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EXISTENCE OF BOUNDARY BLOW UP SOLUTIONS FOR SINGULAR OR DEGENERATE FULLY NONLINEAR EQUATIONS

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ABSTRACT. We prove here the existence of boundary blow up solutions for fully nonlinear equations in general domains, for a nonlinearity satisfying Keller-Osserman type condition. If moreover the nonlinearity is non decreasing, we prove uniqueness for boundary blow up solutions on balls for operators related to Pucci's operators.

1. Introduction. The existence of "large" solutions for the partial differential equation

$$\Delta u = f(u)$$

on bounded domains has been studied by many authors. "Large solution", or boundary blow up solution, means a solution which is infinite on the boundary of the domain. Throughout this article we refer to such solutions as BBUS. We emphasize the interest of BBUS for nonlinear partial differential equations that satisfy some comparison or maximum principle; actually BBUS provide upper bound for any solution of the PDE on a given bounded domain, regardless of boundary conditions.

The existence of such solutions is linked to whether or not f satisfies some property known as the "Keller Osserman condition". More precisely suppose that F is the primitive of $f \geq 0$, which is zero on zero, and define $\phi(a) = \int_a^\infty \frac{1}{\sqrt{2(F(s)-F(a))}} ds$, then f is said to satisfy the Keller Osserman condition if there exists some $a > 0$ for which $\phi(a) < \infty$.

There is a great literature on the subject, and the first papers on that question considered particular case of function f : In 1916, Bieberbach [6] solved the problem for $f(u) = e^u$ and $\Omega \subset \mathbb{R}^2$. In 1940, Rademacher extended its result to higher dimension. Wittich established the connection with the existence of entire solutions for the equation $\Delta u = e^u$. In 1957, Keller [28] and Osserman [36] proved

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existence's results for more general nonlinearity f . In 1974 Loewner and Nirenberg [30] investigated the connection of the problem with differential geometry.

A series of papers bring different contributions to the problem : in [3] the authors consider more general nonlinearity of the form $f(x, u)$ in place of f . In [39] Veron proved similar result by a new method. The authors in [3] specify the blow up behavior of the function and later also those of the gradient. Uniqueness and asymptotic behavior in non smooth domains is studied in the papers of Marcus and Veron [31], [32]. In [33] the author considers the same problem with the p -Laplace operator.

We want to point out that most of the authors which worked on this subject until 2007 assumed that the function f is non decreasing. In particular this assumption is sufficient to ensure the uniqueness for classical problems with Dirichlet boundary conditions, and is a crucial point for some of the results obtained. In 2007, Dumont, Dupaigne, Goubet and Radulescu in [25] consider the case where f is only C^1 , $f \geq 0$, and $f(0) = 0$. They introduced the concept of minimal solution, which replaces in some way the uniqueness. Furthermore they prove that if f satisfies the Keller Osserman condition (there exists a such that $\phi(a) < \infty$) then there exists some ball on which there exists a blow up (large) solution. They extend the existence's result to general regular domains and provide a rate of blow up of the solutions obtained, as well as a numerical method, and numerical calculations. In [21] the author extends some of the results of [25] to the p -Laplace operator. Costin and Dupaigne in [19], provide a more precise description of properties of BBUS, for the asymptotic behavior, uniqueness, and symmetry properties while the domain is a ball. See also [20] for miscellaneous uniqueness results.

Here we treat the case of fully nonlinear equations singular or degenerate. Since these equations are not in divergence form, we need to use the concept of viscosity solutions. So in general the variational technics cannot be employed. In a recent paper [22] Davila, Felmer and Quaas, and in [26], Esteban, Felmer and Quaas consider the case where $f(u) = -f + |u|^{s-1}u$ and $s > 1 + \alpha$, which can be seen as a particular case where $u \mapsto f(u)$ is increasing. To our knowledge, in any other situations, nothing has been done for these operators in the direction of the present paper. We attack here the existence of BBUS for fully non linear elliptic equations, without assuming that the function f is non decreasing. Then we discuss the uniqueness result assuming that f is increasing. We provide also the behavior of BBUS near the boundary in an implicit form.

Assume in the sequel that $N \geq 2$. We consider the problem (P)

$$\begin{cases} H(\nabla u, D^2u) = f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where the operator H is supposed to be continuous on $\mathbb{R}^N \setminus \{0\} \times S$, -where S is the space of symmetric matrix on \mathbb{R}^N -, and to satisfy for some $\alpha \in]-1, \infty[$

- (H1) H satisfies for all $p \neq 0$ in \mathbb{R}^N and $M \in S$, and for all $t \neq 0$, $\mu > 0$

$$H(tp, \mu M) = |t|^\alpha \mu H(p, M)$$

- (H2) There exist some constants $0 < a < A$, such that for all M and $N \geq 0$ in S , and for all $p \neq 0$

$$a|p|^{\alpha \text{tr}(N)} \leq H(p, M + N) - H(p, M) \leq A|p|^{\alpha \text{tr}(N)}.$$

We shall sometimes assume in addition that

- (H3) $H(p, M) = |p|^\alpha \tilde{H}(M)$, where \tilde{H} is invariant under rotation, i.e. is such that for any orthogonal matrix O , then $\tilde{H}({}^tOMO) = \tilde{H}(M)$.

The function f is supposed to be non negative, C^1 and such that $f(0) = 0$, and $f(r) > 0$ for $r > 0$. The regularity assumption $f \in C^1$ can be relaxed in some places. These assumptions, together with a suitable Keller-Osserman condition (see below), allow us to prove the existence of BBUS. We assume that the function f is non decreasing for uniqueness results. The solutions that we consider are intended in the viscosity sense. We recall that notion in the second section.

Remark 1. When \tilde{H} satisfies (H1), (H2), (H3), due to the rotational invariance, $\tilde{H}(M)$ depends only on the eigenvalues of M . For instance while $u(x) = g(r)$ is radial, the eigenvalues of D^2u are $\frac{g'(r)}{r}$ with the multiplicity $N - 1$ and g'' of multiplicity one. Then the hypothesis on \tilde{H} imply that $\tilde{H}(D^2u)$ is bounded by below by

$$A \left((g'')^- + \frac{N-1}{r} (g')^- \right) + a \left((g'')^+ + \frac{N-1}{r} (g')^+ \right),$$

and by above by

$$A \left((g'')^+ + \frac{N-1}{r} (g')^+ \right) + a \left((g'')^- + \frac{N-1}{r} (g')^- \right).$$

This fact concerning radial solutions will be used in the sequel without notice.

