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One-dimensional symmetry for solutions of Allen Cahn fully nonlinear equations.

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Abstract

This article presents some qualitative results for solutions of the fully nonlinear elliptic equation $F(\nabla u, D^2 u) + f(u) = 0$ in \mathbb{R}^N . Precisely under some assumptions on f , if $-1 \leq u \leq 1$ and $\lim_{x_1 \rightarrow \pm\infty} u(x_1, x') = \pm 1$ uniformly with respect to x' , then the solution depends only on x_1 .

1 Introduction

The sliding method was introduced in [6] by Berestycki and Nirenberg in order to prove monotonicity of solutions of

$$\Delta u + f(u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^N. \quad (1.1)$$

This powerful method uses two features of the Laplacian, comparison principle and invariance with respect to translation. The idea in general is the following: Fix any direction ν ; first slide of $t\nu$ the solution of (1.1) with $t \in \mathbb{R}$ large enough that the intersection of the slided domain with Ω is small enough or "narrow enough" for the maximum principle to hold in that intersection. Since the Laplacian is invariant by translation the slided solution satisfies the same equation then u and this allows to compare the values of the slided solution with the original solution. Then continue "sliding" i.e. decrease t until reaching a critical position.

Coupling simplicity with ductility, the sliding method of [6] has been incredibly influential, it is possible to count over two hundred citations of the work (e.g.

through google scholar). We shall here only recall the work by Berestycki, Hamel and Monneau [5] where the method is used to prove the so called Gibbons conjecture . This was simultaneously and independently solved by Barlow, Bass and Gui [3] and Farina [20]. Precisely in [5], the authors prove that if f is a $C^1([-1, 1])$ function decreasing near -1 and 1 , with $f(-1) = f(1) = 0$ (typically, $f(u) = u - u^3$) then the solutions of (1.1) in \mathbb{R}^N that converge uniformly to 1 or -1 at infinity in some fixed direction, say x_1 , are in fact one dimensional i.e functions of x_1 alone. In [5], the sliding method is coupled with a maximum principle (comparison principle) in unbounded domains contained in some cone. This equation is named after Allen-Cahn who used it [1] to describe the interfaces of gasses or solids.

As is well known the Gibbons conjecture is a weak form of the famous De Giorgi's conjecture which states that for $f(u) = u - u^3$, the level sets of monotone, entire solutions of (1.1) are hyperplanes for $N \leq 8$. This result has been proved in dimension 2 and 3 respectively by Ghoussoub and Gui [24] and by Ambrosio, Cabré [2], while Del Pino, Kowalczyk and Wei [18] have proved that it does not hold for $N > 8$ by constructing a counter example. Savin has proved the case $4 \leq N \leq 8$, with the further condition that the limit be ± 1 in a direction at infinity, in that case this condition is not assumed to be uniform with respect to the other variables. See also [29] for analogous results concerning the p -Laplacian.

In the present note for $F(\nabla u, D^2 u) := |\nabla u|^\alpha \tilde{F}(D^2 u)$ with $\alpha > -1$ and \tilde{F} uniformly elliptic (for precise assumptions see section 2) we prove an analogue of the results just discussed i.e.

Theorem 1.1 *Let f be defined on $[-1, 1]$, C^1 and such that f is nonincreasing near -1 and 1 , with $f(-1) = f(1) = 0$. Let u be a viscosity solution of*

$$F(\nabla u, D^2 u) + f(u) = 0 \quad \text{in } \mathbb{R}^N,$$

with values in $[-1, 1]$. Suppose that $\lim_{x_1 \rightarrow \pm\infty} u(x_1, x') = \pm 1$, uniformly with respect to x' . When $\alpha \neq 0$ we also suppose that for any $b < c$ there exists $m > 0$ such that $|\nabla u(x)| \geq m > 0$ in $[b, c] \times \mathbb{R}^{N-1}$ in the viscosity sense. Then u does not depend on x' i.e. $u(x_1, x') = v(x_1)$ where

$$\begin{cases} F(v'e_1, v''e_1 \otimes e_1) + f(v) = 0 & \text{in } \mathbb{R}, \\ |v| \leq 1, \quad \lim_{x \rightarrow \pm\infty} v = \pm 1 \end{cases} \quad (1.2)$$

and v is increasing.

Remark 1.2 *This theorem proves that in the case $\alpha > 0$, and for a large class of operators, there is no solution which satisfies the assumptions. Indeed, we shall prove in section 5 that every solution of*

$$-|u'|^\alpha u'' = f(u)$$

with $|u| \leq 1$, $\lim_{x \rightarrow \pm\infty} u = \pm 1$ and 1 and -1 are simple roots of f , cannot satisfy $u' > 0$ on \mathbb{R} .

Many remarks are in order. Let us note that in the case $\alpha \leq 0$, some recent regularity results [9] prove that locally Lipschitz solutions are in fact $\mathcal{C}^{1,\beta}$ for some $\beta < 1$, and this regularity is sufficient to prove the results enclosed here. For $\alpha > 0$ the \mathcal{C}^1 regularity is a consequence of the hypothesis on the positivity of the norm of the gradient.

A key ingredient in the proof of this result, which is of independent interest, is the following, strong comparison principle.

Proposition 1.3 *Suppose that Ω is some open set, and x_o, r such that $B(x_o, r) \subset \Omega$.*

Suppose that f is \mathcal{C}^1 on \mathbb{R} , and that u and v are, respectively, \mathcal{C}^1 bounded sub- and super-solutions of

$$F(\nabla w, D^2 w) + f(w) = 0 \quad \text{in } \Omega$$

such that $u \geq v$ and $\nabla v \neq 0$ (or $\nabla u \neq 0$) in $B(x_o, r)$, then, either $u > v$ or $u \equiv v$ in $B(x_o, r)$.

Observe that in Proposition 1.3 the condition that the gradient needs to be different from zero cannot be removed and this is why we need the condition in Theorem 1.1. Indeed, for any $m, k \in \mathbb{Z}$ with $k \leq m$ the functions

$$u_{k,m}(x) = \begin{cases} 1 & \text{for } x_1 \geq (2m+2)\pi \\ \cos x_1 & \text{for } (2k+1)\pi \leq x_1 \leq (2m+2)\pi \\ -1 & \text{for } x_1 \leq (2k+1)\pi \end{cases}$$

are viscosity solutions of

$$|\nabla u|^2(\Delta u) + (u - u^3) = 0,$$

and they are $\mathcal{C}^{1,\beta}$ for all $\beta < 1$.

