tw(1 + u^2))I_n + twu(1 - u^2)A^{(2)} - tu(1 + uw^2)(A^{(3)} - uA^{(1)}) - wu(1 + ut^2)(A^{(1)} - uA^{(3)}).$$

Then the series  $P_\Gamma(t, u, w)$  satisfies the following relation

$$J(t, u, w)v_{0,0,0} = \left( (1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4) (1 - wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4) \right. \\ \left. \left( (1 + u^2)(1 - u^2)^2 - u(1 - u^2)^2A^{(2)} + u^2(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)}) \right) \right) P_\Gamma(t, u, w).$$

Proof:

• Set  $x := P_\Gamma(t, u, w)$ . Set also  $v_{p,q,-1} := 0$ ,  $v_{p,-1,r} := 0$  and  $v_{-1,q,r} := 0$  for  $(p, q, r) \in \mathbb{N}^3$ , such that, according to the Clebsch-Gordan formula, the formulae of the preceding corollary are still true for  $(p, q, r) \in \mathbb{N}^3$ . So we have (by denoting  $\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty}$  by  $\sum_{pqr}$ )

$$(1 - wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4)x \\ = \sum_{pqr} v_{p,q,r} t^p u^q w^r - \sum_{pqr} (v_{p,q,r+1} + v_{p-1,q,r} + v_{p,q+1,r-1} + v_{p+1,q-1,r}) t^p u^q w^{r+1} \\ + \sum_{pqr} (v_{p,q+1,r} + v_{p,q-1,r} + v_{p+1,q-1,r+1} + v_{p-1,q+1,r-1} + v_{p-1,q,r+1} + v_{p+1,q,r-1}) t^p u^q w^{r+2} \\ - \sum_{pqr} (v_{p+1,q,r} + v_{p,q,r-1} + v_{p-1,q+1,r} + v_{p,q-1,r+1}) t^p u^q w^{r+3} + \sum_{pqr} v_{p,q,r} t^p u^q w^{r+4},$$

hence

$$(1 - wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4)x = (1 - tw + uw^2 - t^{-1}uw) \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} v_{p,q,0} t^p u^q + t^{-1}uw \sum_{q=0}^{\infty} v_{0,q,0} u^q. \quad (2)$$

• In the same way (by denoting  $\sum_{p=0}^{\infty} \sum_{q=0}^{\infty}$  by  $\sum_{pq}$ )

$$(1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4) \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} v_{p,q,0} t^p u^q \\ = \sum_{pq} v_{p,q,0} t^p u^q - \sum_{pq} (v_{p+1,q,0} + v_{p-1,q+1,0} + v_{p,q-1,1}) t^{p+1} u^q \\ + \sum_{pq} (v_{p,q+1,0} + v_{p,q-1,0} + v_{p+1,q-1,1} + v_{p-1,q,1}) t^{p+2} u^q \\ - \sum_{pq} (v_{p,q,1} + v_{p-1,q,0} + v_{p+1,q-1,0}) t^{p+3} u^q + \sum_{pq} v_{p,q,0} t^{p+4} u^q,$$

hence

$$(1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4) \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} v_{p,q,0} t^p u^q = (1 + t^2u) \sum_{q=0}^{\infty} v_{0,q,0} u^q - tu \sum_{q=0}^{\infty} v_{0,q,1} u^q. \quad (3)$$

Moreover, we have

$$(1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4) \sum_{q=0}^{\infty} v_{0,q,0} u^q \\ = \sum_{q=0}^{\infty} v_{0,q,0} u^q - \sum_q (v_{1,q,0} + v_{0,q-1,1}) tu^q + \sum_q (v_{0,q+1,0} + v_{0,q-1,0} + v_{1,q-1,1}) t^2 u^q \\ - \sum_q (v_{0,q,1} + v_{1,q-1,0}) t^3 u^q + \sum_q v_{0,q,0} t^4 u^q,$$

hence

$$\begin{aligned}
& (1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4) \sum_{p=0}^{\infty} v_{0,q,0}u^q \\
= & (1 + t^4 + t^2u^{-1} + t^2u) \sum_{q=0}^{\infty} v_{0,q,0}u^q - t^2u^{-1}v_{0,0,0} - (t + t^3u) \sum_{q=0}^{\infty} v_{1,q,0}u^q \\
& - (tu + t^3) \sum_{q=0}^{\infty} v_{0,q,1}u^q + t^2u \sum_{q=0}^{\infty} v_{1,q,1}u^q.
\end{aligned} \tag{4}$$

