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Regularity for radial solutions of degenerate fully nonlinear equations.

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Abstract

In this paper we prove the $\mathcal{C}^{1,\beta}$ regularity of radial solutions for fully nonlinear degenerate elliptic equations.

1 Introduction

In this paper, for f a continuous radial function, we prove the regularity of radial solutions of

$$F(x, \nabla u, D^2 u) = f,$$

when F is a fully nonlinear degenerate elliptic operator, homogenous of degree 1 in the Hessian and homogenous of some degree $\alpha > -1$ in the gradient, which is elliptic when the gradient is not null.

Precise conditions on F will be stated in the next section. For the moment we list some operators that belong to the class of operators we consider for the regularity results. Maybe the most typical is

$$F(\nabla u, D^2 u) = |\nabla u|^\alpha \mathcal{M}_{a,A}(D^2 u)$$

where $\mathcal{M}_{a,A}$ is one of the Pucci operators (i.e. either $\mathcal{M}_{a,A} = \mathcal{M}_{a,A}^+$ or $\mathcal{M}_{a,A} = \mathcal{M}_{a,A}^-$). So it is fully nonlinear in the Hessian, but "quasilinear" in the gradient.

Another important example is the $\alpha + 2$ -Laplacian i.e.

$$F(\nabla u, D^2 u) = \Delta_{\alpha+2} u = \operatorname{div}(|\nabla u|^\alpha \nabla u)$$

or, more in general,

$$F(\nabla u, D^2u) = |\nabla u|^\alpha (p_1 \text{tr}(D^2u) + p_2 \langle D^2u \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \rangle)$$

with $p_1 > 0$ and $p_1 + p_2 > 0$.

Before stating our main result let us mention the state of the art concerning regularity results of fully nonlinear equations. Of course it would be too long to state all that is known so we will just recall those closer to the scope of this paper.

We begin by recalling that when the operator F is fully nonlinear but uniformly elliptic and independent of the gradient, it is known (see e.g. Evans [11], Cabré, Caffarelli [7], [5], [8]) that u is $C^{1,\beta}$ for some $\beta \in (0, 1)$; when F is concave in the Hessian the solutions are $C^{2,\beta}$.

In the radial case, N. Nadirashvili, S. Vladut in [15] have proved C^2 regularity of radial solutions for a more general class of fully nonlinear operators uniformly elliptic. In particular for $N \leq 3$ the solutions are C^2 even without requiring the convexity assumption of the operator.

On the other hand in the quasi linear case, but when the operator is in divergence form, for example when F is the $(\alpha + 2)$ -Laplacian, the $C^{1,\beta}$ regularity of weak solutions was proved by Evans [12] and Di Benedetto [10]. This regularity will hold for radial viscosity solutions, since it is possible to prove that radial viscosity solutions and weak solutions coincide. The proof given here provides an elementary way to obtain the same result in the radial case.

Finally let us recall that in a previous paper [3] we proved that for F such that $F(\nabla u, D^2u) := |\nabla u|^\alpha \tilde{F}(D^2u)$ and for $\alpha \in (-1, 0]$ all solutions of

$$|\nabla u|^\alpha \tilde{F}(D^2u) = f(x), \tag{1.1}$$

are $C^{1,\beta}$.

(if \tilde{F} is concave we obtained that the solutions are $C^{2,\beta}$). In that case, differently from here, the operator is "more elliptic". Briefly speaking, the result is obtained by proving that the solution of (1.1), are in fact solutions of

$$\tilde{F}(D^2u) = f(x) |\nabla u|^{-\alpha}$$

in the L^p viscosity sense. This uses the Lipschitz regularity of u and a fixed point argument. Of course, this argument fails in the case $\alpha \geq 0$.

So that the case $\alpha \geq 0$ was completely open. Let us emphasize that regularity results for degenerate elliptic operators that are not in divergence form are in

general very difficult. The difficulty comes from the fact that difference of solutions are not sub or super solutions of some elliptic equation. As an example, let us recall that for the infinity Laplacian Δ_∞ the solutions are known to be in $\mathcal{C}^{1,\beta}$ for small $\beta > 0$ only in dimension 2, and only for $f \equiv 0$, see [13].

We now state the main result of this paper, which holds under the hypothesis described in section 2.

Theorem 1.1 *If Ω is either a ball or an annulus, any radial viscosity solution of*

$$F(x, \nabla u, D^2 u) = f(|x|) \text{ in } \Omega$$

is $C^1(\Omega)$.

Furthermore, suppose that either $N \leq 3$ or, for any $N > 3$, suppose that $M \mapsto F(x, p, M)$ is convex or concave, then u is $\mathcal{C}^{1, \frac{1}{1+\alpha}}$ everywhere and it is \mathcal{C}^2 on points where the derivative is different from zero.

Remark 1.2 *Observe that where the radial derivative is zero the Hölder continuity of the first derivative is optimal. Indeed it is easy to see that $u(x) = |x|^{\frac{\alpha+2}{\alpha+1}}$ is a viscosity solution of*

$$|\nabla u|^\alpha \mathcal{M}_{a,A}^+(D^2 u) = c$$

for $c = \left(\frac{\alpha+2}{\alpha+1}\right)^{\alpha+1} A\left(\frac{1}{1+\alpha} + N - 1\right)$.

Beside its intrinsic interest, the regularity question was raised naturally while proving the simplicity of the principal eigenfunctions : In recent years, the concept of principal eigenvalue has been extended to fully nonlinear operators, by means of the maximum principle (see [1]). The values

$$\lambda^+(\Omega) = \sup\{\lambda, \exists \phi > 0 \text{ in } \Omega, F(x, \nabla \phi, D^2 \phi) + \lambda \phi^{1+\alpha} \leq 0 \text{ in } \Omega\}$$

$$\lambda^-(\Omega) = \sup\{\lambda, \exists \psi < 0 \text{ in } \Omega, F(x, \nabla \psi, D^2 \psi) + \lambda |\psi|^\alpha \psi \geq 0 \text{ in } \Omega\}$$

are generalized eigenvalues in the sense that there exists a non trivial solution to the Dirichlet problem

$$F(\nabla \phi, D^2 \phi) + \lambda^\pm(\Omega) |\phi|^\alpha \phi = 0 \text{ in } \Omega, \phi = 0 \text{ on } \partial\Omega.$$

In [3], we proved that, for F satisfying (1.1), these eigenfunctions are simple as long as $\partial\Omega$ has only one connected component. This result extends to the situation where $\partial\Omega$ has at most two connected components when the dimension is 2, and

when \tilde{F} is convex or concave. In the first case, the proof uses the fact that the eigenfunctions are $\mathcal{C}^{1,\beta}$, in the second one, that they are \mathcal{C}^2 .