Remark 2. Introduce a and A that are some numbers such that $0 < a < A$. Let $M_{a,A}^+(N) = Atr(N^+) - atr(N^-)$ and $M_{a,A}^-(N) = atr(N^+) - Atr(N^-)$ be the Pucci's operators, where $N = N^+ - N^-$ is the minimal decomposition of the symmetric matrix N into the difference of two positive symmetric matrices. Then the operator $H^\pm(p, M) = |p|^\alpha M_{a,A}^\pm(M)$ satisfies the above assumptions. In the sequel we refer to these operators as operators related to Pucci's operator, or, for shorthand, Pucci's operators. The operators which satisfy (H1)-(H2) are sandwiched between the operators H^+ and H^- . This will allow us to prove estimate on solutions to fully non linear equations from equations involving Pucci's operators. In the sequel we shall denote for simplicity $H^\pm[u] \equiv |\nabla u|^\alpha M_{a,A}^\pm(D^2u)$.

Remark 3. The class of operators related to Pucci's operators, or more generally operators that satisfy (H1), (H2), (H3) and

- (H4) \tilde{H} is convex or concave,

ensure more regularity for viscosity solutions (see the definition below). This assumption is not needed for the existence result of BBUS; nevertheless, we shall use in the sequel this property of Pucci's operators to ensure that radial solutions on small balls are smooth enough. The assumption (H4) is a sufficient condition to get the C^2 regularity of the solutions on points where the gradient is different from zero. For the sake of completeness, let us point out that in [17] the authors present a more general class of operators for which the C^2 regularity of the solutions hold. Let us cite also a recent paper of Nadirashvili and Vladut [34] which prove that for operators Hessian (which satisfy $\tilde{H}({}^tOMO) = \tilde{H}(M)$) and for $N \leq 3$ the axially symmetric solutions are C^2 .

As in the case of the Laplacian, the existence of solutions for (P) is related to some condition on the primitive F of f which is zero on 0. This condition is the following : Suppose that $F(x) = \int_0^x f(s)ds$ and define

$$\Phi(\beta) = \int_{\beta}^{\infty} \frac{ds}{(F(s) - F(\beta))^{\frac{1}{2+\alpha}}} \quad (1)$$

Definition 1. We shall say that f satisfies the Keller Osserman condition whenever there exists β such that $\Phi(\beta) < \infty$.

Definition 2. We shall say that f satisfies the sharpened Keller Osserman condition whenever $\liminf_{\beta \rightarrow +\infty} \Phi(\beta) = 0$.

In this paper we shall prove the following theorems. The first one is concerned with some existence result.

Theorem 1.1. Suppose that f is C^1 on \mathbb{R}^+ , $f(0) = 0$, $f(r) > 0$ for $r > 0$. The following assertions are equivalent

1. f satisfies the Keller Osserman condition.
2. For any operator H which satisfies (H1) and (H2), there exists some ball where there exists a boundary blow up solution.
3. f satisfies the sharpened Keller Osserman condition.
4. For all operators H which satisfy (H1) and (H2), there exists a boundary blow up solution on all arbitrary smooth bounded domain.

The second one describes a uniqueness result.

Theorem 1.2. Consider the particular case $|\nabla u|^\alpha \Delta u = f(u)$ on a ball. Assume that f is non decreasing, C^1 and satisfies the Keller Osserman condition. Then there exists a unique BBUS for this problem.

Remark 4. It is worth to point out that the statement of Theorem 1.2 is valid for Pucci's operator H^+ and H^- . In this case, since the function f is non decreasing, then $H^+[u] = f(u)$ (respectively $H^-[u] = f(u)$) amounts to solve $A|\nabla u|^\alpha \Delta u = f(u)$ (respectively $a|\nabla u|^\alpha \Delta u = f(u)$). It is also worth to emphasize that classical results in the literature are proved under the extra assumption that $f(u)/u$ is increasing (or f is convex) in a neighborhood of ∞ . The first result removing this assumption appears in [19].

This paper is organized as follows : in the second section we recall some basic tools for the existence of viscosity solutions for such partial differential equations, in section 3 we give some local existence's and uniqueness result for radial solutions of $H[u] = f(u)$, together with some qualitative results. Section 4 is devoted to the proof of Theorem 1.1. In section 5 we study the blow up rate of convergence of blow up solutions, under some additional assumption on f , which will be specified in the sequel. In section 6 we prove that when f is non decreasing there is uniqueness of Boundary Blow Up Solutions while the domain is a ball.

2. Some definitions and comparison principle. We begin with recalling the definition of viscosity solution in the present context.

Definition 3. Let Ω be a domain in \mathbb{R}^N , let $f(x, u)$ be a continuous function on $\Omega \times \mathbb{R}$, suppose that H satisfies (H1) and (H2), then v , continuous in Ω is called

a viscosity super-solution (respectively sub-solution) of $H(\nabla u, D^2u) = f(x, u)$ if for all $x_0 \in \Omega$,

-Either there exists an open ball $B(x_0, \delta)$, $\delta > 0$ in Ω on which v equals some constant c and $0 \leq f(x, c)$, for all $x \in B(x_0, \delta)$ (respectively $0 \geq f(x, c)$ for all $x \in B(x_0, \delta)$)

-Or $\forall \varphi \in C^2(\Omega)$, such that $v - \varphi$ has a local minimum (respectively local maximum) at x_0 and $\nabla \varphi(x_0) \neq 0$, one has $H(\nabla \varphi(x_0), D^2 \varphi(x_0)) \leq f(x_0, v(x_0))$ (respectively $H(\nabla \varphi(x_0), D^2 \varphi(x_0)) \geq f(x_0, v(x_0))$).

A viscosity solution is a function which is both a super-solution and a sub-solution.

With this definition we have the following comparison principle.

Theorem 2.1. [9] *Let Ω be a bounded open set in \mathbb{R}^N . Suppose that H satisfies conditions (H1) and (H2). Suppose that f is some continuous and increasing function on \mathbb{R} , such that $f(0) = 0$. Suppose that $u \in C(\bar{\Omega})$ is a viscosity sub-solution of $H = f$ and $v \in C(\bar{\Omega})$ is a viscosity super-solution of $H = f$. Then if $u \leq v$ on $\partial\Omega$, $u \leq v$ in Ω .*

If f is nondecreasing, the same result holds when furthermore v is a strict super-solution or vice versa when u is a strict sub-solution.

Remark 5. *If Ω is a ball, when H satisfies in addition (H3) and f is nondecreasing, the comparison principle is valid for radial sub and super-solution, not only for strict sub and super-solution.*

This comparison theorem permits to prove the uniqueness of viscosity solution for Dirichlet problem when f is increasing. Furthermore using Perron's method adapted to the present context one gets the existence of solution as soon as there exists a sub and a super-solution that are ordered.

As a consequence, one can solve the Dirichlet problem on regular domains, by using some function of the distance to the boundary as a super-solution. Of course the solutions of $H[u] - f(u) \leq 0$ with non negative boundary conditions are non negative.

In the paper we will frequently use the compactness of sequences of bounded solutions. We specify the result that we use, which is a consequence of Harnack's inequality, [22], [14], [27].