Observe that e.g. $u_{0,0} \geq u_{0,i}$ for all $i \geq 1$ and $u_{0,0}(2\pi, y) = u_{0,i}(2\pi, y)$ but the functions don't coincide.

When $\alpha = 0$, De Silva and Savin in [19], have proved the analogue of De Giorgi's conjecture for uniformly elliptic operators in dimension 2. With f as above, they prove that if there exists a one dimensional monotone solution i.e. $g : \mathbb{R} \mapsto [-1, 1]$ such that $u(x) = g(\eta \cdot x)$ is a solution of

$$\tilde{F}(D^2u) + f(u) = 0 \quad \text{in } \mathbb{R}^2 \quad (1.3)$$

satisfying $\lim_{t \rightarrow \pm\infty} g(t) = \pm 1$ then, all monotone bounded solutions of (1.3) are one dimensional, i.e. their level sets are straight lines.

In the last section of this paper we prove the existence of one dimensional solutions in the case $\alpha \leq 0$ i.e. we prove that there exist solutions of (1.2). Precisely we give conditions on f that guarantee existence of solutions of the ODE

$$|u'|^\alpha \mathcal{M}_{a,A}^+(u'') + f(u) = 0$$

that satisfy $\lim_{x \rightarrow \pm\infty} u(x) = \pm 1$.

Observe for f satisfying the conditions of (1.1), in general, the solution of the ODE may not exist. Indeed, let $\tilde{F}(D^2u) = \mathcal{M}_{a,A}^+(D^2u)$ where, for any symmetric matrix M with eigenvalues e_i ,

$$\mathcal{M}_{a,A}^+(M) = a \sum_{e_i < 0} e_i + A \sum_{e_i > 0} e_i.$$

Then, as shown in the last section, for $a < A$ there are no one dimensional solutions of

$$\mathcal{M}_{a,A}^+(D^2u) + u - u^3 = 0,$$

that satisfy the asymptotic conditions.

While completing this work, we have received an interesting preprint by Farina and Valdinoci, [21], who treats Gibbons conjecture in a very general setting that includes the case $\alpha = 0$ in this note.

2 Assumptions and known results

In the whole paper we shall suppose the following hypotheses on the operator F .

Let S be the set of $N \times N$ symmetric matrices, and let $\alpha > -1$. Then F is defined on $\mathbb{R}^N \setminus \{0\} \times S$ by

$$F(p, M) = |p|^\alpha \tilde{F}(M), \quad (2.4)$$

where \tilde{F} satisfies

$$\tilde{F}(tM) = t\tilde{F}(M) \quad \text{for any } t \in \mathbb{R}^+, M \in S,$$

and there exist $A \geq a > 0$ such that for any M and any $N \in S$ such that $N \geq 0$

$$\text{atr}(N) \leq \tilde{F}(M + N) - \tilde{F}(M) \leq \text{Atr}(N). \quad (2.5)$$

Example 2.1 1) Let $0 < a < A$ and $\mathcal{M}_{a,A}^+(M)$ be the Pucci's operator $\mathcal{M}_{a,A}^+(M) = \text{Atr}(M^+) - \text{atr}(M^-)$ where M^\pm are the positive and negative part of M , and $\mathcal{M}_{a,A}^-(M) = -\mathcal{M}_{a,A}^+(-M)$. Then F defined as

$$F(p, M) = |p|^\alpha \mathcal{M}_{a,A}^\pm(M)$$

satisfies the assumptions.

2) Let B be a symmetric positive definite matrix then $F(p, M) = |p|^\alpha (\text{tr}(BM))$, is another example of operator satisfying the assumptions.

We now recall what we mean by viscosity solutions in our context :

Definition 2.1 Let Ω be a bounded domain in \mathbb{R}^N , let g be a continuous function on $\Omega \times \mathbb{R}$, then v , continuous on $\bar{\Omega}$ is called a viscosity super-solution (respectively sub-solution) of $F(\nabla u, D^2u) = g(x, u)$ if for all $x_0 \in \Omega$,

-Either there exists an open ball $B(x_0, \delta)$, $\delta > 0$ in Ω on which v is a constant c and $0 \leq g(x, c)$, for all $x \in B(x_0, \delta)$ (respectively $0 \geq g(x, c)$ for all $x \in B(x_0, \delta)$)

-Or $\forall \varphi \in \mathcal{C}^2(\Omega)$, such that $v - \varphi$ has a local minimum (respectively local maximum) at x_0 and $\nabla \varphi(x_0) \neq 0$, one has

$$F(\nabla \varphi(x_0), D^2 \varphi(x_0)) \leq g(x_0, v(x_0)).$$

(respectively

$$F(\nabla \varphi(x_0), D^2 \varphi(x_0)) \geq g(x_0, v(x_0)).$$

A viscosity solution is a function which is both a super-solution and a sub-solution.

Remark 2.2 When F is continuous in p , and $F(0, 0) = 0$, this definition is equivalent to the classical definition of viscosity solutions, as in the User's guide [15].

We now give a definition that will be needed in the statement of our main theorem.

Definition 2.3 We shall say that $|\nabla u| \geq m > 0$ in Ω in the viscosity sense, if for all $\varphi \in \mathcal{C}^2(\Omega)$, such that $u - \varphi$ has a local minimum or a local maximum at some $x_0 \in \Omega$,

$$|\nabla \varphi(x_0)| \geq m.$$

In our context, since the solutions considered have their gradient different from zero everywhere, the viscosity solutions can be intended in the classical meaning.

We begin to recall some of the results obtained in [8] which will be needed in this article.

Theorem 2.4 Suppose that c is a continuous and bounded function satisfying $c \leq 0$.

Suppose that f_1 and f_2 are continuous and bounded and that u and v satisfy

$$\begin{aligned} F(\nabla u, D^2 u) + c(x)|u|^\alpha u &\geq f_1 && \text{in } \Omega, \\ F(\nabla v, D^2 v) + c(x)|v|^\alpha v &\leq f_2 && \text{in } \Omega, \\ u &\leq v && \text{on } \partial\Omega. \end{aligned}$$

If $f_2 < f_1$ then $u \leq v$ in Ω . Furthermore, if $c < 0$ in Ω and $f_2 \leq f_1$ then $u \leq v$ in Ω .