Of course we use viscosity solutions, and then, as can easily be imagined, the difficulties arise where the derivative is zero. So we will check that if $u' \neq 0$ in the viscosity sense at some point, then this holds in a neighborhood, and furthermore, in that neighborhood the solution is \mathcal{C}^1 . Then we treat the points where $u' = 0$ in the viscosity sense. The proof relies only on the comparison principle, Hopf principle, the regularity results of [9] and [6], together with some classical analysis.

The Hölder regularity of the derivative of the radial solution will require a more involved argument which is somehow less viscous and the proof is in section 4.

In the appendix we prove the equivalence of the notion of viscosity solutions.

After submitting this paper, we received a nice preprint from Imbert and Silvestre [14] where they prove interior Hölder regularity of the derivatives for any solution of (1.1) with $\alpha > 0$. Their technics is very different from our and uses an "improvement of flatness lemma" in order to estimate the oscillation of the gradient.

2 Hypothesis and known results.

In all the paper we suppose that either Ω is a ball or an annulus. We shall consider solutions of the following equation

$$F(x, \nabla u, D^2 u) = f(|x|). \quad (2.2)$$

The function f is supposed to be continuous on $[0, R]$ when $\Omega = B(0, R)$, on $[R_1, R_2]$ when $\Omega = B(0, R_2) \setminus \overline{B(0, R_1)}$.

The operator F is supposed to be continuous on $\Omega \times (\mathbb{R}^N)^* \times S$, where S is the space of $N \times N$ symmetric matrices and to satisfy:

(H1) For some $\alpha > -1$, for all $x \in \Omega$, for all $p \neq 0$ and $N \in S$ and for all $t \in \mathbb{R}$ and $\mu > 0$, $F(x, tp, \mu N) = |t|^\alpha \mu F(x, p, N)$.

(H2) F is fully nonlinear elliptic, i.e there exist some positive constants a and A , such that for any $M \in S$ and $N \geq 0$ in S , one has

$$a|p|^\alpha \text{tr}(N) \leq F(x, p, M + N) - F(x, p, M) \leq A|p|^\alpha \text{tr}(N).$$

(H3) F is an Hessian operator, i.e. for any $M \in S$ and $O \in \mathcal{O}(n, \mathbb{R})$, $F(Ox, {}^tOp, {}^tOMO) = F(x, p, M)$.

(H4) There exists $\nu > 0$ and $\kappa \in]1/2, 1]$ such that for all $|p| = 1$, $|q| \leq \frac{1}{2}$, $M \in S$

$$|F(x, p+q, M) - F(x, p, M)| \leq \nu |q|^\kappa |M|$$

which implies by homogeneity that for all $p \neq 0$, $|q| \leq \frac{|p|}{2}$, $M \in S$

$$|F(x, p+q, M) - F(x, p, M)| \leq \nu |q|^\kappa |p|^{\alpha-\kappa} |M|$$

We need to precise what we mean by viscosity solutions :

Definition 2.1 Let Ω be a domain in \mathbb{R}^N , let g be a continuous function on $\Omega \times \mathbb{R}$, then v , continuous in Ω is called a viscosity super-solution (respectively sub-solution) of $F(x, \nabla u, D^2u) = g(x, u)$ if for all $x_0 \in \Omega$,

- Either there exists an open ball $B(x_0, \delta)$, $\delta > 0$ in Ω on which v equals to a constant c and $0 \leq g(x, c)$, for all $x \in B(x_0, \delta)$ (respectively $0 \geq g(x, c)$ for all $x \in B(x_0, \delta)$).

- Or $\forall \varphi \in \mathcal{C}^2(\Omega)$, such that $v - \varphi$ has a local minimum (respectively local maximum) at x_0 and $\nabla \varphi(x_0) \neq 0$, one has $F(x_0, \nabla \varphi(x_0), D^2 \varphi(x_0)) \leq g(x_0, v(x_0))$. (respectively $F(x_0, \nabla \varphi(x_0), D^2 \varphi(x_0)) \geq g(x_0, v(x_0))$).

A viscosity solution is a function which is both a super-solution and a sub-solution.

Let us observe that in the case where $\alpha > 0$, the operator is well defined everywhere, and then it is a natural question to ask if the viscosity solutions in the sense above are the same as the viscosity solutions in the usual sense. The answer is yes as is proved in the appendix of this paper. However, the Definition 2.1 presents the advantage not to test points where the gradient of the test function is zero.

From now on we suppose that $\alpha \geq 0$ and that the solutions are "radial". Let us observe that the hypothesis (H3) that F be a fully nonlinear elliptic hessian operator, implies that there exists some operator H defined on $\mathbb{R}^+ \times \mathbb{R}^2$, such that if $u(x) = g(|x|)$ is radial and \mathcal{C}^2 , then, for $r = |x|$,

$$F(x, \nabla u, D^2u) = H(r, g'', g')$$

with

$$|g'|^\alpha \left(A \left((g'')^- + \frac{N-1}{r} (g')^- \right) + a \left((g'')^+ + \frac{N-1}{r} (g')^+ \right) \right)$$

$$\begin{aligned}
&\leq H(r, g'', g') \\
&\leq |g'|^\alpha \left(A \left((g'')^+ + \frac{N-1}{r} (g')^+ \right) \right. \\
&\quad \left. + a \left((g'')^- + \frac{N-1}{r} (g')^- \right) \right).
\end{aligned}$$

We now recall some known results concerning the operators considered :

Proposition 2.2 ([2]) *Suppose that Ω is a bounded open set. Suppose that f and g are continuous on $\overline{\Omega}$ such that $f \geq g$ and suppose that F satisfies hypothesis (H1) and (H2). Assume that β is some continuous and non decreasing function on \mathbb{R} , and that u and v are continuous respectively sub- and super-solutions of the equation*

$$\begin{aligned}
F(x, \nabla u, D^2 u) - \beta(u) &\geq f \text{ in } \Omega, \\
F(x, \nabla v, D^2 v) - \beta(v) &\leq g \text{ in } \Omega,
\end{aligned}$$

with $u \leq v$ on $\partial\Omega$.

If $f > g$ in Ω , or if $f \geq g$ but β is increasing, then $u \leq v$ in Ω .

We shall also need the Lipschitz regularity of the solutions:

Proposition 2.3 ([2]) *Suppose that Ω is an bounded regular domain of \mathbb{R}^N . Suppose that F satisfies (H1), (H2) and (H4). Let u be a $\mathcal{C}(\overline{\Omega})$ solution of*

$$\begin{cases} F(x, \nabla u, D^2 u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then u is Lipschitz continuous in Ω .