By combining Equations (2), (3) and (4), we get

$$\begin{aligned}
& (1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4)(1 - wA^{(3)} + w^2A^{(2)} - w^3A^{(3)} + w^4)x \\
= & (1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4) \left( (1 - tw + uw^2 - t^{-1}uw) \sum_{pq} v_{p,q,0}t^p u^q + t^{-1}uw \sum_{q=0}^{\infty} v_{0,q,0}u^q \right) \\
= & (1 - tw + uw^2 - t^{-1}uw) \left( (1 + t^2u) \sum_{q=0}^{\infty} v_{0,q,0}u^q - tu \sum_{q=0}^{\infty} v_{0,q,1}u^q \right) \\
& + (1 + t^4 + t^2u^{-1} + t^2u)t^{-1}uw \sum_{q=0}^{\infty} v_{0,q,0}u^q - twv_{0,0,0} - (1 + t^2u)uw \sum_{q=0}^{\infty} v_{1,q,0}u^q \\
& - (u + t^2)uw \sum_{q=0}^{\infty} v_{0,q,1}u^q + tu^2w \sum_{q=0}^{\infty} v_{1,q,1}u^q,
\end{aligned}$$

hence

$$\begin{aligned}
& (1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4)(1 - wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4)x \\
= & (1 + ut^2)(1 + uw^2) \sum_{q=0}^{\infty} v_{0,q,0}u^q - tu(1 + uw^2) \sum_{q=0}^{\infty} v_{0,q,1}u^q \\
& - wu(1 + ut^2) \sum_{q=0}^{\infty} v_{1,q,0}u^q - twv_{0,0,0} + tu^2w \sum_{q=0}^{\infty} v_{1,q,1}u^q.
\end{aligned} \tag{5}$$

Besides, we have the two following equations

$$A^{(1)} \sum_{q=0}^{\infty} v_{0,q,0}u^q = \sum_{q=0}^{\infty} v_{1,q,0}u^q + u \sum_{q=0}^{\infty} v_{0,q,1}u^q, \tag{6}$$

and

$$A^{(3)} \sum_{q=0}^{\infty} v_{0,q,0}u^q = \sum_{q=0}^{\infty} v_{0,q,1}u^q + u \sum_{q=0}^{\infty} v_{1,q,0}u^q. \tag{7}$$

From these two equations, we deduce

$$\sum_{q=0}^{\infty} v_{0,q,1}u^q = (1 - u^2)^{-1}(A^{(3)} - uA^{(1)}) \sum_{q=0}^{\infty} v_{0,q,0}u^q. \tag{8}$$

Now, we have

$$A^{(1)} \sum_{q=0}^{\infty} v_{0,q,1}u^q = \sum_{q=0}^{\infty} v_{1,q,1}u^q + \sum_{q=0}^{\infty} v_{0,q,0}u^q + u \sum_{q=0}^{\infty} v_{0,q,2}u^q, \tag{9}$$

and

$$A^{(3)} \sum_{q=0}^{\infty} v_{0,q,1}u^q = \sum_{q=0}^{\infty} v_{0,q,2}u^q + u^{-1} \sum_{q=0}^{\infty} v_{0,q,0}u^q + u \sum_{q=0}^{\infty} v_{1,q,1}u^q - u^{-1}v_{0,0,0}, \tag{10}$$

therefore

$$\sum_{q=0}^{\infty} v_{1,q,1}u^q = (1-u^2)^{-1}(A^{(1)} - uA^{(3)}) \sum_{q=0}^{\infty} v_{0,q,1}u^q - (1-u^2)^{-1}v_{0,0,0}.$$

So, according to Equation (8), we deduce

$$\sum_{q=0}^{\infty} v_{1,q,1}u^q = (1-u^2)^{-2}(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)}) \sum_{q=0}^{\infty} v_{0,q,0}u^q - (1-u^2)^{-1}v_{0,0,0}. \quad (11)$$