Proposition 2.5 Suppose that \mathcal{O} is a smooth bounded domain. Let u be a solution of

$$F(\nabla u, D^2 u) \leq 0 \quad \text{in } \mathcal{O}. \tag{2.6}$$

If there exists some constant c_o , such that $u \geq c_o$ inside \mathcal{O} and $u(\bar{x}) = c_o$ with $\bar{x} \in \partial\mathcal{O}$, then either $u \equiv c_o$ in \mathcal{O} or

$$\liminf_{t \rightarrow 0^+} \frac{u(\bar{x} - t\vec{n}) - u(\bar{x})}{t} > 0,$$

where \vec{n} is the outer normal to $\partial\mathcal{O}$ at \bar{x} .

Remark 2.6 *In particular Proposition 2.5 implies that a non constant supersolution of (2.6) in a domain Ω has no interior minimum.*

If $c_o = 0$, the result can be extended in the following manner : Suppose that $\beta \geq \alpha$, that c is continuous and bounded, and u is a nonnegative solution of

$$F(\nabla u, D^2 u) + c(x)u^{1+\beta} \leq 0$$

then either $u \equiv 0$ or $u > 0$ in Ω . In that last case, if $u = 0$ on some point $x_o \in \partial\Omega$, then $\partial_{\bar{n}} u(x_o) < 0$.

We now recall the regularity results obtained in [9].

Theorem 2.7 *Suppose that Ω is a bounded C^2 domain and $\alpha \leq 0$. Suppose that g is continuous on $\Omega \times \mathbb{R}$. Then the bounded solutions of*

$$\begin{cases} F(\nabla u, D^2 u) = g(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.7)$$

satisfy $u \in C^{1,\beta}(\bar{\Omega})$, for some $\beta \in (0, 1)$.

Furthermore if Ω is a domain (possibly unbounded) of \mathbb{R}^N and if u is bounded and locally Lipschitz then $u \in C_{loc}^{1,\beta}(\Omega)$ for some $\beta \in (0, 1)$.

When $\alpha > 0$, C^1 regularity results are not known except for the one dimensional case or the radial case, however here, since the solutions that we consider have the gradient bounded away from zero, this regularity is just a consequence of classical results and a priori estimates. Indeed next theorem is just an application of Theorem 1.2 of [14], which in turn is the extension of Caffarelli's classical result:

Theorem 2.8 *Suppose that Ω is a (possibly unbounded) domain, and that g is C^1 and bounded. Let u be a bounded solution of*

$$F(\nabla u, D^2 u) = g(u) \quad \text{in } \Omega. \quad (2.8)$$

If $|\nabla u| \geq m > 0$ in Ω in the sense of Definition 2.3 then $\forall y, \forall \rho$ such that $B(y, \rho) \subset \Omega$, there exist $\beta \in (0, 1)$ and $C = C(a, A, N, |g(u)|_\infty, m)$ such that

$$\|u\|_{C^{1,\beta}(B(y, \frac{\rho}{2}))} \leq C \sup_{B(y, \rho)} |u|. \quad (2.9)$$

Proof. We introduce the operator:

$$G(v, \nabla v, D^2v) := \tilde{F}(D^2v) - g(v) \sup \left(|\nabla v|, \frac{m}{2} \right)^{-\alpha}.$$

If u is a solution of (2.8) such that in the viscosity sense $|\nabla u| \geq m > 0$, then it is a solution of

$$G(u, \nabla u, D^2u) = 0 \quad \text{in } \Omega.$$

Indeed, e.g. if $\varphi \in \mathcal{C}^2$ is such that $(u - \varphi)(x) \geq (u - \varphi)(\bar{x})$ for some $\bar{x} \in \Omega$, then $|\nabla \varphi|(\bar{x}) \geq m$ and

$$|\nabla \varphi|^\alpha(\bar{x}) \tilde{F}(D^2\varphi(\bar{x})) \geq g(u(\bar{x})) \Rightarrow \tilde{F}(D^2\varphi(\bar{x})) - |\nabla \varphi(\bar{x})|^{-\alpha} g(u(\bar{x})) \geq 0.$$

In order to apply Theorem 1.2 of [14], it is enough to remark that G does not depend on x and therefore the condition on the modulus of continuity is automatically satisfied.

Furthermore, the dependence on the gradient is Lipschitz, where the Lipschitz constant depends on m and $|g(u)|_\infty$. Applying Theorem 1.2 of [14] we have obtained the above estimate and $u \in C^{1,\beta}(\Omega)$. This ends the proof.

3 Comparison principles

As mentioned in the introduction, we begin by proving a strong comparison principle, that extends the one obtained in [9].

Proposition 3.1 *Suppose that Ω is some open subset of \mathbb{R}^N , f is \mathcal{C}^1 on \mathbb{R} . Let u and v be \mathcal{C}^1 bounded sub-solution and super-solution of*

$$F(\nabla u, D^2u) + f(u) = 0 \quad \text{in } \Omega.$$

Suppose that \mathcal{O} is some connected subset of Ω , with $u \geq v$ and $\nabla v \neq 0$ (or $\nabla u \neq 0$) on \mathcal{O} , then either $u > v$ or $u \equiv v$ in \mathcal{O} .

Remark 3.2 *Of course when $\alpha = 0$ the strong comparison principle is classical and holds without requiring that the gradient be different from zero.*

Proof of Proposition 3.1. We write the proof in the case $\alpha < 0$, the changes to bring when $\alpha > 0$ being obvious.

We argue as in [9]. Suppose that x_o is some point where $u(x_o) > v(x_o)$ (if such point doesn't exist we have nothing to prove).

Suppose by contradiction that there exists some point x_1 such that $u(x_1) = v(x_1)$. It is clear that it can be chosen in such a way that, for $R = |x_1 - x_o|$, $u > v$ in $B(x_o, R)$ and x_1 is the only point in the closure of that ball on which u and v coincide. Without loss of generality, one can also assume that $B(x_o, \frac{3R}{2}) \subset \mathcal{O}$.

