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EIGENVALUE, MAXIMUM PRINCIPLE AND REGULARITY FOR FULLY NON LINEAR HOMOGENEOUS OPERATORS.

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ABSTRACT. The main scope of this article is to define the concept of principal eigenvalue for fully non linear second order operators in bounded domains that are elliptic, homogenous with lower order terms. In particular we prove maximum and comparison principle, Hölder and Lipschitz regularity. This leads to the existence of a first eigenvalue and eigenfunction and to the existence of solutions of Dirichlet problems within this class of operators.

1. INTRODUCTION

In [5], inspired by the acclaimed work of Berestycki, Nirenberg and Varadhan [3], we extended the definition of principal eigenvalue to Dirichlet problems for fully-non linear second order elliptic operators.

Precisely, given a bounded domain Ω , given $\alpha > -1$ we defined the "principal eigenvalue" for $F(\nabla u, D^2 u)$ satisfying:

- (H1) $F(tp, \mu X) = |t|^\alpha \mu F(p, X), \forall t \in \mathbb{R}^*, \mu \in \mathbb{R}^+$
- (H2) $a|p|^{\alpha \text{tr} N} \leq F(p, M + N) - F(p, M) \leq A|p|^{\alpha \text{tr} N}$ for $0 < a \leq A, \alpha > -1$ and $N \geq 0$.

Indeed we showed that

$$\bar{\lambda} = \sup \{ \lambda \in \mathbb{R}, \exists \phi > 0 \text{ in } \overline{\Omega}, F(\nabla \phi, D^2 \phi) + \lambda \phi^{\alpha+1} \leq 0 \text{ in the viscosity sense } \}$$

is well defined and it satisfies the following properties:

(I) *There exists ϕ a continuous positive viscosity solution of*

$$\begin{cases} F(\nabla \phi, D^2 \phi) + \bar{\lambda} \phi^{\alpha+1} = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

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(II) Furthermore, suppose that $\lambda < \bar{\lambda}$. If $f \leq 0$ in Ω is bounded and continuous, then there exists a non-negative, viscosity solution of

$$(1) \quad \begin{cases} F(\nabla u, D^2 u) + \lambda u^{\alpha+1} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If moreover f is negative in Ω , the solution is unique.

Hence $\bar{\lambda}$ was denoted *principal eigenvalue* of $-F$ in Ω .

In the case $\alpha = 0$, and for F a linear uniformly elliptic second order operator, these results are included in [3]; when F is one of the Pucci operators the problem has been treated by Quaas [25] and Busca, Esteban and Quaas [8]. Their papers give a more complete description of the spectrum and also treat bifurcation problems. In [5] and in this note the situation is complicated by the fact that there are no known results about the regularity of the solution, or the existence of the solution, even without the zero order term.

Clearly the operator F can be seen as a non-variational extension of the p -Laplacian: $\Delta_p = \operatorname{div}(|\nabla \cdot|^{p-2} \nabla \cdot)$ with $\alpha = p - 2$.

The scope of this article is to complete the results of [5]; indeed we consider operators that depend explicitly on x , we include lower order terms, moreover we define $\bar{\lambda}$ in a more suitable way i.e. without requiring that super-solutions are positive up to the boundary. Precisely we shall study existence of solutions, eigenvalue problems and regularity of the solutions for operators of the following type:

$$G(x, u, \nabla u, D^2 u) := F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha + c(x) |u|^\alpha u$$

where F satisfies assumptions as in [4] i.e. the above assumption (H1) and (H2), plus some continuity with respect to the x variable. See e.g. [16] for similar conditions. Because of the new setting the proofs differ in nature from [5].

The hypothesis on b and c are quite standard and they will be described in the next section.

As mention above we define $\bar{\lambda}$ in a more "correct" way i.e. :

$$\bar{\lambda} := \sup \{ \lambda \in \mathbb{R}, \exists \phi > 0 \text{ in } \Omega, G(x, \phi, \nabla \phi, D^2 \phi) + \lambda \phi^{\alpha+1} \leq 0 \text{ in the viscosity sense} \}.$$

The main aim of this paper is to prove the two following existence results:
Suppose that $f \leq 0$, bounded and continuous, that $\lambda < \bar{\lambda}$, then there exists a non-negative solution of

$$\begin{cases} G(x, u, \nabla u, D^2 u) + \lambda u^{1+\alpha} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore there exists $\phi > 0$ in Ω such that ϕ is a viscosity solution of

$$\begin{cases} G(x, \phi, \nabla \phi, D^2 \phi) + \bar{\lambda} \phi^{1+\alpha} = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

ϕ is γ -Hölder continuous for all $\gamma \in]0, 1[$ and locally Lipschitz.

Let us mention that it is possible to define another "eigenvalue" : Indeed let

$$\underline{\lambda} = \sup \{ \mu, \exists \phi < 0 \text{ in } \Omega, G(x, \phi, \nabla \phi, D^2 \phi) + \mu |\phi|^\alpha \phi \geq 0 \text{ in the viscosity sense} \}.$$

If $\underline{G}(x, u, p, X) = -F(x, p, -X) + b(x) \cdot p |p|^{\alpha+1} + c(x) |u|^\alpha u$ then $\underline{\lambda} = \bar{\lambda}(\underline{G})$. Furthermore if F satisfies (H2) then so does $\underline{F}(x, p, X) = -F(x, p, -X)$. Hence it is

possible to prove for $\underline{\lambda}$ the same results than for $\bar{\lambda}$. It is important to remark that in general $\underline{G} \neq G$ and hence $\underline{\lambda}$ can be different from $\bar{\lambda}$.

While we were completing this paper, we received a very interesting preprint of Ishii and Yoshimura [18] where similar results are obtained in the case $\alpha = 0$. Let us mention that they call the eigenvalue a demi-eigenvalue as in the paper of P.L. Lions [23], and they characterize it as the supremum of those $\lambda \in \mathbb{R}$ for which there is a viscosity supersolution $u \in C(\Omega)$ of $F[u] = \lambda u + 1$ in Ω which satisfies $u \geq 0$ in $\bar{\Omega}$. " (their F is our $-G$).

In the next section we state precisely the conditions on G and the definition of viscosity solution in this setting. In section 3 we prove a comparison principle and some boundary estimates that allow to prove that for $\lambda < \bar{\lambda}$ the maximum principle holds. This will be done in the fourth section, where we also provide some estimates on $\bar{\lambda}$ and a further comparison principle when $\lambda < \bar{\lambda}$. In section 5, using Ishii-Lions technique we prove regularity results, these in particular give the required relative compactness for the sequence of solutions that are used to prove the main existence's results in the last section.

2. MAIN ASSUMPTIONS AND DEFINITIONS.

In this section, we state the assumptions on the operators

$$G(x, u, \nabla u, D^2u) = F(x, \nabla u, D^2u) + b(x) \cdot \nabla u |\nabla u|^\alpha + c(x) |u|^\alpha u$$

treated in this note and the notion of viscosity solution.

The operator F is continuous on $\mathbb{R}^N \times (\mathbb{R}^N)^* \times S$, where S denotes the space of symmetric matrices on \mathbb{R}^N .

The following hypothesis will be considered

- (H1) $F : \Omega \times \mathbb{R}^N \setminus \{0\} \times S \rightarrow \mathbb{R}$, and $\forall t \in \mathbb{R}^*, \mu \geq 0, F(x, tp, \mu X) = |t|^\alpha \mu F(x, p, X)$.
- (H2) For $x \in \bar{\Omega}, p \in \mathbb{R}^N \setminus \{0\}, M \in S, N \geq 0$
- (2) $a|p|^\alpha \text{tr}(N) \leq F(x, p, M + N) - F(x, p, M) \leq A|p|^\alpha \text{tr}(N)$.
- (H3) There exists a continuous function $\tilde{\omega}, \tilde{\omega}(0) = 0$ such that for all $x, y, p \neq 0, \forall X \in S$

$$|F(x, p, X) - F(y, p, X)| \leq \tilde{\omega}(|x - y|) |p|^\alpha |X|.$$

- (H4) There exists a continuous function ω with $\omega(0) = 0$, such that if $(X, Y) \in S^2$ and $\zeta \in \mathbb{R}$ satisfy

$$-\zeta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 4\zeta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and I is the identity matrix in \mathbb{R}^N , then for all $(x, y) \in \mathbb{R}^N, x \neq y$

$$F(x, \zeta(x - y), p, X) - F(y, \zeta(x - y), p, -Y) \leq \omega(\zeta|x - y|^2).$$

The condition (H2), usually called uniformly elliptic condition, will be in some cases replaced by the much weaker condition

- (H2') for all $x \in \Omega, p \in \mathbb{R}^N \setminus \{0\}, M \in S, N \geq 0$,

$$F(x, p, N + M) \geq F(x, p, M).$$

Remark 1. The assumption (H2) and the fact that $F(x, p, 0) = 0$ implies that

$$|p|^\alpha (\text{atr}(M^+) - \text{Atr}(M^-)) \leq F(x, p, M) \leq |p|^\alpha (\text{Atr}(M^+) - \text{atr}(M^-))$$

where $M = M^+ - M^-$ is a minimal decomposition of M into positive and negative symmetric matrices.

We now assume conditions for the lower order terms i.e. we shall suppose that $b : \Omega \mapsto \mathbb{R}^N$ and $c : \Omega \mapsto \mathbb{R}$ are continuous and bounded.

We shall sometimes require (for example for the comparison and the maximum principle) that b satisfies:

- (H5) -Either $\alpha < 0$ and b is Hölderian of exponent $1 + \alpha$,
 - or $\alpha \geq 0$ and, for all x and y ,

$$\langle b(x) - b(y), x - y \rangle \leq 0$$

Let us recall what we mean by *viscosity solutions*, adapted to our context.

It is well known that in dealing with viscosity respectively sub and super solutions one works with

$$u^*(x) = \limsup_{y, |y-x| \leq r} u(y)$$

and

$$u_*(x) = \liminf_{y, |y-x| \leq r} u(y).$$

It is easy to see that $u_* \leq u \leq u^*$ and u^* is upper semicontinuous (USC) u_* is lower semicontinuous (LSC). See e.g. [11, 16].

Definition 1. Let Ω be a bounded domain in \mathbb{R}^N , then v , bounded on $\bar{\Omega}$ is called a viscosity super solution of $\mathcal{G}(x, \nabla u, D^2 u) = g(x, u)$ if for all $x_0 \in \Omega$,

-Either there exists an open ball $B(x_0, \delta)$, $\delta > 0$ in Ω on which $v = cte = c$ and $0 \leq g(x, c)$, for all $x \in B(x_0, \delta)$

-Or $\forall \varphi \in \mathcal{C}^2(\Omega)$, such that $v_* - \varphi$ has a local minimum on x_0 and $\nabla \varphi(x_0) \neq 0$, one has

$$(3) \quad \mathcal{G}(x_0, \nabla \varphi(x_0), D^2 \varphi(x_0)) \leq g(x_0, v_*(x_0)).$$

Of course u is a viscosity sub solution if for all $x_0 \in \Omega$,

-Either there exists a ball $B(x_0, \delta)$, $\delta > 0$ on which $u = cte = c$ and $0 \geq g(x, c)$, for all $x \in B(x_0, \delta)$

-Or $\forall \varphi \in \mathcal{C}^2(\Omega)$, such that $u^* - \varphi$ has a local maximum on x_0 and $\nabla \varphi(x_0) \neq 0$, one has

$$(4) \quad \mathcal{G}(x_0, \nabla \varphi(x_0), D^2 \varphi(x_0)) \geq g(x_0, u^*(x_0)).$$

A viscosity solution is a function which is both a super-solution and a sub-solution.

In particular we shall use this definition with $\mathcal{G}(x, p, X) = F(x, p, X) + b(x) \cdot p |p|^\alpha$.

See e.g. [10] for similar definition of viscosity solution for equations with singular operators.

For convenience we recall the definition of semi-jets given e.g. in [11]

$$\begin{aligned} J^{2,+}u(\bar{x}) &= \{(p, X) \in \mathbb{R}^N \times S, u(x) \leq u(\bar{x}) + \langle p, x - \bar{x} \rangle + \\ &\quad + \frac{1}{2} \langle X(x - \bar{x}), x - \bar{x} \rangle + o(|x - \bar{x}|^2)\} \end{aligned}$$

and

$$\begin{aligned} J^{2,-}u(\bar{x}) &= \{(p, X) \in \mathbb{R}^N \times S, u(x) \geq u(\bar{x}) + \langle p, x - \bar{x} \rangle + \\ &\quad + \frac{1}{2} \langle X(x - \bar{x}), x - \bar{x} \rangle + o(|x - \bar{x}|^2)\}. \end{aligned}$$

In the definition of viscosity solutions the test functions can be substituted by the elements of the semi-jets in the sense that in the definition above one can restrict to the functions ϕ defined by $\phi(x) = u(\bar{x}) + \langle p, x - \bar{x} \rangle + \frac{1}{2} \langle X(x - \bar{x}), x - \bar{x} \rangle$ with $(p, X) \in J^{2,-}u(\bar{x})$ when u is a super solution and $(p, X) \in J^{2,+}u(\bar{x})$ when u is a sub solution.

Remark 2. In all the paper we shall consider that Ω is a bounded \mathcal{C}^2 domain. In particular we shall use several times the fact that this implies that the distance to the boundary:

$$d(x, \partial\Omega) := d(x) := \inf\{|x - y|, y \in \partial\Omega\}$$

satisfies the following properties:

- (1) d is Lipschitz continuous
- (2) There exists $\delta > 0$ such that in $\Omega_\delta = \{x \in \Omega \text{ such that } d(x) \leq \delta\}$, d is $\mathcal{C}^{1,1}$.
- (3) d is semi-concave, i.e. there exists $C_1 > 0$ such that $d(x) - C_1|x|^2$ is concave and then $J^{2,+}d(x) \neq \emptyset$.
- (4) If $J^{2,-}d(x) \neq \emptyset$, d is differentiable at x and $|\nabla d(x)| = 1$.