By using Equation (11), we may write Equation (5) as

$$\begin{aligned} & (1-tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4)(1-wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4)x \\ &= \left( (1+ut^2)(1+uw^2) + tu^2w(1-u^2)^{-2}(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)}) \right) \sum_{q=0}^{\infty} v_{0,q,0}u^q \\ & \quad - tu(1+uw^2) \sum_{q=0}^{\infty} v_{0,q,1}u^q - wu(1+ut^2) \sum_{q=0}^{\infty} v_{1,q,0}u^q - (tw + tu^2w(1-u^2)^{-1})v_{0,0,0}. \end{aligned} \quad (12)$$

From Equations (6) and (7), we also deduce

$$\sum_{q=0}^{\infty} v_{1,q,0}u^q = (1-u^2)^{-1}(A^{(1)} - uA^{(3)}) \sum_{q=0}^{\infty} v_{0,q,0}u^q. \quad (13)$$

So, by using Equations (8) and (13), we obtain

$$\begin{aligned} & (1-tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4)(1-wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4)x \\ &= \left( (1+ut^2)(1+uw^2) - tu(1+uw^2)(1-u^2)^{-1}(A^{(3)} - uA^{(1)}) \right. \\ & \quad \left. - wu(1+ut^2)(1-u^2)^{-1}(A^{(1)} - uA^{(3)}) + tu^2w(1-u^2)^{-2}(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)}) \right) \sum_{q=0}^{\infty} v_{0,q,0}u^q \\ & \quad - (tw + tu^2w(1-u^2)^{-1})v_{0,0,0}, \end{aligned} \quad (14)$$

i.e., by multiplying (14) by  $(1-u^2)^2$ ,

$$\begin{aligned} & (1-u^2)^2(1-tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4)(1-wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4)x \\ &= \left( (1-u^2)^2(1+ut^2)(1+uw^2) - tu(1+uw^2)(1-u^2)(A^{(3)} - uA^{(1)}) \right. \\ & \quad \left. - wu(1+ut^2)(1-u^2)(A^{(1)} - uA^{(3)}) + tu^2w(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)}) \right) \sum_{q=0}^{\infty} v_{0,q,0}u^q \\ & \quad - (tw(1-u^2)^2 + tu^2w(1-u^2))v_{0,0,0}. \end{aligned} \quad (15)$$

• Consider now the following equation

$$A^{(2)} \sum_{q=0}^{\infty} v_{0,q,0}u^q = u^{-1} \sum_{q=0}^{\infty} v_{0,q,0}u^q + u \sum_{q=0}^{\infty} v_{0,q,0}u^q + u \sum_{q=0}^{\infty} v_{1,q,1}u^q - u^{-1}v_{0,0,0}. \quad (16)$$

Then, according to Equation (11), we have

$$\begin{aligned} A^{(2)} \sum_{q=0}^{\infty} v_{0,q,0}u^q &= u^{-1} \sum_{q=0}^{\infty} v_{0,q,0}u^q + u \sum_{q=0}^{\infty} v_{0,q,0}u^q \\ & \quad + u(1-u^2)^{-2}(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)}) \sum_{q=0}^{\infty} v_{0,q,0}u^q - u(1-u^2)^{-1}v_{0,0,0} - u^{-1}v_{0,0,0}, \end{aligned}$$

i.e.

$$\left( A^{(2)} - u^{-1} - u - u(1 - u^2)^{-2}(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)}) \right) \sum_{q=0}^{\infty} v_{0,q,0} u^q = -(u(1 - u^2)^{-1} + u^{-1})v_{0,0,0}. \quad (17)$$

This last equation reads

$$\left( -u(1 - u^2)^2 A^{(2)} + (1 + u^2)(1 - u^2)^2 + u^2(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)}) \right) \sum_{q=0}^{\infty} v_{0,q,0} u^q = (1 - u^2)v_{0,0,0}. \quad (18)$$