We may suppose without loss of generality that v is the function whose gradient is bounded away from zero. Let then $L_1 = \inf_{B(x_o, \frac{3R}{2})} |\nabla v| > 0$, $L_2 = \sup_{B(x_o, \frac{3R}{2})} |\nabla v|$. We will prove that there exist two constants $c > 0$ and $\delta > 0$ such that

$$u \geq v + \delta(e^{-c|x-x_o|} - e^{-\frac{3cR}{2}}) \equiv v + w \quad \text{in} \quad \frac{R}{2} \leq |x - x_o| = r \leq \frac{3R}{2}.$$

This will contradict the fact that $u(x_1) = v(x_1)$.

Let $\delta \leq \min_{|x-x_o|=\frac{R}{2}} (u - v)$, so that

$$u \geq v + w \quad \text{on} \quad \partial \left(B(x_o, \frac{3R}{2}) \setminus \overline{B(x_o, \frac{R}{2})} \right).$$

Define

$$\gamma(x) = \begin{cases} \frac{f(u(x)) - f(v(x))}{u(x) - v(x)} & \text{if } u(x) \neq v(x) \\ f'(u(x)) & \text{if } u(x) = v(x). \end{cases}$$

Since f is \mathcal{C}^1 and the functions u and v are bounded, γ is continuous and bounded. We write

$$f(u) = \gamma(x)(u - v) + f(v),$$

$$F(\nabla u, D^2 u) - (|\gamma|_\infty + 1)(u - v) = -f(v) + (-\gamma - |\gamma|_\infty - 1)(u - v) \leq F(\nabla v, D^2 v).$$

We shall prove that, for c chosen conveniently,

$$F(\nabla v, D^2 v) < F(\nabla(v + w), D^2(v + w)) - (|\gamma|_\infty + 1)w,$$

this will imply that

$$F(\nabla u, D^2 u) - (|\gamma|_\infty + 1)u \leq F(\nabla(v + w), D^2(v + w)) - (|\gamma|_\infty + 1)(v + w).$$

Let φ be some test function for v from above, a simple calculation on w implies that, if $c \geq \frac{1}{a}(\frac{2(2A(N-1))}{R})$ then

$$\begin{aligned}
|\nabla\varphi + \nabla w|^\alpha & \cdot \tilde{F}(D^2\varphi + D^2w) - (|\gamma|_\infty + 1)w \\
& \geq |\nabla\varphi + \nabla w|^\alpha \tilde{F}(D^2\varphi) + |\nabla\varphi + \nabla w|^\alpha \mathcal{M}^-(D^2w) - (|\gamma|_\infty + 1)w \\
& \geq |\nabla\varphi + \nabla w|^\alpha \frac{F(\nabla\varphi, D^2\varphi)}{|\nabla\varphi|^\alpha} + \\
& \quad + |\nabla\varphi + \nabla w|^\alpha \frac{ac^2}{2} \delta e^{-cr} - (|\gamma|_\infty + 1)\delta e^{-cr}.
\end{aligned}$$

We also impose $\delta < \frac{RL_1e}{16}$ so that $|\nabla w| \leq \frac{|\nabla\varphi|}{8}$; then the inequalities

$$||\nabla\varphi + \nabla w|^\alpha - |\nabla\varphi|^\alpha| \leq |\alpha||\nabla w||\nabla\varphi|^{\alpha-1} \left(\frac{1}{2}\right)^{\alpha-1} \leq \frac{|\nabla\varphi|^\alpha}{2}$$

imply that

$$|\nabla\varphi + \nabla w|^\alpha \left(\tilde{F}(D^2\varphi + D^2w) \right) \geq -f(v) - |f(v)|_\infty |\nabla\varphi|^{-1} |\alpha| 2^{1-\alpha} c \delta e^{-cr} + L_2^\alpha \frac{ac^2}{4} \delta e^{-cr}.$$

It is now enough to choose

$$c \geq \frac{4A(N-1)}{R} + \frac{|\alpha||f(v)|_\infty 2^{2-\alpha}}{aL_2^{1+\alpha}} + \left(\frac{16(|\gamma|_\infty + 1)}{aL_2^\alpha} \right)^{\frac{1}{2}}$$

to finally obtain

$$|\nabla\varphi + \nabla w|^\alpha \tilde{F}(D^2\varphi + D^2w) - (|\gamma|_\infty + 1)w \geq f(v) + \frac{ac^2\delta L_2^\alpha e^{-cr}}{8} - (|\gamma|_\infty + 1)\delta e^{-cr}$$

i.e.

$$F(\nabla(v+w), D^2(v+w)) - (|\gamma|_\infty + 1)w > F(\nabla v, D^2v).$$

Hence the comparison principle, Theorem 2.4, gives that

$$u \geq v + w \quad \text{in } B(x_o, \frac{3R}{2}) \setminus \overline{B(x_o, \frac{R}{2})},$$

the desired contradiction. This ends the proof of Proposition 3.1.

From now f will denote a \mathcal{C}^1 function defined on $[-1, 1]$, such that $f(-1) = f(1) = 0$, and nonincreasing on the set $[-1, -1 + \delta] \cup [1 - \delta, 1]$ for some $\delta \in]0, 1[$.

Next is a comparison principle in unbounded domains that are "strip" like.

Proposition 3.3 *Suppose that u and v are \mathcal{C}^1 , have values in $[-1, 1]$ and are respectively sub and super solutions of*

$$F(\nabla w, D^2 w) + f(w) = 0 \quad \text{in } \mathbb{R}^N$$

with $F(\nabla u, D^2 u) \in L^\infty$, $F(\nabla v, D^2 v) \in L^\infty$. If $b, c \in \mathbb{R}$ are such that $b < c$, $\Omega = [b, c] \times \mathbb{R}^{N-1}$, $|\nabla u|$ and $|\nabla v| \geq m > 0$ and either $u \leq -1 + \delta$ or $v \geq 1 - \delta$ in Ω , then

$$u - v \leq \sup_{\partial\Omega} (u - v)^+.$$

Proof of Proposition 3.3.

Without loss of generality f can be extended outside of $[-1, 1]$ in order that f be still \mathcal{C}^1 , bounded, and nonincreasing after $1 - \delta$ and before $-1 + \delta$. Suppose, to fix the ideas, that $v \geq 1 - \delta$ in Ω .