3. A COMPARISON PRINCIPLE AND SOME BOUNDARY ESTIMATES.

We start by establishing a comparison result which is a sort of extension of the comparison Theorem 2.1 in [4].

Theorem 1. *Suppose that F satisfies (H1), (H2'), and (H4), that b is continuous and bounded and b satisfies (H5). Let f and g be respectively upper and lower semi continuous.*

Suppose that β is some continuous function on \mathbb{R}^+ such that $\beta(0) = 0$. Suppose that $\phi > 0$ in Ω lower semicontinuous and σ upper semicontinuous, satisfy, respectively, in the viscosity sense,

$$\begin{aligned} F(x, \nabla\phi, D^2\phi) + b(x) \cdot \nabla\phi |\nabla\phi|^\alpha - \beta(\phi) &\leq f \\ F(x, \nabla\sigma, D^2\sigma) + b(x) \cdot \nabla\sigma |\nabla\sigma|^\alpha - \beta(\sigma) &\geq g. \end{aligned}$$

Suppose that β is increasing on \mathbb{R}^+ and $f \leq g$, or β is nondecreasing and $f < g$. If $\sigma \leq \phi$ on $\partial\Omega$, then $\sigma \leq \phi$ in Ω .

Before starting the proof, for convenience, of the reader let us recall the following lemmata proved in [4], the first one being an extension of Ishii's acclaimed result.

Lemma 1. *Let Ω be a bounded open set in \mathbb{R}^N , which is piecewise \mathcal{C}^1 . Let u upper semi-continous in $\bar{\Omega}$, v lower semicontinuous in $\bar{\Omega}$, $(x_j, y_j) \in \Omega^2$, $x_j \neq y_j$, and $q \geq \sup\{2, \frac{\alpha+2}{\alpha+1}\}$.*

We assume that the function

$$\psi_j(x, y) = u(x) - v(y) - \frac{j}{q} |x - y|^q$$

has a local maximum on (x_j, y_j) , with $x_j \neq y_j$. Then, there are $X_j, Y_j \in \mathcal{S}^N$ such that

$$j(|x_j - y_j|^{q-2}(x_j - y_j), X_j) \in J^{2,+}u(x_j)$$

$$j(|x_j - y_j|^{q-2}(x_j - y_j), -Y_j) \in J^{2,-}v(y_j)$$

and

$$-4jk_j \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix} \leq 3jk_j \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

where

$$k_j = 2^{q-3}q(q-1)|x_j - y_j|^{q-2}.$$

Lemma 2. *Under the previous assumptions on F , let v be a lower semicontinuous, viscosity supersolution of*

$$F(x, \nabla v, D^2v(x)) + b(x) \cdot \nabla v |\nabla v|^\alpha - \beta(v(x)) \leq f(x)$$

for some functions (f, β) upper semi continuous in Ω . Suppose that \bar{x} is some point in Ω such that

$$v(x) + C|x - \bar{x}|^q \geq v(\bar{x}),$$

where \bar{x} is a strict local minimum of the left hand side and v is not locally constant around \bar{x} . Then,

$$-\beta(v(\bar{x})) \leq f(\bar{x}).$$

Remark: This Lemma was stated and proved for continuous super solutions in [4]. The proof is similar but it is adjusted to lower semi continuous super solutions and is given here for the convenience of the reader.

Proof. Without loss of generality we can suppose that $\bar{x} = 0$.

Since the infimum is strict, for $\epsilon > 0$, there exists N such that for any $n > N$

$$\inf_{\frac{1}{n} \leq |x| \leq R} (v(x) + C|x|^q) \geq m_n > v(0) + \epsilon$$

We take in the following also N large enough in order that

$$\left(\frac{1}{N}\right)^q C \leq \epsilon/4$$

and such that $C(\text{diam}\Omega + 1)^{q-1} \frac{1}{N} < \epsilon/4$

Since v is not locally constant for any n , there exists (t_n, z_n) in $B(0, \frac{1}{n})^2$ such that

$$v(t_n) > v(z_n) + C|z_n - t_n|^q$$

We consider

$$\inf_{|x| \leq R} v(x) + C|x - t_n|^q.$$

We prove in what follows that the infimum is achieved in $B(0, \frac{1}{n})$ and that it is not achieved on t_n .

Let us observe indeed

$$\inf_{|x| \leq \frac{1}{n}} v(x) + C|x - t_n|^q \leq v(0) + \epsilon/4$$

and since

$$\inf_{|x| \leq \frac{1}{n}} v(x) + C|x - t_n|^q \leq v(z_n) + C|z_n - t_n|^q < v(t_n)$$

the infimum on $|x| \leq 1/n$ cannot be achieved on t_n .

Moreover

$$\begin{aligned} \inf_{\frac{1}{n} \leq |x| \leq R} (v(x) + C|x - t_n|^q) &\geq \inf_{\frac{1}{n} \leq |x| \leq R} (v(x) + C|x|^q + C|x - t_n|^q - C|x|^q) \\ &\geq m_n - Cq|t_n||x - \theta t_n|^{q-1} \geq m_n - C(\text{diam}\Omega + 1)^{q-1}|t_n| \\ &\geq m_n - \epsilon/4 \geq v(0) + \frac{3\epsilon}{4} \end{aligned}$$

Since the infimum cannot be achieved on t_n , let y_n , $|y_n| \leq \frac{1}{n}$ be a point such that the infimum is achieved on y_n , then

$$v(x) + C|x - t_n|^q \geq v(y_n) + C|y_n - t_n|^q$$

and then

$$\varphi(x) = v(y_n) + C|y_n - t_n|^q - C|x - t_n|^q$$

is a test function for v on y_n with a gradient $\neq 0$ on that point. Since v is a supersolution one gets

$$-AC^{\alpha+1}|y_n - t_n|^{q(\alpha+1)-\alpha-2} - \beta(v(y_n)) \leq f(y_n)$$

Let us observe that $v(y_n) \rightarrow v(0)$. Indeed one has by the lower semicontinuity of v

$$v(0) \leq \liminf v(y_n)$$

and using

$$v(0) + C|t_n|^q \geq v(y_n) + C|y_n - t_n|^q$$

one has the reverse inequality.

Then by the uppersemicontinuity of f and β one gets that

$$-\beta(v(0)) \leq f(0)$$

which is the desired conclusion. \square

Proof. of Theorem 1.

Suppose by contradiction that $\max(\sigma - \phi) > 0$ in Ω . Since $\sigma \leq \phi$ on the boundary, the supremum can only be achieved inside Ω .

Let us consider for $j \in \mathbf{N}$ and for some $q > \max(2, \frac{\alpha+2}{\alpha+1})$

$$\psi_j(x, y) = \sigma(x) - \phi(y) - \frac{j}{q}|x - y|^q.$$

Suppose that (x_j, y_j) is a maximum for ψ_j . Then

(i) from the boundedness of σ and ϕ one deduces that $|x_j - y_j| \rightarrow 0$ as $j \rightarrow \infty$.

Thus up to subsequence $x_j, y_j \rightarrow \bar{x}$

(ii) One has $\liminf \psi_j(x_j, y_j) \geq \sup(\sigma - \phi)$;

(iii) $\limsup \psi_j(x_j, y_j) \leq \limsup \sigma(x_j) - \phi(y_j) = \sigma(\bar{x}) - \phi(\bar{x})$

(iv) Thus $j|x_j - y_j|^q \rightarrow 0$ as $j \rightarrow +\infty$ and \bar{x} is a maximum point for $\sigma - \phi$.

Claim: For j large enough, there exist x_j and y_j such that (x_j, y_j) is a maximum pair for ψ_j and $x_j \neq y_j$.

Indeed suppose that $x_j = y_j$. Then one would have

$$\begin{aligned}\psi_j(x_j, x_j) &= \sigma(x_j) - \phi(x_j) \\ &\geq \sigma(x_j) - \phi(y) - \frac{j}{q}|x_j - y|^q; \\ \psi_j(x_j, x_j) &= \sigma(x_j) - \phi(x_j) \\ &\geq \sigma(x) - \phi(x_j) - \frac{j}{q}|x - x_j|^q;\end{aligned}$$

and then x_j would be a local maximum for

$$\Phi := \phi(y) + \frac{j}{q}|x_j - y|^q.$$

and similarly a local minimum for

$$\Sigma := \sigma(x) - \frac{j}{q}|x_j - x|^q.$$

We first exclude that x_j is both a strict local maximum and a strict local minimum. Indeed in that case, by Lemma 2

$$-\beta(\phi(x_j)) \leq f(x_j)$$

$$-\beta(\sigma(x_j)) \geq g(x_j)$$

This is a contradiction because either β is increasing

$$-g(x_j) \geq \beta(\sigma(x_j)) > \beta(\phi(x_j)) \geq -f(x_j) \geq -g(x_j)$$

or

$$-g(x_j) \geq \beta(\sigma(x_j)) \geq \beta(\phi(x_j)) \geq -f(x_j) > -g(x_j).$$

Hence x_j cannot be both a strict minimum for Φ and a strict maximum for Σ . In the first case there exist $\delta > 0$ and $R > \delta$ such that $B(x_j, R) \subset \Omega$ and

$$\phi(x_j) = \inf_{\delta \leq |x - x_j| \leq R} \left\{ \phi(x) + \frac{j}{q}|x - x_j|^q \right\}.$$

Then if y_j is a point on which the minimum above is achieved, one has

$$\phi(x_j) = \phi(y_j) + \frac{j}{q}|x_j - y_j|^q,$$

and (x_j, y_j) is still a maximum point for ψ_j since for all $(x, y) \in \Omega^2$

$$\sigma(x_j) - \phi(y_j) - \frac{j}{q}|x_j - y_j|^q = \sigma(x_j) - \phi(x_j) \geq \sigma(x) - \phi(y) - \frac{j}{q}|x - y|^q.$$

This concludes the Claim. In the other case, similarly, one can replace x_j by a point y_j near x_j with

$$\sigma(x_j) = \sigma(y_j) + \frac{j}{q}|x_j - y_j|^q,$$

and (y_j, x_j) is still a maximum point for ψ_j .

We can now conclude. By Lemma 1 there exist X_j and Y_j such that

$$(j|x_j - y_j|^{q-2}(x_j - y_j), X_j) \in J^{2,+}\sigma(x_j)$$

and

$$(j|x_j - y_j|^{q-2}(x_j - y_j), -Y_j) \in J^{2,-}\phi(y_j).$$

We can use the fact that σ and ϕ are respectively sub and super solution to obtain:

$$\begin{aligned} g(y_j) &\leq F(y_j, j|x_j - y_j|^{q-2}(y_j - x_j), -Y_j) + \\ &+ b(y_j) \cdot j^{1+\alpha} |x_j - y_j|^{(q-1)(1+\alpha)-1} (y_j - x_j) - \beta(\phi(y_j)) \\ &\leq F(x_j, j|x_j - y_j|^{q-2}(x_j - y_j), X_j) + \\ &+ b(x_j) \cdot j^{1+\alpha} |x_j - y_j|^{(q-1)(1+\alpha)-1} (x_j - y_j) + \\ &+ C(j|x_j - y_j|^q)^{1+\alpha} + \omega(j|x_j - y_j|^q + \frac{1}{j}) - \beta(\phi(y_j)) \\ &\leq f(x_j) + C(j|x_j - y_j|^q)^{1+\alpha} + \omega(j|x_j - y_j|^q + \frac{1}{j}) + \beta(\sigma(x_j)) - \beta(\phi(y_j)). \end{aligned}$$

Passing to the limit and using the fact that g and f are respectively lower and upper semi continuous and β is continuous, we obtain

$$g(\bar{x}) \leq f(\bar{x}) + \beta(\sigma(\bar{x})) - \beta(\phi(\bar{x}))$$

which contradicts our hypotheses in all cases and $\sigma \leq 0$ in Ω . \square

As an application of the comparison theorem we will state bounds for sub and super solutions near the boundary. The conclusions given in Proposition 1 and Corollary 2 will be used in the proof of the maximum principle Theorem 3.

Proposition 1. *Suppose that F satisfies (H1) and (H2), and that b is bounded.*

Let u be uppersemicontinuous subsolution of

$$\begin{cases} F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha \geq -m & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for some constant $m \geq 0$. Then there exists $\delta > 0$ and some constant C_3 that depends only on the structural data such that

$$u(x) \leq C_3 d(x, \partial\Omega)$$

if the distance to the boundary $d(x, \partial\Omega) < \delta$.

Proof. Let $d(x) = d(x, \partial\Omega)$. First let us observe that one can assume that there exists d_0 such that $\Omega_{d_0} = \{x \in \Omega \text{ such that } d(x) < d_0\}$ the supremum of u is positive because otherwise there is nothing to prove.