Remark 2.4 *We shall use this Proposition for radial solutions, so it is clear that the result holds when u is constant on the boundary, even if the constant is not zero.*

3 \mathcal{C}^1 regularity

In the sequel, since we only consider radial solution, we shall denote, for simplicity,

$$F[u](r) = H(r, u'', \frac{u'}{r})$$

where, with a classical abuse of notation, $u(x)$ will be identify with $u(r)$ for $r = |x|$. We will study solutions of $F[u](r) = f(r)$ on either the interval $[0, R)$ or the interval (R_1, R) .

The following two definitions will be useful.

Definition 3.1 For any $(p, q) \in \mathbb{R}^2$ and any $r \in [0, R]$, we define the paraboloid

$$w[p, q, r](s) = u(r) + p(s - r) + \frac{q}{2}(s - r)^2.$$

We also give the following

Definition 3.2 For a Lipschitz continuous function u , we define the following so called derivative numbers of u :

$$\lambda_g(r_1) = \liminf_{r \rightarrow r_1, r < r_1} \frac{u(r) - u(r_1)}{r - r_1},$$

$$\Lambda_g(r_1) = \limsup_{r \rightarrow r_1, r < r_1} \frac{u(r) - u(r_1)}{r - r_1},$$

$$\lambda_d(r_1) = \liminf_{r \rightarrow r_1, r > r_1} \frac{u(r) - u(r_1)}{r - r_1},$$

$$\Lambda_d(r_1) = \limsup_{r \rightarrow r_1, r > r_1} \frac{u(r) - u(r_1)}{r - r_1}.$$

(Here "g" stands for "gauche" and "d" stands for "droite"). Clearly for $r_1 = 0$, only the "right" derivatives are defined.

We shall say that $u'(\bar{r}) > 0$ in the viscosity sense (respectively $u'(\bar{r}) < 0$) if $\inf(\lambda_g(\bar{r}), \lambda_d(\bar{r})) > 0$ (respectively if $\sup(\Lambda_d(\bar{r}), \Lambda_g(\bar{r})) < 0$).

On the opposite we shall say that $u'(\bar{r}) = 0$ in the viscosity sense if one has

$$\lambda_g(\bar{r})\lambda_d(\bar{r}) \leq 0 \text{ and } \Lambda_d(\bar{r})\Lambda_g(\bar{r}) \geq 0.$$

Remark 3.3 Let us note first that for u solution of (2.2) all the numbers defined above exist and are finite since the solutions are Lipschitz; furthermore we proved in [4] that since f is bounded, $\Lambda_g \geq \lambda_d$, $\Lambda_d \geq \lambda_g$. Finally, if all these numbers coincide on \bar{r} , $u'(\bar{r})$ exists in the classical sense.

We begin with a lemma which legitimates the fact that radial viscosity solutions satisfy $u'(0) = 0$.

Lemma 3.4 *Suppose that u is a radial continuous viscosity solution of $F[u] = f$ on $[0, R)$, then $u'(0)$ exists and it is zero.*

Remark 3.5 *We want to point out that for radial function, i.e. for the continuous functions u defined on some ball of \mathbb{R}^N , such that there exists v continuous on $[0, r]$, with $u(x) = v(|x|)$, in order to test on points $x \neq 0$, it is sufficient to use test functions which are radial. A consequence of Lemma 3.4 and Definition 2.1 is that we do not need to test at the point zero. As a consequence u is a viscosity supersolution of $F[u] = f$ in $B(0, R)$ if and only if $u'(0)$ exists and is zero, and for all $r \neq 0$ and for all φ which is C^2 around $\bar{r} \neq 0$ which touches u by below on \bar{r}*

$$F[\varphi](\bar{r}) \leq f(\bar{r}).$$

Proof of Lemma 3.4. The scope is to prove that

$$\Lambda_d(0) = \lambda_d(0) = 0.$$

We suppose first that $\Lambda_d(0) > 0$. Let $m' < \Lambda_d(0)$ arbitrary close to it. Choose δ small enough in order that

$$a(m')^{1+\alpha} \frac{N-1}{\delta} > |f|_\infty.$$

For this choice of m' , there exists $r \in]0, \delta]$ such that

$$\frac{u(r) - u(0)}{r} \geq m' \text{ i.e. } u(r) \geq u(0) + m'r.$$

Let $w := w[m', 0, 0]$ so that $w(0) = u(0)$, $w(r) \leq u(r)$, and

$$F[w](s) \geq a(m')^{1+\alpha} \frac{N-1}{s} > f(s) \text{ on } (0, r].$$

Then, by Proposition 2.2, $w(s) \leq u(s)$, in $[0, r]$. Hence

$$\frac{u(s) - u(0)}{s} \geq m', \text{ and then } \lambda_d(0) \geq m'.$$

This implies, by the arbitrariness of m' , that $\Lambda_d(0) = \lambda_d(0)$.

We now suppose that $\Lambda_d(0) \leq 0$. If $\lambda_d(0) = 0$ then so does $\Lambda_d(0)$ and the thesis is proved, so we suppose that $\lambda_d(0) < 0$. We proceed with a proof which is similar to the one above.

Let $0 < m' < -\lambda_d(0)$ arbitrarily close to it. Choose δ as above, by hypothesis, there exists $r \in]0, \delta]$, such that

$$\frac{u(r) - u(0)}{r} \leq -m'.$$

Let $w \equiv w[-m', 0, 0]$, then $w(0) = u(0)$, $w(r) \geq u(r)$ and

$$F[w](s) \leq -a(m')^{1+\alpha} \frac{N-1}{s} \leq f(s) \text{ on } (0, \delta].$$

Then by Proposition 2.2, $w(s) \geq u(s)$ and then for all $s \in [0, r]$,

$$\frac{u(s) - u(0)}{s} \leq -m' \text{ and } \Lambda_d(0) \leq -m'.$$

By the arbitrariness of $-m'$ we obtain that $\Lambda_d(0) = \lambda_d(0)$. And the existence of the derivative at zero is proved.

We now prove that it is zero. Suppose by contradiction that it is not, one can suppose that it is positive, the other case can be done with obvious changes. Let δ be small enough that

$$\frac{a(u'(0))^{1+\alpha}}{\delta 2^{1+\alpha}} > f(r) \text{ for } r < \delta.$$

The function

$$\varphi(x) = u(0) + \frac{u'(0)}{2} x_1 + \frac{u'(0)}{4\delta} x_1^2.$$

touches u from below at zero and can be used as a test function and by definition of supersolution

$$\frac{a(u'(0))^{1+\alpha}}{\delta 2^{1+\alpha}} \leq F(0, \nabla\varphi(0), D^2\phi(0)) \leq f(0),$$

a contradiction with the choice of δ . This ends the proof.

We now state the main result of this section i.e. the first part of Theorem 1.1.