We can also assume that $u \leq v$ on $\partial\Omega$. Indeed, since f is decreasing after $1 - \delta$, $w = v + \sup_{\partial\Omega} (u - v)^+$ is a super-solution which satisfies $F(\nabla w, D^2 w) \in L^\infty$. Suppose by contradiction that $\sup_{\Omega} (u - v) = \lambda$ for some $\lambda > 0$. Let then $(x^k)_k$ be some sequence such that $(u - v)(x^k) \rightarrow \lambda$. Eventually extracting from $(x^k)_k$ a subsequence, still denoted $(x^k)_k$, we have $x_1^k \rightarrow \bar{x}_1 \in [b, c]$. For any $x = (x_1, x')$ let

$$u^k(x_1, x') = u(x_1, x' + (x')^k)$$

and

$$v^k(x_1, x') = v(x_1, x' + (x')^k).$$

By the uniform estimates (2.9) in Theorem 2.8 one can extract from $(u^k)_k$ and $(v^k)_k$ some subsequences, denoted in the same way, such that $u^k \rightarrow \bar{u}$ and $v^k \rightarrow \bar{v}$ uniformly on every compact set of $[b, c] \times \mathbb{R}^{N-1}$ and \bar{u} and $\bar{v} + \lambda$ are solutions of

$$F(\nabla \bar{u}, D^2 \bar{u}) \geq -f(\bar{u}),$$

$$F(\nabla(\bar{v} + \lambda), D^2(\bar{v} + \lambda)) \leq -f(\bar{v}) \leq -f(\bar{v} + \lambda).$$

Furthermore, $\bar{u} \leq \bar{v} + \lambda$, and through the uniform convergence on the compact set $[b, c] \times \{0\}^{N-1}$, $\lim_k u^k(\bar{x}_1, 0) = \lim_k u^k(x_1^k, 0)$ and $\lim_k v^k(\bar{x}_1, 0) = \lim_k v^k(x_1^k, 0)$. This implies that

$$\begin{aligned} \bar{u}(\bar{x}_1, 0) &= \lim_k u(x_1^k, 0 + x'^k) \\ &= \lim_k v(x_1^k, 0 + x'^k) + \lambda = \bar{v}(\bar{x}_1, 0) + \lambda. \end{aligned}$$

Now using the fact that $|\nabla u| > m$ and $|\nabla v| > m$ on $[b, c] \times \mathbb{R}^{N-1}$, by passing to the limit one gets that $|\nabla \bar{u}| \geq m > 0$ and $|\nabla \bar{v}| \geq m$ on that strip, and the strong comparison principle in Proposition 3.1, implies that $\bar{u} \equiv \bar{v} + \lambda$.

On the other hand,

$$u(b, x' + x'^k) \leq v(b, x' + x'^k)$$

implies, by passing to the limit that

$$\bar{u}(b, x') \leq \bar{v}(b, x')$$

a contradiction.

4 Proof of the one dimensionality.

We are now in a position to prove the main result of this paper:

Proof of Theorem 1.1. We proceed analogously to the proof given in [5]. First observe that by Theorem 2.8 the solution u is in $\mathcal{C}_{loc}^{1,\beta}(\mathbb{R}^N)$, so that the condition on the gradient is pointwise and not only in the viscosity sense.

Let δ be such that f is nonincreasing on $[-1, -1 + \delta] \cup [1 - \delta, 1]$. Define

$$\Sigma_M^+ := \{x = (x_1, x') \in \mathbb{R}^N, x_1 \geq M\} \quad \text{and} \quad \Sigma_M^- := \{x = (x_1, x') \in \mathbb{R}^N, x_1 \leq M\}.$$

By the uniform behavior of the solution in the x_1 direction, there exists $M_1 > 0$ such that

$$u(x) \geq 1 - \delta \quad \text{in} \quad \Sigma_{M_1}^+, \quad u(x) \leq -1 + \delta \quad \text{in} \quad \Sigma_{(-M_1)}^-.$$

Fix any $\nu = (\nu_1, \dots, \nu_n)$ such that $\nu_1 > 0$ and let $u_t(x) := u(x + t\vec{\nu})$.

Claim 1 : For t large enough, $u_t \geq u$ in \mathbb{R}^N .

For $x \in \Sigma_{(-M_1)}^+$ and for t large enough, say $t > \frac{2M_1}{\nu_1}$,

$$u(x + t\vec{\nu}) \geq 1 - \delta \quad \text{and} \quad u^t \geq u \quad \text{on} \quad x_1 = -M_1.$$

We begin to prove that $u_t \geq u$ in $\Sigma_{(-M_1)}^+$.

Suppose by contradiction that $\sup_{\Sigma_{(-M_1)}^+} (u - u_t) = m_o > 0$.

Observe that since $\lim_{x_1 \rightarrow +\infty} u = \lim_{x_1 \rightarrow +\infty} u_t = 1$ uniformly, there exists M_2 such that for $x_1 > M_2 \geq -M_1$, $|u_t - u| < \frac{m_o}{2}$. Then $\sup_{\Sigma_{(-M_1)}^+} (u - u_t) = m_o$ is achieved inside $[-M_1, M_2] \times \mathbb{R}^{N-1}$.

On that strip, by hypothesis, there exists $m > 0$ such that $|\nabla u|, |\nabla u_t| \geq m$, and also $u_t \geq 1 - \delta$. Then one can apply the strong comparison principle in Proposition 3.3 with $b = -M_1$ and $c = M_2$ and obtain that

$$u - u_t \leq \sup_{\{x_1 = -M_1\} \cup \{x_1 = M_2\}} (u - u_t)^+ < \frac{m_o}{2},$$

a contradiction. Finally we have $u \leq u_t$ in $\Sigma_{(-M_1)}^+$.

We can do the same in $\Sigma_{\{-M_1\}}^-$ by observing that, in that case, $u \leq -1 + \delta$.

This ends the proof of Claim 1.

Let $\tau = \inf\{t > 0, \text{ such that } u_t \geq u \in \mathbb{R}^N\}$, by Claim 1, τ is finite.

Claim 2: $\tau = 0$.

To prove this claim, we argue by contradiction, assuming that it is positive. We suppose first that

$$\eta := \inf_{[-M_1, M_1] \times \mathbb{R}^{N-1}} (u_\tau - u) > 0,$$

and we prove then that there exists $\epsilon > 0$ such that $u_{\tau-\epsilon} \geq u$ in \mathbb{R}^N . This will contradict the definition of τ .