We recall that Ω is a bounded C^2 domain and using the properties of the distance function stated in Remark 2 we know that $D^2 d \leq C_1 Id$ for some constant C_1 that depends on Ω . Let $\Omega_\delta = \{x \in \Omega; d(x) < \delta\}$. Suppose that $\delta < \frac{a}{4(C_1(A+a)N+|b|_\infty)}$. For some constants γ and C_2 that will be chosen later we introduce

$$\psi(x) = \gamma \log(1 + C_2 d(x)).$$

We use the inequalities

$$tr(D^2 \psi)^+ \leq \frac{\gamma C_2 N C_1}{1 + C_2 d}$$

and

$$tr(D^2 \psi)^- \geq \frac{\gamma C_2^2}{(1 + C_2 d)^2} - \frac{\gamma C_2 C_1 N}{1 + C_2 d}$$

Then choosing $C_2 > 4 \frac{C_1(A+a)N+|b|_\infty}{a}$, one has

$$\begin{aligned} & F(x, \nabla\psi, D^2\psi) + b(x) \cdot \nabla\psi |\nabla\psi|^\alpha \\ & \leq \gamma^{\alpha+1} \left(\frac{C_2}{1+C_2d} \right)^{\alpha+1} \left[\frac{-aC_2}{1+C_2d} + C_1(A+a)N + |b|_\infty \right] \\ & \leq -\frac{a}{2} \gamma^{\alpha+1} \left(\frac{C_2}{1+C_2\delta} \right)^{\alpha+2}. \end{aligned}$$

Now we choose γ sufficiently large that $\gamma > \sup_{\{x, d(x) \leq \delta\}} u(x)$ and

$$\frac{a}{4} \gamma^{\alpha+1} \left(\frac{C_2}{1+C_2\delta} \right)^{\alpha+2} \geq m$$

i.e.

$$\gamma = \max \left\{ \left(\frac{am}{4} \right)^{\frac{1}{\alpha+1}} \left(\frac{1+C_2\delta}{C_2} \right)^{\frac{\alpha+2}{\alpha+1}}, \sup_{\{x, d(x) \leq \delta\}} u(x) \right\}.$$

With this choice of constants we have obtained that

$$\begin{aligned} & F(x, \nabla\psi, D^2\psi) + b(x) \cdot \nabla\psi |\nabla\psi|^\alpha \\ & \leq -\frac{a}{2} \gamma^{\alpha+1} \left(\frac{C_2}{1+C_2\delta} \right)^{\alpha+2} \\ & < -\frac{a}{4} \gamma^{\alpha+1} \left(\frac{C_2}{1+C_2\delta} \right)^{\alpha+2} \\ & \leq -2m \\ & \leq F(x, \nabla u, D^2u) + b(x) \cdot \nabla u |\nabla u|^{\alpha+1} \end{aligned}$$

and furthermore $u \leq \psi$ on $\partial\Omega_\delta$.

Hence by Theorem 1 with $f = -\frac{a}{2} \gamma^{\alpha+1} \left(\frac{C_2}{1+C_2\delta} \right)^{\alpha+2}$, $g(x) = -m$ we obtain

$$u(x) \leq \gamma \log(1 + C_2 d(x)) \leq \gamma C_2 d(x)$$

since $u \leq \psi$ in Ω_δ . □

The comparison principle in [4] allows also to establish a strict maximum principle:

Theorem 2. *Suppose that F satisfies (H2), b and c are continuous and bounded and b satisfies (H5). Let u be a viscosity non-negative lowersemicontinuous super solution of*

$$F(x, \nabla u, D^2u) + b(x) \cdot \nabla u |\nabla u|^\alpha + c(x) u^{\alpha+1} \leq 0.$$

Then either $u \equiv 0$ or $u > 0$ in Ω .

Remark: Other strong maximum principles and strong minimal principles have been established in [2] for a more general class of fully nonlinear operators that are "proper".

Proof. Using the inequality in (H2), let us recall, using Remark 1, that

$$\begin{aligned} F(x, p, M) &\geq |p|^\alpha (atr(M)^+ - Atr(M)^-) \\ &:= H(p, M). \end{aligned}$$

Hence it is sufficient to prove the proposition when u is a super solution of

$$H(\nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha + c(x) u^{1+\alpha} = 0.$$

H does not depend on x and it satisfies the hypothesis of Theorem 1.

Moreover one can assume that c is some negative constant. Indeed, suppose that we have proved that for any $u \geq 0$ super solution of

$$(5) \quad H(\nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha - |c|_\infty u^{\alpha+1} \leq 0,$$

we have that $u > 0$ in Ω . Then if

$$F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha + c(x) u^{\alpha+1} \leq 0$$

for some $u \geq 0$ we have that u is a non negative super solution of (5) and then $u > 0$ and that would conclude the proof.

Hence we suppose by contradiction that x_0 is some point inside Ω on which $u(x_0) = 0$. Following e.g. Vazquez [27], one can assume that on the ball $|x - x_1| = |x - x_0| = R$, x_0 is the only point on which u is zero and that $B(x_1, \frac{3R}{2}) \subset \Omega$. Let $u_1 = \inf_{|x-x_1|=\frac{R}{2}} u > 0$, by the lower semicontinuity of u . Let us construct a sub solution on the annulus $\frac{R}{2} \leq |x - x_1| = \rho < \frac{3R}{2}$.

Let us recall that if $\phi(\rho) = e^{-k\rho}$, the eigenvalues of $D^2\phi$ are $\phi''(\rho)$ of multiplicity 1 and $\frac{\phi'}{\rho}$ of multiplicity $N-1$.

Then take k such that

$$k^{\alpha+2} > \left(\frac{2(N-1)A}{Ra} + |b|_\infty \right) k^{\alpha+1} + |c|_\infty.$$

If k is as above, let m be chosen such that

$$m(e^{-kR/2} - e^{-kR}) = u_1$$

and define $v(x) = m(e^{-k\rho} - e^{-kR})$ with $\rho = |x|$. The function v is a strict subsolution in the annulus, in the sense that it satisfies $H(\nabla v, D^2 v) + b(x) \cdot \nabla v |\nabla v|^\alpha - |c|_\infty v^{\alpha+1} > 0$ in the annulus. Furthermore

$$\begin{cases} v \leq u & \text{on } |x - x_1| = \frac{R}{2} \\ v \leq 0 \leq u & \text{on } |x - x_1| = \frac{3R}{2}. \end{cases}$$

Hence $u \geq v$ everywhere on the boundary of the annulus. In fact $u \geq v$ everywhere in the annulus, since we can use the comparison principle Theorem 1 for the operator $H + b(x) \cdot \nabla \cdot |\nabla \cdot|^\alpha - |c|_\infty |\cdot|^\alpha$. Then v is a test function for u at x_0 . Then, since u is a super solution and $\nabla v(x_0) \neq 0$:

$$H(\nabla v(x_0), D^2 v(x_0)) + b(x_0) \cdot \nabla v(x_0) |\nabla v(x_0)|^\alpha - |c|_\infty v^{\alpha+1}(x_0) \leq 0$$

which clearly contradicts the definition of v . Finally u cannot be zero inside Ω . \square

Corollary 1 (Hopf). *Let v be a viscosity continuous super solution of*

$$F(x, \nabla v, D^2 v) + b(x) \cdot \nabla v |\nabla v|^\alpha + c(x) |v|^\alpha v \leq 0.$$

Suppose that v is positive in a neighborhood of $x_o \in \partial\Omega$ and $v(x_o) = 0$ then there exist $C > 0$ and $\delta > 0$ such that

$$v(x) \geq C|x - x_o|$$

for $|x - x_o| \leq \delta$.

To prove this corollary just proceed as in the proof of Theorem 2 and remark that $e^{-k\rho} - e^{-kR} \geq C(R - \rho)$.

In fact, one can get a better estimate about supersolutions near the boundary i.e. some sort of limited expansion at the order two. We still denote by $d(x)$ the distance to the boundary of Ω and, for $d > 0$, $\Omega_d = \{x \in \Omega, \quad d(x) \leq d\}$.

Proposition 2. *Suppose that v is a lowersemicontinuous supersolution of*

$$F(x, \nabla v, D^2 v) + b \cdot \nabla v |\nabla v|^\alpha + c |v|^\alpha v \leq 0$$

in Ω_{d_1} , for some $d_1 > 0$, and v is ≥ 0 on the boundary, $v > 0$. Then there exists $d_2 \leq d_1$, $d_2 > 0$ and some constants $\gamma, C > 0$ such that on Ω_{d_2}

$$v(x) \geq \gamma(d(x) + \log(1 + Cd(x)^2)) \geq \gamma(d(x) + \frac{Cd(x)^2}{2}).$$

Proof. We start by proving the following

Claim: For some constant $C > 0$ large enough, there exists a neighborhood of $\partial\Omega$ such that

$$\varphi(x) = d(x) + \log(1 + Cd^2(x))$$

satisfies

$$F(x, \nabla \varphi, D^2 \varphi) + b \cdot \nabla \varphi |\nabla \varphi|^\alpha + c |\varphi|^\alpha \varphi > m > 0$$

for some constant $m > 0$.

Let d_0 be such that in $\Omega_{d_0} := \{x \in \Omega : \quad d(x) < d_0\}$ the distance is smooth and there exists C_1 such that $|D^2 d|_\infty \leq C_1$ as seen in Remark 2. Note that this implies that $tr(D^2 d)^+ + tr(D^2 d)^- \leq C_1 N$.

Let $C > (\frac{25}{6} + \frac{5A}{a})NC_1 + 25 \frac{|b|_\infty 2^{1+\alpha} + |c|_\infty}{\inf(1, 2^\alpha)}$ and $d < \inf(\frac{1}{2C}, \frac{1}{2}, d_0)$.

In Ω_{d_0}

$$|\varphi| \leq d + Cd^2 \leq \frac{1}{2} + \frac{1}{4} \leq 1$$

We compute the two first derivatives of φ :

$$\nabla \varphi = \nabla d (1 + \frac{2Cd}{1 + Cd^2})$$

and then

$$D^2 \varphi = D^2 d (1 + \frac{2Cd}{1 + Cd^2}) + \frac{2C(\nabla d \otimes \nabla d)(1 - Cd^2)}{(1 + Cd^2)^2}.$$

In particular

$$(D^2 \varphi)^- \leq (D^2 d)^- (1 + \frac{2Cd}{1 + Cd^2}),$$

and

$$(D^2 \varphi)^+ \geq 2C(\nabla d \otimes \nabla d) \frac{(1 - Cd^2)}{(1 + Cd^2)^2} - (D^2 d)^- (1 + \frac{2Cd}{1 + Cd^2}), .$$

These imply that

$$\text{tr}(D^2\varphi)^- \leq 2C_1N$$

and

$$\text{tr}(D^2\varphi)^+ \geq C\frac{24}{25} - 2NC_1 \geq \frac{12C}{25}.$$

Hence we obtain

$$\begin{aligned} F(x, \nabla\varphi, D^2\varphi) &+ b(x) \cdot \nabla\varphi |\nabla\varphi|^\alpha + c(x)\varphi^{1+\alpha} \\ &\geq \inf(1, 2^\alpha) \left(\frac{12aC}{25} - 2ANC_1 \right) - |b|_\infty |\nabla\varphi|^{1+\alpha} - |c|_\infty |\varphi|^{1+\alpha} \\ &\geq \inf(1, 2^\alpha) \frac{2aC}{25} - |b|_\infty 2^{1+\alpha} - |c|_\infty \\ &\geq \inf(1, 2^\alpha) \left(\frac{aC}{25} \right) > 0 \end{aligned}$$

This ends the proof of the Claim.

To conclude the proof of the proposition we choose C and d_0 as in the claim, $d_2 \leq (d_1, d_0)$. Since $v > 0$ inside $\{x, d(x) < d_1\}$ let γ be such that $\gamma(d_2 + \log(1 + Cd_2^2)) \leq \min_{d(x)=d_2} v$. Then $v \geq \gamma(d + \log(1 + Cd^2))$ on the boundary of the "crown" $\{x, 0 < d(x) < d_2\}$ in Ω . Since in addition v satisfies

$$F(x, \nabla v, D^2v) + b \cdot \nabla v |\nabla v|^\alpha + c|v|^\alpha v \leq 0$$

and $\varphi = \gamma(d + \log(1 + Cd^2))$ satisfies

$$F(x, \nabla\varphi, D^2\varphi) + b \cdot \nabla\varphi |\nabla\varphi|^\alpha + c|\varphi|^\alpha \varphi > 0,$$

the comparison principle implies that $v \geq \gamma(d + \log(1 + Cd^2))$ in $\{x, d(x) \leq d_2\}$. \square

4. MAXIMUM PRINCIPLE FOR $\lambda < \bar{\lambda}$; BOUNDS FOR $\bar{\lambda}$.

4.1. Maximum principle. We can now state and prove the following Maximum principle:

Theorem 3. *Let Ω be a bounded domain of \mathbb{R}^n . Suppose that F satisfies (H1), (H2'), (H4), that b and c are continuous and b satisfies (H5). Suppose that $\tau < \bar{\lambda}$ and that u is a viscosity sub solution of*

$$G(x, u, \nabla u, D^2u) + \tau|u|^\alpha u \geq 0 \quad \text{in } \Omega$$

with $u \leq 0$ on the boundary of Ω , then $u \leq 0$ in Ω .

Remark: Similarly it is possible to prove that if $\tau < \underline{\lambda}$ and v is a super solution of

$$G(x, v, \nabla v, D^2v) + \tau|v|^\alpha v \leq 0 \quad \text{in } \Omega$$

with $v \geq 0$ on the boundary of Ω then $v \geq 0$ in Ω .

Proof. Let $\lambda \in]\tau, \bar{\lambda}[$, and let v be a super solution of

$$G(x, v, \nabla v, D^2v) + \lambda v^{\alpha+1} \leq 0,$$

satisfying $v > 0$ in Ω , which exists by definition of $\bar{\lambda}$.

We assume by contradiction that $\sup u(x) > 0$ in Ω .

Claim : $\sup \frac{u}{v} < +\infty$.

Near the boundary this holds true since from Proposition 1 and Corollary 1 there exists $\delta > 0$ such that $u(x) \leq Cd(x)$ and $v(x) \geq C'd(x)$ for $d(x) \leq \delta$, for some constants C and C' .