Theorem 2.2. *Suppose that Ω is an open set in \mathbb{R}^N . Suppose that H satisfies (H1) and (H2). Suppose that $(f_n)_n$ is a sequence of continuous functions, and uniformly bounded on compact sets of Ω and $(u_n)_n$ is a sequence of continuous viscosity solutions, uniformly bounded on compact sets of Ω , of*

$$H[u_n] = f_n.$$

Then for every compact set K in Ω , the sequence (u_n) is relatively compact in $C(K)$.

3. The minimality principle. We begin to enounce an existence's result as soon as one has a sub- and a super-solution of the equation.

Proposition 1. [9], [10] *Suppose that Ω is some bounded C^2 domain in \mathbb{R}^N , that $f \in C^1(\mathbb{R})$ and $g \in W^{2,\infty}(\partial\Omega)$, and H satisfies (H1) and (H2).*

Suppose that there exists \underline{u} and \bar{u} continuous on $\bar{\Omega}$, such that $\underline{u} \leq \bar{u}$ and \underline{u} and \bar{u} are respectively nonnegative solutions of

$$\begin{cases} H[\underline{u}] \geq f(\underline{u}) & \text{in } \Omega \\ \underline{u} \leq g & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} H[\bar{u}] \leq f(\bar{u}) & \text{in } \Omega \\ \bar{u} \geq g & \text{on } \partial\Omega. \end{cases}$$

Then there exists u , $\underline{u} \leq u \leq \bar{u}$ which is a solution of the equation.

Remark 6. This result uses some existence's result for Dirichlet problem, [10]. In [22] the authors extend these existence's results to Hölder's continuous boundary data.

The proof of Proposition 1 can be found in [10]. We give a short proof for the convenience of the reader. Let $\Lambda = \sup_{[\underline{u}, \bar{u}]} |f'|$. We define a sequence u_k by a recursive way

$$\begin{cases} H[u_{k+1}] - \Lambda u_{k+1} = f(u_k) - \Lambda u_k & \text{in } \Omega \\ u_{k+1} = g & \text{on } \partial\Omega \end{cases}$$

beginning with $u_0 = \underline{u}$. Using the comparison principle in Theorem 2.1 for proper operators of this type one gets that the sequence $(u_k)_k$ is increasing, u_k is a subsolution for all k , u_k lies between \underline{u} and \bar{u} and converges to u . By standard elliptic estimates one gets that u is a solution.

We now define as in [25] the notion of minimal solution.

Proposition 2. Under the previous assumptions, there exists a minimal solution to the Dirichlet problem

$$\begin{cases} H[u] = f(\bar{u}) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

relative to \underline{u} , i.e., a solution u such that $\underline{u} \leq u$, which satisfies for all $\omega \subset \Omega$ smooth and for all v which satisfies $H[v] \leq f(v)$ in ω with $v \geq u$ on $\partial\omega$ and $v \geq \underline{u}$ in ω , then $u \leq v$. We call u the minimal solution relative to \underline{u} .

The proof of this proposition is a carbon copy of the proof of the related proposition in [25]. We check that at each step $u_k \leq v$ in ω and the result holds true thanks to a limiting argument.

Remark 7. It is clear that the minimal solution is unique. Then, assuming that (H3) holds true, if $\Omega = B(R)$ is a ball centered at the origin and \underline{u} is radial the minimal solution relative to \underline{u} is radial, since for every isometry O , $v(x) = u(O(x))$ satisfies the equation and is minimal relative to \underline{u} .

4. Local existence, uniqueness and regularity of radial solutions. We begin with some proposition about existence, uniqueness of the radial solutions to equations involving Pucci's operators, together with some remarks on the sign of their derivatives. In the sequel we denote by $]0, r_0[$ either the interval, or the annulus $\{0 < |x| < r_0\}$; this does not lead to any confusion. This proposition is a particular case of a general proposition that appeared in [9], [11]; see also [15], [16], [17] for other references concerning the regularity of solutions to fully nonlinear PDE.

Proposition 3. *Let f be positive for $r > 0$, $f(0) = 0$ and locally Lipschitz. Suppose that (u_0, u_1) is given, with $u_0 > 0$, and either $u_1 > 0$, $r_0 > 0$ or $u_1 = r_0 = 0$. Then there exists $\delta > 0$ and there exists a unique radial solution of*

$$\begin{cases} H^\pm[\varphi] = f(\varphi) & \text{on }]r_0, r_0 + \delta[\\ \varphi(r_0) = u_0, & \varphi'(r_0) = u_1. \end{cases}$$

Furthermore the solution satisfies $\varphi'(r) > 0$ for $r > 0$, φ is C^2 on $]r_0, r_0 + \delta[$ and $\varphi'' > 0$ on a neighborhood on the right of r_0 .

Proof. We first outline the proof, and we treat the case of H^+ , the other can be done with similar obvious arguments. The first step is devoted to construct a solution of $A|\nabla\varphi|^\alpha\Delta\varphi = f(\varphi)$ on a neighborhood of zero by a fixed point argument. Let us recall that by Remark 3 we know that the solution is C^2 except maybe on zero. It turns out that on some neighborhood of zero, $\varphi'' \geq 0$ and $\varphi' > 0$. Then φ is a solution of the equation with H^+ on a neighborhood of zero. On the other hand, the uniqueness for solutions of Dirichlet problem ensures that this φ is the solution of the equation. In a last step we prove, by classical theorem in ordinary differential equations, that this local solution extends to a global one.

The proof divides into two steps, depending if r_0 is 0 or strictly positive. First consider $r_0 = 0$ and then $u_1 = 0$. To begin with, we tackle the fixed point argument.

We use here a Picard's fixed point argument, but one can also use the Schauder's fixed point theorem, as it is done in [23] for the p -Laplacian. Let $\Phi_\alpha(X) = |X|^{\frac{1}{1+\alpha}-1}X$ and let us introduce the map T defined on $C([0, \delta])$ as

$$T(\varphi)(r) = u_0 + \int_0^r \Phi_\alpha \left(\frac{1+\alpha}{As^{(N-1)(1+\alpha)}} \int_0^s \lambda^{(N-1)(1+\alpha)} f(\varphi(\lambda)) d\lambda \right) ds. \quad (2)$$

Let $\Lambda_f = \sup_{[\frac{2u_0}{3}, \frac{4u_0}{3}]} f'$ and $M_f = \sup_{[\frac{2u_0}{3}, \frac{4u_0}{3}]} f$, $m_f = \inf_{[\frac{2u_0}{3}, \frac{4u_0}{3}]} f$. We choose δ so that

$$\frac{\Lambda_f \delta^{\frac{2+\alpha}{\alpha+1}} \sup(M_f^{\frac{-\alpha}{1+\alpha}}, m_f^{\frac{-\alpha}{1+\alpha}})}{A^{\frac{1}{1+\alpha}}} < \frac{1}{3}, \quad (3)$$

and such that

$$\frac{M_f^{\frac{1}{1+\alpha}} \delta^{\frac{\alpha+2}{\alpha+1}}}{\frac{\alpha+2}{\alpha+1} A^{\frac{1}{1+\alpha}}} \leq \frac{u_0}{3}. \quad (4)$$