By the estimate (2.9) in Theorem 2.8, there exists some constant $c > 0$ such that for all $\epsilon > 0$

$$|u_\tau - u_{\tau-\epsilon}| \leq \epsilon c.$$

Choosing ϵ small enough in order that $\epsilon c \leq \frac{\eta}{2}$ and $\epsilon < \tau$, one gets that $u_{\tau-\epsilon} - u \geq 0$ on $\{x_1 = M_1\}$. The same procedure as in Claim 1 proves that the inequality holds in the whole space \mathbb{R}^N , a contradiction with the definition of τ .

Hence $\eta = 0$ and there exists a sequence $(x_j)_j \in \left([-M_1, M_1] \times \mathbb{R}^{N-1}\right)^{\mathbb{N}}$ such that

$$(u - u_\tau)(x_j) \rightarrow 0.$$

Let $v_j(x) = u(x + x_j)$ and $v_{j,\tau}(x) = u_\tau(x + x_j)$; these are sequences of bounded solutions, by uniform elliptic estimates (consequence of Theorem 2.8), one can extract subsequences, denoted in the same way, such that

$$v_j \rightarrow \bar{v} \quad \text{and} \quad v_{j,\tau} \rightarrow \bar{v}_\tau$$

uniformly on every compact set of \mathbb{R}^N . Moreover, \bar{v} and \bar{v}_τ are solutions of the same equation and $\bar{v} \geq \bar{v}_\tau$. Furthermore $\bar{v}(0) = \lim_{j \rightarrow +\infty} u(x_j) = \lim_{j \rightarrow +\infty} u_\tau(x_j) = \bar{v}_\tau(0)$ and

$$|\nabla \bar{v}|(0) = \lim_{j \rightarrow +\infty} |\nabla u(x_j)| \geq m$$

by the assumption on ∇u .

Since $|\nabla \bar{v}| > 0$ everywhere, by the strong comparison principle in Proposition 3.1, $\bar{v}_\tau = \bar{v}$ on any neighborhood of 0. This would imply that \bar{v} is τ periodic.

By our choice of M_1 , $\forall x \in \Sigma_{2M_1}^+$, $v_j(x) = u(x + x_j) \geq 1 - \delta$ and $\forall x \in \Sigma_{(-2M_1)}^-$, $v_j(x) = u(x + x_j) \leq -1 + \delta$, This contradicts the periodicity. Hence $\tau = 0$ and this ends the proof of Claim 2.

This implies that $\partial_{\vec{\nu}} u(x) \geq 0$, for all $x \in \mathbb{R}^N$ since for all $t > 0$, $u(x + t\vec{\nu}) \geq u(x)$ as long as $\nu_1 > 0$.

Take a sequence $\vec{\nu}_n = (\nu_{1,n}, \nu')$ such that $0 < \nu_{1,n}$ and $\nu_{1,n} \rightarrow 0$. Since u is \mathcal{C}^1 , by passing to the limit,

$$\partial_{\vec{\nu}'} u(x) \geq 0.$$

This is also true by changing $\vec{\nu}'$ in $-\vec{\nu}'$, so finally $\partial_{\vec{\nu}'} u(x) = 0$. This ends the proof of Theorem 1.1.

5 Existence's and non existence's results for the ODE.

5.1 The case $\alpha \leq 0$

In this section we prove that, when $\alpha \leq 0$, and some compatibility assumptions on f , the one dimensional problem (1.2) when \tilde{F} is one of the Pucci's operators, admits a solution. This solution is unique up to translation.

We introduce the function $f_{a,A}(t) = \begin{cases} \frac{f(t)}{A} & \text{if } f(t) > 0 \\ \frac{f(t)}{A} & \text{if } f(t) < 0 \end{cases}$. We have the following existence and uniqueness result :

Proposition 5.1 *Let $\alpha \in]-1, 0]$. Suppose that f is \mathcal{C}^1 on $[-1, 1]$ with $f(-1) = f(0) = f(1) = 0$, and :*

1. $f'(\pm 1) < 0$,
2. $\int_{-1}^1 f_{a,A}(s) ds = 0$,
3. for all $t \in (-1, 0]$, $\int_t^1 f_{a,A}(s) ds > 0$.

Then the equation

$$\begin{cases} |v'|^\alpha \mathcal{M}_{a,A}^\pm(v'') + f(v) = 0 & \text{in } \mathbb{R}, \\ |v| \leq 1, \quad \lim_{x \rightarrow \pm\infty} v = \pm 1 \end{cases} \quad (5.10)$$

admits a solution, unique up to translations and satisfies $v' > 0$ on \mathbb{R} .

The solution will be obtained using the existence of solution for some ODE, choosing the initial data for which the corresponding solution satisfies the required asymptotic behaviour. The Cauchy problem that we consider is

$$\begin{cases} -u''|u'|^\alpha = f_{a,A}(u), & \text{in } \mathbb{R} \\ u(0) = 0, u'(0) = \delta. \end{cases} \quad (5.11)$$

The equation in (5.11) is, of course, intended in the viscosity sense given in Definition 2.1 in the particular case of the dimension one.

It will be useful that f be defined on \mathbb{R} . We then extend f outside of $[-1, 1]$ so that $f \in \mathcal{C}^{0,1}(\mathbb{R})$, $f \geq 0$ on $(-\infty, -1)$, $f \leq 0$ on $[1, +\infty)$. Then the extension satisfies also for all $t \in \mathbb{R} \setminus \{\pm 1\}$

$$\int_t^1 f_{a,A}(s) ds > 0.$$

Let us observe that since we are in the one dimensional case, the equation can be written in divergence's form, and then solutions in the variational sense can be considered. More precisely we shall prove existence and uniqueness of weak -or variational- solutions (see Definition (5.2) below) and remark that they are viscosity solutions.