In the interior we just use the fact that $v \geq C'\delta > 0$ in $\Omega_\delta = \{x : d(x) \geq \delta\}$ and we can conclude that $\frac{u}{v}$ is bounded in Ω .

We now define $\gamma' = \sup_{x \in \Omega} \frac{u}{v}$ and $w = \gamma v$, where $0 < \gamma < \gamma'$ and γ is sufficiently close to γ' in order that $\frac{\lambda - \tau \left(\frac{\gamma'}{\gamma}\right)^{1+\alpha}}{\left(\frac{\gamma'}{\gamma}\right)^{1+\alpha} - 1} \geq 2|c|_\infty$. Furthermore by definition of the

supremum there exists $\bar{y} \in \bar{\Omega}$ such that $\sup \frac{u(x)}{v(x)} = \frac{u(\bar{y})}{v(\bar{y})} = \gamma'$.

Clearly, by homogeneity, $G(x, w, \nabla w, D^2 w) + \lambda w^{1+\alpha} \leq 0$.

The supremum of $u - w$ is strictly positive, and it is necessarily achieved on $\bar{x} \in \Omega$ since on the boundary $u - w \leq 0$. One has

$$(u - w)(\bar{x}) \geq (u - w)(\bar{y})$$

and then

$$w(\bar{x}) \leq u(\bar{x}) + (w - u)(\bar{y}) < u(\bar{x}).$$

On the other hand

$$(u - w)(\bar{x}) \leq (\gamma' - \gamma)v(\bar{x})$$

and then

$$w(\bar{x}) \geq \frac{\gamma}{\gamma'} u(\bar{x}).$$

As in the comparison principle, we consider, for $j \in \mathbf{N}$ and for some $q > \max(2, \frac{\alpha+2}{\alpha+1})$:

$$\psi_j(x, y) = u(x) - w(y) - \frac{j}{q} |x - y|^q.$$

Since $\sup(u - w) > 0$, the supremum of ψ_j is achieved in $(x_j, y_j) \in \Omega^2$.

For j large enough, ψ_j achieves its positive maximum on some couple $(x_j, y_j) \in \Omega^2$ such that

- 1) $x_j \neq y_j$ for j large enough, (this uses lemma 2 and the definition of γ).
- 2) $(x_j, y_j) \rightarrow (\bar{x}, \bar{x})$ which is a maximum point for $u - w$ and it is an interior point
- 3) $j|x_j - y_j|^q \rightarrow 0$,
- 4) there exist X_j and Y_j in S such that

$$(j|x_j - y_j|^{q-2}(x_j - y_j), X_j) \in J^{2,+}u(x_j)$$

and

$$(j|x_j - y_j|^{q-2}(x_j - y_j), -Y_j) \in J^{2,-}w(y_j)$$

Furthermore

$$\begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix} \leq j \begin{pmatrix} D_j & -D_j \\ -D_j & D_j \end{pmatrix} \leq 2^{q-2} j q (q-1) |x_j - y_j|^{q-2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

with

$$D_j = 2^{q-3} q |x_j - y_j|^{q-2} \left(I + \frac{(q-2)}{|x_j - y_j|^2} (x_j - y_j) \otimes (x_j - y_j) \right).$$

The proof of these facts proceeds similarly to the one given in Theorem 1.

Condition (H4) implies that

$$F(x_j, j(x_j - y_j)|x_j - y_j|^{q-2}, X_j) - F(y_j, j(x_j - y_j)|x_j - y_j|^{q-2}, -Y_j) \leq \omega(j|x_j - y_j|^q).$$

Then, using the above inequality, the properties of the sequence (x_j, y_j) , the condition on b -with C_b below being either its Hölder constant or 0-, and the homogeneity condition (H1), one obtains

$$\begin{aligned}
-(\tau + c(x_j))u(x_j)^{1+\alpha} &\leq F(x_j, j(x_j - y_j)|x_j - y_j|^{q-2}, X_j) \\
&\quad + b(x_j) \cdot j^{(1+\alpha)}|x_j - y_j|^{(q-1)(1+\alpha)-1}(x_j - y_j) \\
&\leq F(y_j, j(x_j - y_j)|x_j - y_j|^{q-2}, -Y_j) \\
&\quad + \omega(j|x_j - y_j|^q) + C_b j^{1+\alpha}|x_j - y_j|^{q(1+\alpha)} \\
&\quad + b(y_j) \cdot j^{1+\alpha}|x_j - y_j|^{(q-1)(1+\alpha)-1}(x_j - y_j) + o(1) \\
&\leq (-\lambda - c(y_j))w(y_j)^{1+\alpha} + o(1).
\end{aligned}$$

By passing to the limit when j goes to infinity, since c is continuous one gets

$$-(\tau + c(\bar{x}))u(\bar{x})^{1+\alpha} \leq -(\lambda + c(\bar{x}))w(\bar{x})^{1+\alpha}$$

If $c(\bar{x}) + \lambda > 0$ one obtains that

$$-(\tau + c(\bar{x}))u(\bar{x})^{1+\alpha} \leq -(\lambda + c(\bar{x})) \left(\frac{\gamma}{\gamma'} \right)^{1+\alpha} u(\bar{x})^{1+\alpha}$$

This contradicts the hypothesis that $\frac{\lambda - \tau \left(\frac{\gamma'}{\gamma} \right)^{1+\alpha}}{\left(\frac{\gamma'}{\gamma} \right)^{1+\alpha} - 1} \geq 2|c|_\infty$.

If $c(\bar{x}) + \lambda = 0$ then $\tau < \lambda$ implies that

$$0 < -(\tau + c(\bar{x}))u(\bar{x})^{1+\alpha} \leq 0$$

a contradiction. Finally if $c(\bar{x}) + \lambda < 0$ we obtain

$$-(\tau + c(\bar{x}))u(\bar{x})^{1+\alpha} \leq -(\lambda + c(\bar{x}))w(\bar{x})^{1+\alpha} \leq -(\lambda + c(\bar{x}))u(\bar{x})^{1+\alpha}$$

once more a contradiction since $\tau < \lambda$.

□

4.2. Bounds on $\bar{\lambda}$.

Proposition 3. *Let $c(x) \equiv 0$. Let F satisfying (H1), (H2). Suppose that Ω is bounded in at least one direction, say e_1 , i.e. there exists R such that $\Omega \subset [-R, R] \times \mathbb{R}^{N-1}$, and that $b_1(x) = \langle b(x), e_1 \rangle$ is bounded. Then there exist some constants $C_1 > 0$, $C_2 > 0$ which depend on a and N such that*

$$\bar{\lambda} > \frac{C_1 e^{-C_2 |b_1|_\infty}}{R^{2+\alpha}}.$$

We deal with the particular case of the dimension 1. In that case we shall use variational techniques and weak solutions to estimate the first eigenvalue, this being justified by the following lemma

Lemma 3. *Suppose that $\Omega =]-R, R[$, $R > 0$, that a is some continuous function such that*

$$0 < a \leq a(x) \leq A$$

in $[-R, R]$, and that b is continuous and bounded. Suppose that g is continuous, then the weak solutions (in $W^{1,2+\alpha}([-R, R])$) and the viscosity solutions of

$$a(x)|u'|^\alpha u'' + b(x)|u'|^\alpha u' = g(x)$$

for $x \in]-R, R[$ coincide.

Proof. Suppose first that $u \in W^{1,2+\alpha}$ is a weak solution.

The previous equation can also be written as

$$\frac{d}{dx}(|u'|^\alpha u' e^{\int_0^x \frac{(\alpha+1)b(t)}{a(t)} dt}) = g(x) e^{\int_0^x \frac{(\alpha+1)b(t)}{a(t)} dt}$$

Then since $|u'|^\alpha u' = h \in L^{\frac{\alpha+2}{\alpha+1}}$ and $e^{\int_0^x \frac{b(t)}{a(t)} dt}$ is continuous, the product is a distribution T which satisfies " T' is continuous". Then T is \mathcal{C}^1 , hence $h(x) = T e^{-\int_0^x \frac{b(t)}{a(t)} dt}$ is \mathcal{C}^1 . Finally $u'(x) = h(x)^{\frac{1}{1+\alpha}}$ is \mathcal{C}^1 on every point where $h(x) \neq 0$, i.e. on each point where $u'(x) \neq 0$. Finally u is \mathcal{C}^2 on such point, and then on those points it satisfies the equation in the classical sense.

We now prove that u is a viscosity solution.

For that aim let φ be such that $(u - \varphi)(x) \geq 0 = (u - \varphi)(\bar{x})$ for all x in a neighborhood of \bar{x} . Since $u \in \mathcal{C}^1$, $\varphi'(\bar{x}) = u'(\bar{x})$. If $\varphi'(\bar{x}) = 0$ then there is nothing to test, if $\varphi'(\bar{x}) \neq 0$ then $u''(\bar{x})$ exists. Moreover $\varphi''(\bar{x}) \leq u''(\bar{x})$, and then since on such points u satisfies the equation in the classical meaning,

$$a(\bar{x})|\varphi'|^\alpha \varphi''(\bar{x}) + b(x)|\varphi'(\bar{x})|^\alpha \varphi'(\bar{x}) \leq a(\bar{x})|u'|^\alpha u''(\bar{x}) + b(x)|u'(\bar{x})|^\alpha u'(\bar{x}) \leq g(\bar{x})$$

one sees that u is a super-solution.

Suppose that φ is some test function by above for u on \bar{x} , again we are only intrested in the case $\varphi'(\bar{x}) \neq 0$ which implies that u' cannot be zero, and $\varphi''(\bar{x}) \geq u''(\bar{x})$. Then since on those points u is a solution in the classical sense

$$a(\bar{x})|\varphi'|^\alpha \varphi''(\bar{x}) + b(x)|\varphi'(\bar{x})|^\alpha \varphi'(\bar{x}) \geq g(\bar{x}).$$

This implies which implies that u is a sub solution .

We prove that the viscosity solutions are weak solutions, in the one dimensional case.

Let v be a weak solution of

$$a(x)|v'|^\alpha v'' + b(x)|v'|^\alpha v' = g(x)$$

$v = 0$ on the boundary.

Let now u be a viscosity solution of the same equation. We want to prove that $u = v$. For that aim let ϵ and let v_ϵ be the weak solution of

$$a(x)|v'_\epsilon|^\alpha v''_\epsilon + b(x)|v'_\epsilon|^\alpha v'_\epsilon = -\epsilon + g(x)$$

$v_\epsilon = 0$ on the boundary. and v^ϵ be the weak solution of

$$a(x)|v^\epsilon|^\alpha (v^\epsilon)' + b(x)|v^\epsilon|^\alpha v^\epsilon = \epsilon + g(x)$$

$v^\epsilon = 0$ on the boundary. By the previous part v_ϵ and v^ϵ are viscosity solutions and by the comparison theorem one gets

$$v^\epsilon \leq u \leq v_\epsilon.$$

Moreover by passing to the limit for weak solutions (for example using variational technics) it is easy to prove that v_ϵ and v^ϵ tend to v weakly in $W^{1,2+\alpha}$ and then in particular in \mathcal{C} . We obtain that

$$v = u.$$

□

Proposition 4. For $x \in]-R, R[$ let

$$G(x, u, u', u'') := a(x)|u'|^\alpha u'' + b(x)|u'|^\alpha u'$$

with $A \geq a(x) \geq a > 0$, continuous on $[-R, R]$ and b bounded, then there exist some constants $C_1 > 0$, $C_2 > 0$ which depend on a and the bound of b such that

$$\bar{\lambda} > \frac{C_1 e^{-C_2 R}}{R^{2+\alpha}}.$$

Proof. Let

$$B(x) = \int_{-R}^x \frac{b(x)(\alpha+1)}{a(x)} dx.$$

Then it is easy to show that

$$\bar{\lambda} \geq \lambda_1 := \inf_{u \in W_0^{1,2+\alpha}([-R, R])} \left\{ \frac{\int_{-R}^R |u'|^{\alpha+2} e^{B(x)} dx}{\int_{-R}^R \frac{\alpha+1}{a(x)} |u|^{\alpha+2} e^{B(x)} dx} \right\}.$$

Indeed, the infimum is achieved and u , a function achieving the infimum, is a weak solution of

$$|u'|^\alpha (a(x)u'' + b(x)u') = -\lambda_1 |u|^\alpha u.$$

Due to the previous lemma u is also a viscosity solution. One can assume that $u \geq 0$, so $u > 0$ in Ω , using strong maximum principle of Vazquez. Hence, by definition, $\bar{\lambda} \geq \lambda_1$.

But one has, for some universal constant C

$$\lambda_1 \geq \frac{a}{\alpha+1} e^{-2|\frac{b(x)}{a(x)}|_\infty R(\alpha+1)} \inf_{u \in W_0^{1,2+\alpha}([-R, R])} \frac{\int_{-R}^R |u'|^{2+\alpha} dx}{\int_{-R}^R |u|^{2+\alpha} dx} = \frac{\frac{Ca}{\alpha+1} e^{-2|\frac{b}{a}|_\infty R(\alpha+1)}}{R^{2+\alpha}}$$

which is the desired result. \square

Proof of Proposition 3.