Theorem 3.6 *Suppose that u is a radial solution of $F[u] = f$ in Ω . Then u is $\mathcal{C}^1(\Omega)$.*

The proof of Theorem 3.6 relies on Proposition 3.7, Corollary 3.8 which give that the derivative exists for any r and that it is continuous when it is not zero. Finally Proposition 3.9 proves that u' is continuous even in points where it is not zero.

Proposition 3.7 *Suppose that u is a radial solution of $F[u] = f$ such that in \bar{r} one of the derivative numbers is different from zero.*

Then in a neighborhood of \bar{r} , u' exists in the classical sense and the function u' is continuous in \bar{r} .

Corollary 3.8 *If, at r_1 , one of the derivative numbers is zero, then $u'(r_1)$ exists and it is zero.*

Proof of Corollary 3.8. By Proposition 3.7 if one of the derivative number has a sign then $u'(r_1)$ exists and it is different from zero, which contradicts our hypothesis, so all four derivative numbers are zero and $u'(r_1) = 0$ in the classical sense.

Proof of Proposition 3.7. There are, in theory, 8 cases to treat, because each of the derivative numbers may be either positive or negative. But in fact considering the function $v = -u$, that satisfies

$$G[v] := -F[-v] = -f,$$

it is enough to consider only half of the cases.

What we want to prove is that, for any \bar{r} as in Proposition 3.7,

$$\exists \delta > 0, \lambda_d(r) = \lambda_g(r) = \Lambda_d(r) = \Lambda_g(r), \forall r \in (\bar{r} - \delta, \bar{r} + \delta). \quad (3.3)$$

Claim 1: $\Lambda_d(\bar{r}) = k < 0$ **implies the thesis (3.3).**

We first prove that

$$\Lambda_d(\bar{r}) < \mu < 0 \Rightarrow \exists \delta > 0, \sup(\lambda_d, \lambda_g, \Lambda_d, \Lambda_g)(r) < \mu < 0, \forall r \in (\bar{r} - \delta, \bar{r} + \delta). \quad (3.4)$$

Let $\mu' \in]\Lambda_d(\bar{r}), \mu[$, and let $\delta_1 > 0$ be small enough in order that $\frac{\mu'}{1 - \sqrt{\delta_1}} > \Lambda_d(\bar{r})$. Let $\delta_2 < \delta_1$ be small enough in order that

$$\frac{a|\mu|^{1+\alpha}}{(1 - \sqrt{\delta_2})\sqrt{\delta_2}} \geq |f|_\infty. \quad (3.5)$$

Let $\delta_3 < \frac{\bar{r}}{2}$ such that for $\delta < \inf(\delta_1, \delta_2, \delta_3)$

$$\frac{u(\bar{r} + \delta) - u(\bar{r})}{\delta} \leq \frac{\mu'}{1 - \sqrt{\delta_1}}.$$

By the continuity of u , for any fixed δ , there exists δ_4 such that for $r \in]\bar{r} - \delta_4, \bar{r} + \delta_4[$

$$\frac{u(r + \delta) - u(r)}{\delta} \leq \frac{\mu}{1 - \sqrt{\delta_1}}. \quad (3.6)$$

For any such r let

$$w := w\left(\mu, \frac{\mu}{(1 - \sqrt{\delta})\sqrt{\delta}}, r\right).$$

Then

$$w(r) = u(r), \quad w(r + \delta) \geq u(r + \delta),$$

and using (3.5) it is easy to check that w is a supersolution in $[r, r + \delta]$. From this, using Proposition 2.2, one gets that $w(s) \geq u(s)$ on $[r, r + \delta]$ and then

$$\frac{u(s) - u(r)}{s - r} \leq \mu$$

which implies that $\Lambda_d(r) \leq \mu$. Exchanging the roles of r and s one obtains also $\Lambda_g(r) \leq \mu$. This proves (3.4).

To complete the proof of **Claim 1** we prove that

$$0 > \Lambda_d(\bar{r}) > \mu \Rightarrow \exists \delta > 0, \quad \inf(\lambda_d, \lambda_g, \Lambda_d, \Lambda_g)(r) > \mu, \quad \forall r \in (\bar{r} - \delta, \bar{r} + \delta). \quad (3.7)$$

Indeed (3.4) and (3.7) imply the thesis (3.3).

The proof of (3.7) proceeds in a fashion similar to the proof of (3.4), we give the details of the computation for completeness sake. Let $\delta_1 < \inf(\frac{\bar{r}}{2}, 1)$ be small enough in order that $\frac{\mu}{1 + \sqrt{\delta_1}} < \Lambda_d(\bar{r})$, that

$$\frac{8A(N - 1)}{\bar{r}} < \frac{a}{\sqrt{\delta_1}} \quad \text{and that} \quad \frac{|\mu|^{1+\alpha} a}{2(1 + \sqrt{\delta_1})^{1+\alpha} \sqrt{\delta_1}} > |f|_\infty. \quad (3.8)$$

As above, there exists δ_4 such that for $\delta < \delta_1$ small enough and for $r \in]\bar{r} - \delta_4, \bar{r} + \delta_4[$,

$$\frac{u(r + \delta) - u(r)}{\delta} > \frac{\mu}{1 + \sqrt{\delta_1}}.$$

We define, on $[r, r + \delta]$,

$$w := w\left[\mu, \frac{|\mu|}{(1 + \sqrt{\delta})\sqrt{\delta}}, r\right]$$

then $w(r) = u(r)$, $w(r + \delta) \leq u(r + \delta)$ and, using (3.8), w is a sub-solution in $[r, r + \delta]$. Then

$$w(s) \leq u(s) \quad \text{on} \quad [r, r + \delta].$$

Finally one gets that for $r \in]\bar{r} - \delta_4, \bar{r} + \delta_4[$, and $s \in [r, r + \delta]$

$$u(s) \geq u(r) + \mu(s - r).$$

Hence

$$\lambda_d(r) \geq \mu.$$

Exchanging the roles of r and s , one gets $\lambda_g(r) \geq \mu$. This proves (3.7) and it ends the proof of **Claim 1**.

Claim 2: $\lambda_d(\bar{r}) = k > 0$ implies the thesis (3.3).

Indeed for $v = -u$,

$$\lambda_{d,u}(\bar{r}) = -\Lambda_{d,v}(\bar{r})$$

(here we have added in the index the function for which the derivative number is computed) so it is enough to use Claim 1.

The proofs of the following claims are similar to the proof of (3.4) or (3.7) but, of course, each case needs a different choice of parabola w , so we give the details for completeness sake.

Claim 3: $\Lambda_d(\bar{r}) = k > 0$ implies the thesis (3.3).