Then

$$\begin{aligned} |T(\varphi(r)) - u_0| &\leq \left| \int_0^r \Phi_\alpha \left(\frac{1+\alpha}{As^{(N-1)(1+\alpha)}} \int_0^s \lambda^{(N-1)(1+\alpha)} f(\varphi(\lambda)) d\lambda \right) ds \right| \\ &\leq \left(\frac{M_f}{A} \right)^{\frac{1}{\alpha+1}} \int_0^r \left(\frac{1+\alpha}{s^{(N-1)(1+\alpha)}} \int_0^s \lambda^{(N-1)(1+\alpha)} d\lambda \right)^{\frac{1}{1+\alpha}} ds \\ &\leq \left(\frac{M_f}{A} \right)^{\frac{1}{1+\alpha}} \left(\frac{(1+\alpha)}{(N-1)(1+\alpha)+1} \right)^{\frac{1}{1+\alpha}} \int_0^r s^{\frac{1}{1+\alpha}} ds \\ &\leq \frac{M_f^{\frac{1}{1+\alpha}} r^{\frac{1}{1+\alpha}+1}}{\frac{\alpha+2}{\alpha+1} A^{\frac{1}{1+\alpha}}} \leq \frac{u_0}{3}. \end{aligned}$$

We now prove that under the assumptions, T is contracting on the ball $B(u_0, \frac{u_0}{3})$. We shall use the inequality for $U, V \in [m_f, M_f]$

$$|\Phi_\alpha(U) - \Phi_\alpha(V)| \leq |U - V| \frac{1}{1 + \alpha} \max(\sup U^{\frac{-\alpha}{\alpha+1}}, \sup V^{\frac{-\alpha}{\alpha+1}}), \quad (5)$$

that we apply here with

$$U(s) = \left(\frac{1 + \alpha}{s^{(N-1)(1+\alpha)}} \int_0^s \lambda^{(N-1)(1+\alpha)} \frac{f(u(\lambda))}{A} d\lambda \right), \quad (6)$$

and

$$V(s) = \left(\frac{1 + \alpha}{s^{(N-1)(1+\alpha)}} \int_0^s \lambda^{(N-1)(1+\alpha)} \frac{f(v(\lambda))}{A} d\lambda \right). \quad (7)$$

Then

$$|U(s) - V(s)| \leq \frac{|f(u) - f(v)|_\infty}{A} \frac{s(1 + \alpha)}{(N-1)(1 + \alpha) + 1} \leq \frac{\Lambda_f |u - v|_\infty s(1 + \alpha)}{A((N-1)(1 + \alpha) + 1)}, \quad (8)$$

and

$$U(s)^{\frac{-\alpha}{\alpha+1}} \leq \sup(M_f^{\frac{-\alpha}{\alpha+1}}, m_f^{\frac{-\alpha}{\alpha+1}}) \left(\frac{s(1 + \alpha)}{A((N-1)(1 + \alpha) + 1)} \right)^{\frac{-\alpha}{\alpha+1}}. \quad (9)$$

Then one has using (5) and (9)

$$\begin{aligned} |T(u) - T(v)|(r) &\leq \left| \int_0^r \Phi_\alpha \left(\frac{1 + \alpha}{s^{(N-1)(1+\alpha)}} \int_0^s \lambda^{(N-1)(1+\alpha)} \frac{f(u(\lambda))}{A} d\lambda \right) dr \right. \\ &\quad \left. - \int_0^r \Phi_\alpha \left(\frac{1 + \alpha}{s^{(N-1)(1+\alpha)}} \int_0^s \lambda^{(N-1)(1+\alpha)} \frac{f(v(\lambda))}{A} d\lambda \right) dr \right| \\ &\leq \frac{(\alpha + 1) \sup(M_f^{\frac{-\alpha}{1+\alpha}}, m_f^{\frac{-\alpha}{1+\alpha}})}{(2 + \alpha) A^{\frac{1}{1+\alpha}}} r^{\frac{\alpha+2}{\alpha+1}} \Lambda_f |u - v|_\infty. \end{aligned}$$

Then by the classical fixed point theorem, there exists a neighborhood on the right of 0 on which there exists a unique solution φ of the equation. Since the fixed point satisfies in a neighborhood of zero $\frac{d}{dr}(r^{(N-1)(1+\alpha)}|\varphi'|^\alpha \varphi') > 0$, one gets that $\varphi'(r) > 0$ for r close to zero.

Let us pretend that $\lim_{r \rightarrow 0} \frac{\varphi'^{1+\alpha}(r)}{r} = \frac{(1+\alpha)f(u_0)}{A(N-1)(1+\alpha)+A}$ is valid; we shall check this in the sequel. This implies that in a neighborhood of zero,

$$|\varphi'|^\alpha \varphi'' \geq \frac{f(\varphi(0))}{A((N-1)(1 + \alpha) + 1)}. \quad (10)$$

Then $\varphi'' > 0$ in a neighborhood on the right of zero, say $[0, \delta_1]$. We argue as in [10]; let v be the unique solution, according to Remark 5, of the problem

$$\begin{cases} H^+[v] = f(\varphi) & \text{on } B(0, \delta_1), \\ v(\delta_1) = \varphi(\delta_1). \end{cases}$$

We get that $v = \varphi$ is the solution of our problem and then $\varphi'' > 0$ on a neighborhood of zero.

To prove that $\lim_{r \rightarrow 0} \frac{\varphi'^{1+\alpha}(r)}{r} = \frac{(1+\alpha)f(u_0)}{A(N-1)(1+\alpha)+A}$, let us note that since φ is a fixed point of T

$$\begin{aligned} r^{(N-1)(1+\alpha)}(\varphi')^{1+\alpha}(r) &= \frac{1 + \alpha}{A} \int_0^r f(\varphi(s)) s^{(N-1)(1+\alpha)} ds \\ &\underset{\sim_{r \rightarrow 0}}{\sim} \frac{(1 + \alpha)f(\varphi(0))}{A} \frac{r^{(N-1)(1+\alpha)+1}}{(N-1)(1 + \alpha) + 1} \end{aligned}$$

This ends the proof by dividing by $r^{(N-1)(1+\alpha)+1}$ on each side. We have proved both the existence of a solution for $H^+[u] = f(u)$, $u(0) = u_0$, $u'(0) = 0$ and in the same time that $u'' \geq 0$ on a neighborhood of zero.

We now consider the case $u_1 > 0$ and $r_0 > 0$. Let us define $h_{a,A}(x) = Ax^+ - ax^-$ and let us observe that the equation can be written as an ordinary differential equation

$$w'' = h_{\frac{1}{A}, \frac{1}{a}} \left(\frac{f(w) - h_{a,A}(w') \frac{(N-1)}{r}}{|w'|^\alpha} \right)$$

with a right hand side which is Lipschitzian around points where $w' \neq 0$. So the Cauchy Lipschitz theorem ensures local existence and uniqueness.