Now, by using Equations (15) and (18), we get

$$\begin{aligned} & \left( (1 - u^2)^2(1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4)(1 - wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4) \right. \\ & \left. \left( -u(1 - u^2)^2 A^{(2)} + (1 + u^2)(1 - u^2)^2 + u^2(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)}) \right) x \right. \\ = & -tw(1 - u^2) \left( -u(1 - u^2)^2 A^{(2)} + (1 + u^2)(1 - u^2)^2 + u^2(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)}) \right) v_{0,0,0} \quad (19) \\ & \left( (1 - u^2)^2(1 + ut^2)(1 + uw^2) - tu(1 + uw^2)(1 - u^2)(A^{(3)} - uA^{(1)}) \right. \\ & \left. - wu(1 + ut^2)(1 - u^2)(A^{(1)} - uA^{(3)}) + tu^2w(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)}) \right) (1 - u^2)v_{0,0,0}, \end{aligned}$$

i.e., after simplification by  $(1 - u^2)$ ,

$$\begin{aligned} & \left( (1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4)(1 - wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4) \right. \\ & \left. \left( (1 + u^2)(1 - u^2)^2 - u(1 - u^2)^2 A^{(2)} + u^2(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)}) \right) x \right. \\ = & \left( (1 - u^2)((1 + ut^2)(1 + uw^2) - tw(1 + u^2)) + twu(1 - u^2)A^{(2)} \right. \\ & \left. - tu(1 + uw^2)(A^{(3)} - uA^{(1)}) - wu(1 + ut^2)(A^{(1)} - uA^{(3)}) \right) v_{0,0,0}. \quad (20) \end{aligned}$$

The proposition is proved. ■

### 3.2 An inversion formula

In order to inverse the relation obtained in Proposition 5 and get an explicit expression for  $P_{\Gamma}(t, u)$ , we need the rational function  $f$  defined by

$$\begin{aligned} f : \mathbb{C}^3 & \rightarrow \mathbb{C}(t, u, w) \\ (d_1, d_2, d_3) & \mapsto (1 - td_1 + t^2d_2 - t^3d_3 + t^4)^{-1} (1 - wd_3 + w^2d_2 - w^3d_1 + w^4)^{-1} \\ & \left( (1 + u^2)(1 - u^2)^2 - u(1 - u^2)^2 d_2 + u^2(d_1 - ud_3)(d_3 - ud_1) \right)^{-1}. \end{aligned}$$

According to Proposition 5, we may write

$$\begin{aligned} J(t, u, w) v_{0,0,0} & = T_{\Gamma} \left( 1 - t\Lambda^{(1)} + t^2\Lambda^{(2)} - t^3\Lambda^{(3)} + t^4 \right) \left( 1 - w\Lambda^{(3)} + w^2\Lambda^{(2)} - w^3\Lambda^{(1)} + w^4 \right) \\ & \left( (1 + u^2)(1 - u^2)^2 - u(1 - ur)^2\Lambda^{(2)} + u^2(\Lambda^{(1)} - u\Lambda^{(3)})(\Lambda^{(3)} - u\Lambda^{(1)}) \right) T_{\Gamma}^{-1} P_{\Gamma}(t, u, w). \end{aligned}$$

We deduce that

$$P_{\Gamma}(t, u, w) = T_{\Gamma} \Delta(t, u, w) T_{\Gamma}^{-1} J(t, u, w) v_{0,0,0} = (T_{\Gamma} \Delta(t, u, w) T_{\Gamma}) (T_{\Gamma}^{-2} J(t, u, w) v_{0,0,0}), \quad (21)$$

where  $\Delta(t, u, w) \in \mathbf{M}_{l+1}\mathbb{C}(t, u, w)$  is the diagonal matrix defined by

$$\Delta(t, u, w)_{jj} = f(\Lambda_{jj}^{(1)}, \Lambda_{jj}^{(2)}, \Lambda_{jj}^{(3)}) = f\left(\overline{\chi(g_j)}, \frac{1}{2}(\chi(g_j)^2 - \chi(g_j^2)), \chi(g_j)\right).$$

This last formula proves Theorem 1.

**Remark 6**

The Poincaré series  $\widehat{P}_\Gamma(t)$  of the algebra of invariants  $\mathbb{C}[z_1, z_2, z_3, z_4]^\Gamma$  is given by

$$\widehat{P}_\Gamma(t) = P_\Gamma(t, 0, 0)_0 = P_\Gamma(0, 0, t)_0.$$

**3.3 Remark for  $\mathbf{SL}_n\mathbb{C}$**

In this section, we consider an integer  $n \geq 2$  and a subgroup  $\Gamma$  of  $\mathbf{SL}_n\mathbb{C}$ . As in paragraph 1, let  $\{\gamma_0, \dots, \gamma_l\}$  be the set of equivalence classes of irreducible finite dimensional complex representations of  $\Gamma$ , where  $\gamma_0$  is the trivial representation. The character associated to  $\gamma_j$  is denoted by  $\chi_j$ .