Suppose that $0 < \mu < \Lambda_d(\bar{r})$, and let $\delta_1 > 0$ such that $\frac{\mu^{1+\alpha} a}{(1-\sqrt{\delta_1})\sqrt{\delta_1}} \geq |f|_\infty$. Let $\delta_2 < \delta_1$ such that $\frac{\mu}{1-\sqrt{\delta_2}} < \Lambda_d(\bar{r})$. As before, let $\delta_3 < \inf(\delta_1, \delta_2)$ such that for $\delta < \delta_3$,

$$\frac{u(\bar{r} + \delta) - u(\bar{r})}{\delta} > \frac{\mu}{1 - \sqrt{\delta_2}}.$$

Finally let δ_4 such that, for $r \in [\bar{r} - \delta_4, \bar{r} + \delta_4]$,

$$\frac{u(r + \delta) - u(r)}{\delta} > \frac{\mu}{1 - \sqrt{\delta_2}}.$$

Define $w := w[\mu, \frac{\mu}{(1-\sqrt{\delta})\sqrt{\delta}}, r]$. Then $w(r) = u(r)$, $w(r + \delta) \leq u(r + \delta)$ and w is a sub-solution.

Hence, by Proposition 2.2, $w(s) \leq u(s)$ on $[r, r + \delta]$, which implies that for $r \in]\bar{r} - \delta_4, \bar{r} + \delta_4[$,

$$\inf(\Lambda_d, \Lambda_g, \lambda_g, \lambda_d)(r) \geq \mu > 0.$$

We are back to the hypothesis of **Claim 2**, which implies the thesis .

Claim 4: $\Lambda_g(\bar{r}) = k < 0$ implies the thesis (3.3).

Let μ be such that $0 > \mu > \Lambda_g(\bar{r})$. Let $\delta_1 < \frac{\bar{r}}{2}$, $\sqrt{\delta_1} < \inf(1, \frac{a\bar{r}}{8A(N-1)})$, $\Lambda_g(\bar{r}) < \frac{\mu}{1-\sqrt{\delta_1}}$, and such that

$$\frac{|\mu|^{1+\alpha} a}{2(1 - \sqrt{\delta_1})\sqrt{\delta_1}} > |f|_\infty.$$

As before there exists $\delta_4 < \delta_1$ such that for $r \in]\bar{r} - \delta_4, \bar{r} + \delta_4[$ and for $\delta < \delta_4$ one has

$$\frac{u(r - \delta) - u(r)}{-\delta} \leq \frac{\mu}{1 - \sqrt{\delta_1}}.$$

Then

$$w(s) := w[\mu, \frac{|\mu|}{(1 - \sqrt{\delta})\sqrt{\delta}}, r](s)$$

is a subsolution which satisfies $w(r) = u(r)$, $w(r - \delta) \leq u(r - \delta)$. Then, by Proposition 2.2, $w(s) \leq u(s)$ in $(r - \delta, r)$. This in turn implies that

$$\sup(\lambda_d, \lambda_g, \Lambda_d, \Lambda_g)(r) < \mu < 0.$$

We are once again in the hypothesis of **Claim 1** and we are done.

Claim 5: $\Lambda_g(\bar{r}) = k > 0$ implies the thesis (3.3).

Let μ be so that $\Lambda_g > \mu > 0$. Let δ_1 be such that $\delta_1 < \frac{\bar{r}}{2}$, $\frac{\mu}{1 - \sqrt{\delta_1}} < \Lambda_g$,

$$\sqrt{\delta_1} < \inf(1, \frac{a\bar{r}}{8A(N-1)}) \quad \text{and} \quad \frac{\mu^{1+\alpha}a}{2(1 - \sqrt{\delta_1})\sqrt{\delta_1}} \geq |f|_\infty.$$

As before for δ fixed, $\delta < \delta_1$, there exists some $\delta_4 < \delta_1$ such that for $r \in [\bar{r} - \delta_4, \bar{r} + \delta_4]$,

$$\frac{u(r - \delta) - u(r)}{-\delta} \geq \frac{\mu}{1 - \sqrt{\delta_1}}.$$

We define $w := w[\mu, \frac{-\mu}{(1 - \sqrt{\delta})\sqrt{\delta}}, r]$, then $w(r) = u(r)$, $w(r - \delta) \geq u(r - \delta)$ and by the assumptions w is a supersolution on $(r - \delta, r)$. Once more this implies that

$$\inf(\lambda_d, \lambda_g, \Lambda_d, \Lambda_g)(r) > \mu > 0, \quad \forall r \in (\bar{r} - \delta, \bar{r} + \delta).$$

And we conclude with **Claim 2**.

Again, using $v = -u$, the **Claims 3, 4 and 5** give that respectively $\lambda_d(\bar{r}) = k < 0$ or $\lambda_g(\bar{r}) = k < 0$ or $\lambda_g(\bar{r}) = k > 0$ imply the thesis (3.3).

Observe that in fact, (3.4) and (3.7) prove more than **Claim 1** because they imply that the derivative of u is continuous in points where the derivative is not zero. And this ends the proof.

We finally give the last step of the proof of Theorem 3.6:

Proposition 3.9 u' is continuous on points \bar{r} where $u'(\bar{r}) = 0$.

Proof. We treat separately the case $\bar{r} = 0$ and $\bar{r} \neq 0$. In the latter case, let

$$\epsilon > 0 \text{ and } \delta_1 < \inf\left(\frac{\bar{r}}{2}, \frac{a\epsilon^{1+\alpha}}{2^{1+\alpha}|f|_\infty}\right), \text{ such that } \frac{u(\bar{r} - \delta) - u(\bar{r})}{(-\delta)} \leq \frac{\epsilon}{4}$$

for $\delta < \delta_1$.

Fixing such δ , let $\delta_2 < \delta_1$ such that for $r \in [\bar{r} - \delta_2, \bar{r} + \delta_2]$, by continuity, $\frac{u(r-\delta)-u(r)}{(-\delta)} \leq \frac{\epsilon}{2}$. Then

$$u(r - \delta) \geq u(r) + \frac{\epsilon}{2}(-\delta).$$

Let $w := w[\epsilon, \frac{\epsilon}{2\delta}, r]$. Then $w(r) = u(r)$, $w(r - \delta) \leq u(r - \delta)$ and, by the hypothesis on δ , w is a sub-solution on $[r - \delta, r]$. Then $w(s) \leq u(s)$ for $r \in [\bar{r} - \delta_2, \bar{r} + \delta_2]$, and $s \in [r - \delta, r]$. By passing to the limit when s goes to r it gives

$$u'(r) \leq \epsilon.$$

We now prove that for all $\epsilon > 0$, there exists a neighborhood of \bar{r} where $u' \geq -\epsilon$.

Let, as above,

$$\delta_1 < \inf\left(\frac{\bar{r}}{2}, \frac{a\epsilon^{1+\alpha}}{2^{1+\alpha}|f|_\infty}\right),$$

such that for $\delta < \delta_1$, $\frac{-\epsilon}{4} \leq \frac{u(\bar{r}-\delta)-u(\bar{r})}{(-\delta)}$.