Suppose that u_1 and u_2 are two radial solutions. If $u_1(0) = u_2(0)$ since $u_1'(0) = u_2'(0) = 0$, using the local uniqueness result one gets that $u_1 = u_2$ in a neighborhood of zero, and using the local uniqueness on every $r_0 \neq 0$ one gets that $u_1 \equiv u_2$. \square

5. Existence of boundary blow up solutions. The strategy is to prove first the existence of BBUS for Pucci's operators on small ball, and then to use these to tackle the more general setting, since the general operators that we consider are sandwiched into two Pucci's operators. In a first subsection we introduce preliminary material, then we attack the core of the proof of Theorem 1.1 in the next two subsections.

5.1. Preliminary results. We begin with an existence's result of boundary blow up solution as soon as one has a boundary blow up super-solution.

Theorem 5.1. *Let Ω be a bounded C^1 domain of \mathbb{R}^N , and $f \in C^1(\mathbb{R})$. Suppose that H satisfies (H1) and (H2). Assume that there exists a function \underline{U} in $C(\bar{\Omega})$ which is a sub-solution of the equation in Ω and a function $v \in C(\Omega)$ which is a boundary blow up super-solution of the equation, and $v \geq \underline{U}$. Then there exists a unique boundary blow up solution of the equation, with $\underline{U} \leq u$ and u is less than any blow up super-solution in $\omega \subset \Omega$.*

We call such a solution the minimal BBUS of (P), relatively to \underline{U} .

Proof. Let k denote an integer $k > |\underline{U}|_\infty$ and let u_k be the minimal solution relative to \underline{U} of

$$\begin{cases} H[u_k] = f(u_k) & \text{in } \Omega, \\ u_k = k & \text{on } \partial\Omega. \end{cases}$$

Using the minimality principle (see the proof of Proposition 2), the sequence u_k is non decreasing. At each step we have by minimality principle that $v \geq u_k$ in Ω . Then $(u_k)_k$ is bounded on every compact set of Ω and by Theorem 2.2, converges to some function u which is such that $\underline{u} \leq u \leq v$. Consider now an open subset ω such that $\bar{\omega} \subset \Omega$, and on which there exists a BBUS \tilde{v} . Using once again the minimality principle, since $\tilde{v}|_{\partial\omega} \geq u|_{\partial\omega}$ then we have that $\tilde{v} \geq u$ on ω . \square

Remark 8. *When H satisfies (H3), the minimal blow up solution in a ball is radial. Indeed, if u is a minimal solution, $v(x) = u(O(x))$ is for all isometry O also a minimal blow up solution. Then by the uniqueness result, $v(x) = u(x)$.*

Remark 9. *Later we shall prove that when f is non decreasing, and $H(p, M) = |p|^\alpha \text{tr}(M)$, there is uniqueness of radial blow up solutions in balls, and then under these assumptions, the radial blow up solution is minimal in the sense defined below.*

We now prove a simple technical lemma for radial solutions for equations involving Pucci's operators.

Proposition 4. *Suppose that $f > 0$ for $r > 0$, $f(0) = 0$ and continuous. Suppose that φ is a radial solution of*

$$H^\pm[\varphi] = f(\varphi(r)) \quad (11)$$

with $\varphi(0) > 0$. Then $\varphi' \geq 0$ and there exists some constant $C(a, A, N, \alpha)$ which depends only on a , A and N and α such that if $N \geq 2$ for any $0 < r_1 < r$

$$\int_{\varphi(r_1)}^{\varphi(r)} \frac{ds}{(F(s) - F(\varphi(r_1)))^{\frac{1}{\alpha+2}}} \geq C(a, A, N, \alpha) r_1 \left(1 - \left(\frac{r_1}{r}\right)^{(N-1)\frac{A}{a}-1}\right).$$

If furthermore f is non decreasing then $\varphi'' \geq 0$.

Proof. We consider the case $H^+[\varphi] = f(\varphi)$, the other case being similar. Let us recall that since φ is radial it satisfies

$$|\varphi'|^\alpha \left(A(\varphi'')^+ - a(\varphi'')^- + A\left(\frac{N-1}{r}\varphi'\right)^+ - a\left(\frac{N-1}{r}\varphi'\right)^- \right) = f(\varphi). \quad (12)$$

Let us recall that according to Proposition 3, $\varphi' > 0$ and φ is C^2 as soon as $r > 0$. The open set $\{r > 0; \varphi'' \neq 0\}$ divides in a countable number of intervals where either $\varphi'' > 0$ or $\varphi'' < 0$. To begin with, suppose that on $[s, r]$, $\varphi'' \leq 0$. Then on that set the equation implies

$$|\varphi'|^\alpha (a\varphi'' + \frac{N-1}{r}A\varphi') \geq f(\varphi). \quad (13)$$

This can also be written as

$$\frac{d}{dr} \left(r^{\frac{(N-1)(1+\alpha)A}{a}} (\varphi')^{1+\alpha} \right) \geq r^{\frac{(N-1)(1+\alpha)A}{a}} \frac{(1+\alpha)f(\varphi)}{a}, \quad (14)$$

and multiplying by $r^{\frac{(N-1)A}{a}} \varphi'$ and integrating on $[s, r]$ one gets

$$\frac{r^{\frac{(N-1)(2+\alpha)A}{a}} (\varphi')^{2+\alpha}}{2+\alpha}(r) - \frac{s^{\frac{(N-1)(2+\alpha)A}{a}} (\varphi')^{2+\alpha}}{2+\alpha}(s) \geq \int_s^r \frac{f(\varphi(t))}{a} t^{\frac{(N-1)(2+\alpha)A}{a}} \varphi'(t) dt.$$

This implies

$$\frac{(\varphi')^{2+\alpha}}{2+\alpha}(r) - \left(\frac{s}{r}\right)^{\frac{(N-1)(2+\alpha)A}{a}} \frac{(\varphi')^{2+\alpha}}{2+\alpha}(s) \geq \int_s^r \frac{f(\varphi(t))}{A} \left(\frac{t}{r}\right)^{\frac{(N-1)(2+\alpha)A}{a}} \varphi'(t) dt. \quad (15)$$

Alternatively, if on $[r, s]$ $\varphi'' \geq 0$ then the equation implies

$$|\varphi'|^\alpha (\varphi'' + \frac{N-1}{r}\varphi') \geq \frac{f(\varphi)}{A}$$

and acting as before

$$\frac{r^{(N-1)(2+\alpha)} (\varphi')^{2+\alpha}}{2+\alpha}(r) - \frac{s^{(N-1)(2+\alpha)} (\varphi')^{2+\alpha}}{2+\alpha}(s) \geq \int_s^r \frac{f(\varphi(t))}{A} t^{(N-1)(2+\alpha)} \varphi'(t) dt.$$

Dividing by $r^{(N-1)(2+\alpha)}$ and using $\frac{t}{r} < 1$ one obtains once again (15).