Consider  $\gamma : \Gamma \rightarrow \mathbf{SL}_n\mathbb{C}$  the natural  $n$ -dimensional representation, and  $\chi$  its character. By complete reducibility we get the decomposition  $\gamma_j \otimes \gamma = \bigoplus_{i=0}^l a_{ij}^{(1)} \gamma_i$  for every  $j \in \llbracket 0, l \rrbracket$ , and we set  $A^{(1)} := \left( a_{ij}^{(1)} \right)_{(i,j) \in \llbracket 0, l \rrbracket^2} \in \mathbf{M}_{l+1}\mathbb{N}$ .

Let  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{sl}_n\mathbb{C}$  and let  $\varpi_1, \dots, \varpi_{n-1}$  be the corresponding fundamental weights, and  $V(p_1\varpi_1 + \dots + p_{n-1}\varpi_{n-1})$  the simple  $\mathfrak{sl}_n\mathbb{C}$ -module of highest weight  $p_1\varpi_1 + \dots + p_{n-1}\varpi_{n-1}$  with  $\mathbf{p} := (p_1, \dots, p_{n-1}) \in \mathbb{N}^{n-1}$ . Then we get an irreducible representation  $\pi_{\mathbf{p}} : \mathbf{SL}_n\mathbb{C} \rightarrow \mathbf{GL}(V(p_1\varpi_1 + \dots + p_{n-1}\varpi_{n-1}))$ . The restriction of  $\pi_{\mathbf{p}}$  to the subgroup  $\Gamma$  is a representation of  $\Gamma$ , and by complete reducibility, we get the decomposition  $\pi_{\mathbf{p}}|_\Gamma = \bigoplus_{i=0}^l m_i(\mathbf{p}) \gamma_i$ , where the  $m_i(\mathbf{p})$ 's are non negative integers. Let  $\mathcal{E} := (e_0, \dots, e_l)$  be the canonical basis of  $\mathbb{C}^{l+1}$ , and

$$v_{\mathbf{p}} := \sum_{i=0}^l m_i(\mathbf{p}) e_i \in \mathbb{C}^{l+1}.$$

As  $\gamma_0$  is the trivial representation, we have  $v_0 = e_0$ . Let us consider the vector (with elements of  $\mathbb{C}[[t_1, \dots, t_{n-1}]] = \mathbb{C}[[\mathbf{t}]]$  as coefficients)

$$P_\Gamma(\mathbf{t}) := \sum_{\mathbf{p} \in \mathbb{N}^{n-1}} v_{\mathbf{p}} \mathbf{t}^{\mathbf{p}} \in (\mathbb{C}[[\mathbf{t}]])^{l+1},$$

and denote by  $P_\Gamma(\mathbf{t})_j$  its  $j$ -th coordinate in the basis  $\mathcal{E}$ .

Given the results from Kostant ([Kos85] and [Kos06]) for  $\mathbf{SL}_2\mathbb{C}$  and our results ([BP09]) about  $\mathbf{SL}_3\mathbb{C}$ , we then formulate the following conjecture:

**Conjecture 7**

The coefficients of the vector  $P_\Gamma(\mathbf{t})$  are rational fractions in  $\mathbf{t}$ , i.e. the formal power series  $P_\Gamma(\mathbf{t})_i$  are rational functions

$$P_\Gamma(\mathbf{t})_i := \frac{N_\Gamma(\mathbf{t})_i}{D_\Gamma(\mathbf{t})}, \quad i \in \llbracket 0, l \rrbracket,$$

where the  $N_\Gamma(\mathbf{t})_i$ 's and  $D_\Gamma(\mathbf{t})$  are elements of  $\mathbb{Q}[[\mathbf{t}]]$ .