Fixing such δ , let $\delta_2 < \delta_1$ such that for $r \in [\bar{r} - \delta_2, \bar{r} + \delta_2]$, by continuity,

$$\frac{u(r - \delta) - u(r)}{(-\delta)} \geq \frac{-\epsilon}{2}.$$

Let $w := w(-\epsilon, \frac{-\epsilon}{2\delta}, r)$. Then $w(r - \delta) \geq u(r - \delta)$, $w(r) = u(r)$ and w is a super-solution on $[r - \delta, r]$. Using again Proposition 2.2

$$w(s) \geq u(s),$$

and passing to the limit $u'(r) \geq -\epsilon$.

We now consider the case where $\bar{r} = 0$. We want to prove the inequality $|u'| \leq \epsilon$ in a neighborhood of zero.

Take any $\epsilon > 0$, by Lemma 3.4 there exists $\delta_\epsilon > 0$ such that

$$\left| \frac{u(\delta) - u(0)}{\delta} \right| \leq \frac{\epsilon}{2}, \quad \forall \delta \in (0, \delta_\epsilon).$$

Let $\delta \leq \min(\frac{a\epsilon^{1+\alpha}(N-1)}{2|f|_\infty}, \delta_\epsilon)$, by the continuity of u , there exists $\delta_1 < \delta$ such that for $r \in [\delta, \delta + \delta_1]$,

$$\frac{u(r) - u(r - \delta)}{\delta} \geq -\epsilon.$$

Let us consider $w := w[-\epsilon, 0, r]$. Then $u(r) = w(r)$, $w(r - \delta) \geq u(r - \delta)$ and, with our choice of δ , w is a supersolution in $[r - \delta, r]$. By Proposition 2.2 we have obtained that

$$u(s) \leq w(s) \text{ on } [r - \delta, r]$$

and then

$$u'(r) \geq -\epsilon, \forall r \in [\delta, \delta + \delta_1].$$

In particular, $u'(\delta) \geq -\epsilon$ for any $\delta \leq \min(\frac{a\epsilon^{1+\alpha}(N-1)}{2|f|_\infty}, \delta_\epsilon)$.

In a similar fashion, for any $\delta \leq \min(\frac{a\epsilon^{1+\alpha}(N-1)}{2|f|_\infty}, \delta_\epsilon)$ and for all $r \in [\delta, \delta + \delta_1]$

$$u'(r) \leq \epsilon.$$

This is the desired result and it ends both the proof of Proposition 3.9 and the proof of Theorem 3.6.

4 $\mathcal{C}^{1,\beta}$ regularity.

We now state the main result of this section which is also the second part of Theorem 1.1.

Theorem 4.1 *Suppose that $N \leq 3$ or for any $N > 3$ that $M \mapsto F(x, p, \cdot)$ is convex or concave and that u is a radial solution of $F[u] = f$ in Ω . Then u is $\mathcal{C}^{1, \frac{1}{1+\alpha}}(\Omega)$ and it is \mathcal{C}^2 where the derivative is different from zero.*

The case where the derivative is different from zero is easy to treat:

Proposition 4.2 *Suppose that for $r_o > 0$, $u'(r_o) \neq 0$ in the viscosity sense. Then, on a neighborhood around r_o , u is $\mathcal{C}^{1,\beta}$ for some β , and if $N \leq 3$ or for any dimension when \tilde{F} is convex or concave, u is $\mathcal{C}^{2,\beta}$.*

Proof of Proposition 4.2. Observe that we now know that u' is continuous, so we can consider $\tilde{F}(x, D^2v) := F(x, \nabla u, D^2v)$ and clearly u is a solution of

$$\tilde{F}(x, D^2v) := f(x).$$

Observe that condition [(H4)] implies that we are in the hypothesis of [7] in $B_{r_o+\delta} \setminus \overline{B_{r_o-\delta}}$ for some $\delta > 0$. Hence u is $\mathcal{C}^{1,\beta}$ on that annulus. Similarly if F is concave or convex we are in the hypothesis of [11] and [7] and u is $\mathcal{C}^{2,\beta}$.

Note that when $N \leq 3$, the $\mathcal{C}^{2,\beta}$ regularity holds without the convexity or concavity assumption thanks to [15].

To prove the $\mathcal{C}^{1,\frac{1}{1+\alpha}}$ regularity result on any point, including those with the derivative equals to zero, we begin to establish a technical lemma.

Lemma 4.3 *Suppose that u is a radial \mathcal{C}^2 viscosity solution of (2.2), and that on $]r, s[, 0 < r < s, u' > 0$. Then*

$$|u'|^\alpha u'(s) \leq |u'|^\alpha u'(r) + (1 + \alpha) \int_r^s \epsilon_{a,A}(f(t)) dt \quad (4.9)$$

where $\epsilon_{a,A}(x) = \frac{x^+}{a} - \frac{x^-}{A}$; furthermore, for $\gamma = \frac{A}{a}(N-1)(1+\alpha)$

$$|u'|^\alpha u'(s) \geq \left(\frac{r}{s}\right)^\gamma |u'|^\alpha u'(r) - \frac{|f|_\infty(1+\alpha)s}{A(N-1)(1+\alpha) + A} \left(1 - \left(\frac{r}{s}\right)^{\gamma+1}\right). \quad (4.10)$$

If $u' < 0$ on $]r, s[, 0 < r < s$, then

$$|u'|^\alpha u'(s) \geq |u'|^\alpha u'(r) + (1 + \alpha) \int_r^s \epsilon_{A,a}(f(t)) dt \quad (4.11)$$

and

$$|u'|^\alpha u'(s) \leq \left(\frac{r}{s}\right)^\gamma |u'|^\alpha u'(r) + \frac{|f|_\infty(1+\alpha)s}{A(N-1)(1+\alpha) + a} \left(1 - \left(\frac{r}{s}\right)^{\gamma+1}\right). \quad (4.12)$$

Proof: We start by supposing that $u' > 0$ in (r, s) . Since u'' is continuous then $(u'')^{-1}(\mathbb{R}^+)$ is an open set of \mathbb{R}^+ and there exists a union of numerable open sets $\cup_{n \in \mathbb{N}}]r_n, r_{n+1}[$, with $]r, s[= \cup_{n \in \mathbb{N}}]r_n, r_{n+1}[$ where u'' is of constant sign on each interval $]r_n, r_{n+1}[$. By redefining in an obvious fashion the end points of the intervals one can suppose that $]r, s[= \cup_{n \in \mathbb{Z}}]r_n, r_{n+1}[$ with $u'' \geq 0$ on $[r_{2p}, r_{2p+1}]$ and $u'' \leq 0$ on $[r_{2p+1}, r_{2p+2}]$.