Suppose that Ω is contained in $[-R, R] \times \mathbb{R}^{N-1}$, let us define

$$u(x) = 3^q R^q - (x_1 + 2R)^q,$$

where $q = \frac{2 \cdot 3^q R |b_1|_\infty}{a} + 2$, then $|\partial_{x_1}| \geq q R^{q-1}$ and

$$a \partial_{x_1 x_1} u + b_1 \partial_{x_1} u \leq -q \frac{a(q-1)}{2} R^{q-2}.$$

Finally, using also $u(x) \leq 3^q R^q$, one gets

$$G(x, u, \nabla u, D^2 u) \leq -c q^{2+\alpha} R^{q(\alpha+1)-\alpha-2} \leq -c R^{-\alpha-2} 3^{-q(\alpha+1)} q^{2+\alpha} u^{1+\alpha},$$

and, by definition,

$$\bar{\lambda} \geq c R^{-\alpha-2} 3^{-q(\alpha+1)} q^{2+\alpha}.$$

Using the expression of q in function of b_1 one gets the announced estimate.

Remark 3. Let us note that in the case $b = cte$ or when there exists some direction e_1 such that $b(x) \cdot e_1 = cte$ and Ω is bounded in this direction one has a better estimate.

Indeed, similarly to [5] one can consider

$$u(x) = u(x_1) = 7R^2 - x_1^2 - 3(\text{sign} b_1) R x_1.$$

This function is positive on $x_1 \in]-R, R]$, its gradient is never zero. Hence one has, for some constant C :

$$\begin{aligned} G(x, u, \nabla u, D^2 u) &= |3(\text{sign} b_1)R + 2x_1|^\alpha (-2 + b_1(-3R(\text{sign} b_1) - 2x_1)) \\ &\leq CR^\alpha (-2 - 3|b_1|R + 2|b_1|R) \\ &\leq CR^\alpha (-2 - |b_1|R). \end{aligned}$$

This implies that

$$G(x, u, \nabla u, D^2 u) \leq -C(2 + |b_1|R)R^{-2-\alpha}u^{1+\alpha}$$

which yields

$$\bar{\lambda} \geq \frac{C_1}{R^{2+\alpha}} + \frac{C_2|b_1|}{R^{1+\alpha}}$$

which is a more accurate lower bound than in the general case.

Proposition 5. *Suppose that R is the radius of the largest ball contained in Ω and suppose that F satisfies assumption (H1) and (H2).*

Furthermore let b and c be bounded functions. Then, there exists some constant C_1 which depends only on N, Ω, α, a and A , such that

$$\bar{\lambda} \leq C_1 \left(\frac{1}{R^{\alpha+2}} + \frac{|b|_\infty}{R^{\alpha+1}} \right) + |c|_\infty.$$

Proof. Without loss of generality we can suppose that the largest ball contained in Ω is $B_R(0)$. Let σ be defined as

$$\sigma(x) = \frac{1}{2q}(|x|^q - R^q)^2$$

with $q = \frac{\alpha+2}{\alpha+1}$, for $x \in B_R(0)$.

We need to compute the supremum in $B_R(0)$ of

$$\frac{-F(x, \nabla \sigma, D^2 \sigma) - b(x) \cdot \nabla \sigma |\nabla \sigma|^\alpha}{\sigma^{\alpha+1}}.$$

Let $\sigma(x) = g(r)$, for $r = |x|$. Clearly $g'(r) = r^{2q-1} - r^{q-1}R^q$ and

$$g''(r) = (2q-1)r^{2q-2} - (q-1)r^{q-2}R^q.$$

Furthermore $g' \leq 0$ while $g'' \leq 0$ for $r \leq \left(\frac{q-1}{2q-1}\right)^{\frac{1}{q}} R$ and positive elsewhere. Hence by condition (H2) and using the fact that for radial functions the eigenvalues of the Hessian are $\frac{g'}{r}$ with multiplicity $N-1$ and g'' (see [?]) we get

$$\text{-for } r \leq \left(\frac{q-1}{2q-1}\right)^{\frac{1}{q}} R$$

$$|F(x, \nabla \sigma, D^2 \sigma)| \leq |g'|^\alpha \left| ag''(r) + a\left(\frac{N-1}{r}\right)g'(r) \right|$$

$$\text{while for } r \geq \left(\frac{q-1}{2q-1}\right)^{\frac{1}{q}} R$$

$$-F(x, \nabla \sigma, D^2 \sigma) \leq -|g'|^\alpha \left[ag''(r) + A\left(\frac{N-1}{r}\right)g'(r) \right]$$

i.e.

$$-F(x, \nabla \sigma, D^2 \sigma) \leq |g'|^\alpha r^{q-2} (-B_1 r^q + B_2 R^q)$$

where $B_1 = a(2q - 1) + A(N - 1)$ and $B_2 = a(q - 1) + A(N - 1)$.

Let R_1 be defined as

$$R_1 = R \left(\frac{B_2 + |b|_\infty R^{(q-1)(\alpha+1)}}{B_1 + |b|_\infty R^{(q-1)(\alpha+1)}} \right)^{\frac{1}{q}} < R$$

then for $r \geq R_1$

$$-F(x, \nabla \sigma, D^2 \sigma) - b(x) \cdot \nabla \sigma |\nabla \sigma|^\alpha \leq 0.$$

Hence the supremum is achieved for $r \leq R_1$. On that set one can use an upper bound for $|-F(x, \nabla \sigma, D^2 \sigma) + b \cdot \nabla \sigma |\nabla \sigma|^\alpha|$ and a lower bound for σ e.g.

$$\sigma \geq \frac{1}{2q} (R^q - R_1^q)^2.$$

More precisely for $r \leq R_1$ and for some constants C_1, C_2, C'_1, C'_2 depending on a, A, N and $|b|_\infty$ one has:

$$\begin{aligned} \frac{|-F(x, \nabla \sigma, D^2 \sigma) - b \cdot \nabla \sigma |\nabla \sigma|^\alpha|}{\sigma^{\alpha+1}} &\leq r^{q(\alpha+1)-\alpha-2} \frac{C_1 R^q}{(R^q - R_1^q)^{\alpha+2}} + \frac{C_2 |b|_\infty R^{(q-1)(\alpha+1)}}{R^{q(\alpha+1)}} \\ &\leq \frac{C'_1}{R^{\alpha+2}} + \frac{C'_2}{R^{\alpha+1}} \end{aligned}$$

Then σ is a subsolution in $B_R(0)$ of

$$F(x, \nabla \sigma, D^2 \sigma) + b \cdot \nabla \sigma |\nabla \sigma|^\alpha + c |\sigma|^\alpha \sigma + \left(\frac{C_1}{R^{\alpha+2}} + \frac{C_2}{R^{\alpha+1}} + |c|_\infty \right) |\sigma|^\alpha \sigma \geq 0,$$

with $\sigma = 0$ on $\partial B_R(0)$. Suppose by contradiction that $\bar{\lambda} > \frac{C_1}{R^{\alpha+2}} + \frac{C_2}{R^{\alpha+1}} + |c|_\infty$. Clearly since $B_R(0) \subset \Omega$, $\bar{\lambda}(B_R(0)) \geq \bar{\lambda} > \frac{C_1}{R^{\alpha+2}} + \frac{C_2}{R^{\alpha+1}} + |c|_\infty$, and then according to the maximum principle, Theorem 3, one should have that $\sigma \leq 0$ in $B_R(0)$, a contradiction. \square

4.3. Comparison theorem for $\lambda < \bar{\lambda}$.

Theorem 4. Suppose that F satisfies (H1), (H2'), and (H4), that b and c are continuous and bounded and b satisfies (H5). Suppose that $\tau < \bar{\lambda}$, $f \leq 0$, f is upper semi-continuous and g is lower semi-continuous with $f \leq g$.

Suppose that there exist σ upper semi continuous, and v non-negative and lower semi continuous, satisfying

$$\begin{aligned} F(x, \nabla v, D^2 v) + b(x) \cdot \nabla v |\nabla v|^\alpha + (c(x) + \tau) v^{1+\alpha} &\leq f \quad \text{in } \Omega \\ F(x, \nabla \sigma, D^2 \sigma) + b(x) \cdot \nabla \sigma |\nabla \sigma|^\alpha + (c(x) + \tau) |\sigma|^\alpha \sigma &\geq g \quad \text{in } \Omega \\ \sigma &\leq v \quad \text{on } \partial \Omega \end{aligned}$$

Then $\sigma \leq v$ in Ω in each of these two cases:

- 1) If $v > 0$ on $\bar{\Omega}$ and either $f < 0$ in Ω , or $g(\bar{x}) > 0$ on every point \bar{x} such that $f(\bar{x}) = 0$,
- 2) If $v > 0$ in Ω , $f < 0$ and $f < g$ on $\bar{\Omega}$

Proof. We act as in the proof of Theorem 3.6 in [4].

- 1) We assume first that $v > 0$ on $\bar{\Omega}$. Suppose by contradiction that $\sigma > v$ somewhere in Ω . The supremum of the function $\frac{\sigma}{v}$ on $\partial \Omega$ is less than 1 since $\sigma \leq v$

on $\partial\Omega$ and $v > 0$ on $\partial\Omega$, then its supremum is achieved inside Ω . Let \bar{x} be a point such that

$$1 < \frac{\sigma(\bar{x})}{v(\bar{x})} = \sup_{x \in \bar{\Omega}} \frac{\sigma(x)}{v(x)}.$$

We define

$$\psi_j(x, y) = \frac{\sigma(x)}{v(y)} - \frac{j}{qv(y)}|x - y|^q.$$

For j large enough, this function achieves its maximum which is greater than 1, on some couple $(x_j, y_j) \in \Omega^2$. It is easy to see that this sequence converges to (\bar{x}, \bar{x}) , a maximum point for $\frac{\sigma}{v}$.

Since on test functions that have zero gradient the definition of viscosity solutions doesn't require to test equation, we need to prove first that x_j, y_j can be chosen such that $x_j \neq y_j$ for j large enough.

Indeed, if $x_j = y_j$ one would have for all $x \in \Omega$

$$\frac{\sigma(x) - \frac{j}{q}|x_j - x|^q}{v(x_j)} \leq C = \frac{\sigma(x_j)}{v(x_j)},$$

which implies that

$$\sigma(x) \leq \sigma(x_j) + \frac{j}{q}|x_j - x|^q.$$

This means that σ has a local maximum on the point x_j . We argue as it is done in the proof of Theorem 1 : If x_j is not a strict local maximum then x_j can be replaced by x'_j close to it and then (x'_j, x_j) is also a maximum point for ψ_j .

If the maximum is strict using Lemma 2 one gets that

$$(c(x_j) + \tau)\sigma(x_j)^{1+\alpha} \geq g(x_j).$$

But one also has

$$v(x) \geq v(x_j) - \frac{j}{qC}|x_j - x|^q.$$

hence x_j is a local minimum for v , and if it is not strict, there exists x'_j which is different from x_j such that (x'_j, x_j) is also a maximum point for ψ_j .

If the minimum is strict using once more Lemma 2 one would have

$$(c(x_j) + \tau)v^{1+\alpha}(x_j) \leq f(x_j).$$

This is a contradiction for j large enough. Indeed, passing to the limit one would get

$$(c(\bar{x}) + \tau)(\sigma(\bar{x})^{1+\alpha} - v(\bar{x})^{1+\alpha}) \geq g(\bar{x}) - f(\bar{x}) \geq 0.$$

Since $\sigma(\bar{x}) > v(\bar{x})$ this implies that

$$c(\bar{x}) + \tau \geq 0.$$

Now there are two cases either $f(\bar{x}) < 0$ or $f(\bar{x}) = 0$ and the above inequality is strict. In both cases it contradicts

$$(c(\bar{x}) + \tau)v(\bar{x})^{1+\alpha} \leq f(\bar{x}).$$

We can take x_j and y_j such that $x_j \neq y_j$.

Moreover there exist X_j and Y_j such that

$$\left(j|x_j - y_j|^{q-2}(x_j - y_j), \frac{X_j}{v(y_j)} \right) \in J^{2,+}\sigma(x_j)$$

and

$$\left(j|x_j - y_j|^{q-2}(x_j - y_j) \frac{v(y_j)}{\beta_j}, \frac{-Y_j}{\beta_j} \right) \in J^{2,-}v(y_j)$$

where $\beta_j = \sigma(x_j) - \frac{j}{q}|x_j - y_j|^q$ and

$$F(x_j, j|x_j - y_j|^{q-2}(x_j - y_j), X_j) - F(y_j, j|x_j - y_j|^{q-2}(x_j - y_j), -Y_j) \leq \omega(v(y_j)j|x_j - y_j|^q).$$

We can use the fact that σ and v are respectively sub and super solution to obtain:

$$\begin{aligned} g(x_j) - \tau\sigma(x_j)^{1+\alpha} &= c(x_j)\sigma(x_j)^{1+\alpha} \leq F(x_j, j|x_j - y_j|^{q-2}(x_j - y_j), \frac{X_j}{v(y_j)}) \\ &\quad + b(x_j) \cdot j^{1+\alpha}|x_j - y_j|^{(q-1)(1+\alpha)-1}(x_j - y_j) \\ &\leq \frac{\beta_j^{1+\alpha}}{v(y_j)^{1+\alpha}} \left\{ F(y_j, j|x_j - y_j|^{q-2}(x_j - y_j) \frac{v(y_j)}{\beta_j}, \frac{-Y_j}{\beta_j}) \right. \\ &\quad \left. + \omega(jv(y_j)|x_j - y_j|^q) \right\} + \\ &\quad + b(x_j) \cdot j^{1+\alpha}|x_j - y_j|^{(q-1)(1+\alpha)-1}(x_j - y_j) \\ &\leq \frac{\beta_j^{1+\alpha}}{v(y_j)^{1+\alpha}} \left\{ F(y_j, j|x_j - y_j|^{q-2}(x_j - y_j) \frac{v(y_j)}{\beta_j}, \frac{-Y_j}{\beta_j}) \right. \\ &\quad \left. + b(y_j) \cdot j^{1+\alpha}|x_j - y_j|^{(q-1)(1+\alpha)-1}(x_j - y_j) \left(\frac{v(y_j)}{\beta_j} \right)^{1+\alpha} \right\} \\ &\quad + \frac{\omega(v(y_j)j|x_j - y_j|^q)}{v(y_j)^{1+\alpha}} + C(j|x_j - y_j|^q)^{1+\alpha} \\ &\leq (-\tau - c(y_j))\beta_j^{1+\alpha} + \frac{\beta_j^{1+\alpha}}{v(y_j)^{1+\alpha}} f(y_j) + o(1). \end{aligned}$$

Passing to the limit, since c is continuous, we get:

$$g(\bar{x}) \leq \left(\frac{\sigma(\bar{x})}{v(\bar{x})} \right)^{\alpha+1} f(\bar{x}).$$

Either $f(\bar{x}) = 0$ and then we have reached a contradiction because, in that case, by hypothesis

$$g(\bar{x}) > 0,$$

or $f(\bar{x}) < 0$, and then we get

$$0 < f(\bar{x}) \left[1 - \left(\frac{\sigma(\bar{x})}{v(\bar{x})} \right)^{\alpha+1} \right] \leq f(\bar{x}) - g(\bar{x}) \leq 0.$$

This concludes the proof of the first part.