Definition 5.2 *A weak solution for (5.11) is a \mathcal{C}^1 function which satisfies in the distribution sense*

$$\begin{cases} -\frac{d}{dx}(|u'|^\alpha u') = (1 + \alpha)f_{a,A}(u) & \text{in } \mathbb{R} \\ u(0) = 0, u'(0) = \delta. \end{cases} \quad (5.12)$$

Remark 5.3 *Let us note that the condition 2 in Proposition 5.1 is necessary for the existence of weak solutions which satisfy*

$$\lim_{x \rightarrow +\infty} u(x) = 1, \quad \lim_{x \rightarrow -\infty} u(x) = -1.$$

Indeed, by continuity, u has a zero and without loss of generality we can suppose that it is in 0. Since the solution u is \mathcal{C}^1 , and bounded, there exists a sequence x_n

diverging to infinity such that $\lim_{x_n \rightarrow +\infty} u'(x_n) = 0$. In particular, multiplying the equation (5.12) by u' and integrating in $[0, x_n]$ and letting n to infinity we obtain

$$|u'(0)|^{2+\alpha} = -(2+\alpha) \int_0^1 \frac{f(s)}{a} ds$$

and similarly considering the solution on $]-\infty, 0]$

$$|u'(0)|^{2+\alpha} = (2+\alpha) \int_0^{-1} \frac{-f(s)}{A} ds = (2+\alpha) \int_{-1}^0 \frac{f(s)}{A} ds.$$

This implies 2.

Proposition 5.4 *For any $\alpha > -1$ there exists $\epsilon > 0$ such that (5.12) admits a solution in $(-\epsilon, \epsilon)$, and for $\alpha \leq 0$ that solution is unique.*

Proof. To prove existence and uniqueness observe that the equation (5.12) can be written, with $u = X$ and $Y = |u'|^\alpha u'$, under the following form

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} |Y|^{\frac{1}{\alpha+1}-1} Y \\ -(1+\alpha)f_{a,A}(X) \end{pmatrix} \quad (5.13)$$

with the initial conditions $X(0) = 0$, $Y(0) = |\delta|^\alpha \delta$. Moreover the map $(X, Y) \mapsto \begin{pmatrix} |Y|^{\frac{1}{\alpha+1}-1} Y \\ -(1+\alpha)f_{a,A}(X) \end{pmatrix}$ is continuous, and if $\alpha \leq 0$ it is Lipschitz continuous, while for $\alpha > 0$ it is Lipschitz continuous around 0, for $Y(0) \neq 0$. Now the result is just an application of the classical Cauchy Peano's Theorem for the existence's view point, and the Cauchy Lipschitz theorem, for the uniqueness result in the case $\alpha \leq 0$. This ends the proof.

Now the existence's result in Proposition 5.1 is a consequence of

Proposition 5.5 *Weak solutions of (5.12) are viscosity solutions of (5.11). When $\alpha \leq 0$ both notions are equivalent.*

Proof: Suppose that u is a solution of (5.12). It is clear that $|u'|^\alpha u'$ is \mathcal{C}^1 , hence if $u' \neq 0$, u' is \mathcal{C}^1 . Finally u is \mathcal{C}^2 on each point where the derivative is different from zero and on such a point the equation is $-|u'|^\alpha u'' = f(u(x))$ so u is a viscosity solution.

We now consider the case where u is constant on $]x_1 - \delta_1, x_1 + \delta_1[$ for some $\delta_1 > 0$. Then the "weak equation" provides $f(u(x_1)) = 0$, then $u(x_1) = 0, 1$ or -1 , and u is a viscosity solution.

We now assume that $\alpha \leq 0$ and recall that according to the regularity results in [10] applied in the one dimensional case, the solutions are \mathcal{C}^2 . We now prove that the viscosity solutions are weak solutions.

When $u'(x) \neq 0$ or when u is locally constant, it is immediate that u is a weak solution in a neighborhood of that point.

Since $u \equiv 0$, $u \equiv 1$ and $u \equiv -1$ are the unique constant solutions of both (5.12) and (5.11), we consider only the case where there exists some point x_1 with $u'(x_1) = 0$, and (without loss of generality), $1 > u(x_1) > 0$. Then, by continuity of u'' and using the equation, there exists $r > 0$ such that

$$u'' \leq 0 \quad \text{in} \quad (x_1 - r, x_1 + r).$$

Furthermore there exists $(x_n)_n$, such that $x_n \in (x_1 - r, x_1)$, $x_n \rightarrow x_1$ and $u'(x_n) \neq 0$; by the equation we obtain that

$$u''(x_n) < 0.$$

Finally, $u'(x) = \int_{x_1}^x u''(t)dt > 0$ for $x \in (x_1 - r, x_1)$. Similarly $u'(x) < 0$ for $x \in (x_1, x_1 + r)$.

Then u satisfies in a neighborhood of x_1 :

$$-\frac{d}{dx}(|u'|^\alpha u') = \frac{(1 + \alpha)f(u(x))}{a}.$$

This proves that u is a weak solution.

We now prove that, for some convenient choice of δ and for $\alpha \leq 0$, the solution of (5.12) provides the solution of (5.10).

In the following δ_1 will denote the positive real

$$\delta_1 = \left((2 + \alpha) \int_0^1 \frac{f(s)}{a} ds \right)^{\frac{1}{2+\alpha}}. \quad (5.14)$$

Proposition 5.6 *Suppose that $\alpha \leq 0$. Let u_δ be the unique solution of (5.11). Then for δ_1 defined in (5.14),*

1) If $\delta > \delta_1$, $|u_\delta(x)| \geq C|x|$ for $C = \delta^{2+\alpha} - \delta_1^{2+\alpha}$. In particular $\lim_{x \rightarrow \pm\infty} u_\delta(x) = \pm\infty$ and $u'_\delta > 0$.

- 2) If $\delta = \delta_1$, $u'_\delta > 0$ in \mathbb{R} and $\lim_{x \rightarrow +\infty} u_\delta(x) = 1$, $\lim_{x \rightarrow -\infty} u_\delta(x) = -1$.
3) If $-\delta_1 \leq \delta < \delta_1$ then $|u_\delta(x)|_\infty < 1$ for any $x \in \mathbb{R}$. The solution can oscillate.
4) If $\delta < -\delta_1$, u_δ is decreasing on \mathbb{R} , hence $u_\delta < 0$ on \mathbb{R}^+ , $u_\delta > 0$ on \mathbb{R}^- .

Remark 5.7 The case 2) in Proposition 5.6 is clearly false in the case $\alpha > 0$. As one can see with the example : $\alpha = 2$, $f(u) = u - u^3$, $u(x) = \sin x$, u satisfies $u'(0) = \delta_1 = 4 \int_0^1 f(s)ds$, but u' is not positive. This observation will be developed in the next subsection.