Consider any interval $[r_0, r]$. Dividing (r_0, r) into a countable union of intervals (r_n, r_{n+1}) where φ'' has a constant sign, and summing the inequalities

$$r_{n+1}^{\frac{(N-1)(2+\alpha)A}{a}} \frac{(\varphi')^{2+\alpha}}{2+\alpha} (r_{n+1}) - r_n^{\frac{(N-1)(2+\alpha)A}{a}} \frac{(\varphi')^{2+\alpha}}{2+\alpha} (r_n) \geq \int_{r_n}^{r_{n+1}} \frac{f(\varphi(t))}{A} t^{\frac{(N-1)(2+\alpha)A}{a}} \varphi'(t) dt,$$

we obtain, eventually dropping $\frac{r_0^{(N-1)(2+\alpha)} (\varphi')^{2+\alpha} (r_0)}{2+\alpha} \geq 0$,

$$\begin{aligned} \frac{r^{\frac{(N-1)(2+\alpha)A}{a}} (\varphi')^{2+\alpha} r}{2+\alpha} &\geq \int_{r_0}^r \frac{f(\varphi(s))}{A} s^{\frac{(N-1)(2+\alpha)A}{a}} \varphi'(s) ds \\ &\geq \frac{r_0^{\frac{(\alpha+2)(N-1)A}{a}}}{A} (F(\varphi(r)) - F(\varphi(r_0))). \end{aligned}$$

Dividing by $(F(\varphi(r)) - F(\varphi(r_0)))$, taking the power $\frac{1}{2+\alpha}$ and integrating on $[r_0, r]$ one gets

$$\int_{\varphi(r_0)}^{\varphi(r)} \frac{ds}{(F(s) - F(\varphi(r_0)))^{\frac{1}{\alpha+2}}} \geq \left(\frac{2+\alpha}{A} \right)^{\frac{1}{\alpha+2}} \frac{r_0}{\frac{(N-1)A}{a} - 1} \left(1 - \left(\frac{r_0}{r} \right)^{\frac{(N-1)A}{a} - 1} \right).$$

This completes the proof of the first assertion. We now assume that f is non decreasing, and to fix the ideas that the operator is H^+ . Due to Proposition 3, we know that φ'' is positive on a neighborhood of 0. Suppose now that there exists r_0 such that $\varphi''(r) > 0$ before r_0 and that $\varphi''(r_0) = 0$. Let us note that for $r < r_0$ the equation is

$$|\varphi'|^\alpha (A\varphi''(r) + \frac{N-1}{r} A\varphi'(r)) = f(\varphi)$$

and multiplying by $\frac{r}{|\varphi'|^\alpha A}$ one has

$$\varphi''(r)r = \frac{f(\varphi(r))r}{A|\varphi'|^\alpha} - (N-1)\varphi'(r).$$

Then subtracting $\varphi''(r_0)r_0 = 0 = \frac{f(\varphi(r_0))r_0}{A|\varphi'|^\alpha} - (N-1)\varphi'(r_0)$, dividing by $r - r_0$ one obtains

$$\begin{aligned} \frac{\varphi''(r)r}{r - r_0} &= \frac{1}{A|\varphi'|^\alpha(r)} \frac{f(\varphi(r))r - f(\varphi(r_0))r_0}{r - r_0} \\ &+ \frac{f(\varphi(r_0))(r_0)}{r - r_0} \left(\frac{1}{A|\varphi'|^\alpha(r)} - \frac{1}{A|\varphi'|^\alpha(r_0)} \right) - (N-1) \frac{\varphi'(r) - \varphi'(r_0)}{r - r_0}. \end{aligned}$$

Taking the limit when r goes to r_0 by the left, using $\varphi''(r_0) = 0$, one gets that $\lim_{r \rightarrow r_0, r < r_0} \frac{\varphi''(r)r}{r - r_0} \geq \frac{1}{A|\varphi'|^\alpha(r_0)} (f(\varphi(r_0)) + r_0 f'(\varphi(r_0)) \varphi'(r_0)) > 0$. This implies that $r \mapsto \varphi''(r)r$ is increasing for $r < r_0$, close to it and contradicts $\varphi''(r_0) = 0$. This ends the proof. □

Remark 10. *The first statement of Proposition 4 is valid for general operators satisfying (H1)-(H2)-(H3), (H4). The proof is similar, replacing the equality (12) by an inequality, since H is sandwiched between Pucci's operators.*

5.2. Proof of Theorem 1.1; existence of BBUS on some ball. We now attack the core of the proof of Theorem 1.1. We begin with 1. \Rightarrow 2., i.e. if f satisfies the Keller Osserman condition, then for all operator H satisfying (H1) and (H2), there exists $R > 0$ and a blow up solution in the ball $B(0, R)$.

We shall prove the existence of a blow up solution u^+ for the operator H^+ on some ball $B(0, R)$, next we will derive from this the existence of some blow up solution for any operator H by using the minimality principle relative to 0 on every compact set of $B(0, R)$.

We use partly the arguments in [25]. Suppose to begin with that β is such that $\Phi(\beta) < C(a, A, N, \alpha)$, and let u be the minimal solution relative to 0 in B_1 of

$$\begin{cases} H^+[u] = f(u) & \text{in } B_1, \\ u = \beta & \text{on } \partial B_1. \end{cases}$$

Then u is radial and $\varphi(r) = u(x)$ for $|x| = r$ satisfies the equation in Proposition 3. Let $\tilde{\beta} = \varphi(0) = u(0)$. By the Cauchy Lipschitz Theorem as employed in the proof of Proposition 3 around the point $r_0 = 1$ where $\varphi'(1) \neq 0$, φ can be extended to a maximal interval $[0, R]$. Then on R , either $\varphi(R) = \infty$ or $\varphi'(R) = \infty$.

Suppose by absurd that $\varphi(R)$ is finite and $\varphi'(R) = \infty$. Then, from the equation

$$|\varphi'|^\alpha \left(A(\varphi'')^+ - a(\varphi'')^- + A\left(\frac{N-1}{r}\varphi'\right) \right) = f(\varphi),$$

we obtain that $\varphi'' < 0$ in a left neighborhood of R , but this would imply that φ' is decreasing in that interval, which contradicts the fact that $\varphi'(R) = +\infty$.

There remains to prove that R is finite in order to be sure that u is a blow up solution.

Assume by contradiction that $R = +\infty$. Apply Proposition 4 with $r_1 = 1$ and r_2 arbitrary large. One obtains

$$\Phi(\varphi(1)) = \Phi(\beta) \geq C(a, A, N, \alpha) \left(1 - \left(\frac{1}{r_2} \right)^{\frac{(N-1)A}{a} - 1} \right).$$

Letting r_2 go to infinity one obtains a contradiction with the assumption.