2) For the second part, let m be such that $f - g \leq -m < 0$, and $f < -\frac{m}{2}$. Let ϵ be given such that by the uniform continuity of the function $(x + \epsilon)^{1+\alpha}$ on $[0, |v|_\infty]$ one has

$$|\lambda + c|_\infty \cdot |(v + \epsilon)^{1+\alpha} - v^{1+\alpha}| \leq \frac{m}{2}.$$

Then $w = v + \epsilon$ is a supersolution of

$$F(x, \nabla w, D^2 w) + (\lambda + c)w^{1+\alpha} \leq f + \frac{m}{2} \leq g - \frac{m}{2} < g \leq F(\sigma, \nabla \sigma, D^2 \sigma) + (\lambda + c)|\sigma|^\alpha \sigma.$$

We are now in a position to use the first part of the theorem, since

$$w = v + \epsilon > 0 \quad \text{and} \quad u \leq v \leq v + \epsilon \quad \text{on} \quad \partial\Omega.$$

and then $u \leq v + \epsilon$ in Ω . Letting ϵ go to zero we get the required conclusion. \square

5. REGULARITY RESULTS

In this section we shall prove that the viscosity solutions are Hölder continuous. Since the Hölder estimates depend only on the bounds of f and the structural constants, this Hölder continuity will allow us to have a compactness criteria that will be useful in the next section. Let us note that we state all the results with $c = 0$. Indeed, one can consider $c(x)|u|^\alpha u$ in the right hand side since it is bounded, and get the same regularity results.

Proposition 6. *Suppose that F satisfies (H1), (H2), (H3). Let f be a bounded function in $\bar{\Omega}$. Let u be a viscosity non-negative bounded solution of*

$$(6) \quad \begin{cases} F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then if b is bounded, for any $\gamma \in (0, 1)$ there exists some constant C which depends only on $|f|_\infty$ and $|b|_\infty$ such that for any $(x, y) \in \bar{\Omega}^2$

$$|u(x) - u(y)| \leq C|x - y|^\gamma.$$

An immediate consequence of the above Proposition is the

Corollary 2. *Suppose that F satisfies (H1), (H2) and (H3). Suppose that f_n is a sequence of continuous and uniformly bounded functions, and u_n is a sequence of bounded viscosity solutions of*

$$F(x, \nabla u_n, D^2 u_n) + b(x) \cdot \nabla u_n |\nabla u_n|^\alpha = f_n(x)$$

with b bounded, $u_n = 0$ on $\partial\Omega$. Then the sequence u_n is relatively compact in $\mathcal{C}(\bar{\Omega})$.

Proof. The proof relies on ideas used to prove Hölder and Lipschitz estimates in [17], as it is done in [5].

We use Proposition 1 in section 3 which implies in particular that there exists M_0 such that

$$(7) \quad u(x) \leq M_0 d(x)^\gamma$$

for $d(x) := d(x, \partial\Omega) \leq \delta$.

We now prove Hölder's regularity inside Ω .

We construct a function Φ as follows: Let M_o and γ be as in (7), $M = \sup(M_o, \frac{2 \sup u}{\delta^\gamma})$ and $\Phi(x) = M|x|^\gamma$. We also define

$$\Delta_\delta = \{(x, y) \in \Omega^2, |x - y| < \delta\}.$$

Claim *For any $(x, y) \in \Delta_\delta$*

$$(8) \quad u^\star(x) - u_\star(y) \leq \Phi(x - y).$$

If the Claim holds this completes the proof, indeed taking $x = y$ we would get that $u^\star = u_\star$ and then u is continuous. Therefore, going back to (8),

$$u(x) - u(y) \leq \frac{2 \sup u}{\delta^\gamma} |x - y|^\gamma,$$

for $(x, y) \in \Delta_\delta$ which is equivalent to the local Hölder continuity.

Let us check that (8) holds on $\partial\Delta_\delta$. On that set:

- either $|x - y| = \delta$ and then $u^*(x) - u_*(y) \leq M\delta^\gamma$ since $M\delta^\gamma \geq 2\sup u$,
 -or $(x, y) \in \partial(\Omega \times \Omega)$. In that case, for $(x, y) \in (\Omega \times \partial\Omega)$ we have just proved that

$$u^*(x) \leq M_\sigma d(x)^\gamma \leq M|y - x|^\gamma,$$

while for $(x, y) \in \partial\Omega \times \Omega$

$$0 - u_*(y) \leq 0 \leq M_0|y - x|^\gamma.$$

Now we consider interior points. Suppose by contradiction that $u^*(x) - u_*(y) > \Phi(x - y)$ for some $(x, y) \in \Delta_\delta$. Then there exists (\bar{x}, \bar{y}) such that

$$u^*(\bar{x}) - u_*(\bar{y}) - \Phi(\bar{x} - \bar{y}) = \sup(u^*(x) - u_*(y) - \Phi(x - y)) > 0.$$

Clearly $\bar{x} \neq \bar{y}$. Then using Ishii's Lemma [16], there exist X and Y such that

$$\begin{aligned} (\gamma M(\bar{x} - \bar{y})|\bar{x} - \bar{y}|^{\gamma-2}, X) &\in J^{2,+}u^*(\bar{x}) \\ (\gamma M(\bar{x} - \bar{y})|\bar{x} - \bar{y}|^{\gamma-2}, -Y) &\in J^{2,-}u_*(\bar{y}) \end{aligned}$$

with

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \begin{pmatrix} B & -B \\ -B & B \end{pmatrix}$$

and $B = D^2\Phi(\bar{x} - \bar{y})$.

We need a more precise estimate, as in [17]. For that aim let :

$$0 \leq P := \frac{(\bar{x} - \bar{y} \otimes \bar{x} - \bar{y})}{|\bar{x} - \bar{y}|^2} \leq I.$$

Using $-(X + Y) \geq 0$ and $(I - P) \geq 0$ and the properties of the symmetric matrices one has

$$\text{tr}(X + Y) \leq \text{tr}(P(X + Y)).$$

Remarking in addition that $X + Y \leq 4B$, one sees that $\text{tr}(X + Y) \leq \text{tr}(P(X + Y)) \leq 4\text{tr}(PB)$. But $\text{tr}(PB) = \gamma M(\gamma - 1)|\bar{x} - \bar{y}|^{\gamma-2} < 0$, hence

$$(9) \quad |\text{tr}(X + Y)| \geq 4\gamma M(1 - \gamma)|\bar{x} - \bar{y}|^{\gamma-2}.$$

Furthermore by Lemma III.1 of [17] there exists a universal constant C such that

$$|X|, |Y| \leq C(|\text{tr}(X + Y)| + |B|^{\frac{1}{2}}|\text{tr}(X + Y)|^{\frac{1}{2}}) \leq C|\text{tr}(X + Y)|,$$

since $|B|$ and $|\text{tr}(X + Y)|$ are of the same order. Now we can use the fact that u is both a sub and a super solution of (6), and applying condition (H2), (H3) concerning F :

$$\begin{aligned} f(\bar{x}) &- (\gamma M)^{1+\alpha}b(\bar{x}).(\bar{x} - \bar{y})|\bar{x} - \bar{y}|^{(\gamma-1)(\alpha+1)-1} \\ &\leq F(\bar{x}, \nabla\Phi(\bar{x} - \bar{y}), X) \\ &\leq F(\bar{y}, \nabla\Phi(\bar{x} - \bar{y}), X) + \tilde{\omega}(|\bar{x} - \bar{y}|)|\nabla\Phi(\bar{x} - \bar{y})|^\alpha|X| \\ &\leq a|\nabla\Phi(\bar{x} - \bar{y})|^\alpha\text{tr}(X + Y) + F(\bar{y}, \nabla_y\Phi(\bar{y}), \text{tr}(-Y)) \\ &\quad + \tilde{\omega}(|\bar{x} - \bar{y}|)|\nabla\Phi(\bar{x} - \bar{y})|^\alpha|X| \\ &\leq f(\bar{y}) + a|\nabla\Phi(\bar{x} - \bar{y})|^\alpha\text{tr}(X + Y) + (\gamma M)^{1+\alpha}|b|_\infty|\bar{x} - \bar{y}|^{(\gamma-1)(\alpha+1)} \\ &\quad + \tilde{\omega}(|\bar{x} - \bar{y}|)|\nabla\Phi(\bar{x} - \bar{y})|^\alpha|\text{tr}(X + Y)|. \end{aligned}$$

Which implies, using (9)

$$|\nabla\Phi(\bar{x}-\bar{y})|^\alpha \gamma M(1-\gamma)|\bar{x}-\bar{y}|^{\gamma-2} \left(a - C\tilde{\omega}(|\bar{x}-\bar{y}|) - 2\frac{|b|_\infty}{(1-\gamma)}|\bar{x}-\bar{y}| \right) \leq f(\bar{y}) - f(\bar{x}).$$

We choose δ small enough in order that $C\tilde{\omega}(\delta) + 2\frac{|b|_\infty}{(1-\gamma)}\delta < \frac{a}{2}$. Recalling that $|\nabla\Phi(\bar{x}-\bar{y})| = \gamma M|\bar{x}-\bar{y}|^{\gamma-1}$ the previous inequality becomes:

$$(10) \quad \frac{a}{2} M^{\alpha+1} \gamma^{1+\alpha} (1-\gamma) |\bar{x}-\bar{y}|^{\gamma(\alpha+1)-(\alpha+2)} \leq 2|f|_\infty.$$

Using $M \geq \frac{2(\sup u)}{\delta^\gamma}$ and $|\bar{x}-\bar{y}| \leq \delta$ one obtains

$$a(2\sup u)^{1+\alpha} \gamma^{1+\alpha} (1-\gamma) \delta^{-(\alpha+2)} \leq 4|f|_\infty.$$

This is clearly false for δ small enough and it concludes the proof. \square

For completeness sake we shall now prove some Lipschitz regularity of the solution. To get Lipschitz regularity we need a further assumption as it was done in [5]. Let us remark that Lipschitz regularity is not necessary to prove the existence results, hence this further assumption will be used only in the present part of the paper.

(H7) *There exists $\nu > 0$ and $\kappa \in]1/2, 1]$ such that for all $|p| = 1$, $|q| \leq \frac{1}{2}$, $B \in \mathcal{S}$*

$$|F(x, p+q, B) - F(x, p, B)| \leq \nu |q|^\kappa |B|$$

which implies by homogeneity that for all $p \neq 0$, $|q| \leq \frac{|p|}{2}$, $B \in \mathcal{S}$

$$|F(x, p+q, B) - F(x, p, B)| \leq \nu |q|^\kappa |p|^{\alpha-\kappa} |B|$$

One has, then, the following regularity result:

Theorem 5. *If F satisfies (H1), (H2), (H3) and (H7) and if b is bounded, then the bounded solutions of*

$$\begin{cases} F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

are Lipschitz continuous inside Ω .

Proof of Theorem 5. The proof proceeds similarly to the proof given by Ishii and Lions in [17] and as it is required in that paper, we use the fact that we already know that u is Hölder continuous, together with the additional assumption (H7).

To simplify the calculation but, without loss of generality we shall suppose that in hypothesis (H2) $a = A = 1$. Let γ be in $]\frac{1}{2\kappa}, 1[$ and c_γ such that by the Hölder's continuity proved before

$$|u(x) - u(y)| \leq c_\gamma |x - y|^\gamma.$$

Let μ be an increasing function such that $\mu(0) = 0$, and $\mu(r) \geq r$, let $l(r) = \int_0^r ds \int_0^s \frac{\mu(\sigma)}{\sigma} d\sigma$, let us note that since $\mu \geq 0$, for $r > 0$:

$$l(r) \leq r l'(r)$$

Let r_0 be such that $l'(r_0) = \frac{1}{2}$. Let also $\delta > 0$ be given, $K = \frac{r_0}{\delta}$, and z be such that $d(z, \partial\Omega) \geq 2\delta$.

We define $\varphi(x, y) = \Phi(x - y) + L|x - z|^k$ where $\Phi(x) = M(K|x| - l(K|x|))$, and

$$\Delta_z = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N, |x - y| < \delta, |x - z| < \delta\}.$$

We shall now choose all the constants above.