However it is not difficult to see that the conclusion in the other cases holds for any α .

Proof of Proposition 5.6.

1 & 4) To fix the ideas we suppose that $\delta > \delta_1$, the proof is identical in the case $\delta < -\delta_1$. For $x > 0$, since $u_\delta > 0$ one has

$$\begin{aligned} |u'_\delta|^{2+\alpha}(x) &= \delta^{2+\alpha} - (2+\alpha) \int_0^{u_\delta(x)} \frac{f(s)}{a} ds \\ &= \delta^{2+\alpha} - \delta_1^{2+\alpha} + (2+\alpha) \int_{u_\delta(x)}^1 \frac{f(s)}{a} ds \\ &\geq \delta^{2+\alpha} - \delta_1^{2+\alpha} := C. \end{aligned}$$

This proves, in particular, that $u'_\delta(x) \neq 0$ for all x and the Cauchy Lipschitz theorem ensures the local existence and uniqueness on every point, hence also the global existence. From this, we also derive that $u'_\delta > 0$ and for $x > 0$, $u_\delta(x) \geq Cx$, and symmetric estimates for $x < 0$ give $u_\delta(x) \leq Cx$.

2) If $\delta = \delta_1$ then $|u'_\delta|^{2+\alpha}(x) = (2+\alpha) \int_{u_\delta(x)}^1 \frac{f(s)}{a} ds > 0$. Suppose that there exists some point \bar{x} such that $u_\delta(\bar{x}) = 1$ then $u'_\delta(\bar{x}) = 0$. By the uniqueness of the solution $u_\delta(x) \equiv 1$, which contradicts the fact that $u'_\delta(0) = \delta_1 \neq 0$.

We have obtained that $u_\delta(x) < 1$ everywhere. Moreover u_δ is increasing and bounded then $\lim_{x \rightarrow +\infty} u'_\delta = 0$. By hypothesis 3. on f , this implies that $\lim_{x \rightarrow +\infty} u_\delta(x) = 1$.

3) Suppose that $0 < \delta < \delta_1$, and let θ^+ be such that

$$(2+\alpha) \int_0^{\theta^+} \frac{f(x)}{a} dx = \delta^{2+\alpha},$$

which exists by the mean value theorem. Either $u_\delta < \theta^+$ for all x , or there exists x_1 such that $u_\delta(x_1) = \theta^+$, and then $u'_\delta(x_1) = 0$. Let us note that $u = \theta^+$ on a

neighborhood of x_1 is not a solution since $f(\theta^+) \neq 0$. So u_δ is not locally constant and in particular, in a right neighborhood of x_1 :

$$\exists \varepsilon_o, \quad u''_\delta(x) \leq 0, \quad u''_\delta \not\equiv 0$$

for all $x \in (x_1, x_1 + \varepsilon_o)$, hence $u'_\delta(x) < 0$ in $(x_1, x_1 + \varepsilon_o)$.

So u is decreasing until it reaches a point where $u'_\delta(x_2) = 0$. Observe that by the equation

$$0 = |u'_\delta|^{2+\alpha}(x_2) = -(2+\alpha) \int_{\theta^+}^{u_\delta(x_2)} f_{a,A}(s) ds.$$

Hence $u(x_2) = \theta^- \in (-1, 0)$.

We can reason as above and obtain that u oscillates between θ^- and θ^+ .

5.2 The case $\alpha > 0$

Proposition 5.8 *Suppose that $\alpha > 0$ and $f \geq 0$ on $[0, 1]$, $f \leq 0$ on $[-1, 0]$, $f(1) = f(-1) = 0$ and $f'(\pm 1) \neq 0$ then the viscosity solutions to the ODE with asymptotic conditions:*

$$\begin{cases} |u'|^\alpha u'' = -f(u), & \text{in } \mathbb{R} \\ -1 \leq u \leq 1, & \lim_{x \rightarrow \pm\infty} u = \pm 1 \end{cases} \quad (5.15)$$

satisfy for some $\hat{x} < \tilde{x}$ in \mathbb{R}

$$u \equiv 1 \text{ on } [\tilde{x}, +\infty[\quad \text{and} \quad u \equiv -1 \text{ on }]-\infty, \hat{x}].$$

Remark 5.9 *The result still holds for the operator $|u'|^\alpha \mathcal{M}_{a,A}^\pm(u)$, the changes to bring being obvious.*

Proof. Suppose without loss of generality that $u(0) = 0$ and $u'(0) > 0$. We need to prove that there exists $x_1 > 0$ such that $u(x_1) = 1$ and $u'(x_1) = 0$ and there exists $x_2 < 0$ such that $u(x_2) = -1$ and $u'(x_2) = 0$.

By the assumptions on f , $u'' \leq 0$ on $[0, \infty[$ and then u' is decreasing on \mathbb{R}^+ hence it has a finite limit at $+\infty$. Since $\lim_{x \rightarrow \pm\infty} u = \pm 1$ this limit is zero. Then one can multiply by u' and integrate between x and $+\infty$, and get

$$|u'(x)|^{\alpha+2} = -F(u(x)) + F(1) \quad (5.16)$$

and then $\int_0^{u(x)} \frac{ds}{(F(1)-F(s))^{\frac{1}{\alpha+2}}} = x$.

Let us recall that $f(1) = 0$, $f'(1) < 0$ and then $F(1) - F(u) \sim (1 - u)^2$ near 1. Let $x_1 = \int_0^1 \frac{ds}{(F(1) - F(s))^{\frac{1}{\alpha+2}}}$, then x_1 is finite and is the first point on which $u(x_1) = 1$, furthermore by (5.16) $u'(x_1) = 0$. The construction of x_2 is analogous.

At this point it is clear that C^2 solutions will oscillate between -1 and 1 , so the only way to construct a viscosity solution of (5.15) that satisfies the asymptotic condition is to impose that on the right of some point \tilde{x} where the C^2 solution satisfies $u(\tilde{x}) = 1$ and $u'(\tilde{x}) = 0$ the solution take the value 1 and on the left of some other point $\hat{x} < \tilde{x}$ the solution takes the value -1 . It is easy to see that these are viscosity solutions.

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