Then, in $[r_{2p}, r_{2p+1}]$,

$$a(u'' + \frac{(N-1)}{r}u')|u'|^\alpha \leq f(r) \leq A(u'' + \frac{(N-1)}{r}u')|u'|^\alpha \quad (4.13)$$

and, in $[r_{2p+1}, r_{2p+2}]$,

$$(Au'' + \frac{a(N-1)}{r}u')|u'|^\alpha \leq f(r) \leq (au'' + \frac{A(N-1)}{r}u')|u'|^\alpha. \quad (4.14)$$

We begin to prove (4.9) using the inequality on the left of f . In each case one can drop the term $|u'|^\alpha u'$, hence integrating and using $\frac{f}{A}, \frac{f}{a} \leq \epsilon_{a,A}(f)$, this imply (4.9) on $[r_{2p}, r_{2p+1}]$ and on $[r_{2p+1}, r_{2p+2}]$.

Let now P arbitrary large negative and N arbitrary large positive, r_P close to r and r_N close to s ,

$$\begin{aligned} |u'|^\alpha u'(r_N) &\leq |u'|^\alpha u'(r_{N-1}) + (1+\alpha) \int_{r_{N-1}}^{r_N} \epsilon_{a,A}(f(t)) dt \\ &\leq |u'|^\alpha u'(r_P) + (1+\alpha) \sum_P^N \int_{r_n}^{r_{n+1}} \epsilon_{a,A}(f(t)) dt \end{aligned}$$

and one obtains (4.9) by passing to the limit when P and N go respectively to $-\infty$ and $+\infty$.

We now prove (4.10). In $[r_{2p+1}, r_{2p+2}]$, since $u'' \leq 0$ we multiply the second inequality of (4.14) by $\frac{(1+\alpha)}{a} r^{\frac{A(N-1)(1+\alpha)}{a}} := \frac{(1+\alpha)}{a} r^\gamma$ and integrating one gets

$$r_{2p+2}^\gamma |u'|^\alpha u'(r_{2p+2}) \geq r_{2p+1}^\gamma |u'|^\alpha u'(r_{2p+1}) + \int_{r_{2p+1}}^{r_{2p+2}} \frac{(1+\alpha)}{a} f(t) t^\gamma dt.$$

Hence dividing by r_{2p+2}^γ , using $f \geq -|f|_\infty$ one gets

$$|u'|^\alpha u'(r_{2p+2}) \geq \left(\frac{r_{2p+1}}{r_{2p+2}} \right)^\gamma |u'|^\alpha u'(r_{2p+1}) - \frac{(1+\alpha)}{a} |f|_\infty \int_{r_{2p+1}}^{r_{2p+2}} \left(\frac{t}{r_{2p+2}} \right)^\gamma dt.$$

Similarly, if $u'' > 0$, multiplying (4.13) by $(1+\alpha)r^{(N-1)(1+\alpha)} := (1+\alpha)r^{\gamma_1}$ one gets

$$\begin{aligned} |u'|^\alpha u'(r_{2p+1}) &\geq \left(\frac{r_{2p}}{r_{2p+1}} \right)^\gamma |u'|^\alpha u'(r_{2p}) - \frac{(1+\alpha)}{A} |f|_\infty \int_{r_{2p}}^{r_{2p+1}} \left(\frac{t}{r_{2p+1}} \right)^{\gamma_1} dt \\ &\geq \left(\frac{r_{2p}}{r_{2p+1}} \right)^\gamma |u'|^\alpha u'(r_{2p}) - \frac{(1+\alpha)}{a} |f|_\infty \int_{r_{2p}}^{r_{2p+1}} \left(\frac{t}{r_{2p+1}} \right)^\gamma dt. \end{aligned}$$

We have used the fact that $\gamma_1 = (N-1)(1+\alpha) < (N-1)(1+\alpha)\frac{A}{a} := \gamma$ and $A(N-1)(1+\alpha) + A \geq A(N-1)(1+\alpha) + a$.

Using the same decomposition of $]r, s[= \cup_{n \in \mathbf{Z}}]r_n, r_{n+1}[$, with u'' of constant sign in each interval, for P large negative and N large positive, r_P close to r and r_N close to s , one has

$$\begin{aligned}
|u'|^\alpha u'(r_N) &\geq \left(\frac{r_{N-1}}{r_N}\right)^\gamma |u'|^\alpha u'(r_{N-1}) - \frac{|f|_\infty(1+\alpha)}{a} \int_{r_{N-1}}^{r_N} \left(\frac{t}{r_N}\right)^\gamma dt \\
&\geq \left(\frac{r_{N-2}}{r_{N-1}}\right)^\gamma \left(\frac{r_{N-1}}{r_N}\right)^\gamma |u'|^\alpha u'(r_{N-2}) \\
&\quad - \left(\frac{r_{N-1}}{r_N}\right)^\gamma \frac{|f|_\infty(1+\alpha)}{a} \int_{r_{N-2}}^{r_{N-1}} \left(\frac{t}{r_{N-1}}\right)^\gamma dt \\
&\quad - \frac{|f|_\infty(1+\alpha)}{a} \int_{r_{N-1}}^{r_N} \left(\frac{t}{r_N}\right)^\gamma dt \\
&= \left(\frac{r_{N-2}}{r_N}\right)^\gamma |u'|^\alpha u'(r_{N-2}) - \frac{|f|_\infty(1+\alpha)}{a} \int_{r_{N-2}}^{r_N} \left(\frac{t}{r_N}\right)^\gamma dt \\
&\geq \left(\frac{r_P}{r_N}\right)^\gamma |u'|^\alpha u'(r_P) - \frac{|f|_\infty(1+\alpha)}{a} \int_{r_P}^{r_N} \left(\frac{t}{r_N}\right)^\gamma dt.
\end{aligned}$$

By passing to the limit when P and N go to $-\infty$ and $+\infty$ one obtains (4.10).

The inequalities (4.11) and (4.12) can of course be proved either in the same manner or considering $v = -u$ as the solution of

$$G[v] = -f$$

and $G[v] = -F[-v]$ which possesses the same properties as F .

End of the proof of Theorem 4.1. We are left to prove that $u \in \mathcal{C}^{1, \frac{1}{1+\alpha}}$ in a neighborhood of r_1 such that $u'(r_1) = 0$.