- k is such that $k = \frac{1}{1 - \frac{1}{2\gamma\kappa}}$

- M and L are such that $M = \frac{4\sup u}{r_0}$ and $L = c_\gamma \delta^{\gamma-k}$, using the Hölder continuity of u , one has

$$u(x) - u(y) \leq \varphi(x, y)$$

on $\partial\Delta_z$. Indeed, the assumption on r_0 implies that $\Phi(x) \geq MK\frac{|x|}{2}$ for $|x| \leq r_0$ and then if $|x - y| = \delta$,

$$\begin{aligned} u(x) - u(y) &\leq 2\sup u \leq \frac{Mr_0}{2} \leq \frac{MK\delta}{2} \\ &\leq \Phi(x - y) \leq \varphi(x, y), \end{aligned}$$

while if $|x - z| = \delta$

$$u(x) - u(y) \leq c_\gamma |x - y|^\gamma \leq c_\gamma \delta^\gamma = L|x - z|^k \leq \varphi(x, y).$$

Suppose by contradiction that for some point (\bar{x}, \bar{y}) one has

$$u(\bar{x}) - u(\bar{y}) > \varphi(\bar{x}, \bar{y}).$$

Clearly $\bar{x} \neq \bar{y}$. Note that

$$L|\bar{x} - z|^k \leq c|\bar{x} - \bar{y}|^\gamma.$$

Proceeding as in the previous proof, there exist X, Y such that

$$(MK(\bar{x} - \bar{y})|\bar{x} - \bar{y}|^{-1}(1 - l'(K|\bar{x} - \bar{y}|)) + kL|\bar{x} - z|^{k-2}(\bar{x} - z), X) \in J^{2,+}u(\bar{x}),$$

and

$$(MK\frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|}(1 - l'(K|\bar{x} - \bar{y}|)), -Y) \in J^{2,-}u(\bar{y}),$$

where the matrices X and Y satisfy

$$(11) \quad \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \begin{pmatrix} B + \tilde{L} & -B \\ -B & B \end{pmatrix}$$

with $B = D^2\Phi(\bar{x} - \bar{y})$ and

$$\tilde{L} = kL|\bar{x} - z|^{k-2} \left(I + (k-2) \frac{(\bar{x} - z \otimes \bar{x} - z)}{|\bar{x} - z|^2} \right).$$

Let us note that similarly to the Hölder case, (11) implies that $X + Y - \tilde{L} \leq 4B$ and then

$$\text{tr}(X + Y - \tilde{L}) \leq 4\text{tr}(PB)$$

with

$$P = \frac{((\bar{x} - \bar{y}) \otimes (\bar{x} - \bar{y}))}{|\bar{x} - \bar{y}|^2}.$$

This gives:

$$\text{tr}(X + Y - \tilde{L}) \leq -\frac{MK\mu(K|\bar{x} - \bar{y}|)}{|\bar{x} - \bar{y}|} \leq -MK^2.$$

Let us note that

$$\nabla_x \varphi(x) = MK(1 - l'(K|\bar{x} - \bar{y}|)) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} + kL|\bar{x} - z|^{k-2}(\bar{x} - z)$$

and

$$L|\bar{x} - z|^{k-1} = O(\delta^{\gamma-k}\delta^{k-1}) = O(K^{1-\gamma}).$$

From this we get in particular that for $\delta > 0$ small enough (or K large enough)

$$2MK \geq (|\nabla_x \varphi(\bar{x}, \bar{y})|, |\nabla_y \varphi(\bar{x}, \bar{y})|) \geq \frac{MK}{4}.$$

Finally observe that $|\tilde{L}| \leq k(k-1)L|\bar{x} - z|^{k-2} \leq (C\delta^{\gamma-k})^{\frac{2}{k}}(\delta)^{\gamma(k-2)/k} = O(\delta^{\gamma-2}) = O(K^{2-\gamma})$, from which we derive that for K large enough $\text{tr}(X + Y) \leq 0$ and

$$|\text{tr}(X + Y)| \geq C(K^2)$$

for some > 0 universal constant C , and $|\tilde{L}| \leq |\text{tr}(X + Y)|$ for K large enough.

In the following we shall need a bound from above for $|X|$. In order to make the reading easier the constants C or c will be constants which depend only on the data, and they may vary from one line to another. Remark that the lemma III.1 in [17] ensures the existence of some universal constant such that

$$|X - \tilde{L}| + |Y| \leq C \left(|B|^{\frac{1}{2}} |\text{tr}(X + Y - \tilde{L})|^{\frac{1}{2}} + |\text{tr}(X + Y - \tilde{L})| \right)$$

with $B = D^2\varphi$, and with the considerations on \tilde{L} with respect to $|\text{tr}(X + Y)|$ one also has

$$|X| + |Y| \leq C \left(|B|^{\frac{1}{2}} |\text{tr}(X + Y)|^{\frac{1}{2}} + |\text{tr}(X + Y)| \right).$$

Let us note that

$$|D^2\varphi| \leq \frac{CK}{|\bar{x} - \bar{y}|},$$

and then with the assumptions on μ , $|\text{tr}(X + Y)| \geq C \geq K|\text{tr}(X + Y)|^{\frac{1}{2}}$ from which one derives that

$$|X| \leq |\text{tr}(X + Y)| \left(1 + \frac{1}{K^{\frac{1}{2}} |\bar{x} - \bar{y}|^{\frac{1}{2}}} \right).$$

We need to prove that

$$|\nabla_x \varphi(\bar{x}, \bar{y})|^{\alpha-\kappa} |\tilde{L}|\bar{x} - z|^{k-1} |X| = o(|\text{tr}(X + Y)| |\nabla \varphi|^\alpha).$$

For that aim we write

$$\begin{aligned} |\nabla_x \varphi(\bar{x}, \bar{y})|^{\alpha-k} (L|\bar{x} - z|^{k-1})^\kappa |X| &\leq cK^{\alpha-\frac{\gamma\kappa}{k}} |\bar{x} - \bar{y}|^{\gamma\kappa(1-\frac{1}{k})} |X| \\ &\leq cK^{\alpha-\frac{\gamma\kappa}{k}} |\bar{x} - \bar{y}|^{\frac{1}{2}} |\text{tr}(X + Y)| \left(1 + \frac{1}{K^{\frac{1}{2}} |\bar{x} - \bar{y}|^{\frac{1}{2}}} \right) \\ &\leq c|\text{tr}(X + Y)| (K^{\alpha-\frac{\gamma\kappa}{k}-\frac{1}{2}}) \\ &= c|\text{tr}(X + Y)| K^\alpha K^{-\gamma\kappa-\frac{1}{2}} = o(|\text{tr}(X + Y)| |\nabla \varphi|^\alpha) \end{aligned}$$

We now obtain using assumption (H2) and (H3) concerning F

$$\begin{aligned} f(\bar{x}) - b(\bar{x}) \cdot \nabla_x \varphi |\nabla_x \varphi|^\alpha &\leq F(\bar{x}, \nabla_x \varphi(\bar{x}, \bar{y}), X) + |b|_\infty O(K^{1+\alpha}) \\ &\leq F(\bar{y}, \nabla_y \varphi(\bar{x}, \bar{y}), X) + \nu |L|\bar{x} - z|^{k-1} |\nabla_x \varphi|^{\alpha-\kappa} |X| \\ &\quad + |b|_\infty O(K^{1+\alpha}) \\ &\leq F(\bar{y}, \nabla_y \varphi(\bar{x}, \bar{y}), -Y) + |\nabla \phi|^\alpha |\text{tr}(X + Y)| \\ &\quad + b(\bar{y}) \cdot \nabla_y \varphi |\nabla_y \varphi|^\alpha + O(K^{-\frac{\gamma\kappa}{k}-\frac{1}{2}} |\nabla \varphi|^\alpha) \text{tr}(X + Y) \\ &\quad + |\nabla \phi|^\alpha \text{tr}(X + Y) + 2|b|_\infty O(K^{1+\alpha}) \\ &\leq f(\bar{y}) + O(K^{2+\alpha-\gamma\kappa}) - C(K^{\alpha+2}) + |b|_\infty O(K^{1+\alpha}) \end{aligned}$$

From this one gets a contradiction for K large. We have proved that for all x such that $d(x, \partial\Omega) \geq 2\delta$ and for y such that $|x - y| \leq \delta$

$$u(x) - u(y) \leq \left(\frac{2 \sup u}{r_0} \right) \left(\frac{|x - y|}{\delta} \right).$$

Recovering the compact set Ω by a finite number of \mathcal{C}^2 sets Ω_i , $\Omega_i \subset \Omega_{i+1}$ such that $d(\partial\Omega_i, \partial\Omega_{i+1}) \leq 2\delta$, the local Lipschitz continuity is proved.

6. EXISTENCE'S RESULTS

We now prove the existence of non negative solutions of

$$\begin{cases} G(x, u, \nabla u, D^2 u) + \lambda u^{1+\alpha} = -f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where f is a given positive function. The steps are the following :

Step 1: Exhibit a sub and a super solution of the equation when the coefficient of the zero order is non positive and f is constant.

Step 2: Under the same conditions on the zero order term, use Perron's method to solve the equation for any negative function $-f$.

Step 3: From the previous steps we construct a solution of the above Dirichlet problem when $\lambda < \bar{\lambda}$ without conditions on the sign of $c(x)$.

Step 4: This will also allow to prove the existence of the associated eigenvalue.

The first step is obtained by remarking that 0 is a sub solution and establishing the following

Proposition 7. *Suppose that F satisfies (H1) and (H2), b and c are bounded; furthermore let c be non-positive in $\bar{\Omega}$. Then there exists a function u which is a nonnegative viscosity super solution of*

$$\begin{cases} F(x, \nabla u, D^2 u) + b(x)|\nabla u|^\alpha \nabla u + c(x)u^{1+\alpha} \leq -1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Proof. Let d be the distance function to $\partial\Omega$, which is well defined in Ω and satisfies the properties stated in Remark 2. Let $K > \text{diam}\Omega$. Then $d \leq K$. Let $\gamma \in]0, 1[$ and let k be a large enough constant to be chosen later. Let u be defined as

$$u(x) = 1 - \frac{1}{(1 + d(x)^\gamma)^k}.$$

Clearly $u = 0$ on the boundary.

Suppose that ψ is a \mathcal{C}^2 function such that $(u - \psi)(x) \geq (u - \psi)(\bar{x}) = 0$, for all x in a small neighbourhood of \bar{x} . Then $J^{2,-}u(\bar{x}) \neq \emptyset$ and then the function ϕ defined as

$$\psi(\cdot) = 1 - \frac{1}{(1 + \phi(\cdot)^\gamma)^k}$$

is a \mathcal{C}^2 function in a neighbourhood of \bar{x} , such that

$$(d - \phi)(x) \geq (d - \phi)(\bar{x}) = 0.$$

This implies that $J^{2,-}d(\bar{x}) \neq \emptyset$. According to some of the properties of d recalled in the introduction, on such a point d is differentiable and then $\nabla\phi(\bar{x}) = \nabla d(\bar{x})$ has modulus 1.

One has

$$\nabla\psi(x) = \frac{k\gamma d(x)^{\gamma-1} \nabla\phi(x)}{(1+d(x)^\gamma)^{k+1}}$$

and

$$D^2\psi = \frac{k\gamma d^{\gamma-2}}{(1+d^\gamma)^{k+2}} [(\gamma-1-(k\gamma+1)d^\gamma)\nabla\phi \otimes \nabla\phi + d(1+d^\gamma)D^2\phi].$$

We need to prove that one can choose k large enough in order that

$$F(x, \nabla\psi, D^2\psi) + b(x) \cdot \nabla\psi |\nabla\psi|^\alpha + c(x)\psi^{\alpha+1} \leq -1.$$

We use Remark 2 on the distance function and the following inequalities on symmetric matrices

$$\text{if } Y \geq 0 \text{ then } (X-Y)^+ \leq X^+ \quad \text{and} \quad (X-Y)^- \geq X^- - Y.$$

Using these with $X = D^2\phi$ and $Y = D\phi \otimes D\phi$, and condition (H2) we obtain

$$F(x, \nabla\psi, D^2\psi) \leq \frac{k^{\alpha+1}\gamma^{1+\alpha}d^{\gamma(\alpha+1)-2-\alpha}}{(1+d^\gamma)^{k(\alpha+1)+\alpha+2}} [a(\gamma-1-d^\gamma(k\gamma+1)) + (A+a)C_1Nd(1+d^\gamma)]$$

The function $\frac{d^{\gamma(\alpha+1)-2-\alpha}}{(1+d^\gamma)^{k(\alpha+1)+\alpha+2}}$ is decreasing hence it is greater than

$$\frac{K^{\gamma(\alpha+1)-2-\alpha}}{(1+K^\gamma)^{k(\alpha+1)+\alpha+2}} = C_4.$$

We shall use this later.