**4 An example of explicit computation**

The classification of finite subgroups of  $\mathbf{SL}_4\mathbb{C}$  is given in [HH01]. It consists in infinite series and 30 exceptional groups (types  $I, II, \dots, XXX$ ). We give here an explicit computation of  $P_\Gamma(t, u, w)$  for one of these exceptional groups. Consider the matrices

$$F_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & j^2 \end{pmatrix}, \quad F'_2 = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -1 & 2 \\ 0 & 2 & 2 & -1 \end{pmatrix}, \quad F'_3 = \frac{1}{4} \begin{pmatrix} -1 & \sqrt{15} & 0 & 0 \\ \sqrt{15} & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix},$$

and the subgroup  $\Gamma = \langle F_1, F'_2, F'_3 \rangle$  of  $\mathbf{SL}_4\mathbb{C}$  (type *II* in [HH01]).  
Here  $l = 4$ ,

$$A^{(1)} = A^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad A^{(2)} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

$\text{rank}(A^{(1)}) = \text{rank}(A^{(2)}) = 4$ , and the eigenvalues of  $A^{(1)}$ ,  $A^{(2)}$ ,  $A^{(3)}$  are  
 $\Theta^{(1)} = \Theta^{(3)} = (4, 0, -1, 1, -1)$ ,  $\Theta^{(2)} = (6, -2, 1, 0, 1)$ ,  $p = 4$ , and  $\tau_0 = s_0s_1$ ,  $\tau_1 = s_2$ ,  $\tau_2 = s_3$ ,  $\tau_3 = s_4$ .

According to formula 21, we get

$$\begin{aligned} D_\Gamma(t, u, w) &= (w-1)^4(u+1)^3(u-1)^5(t-1)^4(w^2+w+1)(w^4+w^3+w^2+w+1) \\ &\quad (w+1)^2(u^4+u^3+u^2+u+1)(u^2+u+1)^2(t^2+t+1)(t^4+t^3+t^2+t+1)(t+1)^2 \\ &= (u-1)(u+1)(u^2+u+1)\tilde{D}_\Gamma(t)\tilde{D}_\Gamma(u)\tilde{D}_\Gamma(w), \end{aligned}$$

with  $\tilde{D}_\Gamma(t) = (t-1)^4(t+1)^2(t^2+t+1)(t^4+t^3+t^2+t+1)$ . Moreover,

$$\hat{P}_\Gamma(t) = \frac{t^8 - t^6 + t^4 - t^2 + 1}{t^{12} - 2t^{10} - t^9 + t^8 + t^7 + t^5 + t^4 - t^3 - 2t^2 + 1}.$$

Because of the too big size of the numerators  $N_\Gamma(t, u, w)_i$ 's, only the denominator is given in the text: all the numerators may be found on the web (<http://math.univ-lyon1.fr/~butin/>).

## References

- [BKR01] Bridgeland T., King A., Reid M., The McKay correspondence as an equivalence of derived categories, *J. Amer. Math. Soc.* **14** (2001), 535–554.
- [BP09] F. Butin, G. S. Perets, McKay correspondence and the branching law for finite subgroups of  $\mathbf{SL}_3\mathbb{C}$ , accepted for publication in *Journal of Group Theory*, arXiv:0909.0578.
- [GNS04] Gomi Y., Nakamura I., Shinoda K., Coinvariant Algebras of Finite Subgroups of  $\mathbf{SL}_3\mathbb{C}$ , *Canad. J. Math.* **56** (3), 495–528, 2004.
- [GSV83] Gonzalez-Sprinberg G., Verdier J.-L., Construction géométrique de la correspondance de McKay, *Annales scientifiques de l'E. N. S., 4ème série* **16** (3), 409–449, 1983.
- [HH01] A. Hanany, Y.-H. He, A monograph on the classification of the discrete subgroups of  $\mathbf{SU}(4)$ , *JHEP* **27**, 2001.
- [Kos85] Kostant B., The McKay Correspondence, the Coxeter Element and Representation Theory, *SMF, Astérisque, hors série*, 209–255, 1985.
- [Kos06] Kostant B., The Coxeter element and the branching law for the finite subgroups of  $\mathbf{SU}(2)$ , The Coxeter legacy, 63–70, *Amer. Math. Soc.*, Providence, RI, 2006, and arXiv:math/0411142v1 [math.RT], 2004.