The previous arguments show that as soon as $\inf \phi < c(a, A, N, \alpha)$ there exists R and u a BBUS defined on $B(0, R)$. Suppose now that $\inf \phi \geq c(a, A, N, \alpha)$, and define K large enough and β such that $\frac{1}{K^{\frac{3+2\alpha}{2+\alpha}}} \inf \phi < c(a, A, N, \alpha)$. Define $\phi_K(x) = \frac{1}{K^{\frac{3+2\alpha}{2+\alpha}}} \phi(Kx)$. $\inf \phi_K < c(a, A, N, \alpha)$. Let $f_K(x) = K^{2+\alpha} f(Kx)$, $F_K(x) = K^{1+\alpha} F(Kx)$, $\phi_K(x) = \phi(F_K(x))$. There exists then v a BBUS for $H^+[v] = f_K(x)$ on $B(0, R)$. Then $u(x) = v(\frac{x}{K})$ is a BBUS in $B(0, RK)$ for the right hand side f .

We now prove that on the same ball, for any operator H satisfying (H1), (H2), there exists a boundary blow up solution. Let u^+ be a blow-up solution for H^+ in some ball. Such a solution is a super-solution for the equation $H[u] = f(u)$ and then according to Theorem 5.1 there exists a minimal blow up solution relative to 0 and less than u^+ in the ball. This completes the proof of 1. yields 2. We now prove that 2. implies 1., i. e we assume that there exists some operator H and some ball $B(0, R)$ such that there exists a blow up solution in that ball and we prove that f satisfies the Keller Osserman condition.

Let u be a blow up solution on $B(0, R)$, then u is a blow up super-solution in the ball for $H^-[u] = f(u)$, and using once more Theorem 5.1 one gets the existence of

a blow up solution for H^- relative to 0, that we denote by φ . We claim that: for all $0 \leq s < t$

$$\frac{|\varphi'|^{\alpha+2}(t) - |\varphi'|^{\alpha+2}(s)}{(\alpha + 2)} \leq \frac{F(\varphi(t)) - F(\varphi(s))}{a}. \quad (16)$$

To prove the claim, dividing $[s, t]$ into a countable number of intervals on which φ'' is positive or negative as it is done in the proof of Proposition 4, one can suppose that φ'' is constant sign on $[s, t]$. Suppose that it is non positive, then the left hand side of the above inequality is ≤ 0 and the right hand side is ≥ 0 so the result is obvious. We now suppose that $\varphi'' \geq 0$ on $[s, t]$, then φ satisfies

$$\frac{d}{dr}(r^{(N-1)(1+\alpha)}(\varphi')^{\alpha+1}) \leq \frac{(1+\alpha)}{a}r^{(N-1)(1+\alpha)}f(\varphi(r)). \quad (17)$$

Multiplying by $r^{N-1}\varphi'$ one gets

$$\frac{d}{dr}\left(\frac{r^{(N-1)(2+\alpha)}(\varphi')^{\alpha+2}}{\alpha + 2}\right) \leq \frac{1+\alpha}{a}r^{(N-1)(2+\alpha)}f(\varphi(r))\varphi'(r), \quad (18)$$

and integrating between t and s one gets

$$\begin{aligned} \frac{t^{(N-1)(2+\alpha)}(\varphi')^{\alpha+2}(t)}{\alpha + 2} - \frac{s^{(N-1)(2+\alpha)}(\varphi')^{\alpha+2}(s)}{\alpha + 2} &\leq \\ \frac{1+\alpha}{a}t^{(N-1)(2+\alpha)}(F(\varphi(t)) - F(\varphi(s))). & \end{aligned} \quad (19)$$

We now use $-s^{(N-1)(2+\alpha)} \geq -t^{(N-1)(2+\alpha)}$ and divide by $t^{(N-1)(2+\alpha)}$ to obtain

$$\frac{|\varphi'|^{\alpha+2}(t) - |\varphi'|^{\alpha+2}(s)}{(\alpha + 2)} \leq (1 + \alpha)\frac{F(\varphi(t)) - F(\varphi(s))}{a}. \quad (20)$$

This completes the proof of (16). Using this inequality with $s = 0$, eventually taking the power $\frac{1}{\alpha+2}$, and next integrating between 0 and R one gets

$$0 \leq \int_0^R \frac{\varphi'(r)}{((\alpha + 2)(F(\varphi(r)) - F(\varphi(0)))^{\frac{1}{2+\alpha}})} dr \leq \frac{R(1 + \alpha)^{\frac{1}{2+\alpha}}}{a^{\frac{1}{2+\alpha}}} \quad (21)$$

which implies the Keller Osserman condition with $\beta = \varphi(0)$. This completes the proof of 2. implies 1.

5.3. Existence of blow up solutions in arbitrary domains and end of the proof of Theorem 1.1. We want to prove the implications concerning items 3. and 4. in Theorem 1.1.

We then assume that f satisfies the sharpened Keller Osserman condition. As in the previous proof we prove first the existence of a blow up solution for the operator H^+ on any domain, and we obtain a blow up solution in the same domain for any operator H satisfying (H1) and (H2) by the arguments already employed in section 5.2.

So we suppose that the operator is H^+ . We begin to establish the following claim. Define

$$R_0 = \inf\{R, \text{there exists a blowup solution in } B_R\}. \text{ Then } R_0 = 0. \quad (22)$$

To prove the claim we assume by contradiction that $R_0 > 0$. Let β_n be a sequence of real numbers going to infinity, such that $\Phi(\beta_n) \rightarrow 0$. Let $\bar{u} = \beta_n$ and $\underline{u} = 0$, and let u be the minimal solution relative to \underline{u} in $B_{\frac{R_0}{2}}$, with the boundary condition

$u = \beta_n$ on $\partial B_{\frac{R_0}{2}}$. Then u is radial and can be extended as a radial solution φ of the equation in $B(0, R_0)$. Indeed, since $\varphi'(\frac{R_0}{2}) \neq 0$, by the Cauchy-Lipschitz theorem as employed in the proof of Proposition 3, there exists around $\frac{R_0}{2}$ some C^2 solution of the equation, and there exists a maximal interval $]0, R[$ on which φ is defined, C^2 and satisfies the equation. If we assume that $R < R_0$, then we prove as before that $\varphi(R) = +\infty$, and this contradicts the definition of R_0 . Finally $R \geq R_0$. We use Proposition 4 with $r_1 = \frac{R_0}{2}$ and $r_2 = R_0$ to get

$$\Phi(\beta_n) \geq \int_{\varphi(\frac{R_0}{2})}^{\varphi(R_0)} \frac{ds}{(F(s) - F(\varphi(\frac{R_0}{2})))^{\frac{1}{2+\alpha}}} \geq C(a, A, N, \alpha) \left(1 - \left(\frac{1}{2}\right)^{\frac{(N-1)A}{\alpha}} - 1\right). \quad (23)$$

We thus have obtained a contradiction. This completes the proof of claim (22).