Now we write

$$\begin{aligned} b(x) \cdot \nabla\psi |\nabla\psi|^\alpha &\leq |b|_\infty k^{1+\alpha} \gamma^{1+\alpha} \frac{d^{(\gamma-1)(1+\alpha)}}{(1+d^\gamma)^{(k+1)(\alpha+1)}} \\ &\leq |b|_\infty \frac{k^{\alpha+1}\gamma^{1+\alpha}d^{\gamma(\alpha+1)-2-\alpha}}{(1+d^\gamma)^{(k+1)(\alpha+1)+1}} d(1+d^\gamma) \end{aligned}$$

We have obtained that there exists a constant $C = C(A, a, |b|_\infty, N)$ such that

$$\begin{aligned} F(x, \nabla\psi, D^2\psi) + b(x) \cdot \nabla\psi |\nabla\psi|^\alpha + c(x)\psi^{1+\alpha} &\leq \\ \frac{k^{\alpha+1}\gamma^{1+\alpha}d^{\gamma(\alpha+1)-2-\alpha}}{(1+d^\gamma)^{k(\alpha+1)+\alpha+2}} [a(\gamma-1-d^\gamma(k\gamma+1)) + Cd(1+d^\gamma)] & . \end{aligned}$$

Clearly since $\gamma < 1$ we can choose k large enough in order that

$$[a(\gamma-1-d^\gamma(k\gamma+1)) + Cd(1+d^\gamma)] < \frac{-1}{C_4 k^{1+\alpha} \gamma^{1+\alpha}} < 0.$$

Then

$$G(x, \psi, \nabla\psi, D^2\psi) \leq k^{\alpha+1}\gamma^{1+\alpha}C_4\left(\frac{-1}{C_4(k\gamma)^{1+\alpha}}\right) = -1$$

which gives the result. \square

Remark 4. Clearly if u is the super solution constructed in the previous Proposition then for any $M > 0$ and any $0 \leq c_o \leq \left(\frac{M}{|c|_\infty}\right)^{\frac{1}{1+\alpha}}$ the function $u_2(x) = M^{\frac{1}{1+\alpha}}u(x) + c_o$ is a super solution of:

$$\begin{cases} F(x, \nabla u_2, D^2 u_2) + b(x) \cdot \nabla u_2 |\nabla u_2|^{\alpha+1} + c(x)u_2^{1+\alpha} \leq -M & \text{in } \Omega \\ u_2 = c_o & \text{on } \partial\Omega. \end{cases}$$

We are now in a position to solve step 2:

Theorem 6. *Suppose that F satisfies (H1) and (H2), that b and c are continuous with $c \leq 0$.*

- (1) *If f is continuous, bounded and $f \leq 0$ on $\overline{\Omega}$, then there exists u a nonnegative viscosity solution of*

$$\begin{cases} F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^{\alpha+1} + c(x) u^{1+\alpha} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

- (2) *For any bounded continuous function $f < -M < 0$ for some positive constant M and any $0 \leq c_o \leq \left(\frac{M}{|c|_\infty}\right)^{\frac{1}{1+\alpha}}$ there exists u a non negative solution of*

$$(12) \quad \begin{cases} F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^{\alpha+1} + c(x) u^{1+\alpha} = f & \text{in } \Omega \\ u = c_o & \text{on } \partial\Omega. \end{cases}$$

Remark 5. In a forthcoming paper,[6] we prove existence's results with general data.

Proof. Let u_2 be the viscosity super solution given in Proposition 7 (see Remark 4), of

$$G(x, u_2, \nabla u_2, D^2 u_2) \leq -|f|_\infty,$$

$u_2 = 0$ on $\partial\Omega$.

We use Perron's method, see Ishii's paper [16]. We define

$$\mathcal{M} = \{u \geq 0, 0 \leq u \leq u_2, u \text{ is a subsolution}\}.$$

Let $v(x) = \sup_{u \in \mathcal{M}} u(x)$. We prove that v is both a sub and a super solution.

We use the same process as in [5] to prove that v^* is a sub solution.

We now prove that v_* is a super solution. If not, there would exist $\bar{x} \in \Omega$, $r > 0$ and $\varphi \in C^2(B(\bar{x}, r))$, with $\nabla \varphi(\bar{x}) \neq 0$, satisfying

$$0 = (v_* - \varphi)(\bar{x}) \leq (v_* - \varphi)(x)$$

on $B(\bar{x}, r)$, such that

$$G(\bar{x}, \varphi(\bar{x}), \nabla \varphi(\bar{x}), D^2 \varphi(\bar{x})) > f(\bar{x}).$$

We prove that $\varphi(\bar{x}) < v_2(\bar{x})$. If not one would have $\varphi(\bar{x}) = v_*(\bar{x}) = v_2(\bar{x})$

$$(v_2 - \varphi)(x) \geq (v_* - \varphi)(x) \geq (v_* - \varphi)(\bar{x}) = (v_2 - \varphi)(\bar{x}) = 0,$$

hence since v_2 is a super solution and φ is a test function for v_2 on \bar{x} ,

$$G(\bar{x}, \varphi(\bar{x}), \nabla \varphi(\bar{x}), D^2 \varphi(\bar{x})) \leq f(\bar{x}),$$

a contradiction. Then $\varphi(\bar{x}) < v_2(\bar{x})$. We construct now a sub solution which is greater than v_* and less than v_2 .

Let $\varepsilon > 0$ be such that

$$G(\bar{x}, \varphi(\bar{x}), \nabla \varphi(\bar{x}), D^2 \varphi(\bar{x})) \geq f(\bar{x}) + \varepsilon,$$

and let δ be such that for $|x - \bar{x}| \leq \delta$:

$$\begin{aligned} & |G(\bar{x}, \varphi(\bar{x}), \nabla \varphi(\bar{x}), D^2 \varphi(\bar{x})) - G(x, \varphi(x), \nabla \varphi(x), D^2 \varphi(x))| \\ & + |f(x) - f(\bar{x})| \leq \frac{\varepsilon}{4}. \end{aligned}$$

Then

$$G(x, \varphi(x), \nabla \varphi(x), D^2 \varphi(x)) \geq f(x) + \frac{\varepsilon}{4}.$$

One can assume that

$$(v_\star - \varphi)(x) \geq |x - \bar{x}|^4.$$

We take $r < \delta^4$ and such that $0 < r < \inf_{|x - \bar{x}| \leq \delta} (v_2(x) - \varphi(x))$, and define

$$w = \sup(\varphi(x) + r, v_\star)$$

w is LSC as it is the supremum of two LSC functions.

One has $w(\bar{x}) = \varphi(\bar{x}) + r$, and $w = v_\star$ for $r < |x - \bar{x}| < \delta$.

w is a sub solution, since when $w = \varphi + r$ one can use $\varphi + r$ as a test function, and using the continuity of c ,

$$G(x, \varphi(x), \nabla \varphi(x), D^2 \varphi(x)) \geq f + \frac{\varepsilon}{4}.$$

Elsewhere $w = v_\star$, hence it is a sub solution. Moreover $w \geq v_\star$, $w \neq v_\star$ and $w \leq g$. This contradicts the fact that v_\star is the supremum of the sub solutions. Using Hölder regularity we get that v_\star is Hölder.

For the proof of the second statement, it is enough to remark that $u = c_o$ is a sub solution of (13) and then proceed as above with u_2 the solution, given in Proposition 7 (see Remark 4), of

$$G(x, u_2, \nabla u_2, D^2 u_2) \leq -|f|_\infty,$$

$u_2 = c_o$ on $\partial\Omega$. □

We now prove an existence result for $\lambda < \bar{\lambda}$ i.e. step 3.

Theorem 7. *Suppose that F , b and c satisfy the assumptions in Theorem 1 and that $\lambda < \bar{\lambda}$,*

- (1) *Suppose that $f \leq 0$, continuous and bounded, then there exists a nonnegative solution of*

$$\begin{cases} F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha + (c(x) + \lambda)u^{1+\alpha} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$
- (2) *For any bounded and continuous function $f \leq -M < 0$ and any $0 \leq c_o \leq \left(\frac{M}{|c|_\infty}\right)^{\frac{1}{1+\alpha}}$ there exists u a non negative solution of*

$$(13) \quad \begin{cases} F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^{\alpha+1} + (c(x) + \lambda)u^{1+\alpha} = f & \text{in } \Omega \\ u = c_o & \text{on } \partial\Omega. \end{cases}$$

Proof. We define a sequence by induction with $u_1 = 0$ and u_{n+1} as the solution of

$$\begin{cases} F(x, \nabla u_{n+1}, D^2 u_{n+1}) + b(x) \cdot \nabla u_{n+1} |\nabla u_{n+1}|^\alpha + (c(x) - |c|_\infty)u_{n+1}^{1+\alpha} = \\ \quad = f - (\lambda + |c|_\infty)u_n^{1+\alpha} & \text{in } \Omega, \\ u_{n+1} = 0 & \text{on } \partial\Omega. \end{cases}$$

which exists by the previous theorem.

The sequence is positive and u_n is increasing, indeed we can use the comparison Theorem 1 with the right hand side equal to $f - (\lambda + |c|_\infty)u_n^{1+\alpha} < 0$ and the function $c(\phi) = (-c + |c|_\infty)\phi^{1+\alpha}$, which is nonnegative and increasing with respect to ϕ . We need to prove that the sequence is bounded : suppose that it is not, then dividing by $|u_{n+1}|_\infty^{1+\alpha}$ and defining $w_n = \frac{u_n}{|u_n|_\infty}$, one gets that w_n satisfies

$$\begin{aligned}
F(x, \nabla w_{n+1}, D^2 w_{n+1}) + b(x) \cdot \nabla w_{n+1} |\nabla w_{n+1}|^\alpha + (c(x) + \lambda - |c + \lambda|_\infty) w_{n+1}^{1+\alpha} = \\
= \frac{f}{|u_{n+1}|_\infty^{1+\alpha}} - |\lambda + c|_\infty \frac{u_n^{1+\alpha}}{|u_{n+1}|_\infty^{1+\alpha}}.
\end{aligned}$$

Since the sequence is increasing the right hand side is bounded and it is greater than $\frac{f}{|u_{n+1}|_\infty^{1+\alpha}} - (|\lambda + c|_\infty) w_{n+1}^{1+\alpha}$. Then w_n converges to w while $\frac{u_n}{|u_{n+1}|_\infty}$ converges to kw for $k = \lim \frac{|u_n|_\infty^{1+\alpha}}{|u_{n+1}|_\infty^{1+\alpha}} \leq 1$.

One gets, by passing to the limit and using the compactness result, that the limit function w satisfies

$$F(x, w, \nabla w, D^2 w) + b(x) \cdot \nabla w |\nabla w|^\alpha + (\lambda + c) w^{1+\alpha} \geq (1 - k) |c + \lambda|_\infty w^{1+\alpha} \geq 0$$

with $w \geq 0$, $|w|_\infty = 1$ and $w = 0$ on the boundary. This contradicts the maximum principle (Theorem 3).

We have obtained that the sequence u_n is bounded. Letting n go to infinity, and using the compactness result (Corollary 2), the sequence being in addition monotone, it converges in its whole to u which is a solution.

The solution is unique if $f \leq -m < 0$ on $\bar{\Omega}$. Indeed suppose that u and v are two solutions then $v(1 + \epsilon)$ is a solution with $f(1 + \epsilon)^{1+\alpha}$ in the right hand side. Since it is strictly less than f one gets by the comparison principle 4 that $v(1 + \epsilon) \geq u$ and since ϵ is arbitrary $v \geq u$. One can of course exchange u and v and obtain that $u = v$ \square

6.1. Existence result for $\lambda = \bar{\lambda}$. We have reached the final step:

Theorem 8. *Let F , b and c as in Theorem 1. Then, there exists $\phi > 0$ in Ω such that ϕ is a viscosity solution of*

$$\begin{cases} F(x, \nabla \phi, D^2 \phi) + b(x) \cdot \nabla \phi |\nabla \phi|^\alpha + (c(x) + \bar{\lambda}) \phi^{1+\alpha} = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover ϕ is γ -Hölder continuous for all $\gamma \in]0, 1[$.

Proof. Let λ_n be an increasing sequence which converges to $\bar{\lambda}$. Let u_n be a non-negative viscosity solution of

$$\begin{cases} F(x, \nabla u_n, D^2 u_n) + b(x) \cdot \nabla u_n |\nabla u_n|^\alpha + (c(x) + \lambda_n) u_n^{1+\alpha} = -1 & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 6 the sequence u_n is well defined. We shall prove that (u_n) is not bounded. Indeed suppose by contradiction that it is. Then by the Hölder's estimate and the compactness result (Corollary 2), one would have that a subsequence, still denoted u_n , tends uniformly to a nonnegative continuous function u which would be a viscosity solution of

$$\begin{cases} F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha + (c(x) + \bar{\lambda}) u^{1+\alpha} = -1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

But then $u > 0$ in Ω and bounded and one can choose ε small enough that

$$F(x, \nabla u, D^2 u) + b(x) \cdot \nabla u |\nabla u|^\alpha + (c(x) + \bar{\lambda} + \varepsilon) u^{1+\alpha} \leq -1 + \varepsilon u^{1+\alpha} \leq 0$$

and this contradicts the definition of $\bar{\lambda}$.

We have obtained that $|u_n|_\infty \rightarrow +\infty$. Then defining $w_n = \frac{u_n}{|u_n|_\infty}$ one has

$$\begin{cases} G(x, w_n, \nabla w_n, D^2 w_n) + \lambda_n w_n^{1+\alpha} = \frac{-1}{|u_n|^{1+\alpha}} & \text{in } \Omega \\ w_n = 0 & \text{on } \partial\Omega. \end{cases}$$

and then extracting as previously a subsequence which converges uniformly, one gets that there exists w , $|w|_\infty = 1$ and

$$\begin{cases} G(x, w, \nabla w, D^2 w) + \bar{\lambda} w^{1+\alpha} = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

The boundary condition is given by the uniform convergence. Clearly w is Hölder continuous, and if F satisfies the assumption (H7), then it is also locally Lipschitz continuous. \square

Remark 6. We have obtained that $\bar{\lambda}$ is also the supremum of the set

$$\{\lambda, \exists \phi > 0 \text{ on } \bar{\Omega}, G(x, \phi, \nabla \phi, D^2 \phi) + \lambda \phi^{1+\alpha} \leq 0 \text{ in the viscosity sense}\}.$$

Indeed, for $\lambda < \bar{\lambda}$ there exists v which is zero on the boundary, such that

$$G(x, v, Dv, D^2 v) + \lambda v^{1+\alpha} = -1.$$

Then using a continuity argument, one gets that for ϵ small enough $w = v + \epsilon$ is a supersolution of

$$G(x, w, Dw, D^2 w) + \lambda w^{1+\alpha} \leq -1/2.$$

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