Suppose first that $r_1 \neq 0$ and take $\frac{r_1}{2} < r < r_1$. We shall prove that

$$|u'|^{\alpha+1}(r) \leq \frac{2^{\gamma-1}(\gamma+1)|f|_\infty(1+\alpha)}{A}(r_1 - r). \quad (4.15)$$

For that aim, suppose first that $u'(r) > 0$ and let s be the first point between r_1 and r , so that $u'(s) = 0$. Then $u' > 0$ between s and r and inequality (4.10), with $\gamma = \frac{A(N-1)(1+\alpha)}{a}$, becomes

$$|u'|^\alpha u'(s) = 0 \geq \left(\frac{r}{s}\right)^\gamma |u'|^\alpha u'(r) - \frac{|f|_\infty(1+\alpha)s}{A(N-1)(1+\alpha) + a} \left(1 - \left(\frac{r}{s}\right)^{\gamma+1}\right)$$

and then

$$\begin{aligned}
|u'|^\alpha u'(r) &\leq \left(\frac{s}{r}\right)^\gamma \frac{|f|_\infty(1+\alpha)}{A(N-1)(1+\alpha)+a} \left(1 - \left(\frac{r}{s}\right)^{\gamma+1}\right) \\
&\leq (\gamma+1) \frac{s^{\gamma-1}}{r^{\gamma-1}} \frac{|f|_\infty(1+\alpha)}{A(N-1)(1+\alpha)+a} (s-r) \\
&\leq 2^{\gamma-1}(\gamma+1) \frac{|f|_\infty(1+\alpha)}{A(N-1)(1+\alpha)+a} (r_1-r).
\end{aligned}$$

From this one gets that

$$|u'(r)| \leq C(r_1 - r)^{\frac{1}{1+\alpha}}.$$

The case where $u'(r) < 0$ can be done similarly.

We now consider the right of r_1 . This proof still holds when $r_1 = 0$. Suppose $s > r_1$, we want to prove that

$$|u'|^{\alpha+1}(s) \leq \frac{(1+\alpha)|f|_\infty}{a}(s-r).$$

For that aim suppose that $u'(s) > 0$. Let r be the last point in $]r_1, s[$ such that $u'(r) = 0$. Then $u' > 0$ on $]r, s[$ and by (4.13)

$$|u'|^\alpha u'(s) \leq 0 + (1+\alpha) \int_r^s \epsilon_{a,A}(f).$$

This implies

$$|u'|^{\alpha+1}(s) \leq \frac{(1+\alpha)|f|_\infty}{a}(s-r) \leq \frac{(1+\alpha)|f|_\infty}{a}(s-r_1).$$

5 Appendix : The equivalence of definitions of viscosity solution in the case $\alpha > 0$.

We have the following equivalence result

Proposition 5.1 *If F satisfies (H1) and (H2) with $\alpha \geq 0$, the viscosity solutions in the classical meaning are the same as the viscosity solutions in the sense of Definition 2.1.*

Proof of Proposition 5.1.

We assume that u is a supersolution in the sense of Definition 2.1 and we want to prove that it is a supersolution in the classical sense. We suppose that for $x_o \in \Omega$ there exists $M \in S$ such that

$$u(x) \geq u(x_o) + \frac{1}{2} \langle M(x - x_o), (x - x_o) \rangle + o(|x - x_o|^2) := \phi(x). \quad (5.16)$$

Let us observe first that one can suppose that M is invertible, since if it is not, it can be replaced by $M_n = M - \frac{1}{n}I$ which satisfies (5.16) and tends to M . Without loss of generality we will suppose that $x_o = 0$.

Let $k > 2$ and $R > 0$ such that

$$\inf_{x \in B(0, R)} \left(u(x) - \frac{1}{2} \langle Mx, x \rangle + |x|^k \right) = u(0)$$

where the infimum is strict. We choose $\delta < R$ such that $(2\delta)^{k-2} < \frac{\inf_i |\lambda_i(M)|}{2k}$ where $\lambda_i(M)$ are the eigenvalues of M .

Let ϵ be such that

$$\inf_{|x| > \delta} \left(u(x) - \frac{1}{2} \langle Mx, x \rangle + |x|^k \right) = u(0) + \epsilon$$

and let $\delta_2 < \delta$ and such that $k(2\delta)^{k-1}\delta_2 + |M|_\infty(\delta_2^2 + 2\delta_2\delta) < \frac{\epsilon}{4}$. Then, for x such that $|x| < \delta_2$,

$$\begin{aligned} \inf_{|y| \leq \delta} \left\{ u(y) - \frac{1}{2} \langle M(y - x), y - x \rangle + |y - x|^k \right\} &\leq \inf_{|y| \leq \delta} \left\{ u(y) - \frac{1}{2} \langle My, y \rangle + |y|^k \right\} + \frac{\epsilon}{4} \\ &= u(0) + \epsilon/4 \end{aligned}$$

and on the opposite

$$\begin{aligned} \inf_{|y| > \delta} \left\{ u(y) - \frac{1}{2} \langle M(y - x), y - x \rangle + |y - x|^k \right\} &\geq \inf_{|y| > \delta} \left\{ u(y) - \frac{1}{2} \langle My, y \rangle + |y|^k \right\} - \frac{\epsilon}{4} \\ &> u(0) + \frac{3\epsilon}{4}. \end{aligned}$$

Since the function u is supposed to be non locally constant, there exist x_δ and y_δ in $B(0, \delta_2)$ such that

$$u(x_\delta) > u(y_\delta) - \frac{1}{2} \langle M(x_\delta - y_\delta), x_\delta - y_\delta \rangle + |x_\delta - y_\delta|^k$$

and then the infimum $\inf_{y, |y| \leq \delta} \{u(y) - \frac{1}{2} \langle M(x_\delta - y), x_\delta - y \rangle + |x_\delta - y|^k\}$ is achieved on some point z_δ different from x_δ . This implies that the function

$$\varphi(z) := u(z_\delta) + \frac{1}{2} \langle M(x_\delta - z), x_\delta - z \rangle - |x_\delta - z|^k - \frac{1}{2} \langle M(x_\delta - z_\delta), x_\delta - z_\delta \rangle + |x_\delta - z_\delta|^k$$

touches u by below at the point z_δ . But

$$\nabla \varphi(z_\delta) = M(z_\delta - x_\delta) - k|x_\delta - z_\delta|^{k-2}(z_\delta - x_\delta),$$

cannot be zero since, if it was, $z_\delta - x_\delta$ would be an eigenvector for the eigenvalue $k|x_\delta - z_\delta|^{k-2}$ which is supposed to be strictly less than the modulus of any eigenvalue of M .

We have obtained that $\nabla \varphi(z_\delta) \neq 0$ and then, since u is a supersolution in the sense of Definition 2.1,

$$F(z_\delta, M(z_\delta - x_\delta) - k|x_\delta - z_\delta|^{k-2}(z_\delta - x_\delta), M - \frac{d^2}{dz^2}(|x_\delta - z|^k)(z_\delta)) \leq g(z_\delta, u(z_\delta)).$$

By passing to the limit we obtain

$$0 \leq g(0, u(0)),$$

which is the desired conclusion.

Of course we can do the same for sub-solutions.

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