We now prove the existence of a blow up solution for H^+ in any domain Ω . We shall obtain u as the limit of solutions which satisfy $u_k = k$ on the boundary. So let u_k be the minimal solution relative to $\underline{u} = 0$ with the boundary condition $u_k = k$ on $\partial\Omega$.

For $x \in \Omega$ there exists some ball $B(x, r) \subset \Omega$ and some blow up solution u_r in $B(x, r)$. By the minimality of u_k , $u_k \leq u_r$. As a consequence the sequence $(u_k)_k$ is uniformly bounded in $B(x, \frac{r}{2})$. Let K be a compact set in Ω . Recovering K by a finite number of balls $B(x_i, \frac{r_i}{2})$ one gets that $(u_k)_k$ is uniformly bounded on K . Moreover by the minimality principle $u_k \leq u_{k+1}$, which implies using in addition properties of limit of viscosity solutions as enounced in Theorem 2.2, that $(u_k)_k$ converges uniformly on K to some solution u of the equation. Since K is arbitrary chosen we have obtained a solution in Ω . There remains to prove that u is $+\infty$ on the boundary. Let $x_0 \in \partial\Omega$ and let x_l be some sequence of points in Ω such that $x_l \rightarrow x_0$ when l goes to infinity. Since $u \geq u_k$, $\liminf_{l \rightarrow +\infty} u(x_l) \geq \liminf_{l \rightarrow +\infty} u_k(x_l) = k$ and letting k go to infinity one gets the result.

As we said at the beginning of the proof one can use the arguments already employed in subsection 5.2 to get that for any operator H satisfying (H1), (H2), and for any smooth bounded domain Ω , there exists a boundary blow up solution. This is 3. implies 4. We now prove that if there exists an operator H satisfying (H1), (H2) for which there exists a boundary blow up solution on every domain Ω , then the sharpened Keller Osserman condition is satisfied. As in section 5.2 the assumptions imply that there exists a blow up solution in any domain for H^- . Let u_n be the minimal blow up solution as defined in Theorem 5.1 for the operator H^- and relative to 0 in the ball $B(0, \frac{1}{n})$. Then u_n is radial. Let $\beta_n = u_n(0)$. If we can prove that β_n is unbounded, then, extracting a subsequence one gets the sharpened Keller Osserman condition with such β_n .

To prove that β_n is unbounded, suppose that, up to a subsequence, β_n converges to some $\beta > 0$. Let us apply the estimate (21) with $R = \frac{1}{n}$ one obtains

$$0 \leq \int_{\beta_n}^{\infty} \frac{dt}{((\alpha + 2)(F(t) - F(\beta_n)))^{\frac{1}{\alpha+2}}} \leq \frac{1}{n}. \quad (24)$$

One concludes by Fatou's lemma that

$$\int_{\beta}^{\infty} \frac{dt}{((\alpha + 2)(F(t) - F(\beta)))^{\frac{1}{\alpha+2}}} = 0, \quad (25)$$

which is absurd. Therefore 4. \Rightarrow 3.

We end this section by proving that Keller Osserman condition and sharpened Keller Osserman condition are equivalent, this will complete the proof of Theorem 1.1. We focus on the proof of 1 implies 3, since the converse statement is obvious. Let F^{-1} denote the reciprocal function of F and $g = (F^{-1})'$. Then one has

$$g'(F(u)) = (F^{-1})''(F(u)) = -\frac{F''(u)}{(F'(u))^3}. \quad (26)$$

Up to translation one can assume that $\int_0^\infty \frac{dt}{(F(t))^{2+\alpha}} < \infty$. This can also be written as

$$\int_0^\infty \frac{g(u)du}{u^{\frac{1}{2+\alpha}}} < \infty, \quad (27)$$

whereas the Sharpened Keller Osserman condition can be written as

$$\liminf \int_\beta^\infty \frac{g(u)}{(u-\beta)^{\frac{1}{2+\alpha}}} du = 0. \quad (28)$$

Let us observe that

$$\limsup_{\beta \rightarrow +\infty} \int_{2\beta}^\infty \frac{g(u)}{(u-\beta)^{\frac{1}{2+\alpha}}} du = 0. \quad (29)$$

Indeed, $u \geq 2\beta$ implies $u \leq 2(u-\beta)$ and then

$$\limsup_{\beta \rightarrow +\infty} \int_{2\beta}^\infty \frac{g(u)}{(u-\beta)^{\frac{1}{2+\alpha}}} du \leq 2^{\frac{1}{2+\alpha}} \limsup_{\beta \rightarrow +\infty} \int_{2\beta}^\infty \frac{g(u)}{u^{\frac{1}{2+\alpha}}} du = 0. \quad (30)$$

Hence (29) is proved. So we need to prove that

$$\liminf_{\beta \rightarrow +\infty} \int_\beta^{2\beta} \frac{g(u)}{(u-\beta)^{\frac{1}{2+\alpha}}} du = 0. \quad (31)$$

We assume by contradiction that there exists some constant $C > 0$ and some B large such that for $\beta > B$, $\int_\beta^{2\beta} \frac{g(u)}{(u-\beta)^{\frac{1}{2+\alpha}}} du \geq C$. We make a first change of variables to get $\int_0^{\beta^{\frac{2+\alpha}{2+\alpha}}} g(u^{2+\alpha} + \beta)u^\alpha du \geq C$ or equivalently

$$\int_0^\beta g(u^{2+\alpha} + \beta^{2+\alpha})u^\alpha du \geq C. \quad (32)$$

Integrating this inequality on $[B, R]$ with $R > 2B$ one obtains

$$\begin{aligned} C \frac{R}{2} \leq C(R-B) &\leq \int_B^R \int_0^\beta g(u^{2+\alpha} + \beta^{2+\alpha})u^\alpha dud\beta \\ &\leq \int_0^R \int_0^R g(u^{2+\alpha} + \beta^{2+\alpha})u^\alpha dud\beta. \end{aligned}$$

Let us define for $\theta \in [0, \frac{\pi}{2}]$

$$\begin{aligned} u &= \rho \cos^{\frac{2}{2+\alpha}} \theta, \\ \beta &= \rho \sin^{\frac{2}{2+\alpha}} \theta, \end{aligned}$$

so that with $c = \cos \theta$ and $s = \sin \theta$,

$$du \wedge d\beta = \frac{2}{2+\alpha} \rho (cs)^{\frac{2}{2+\alpha}-1} d\rho d\theta.$$

Let us note that the set $[0, R] \times [0, R]$ is contained in the set $\{\rho \in [0, 2^{\frac{1}{2+\alpha}} R], \theta \in [0, \frac{\pi}{2}]\}$. We have obtained

$$\begin{aligned} \frac{(2+\alpha)CR}{4} &\leq \int_0^{2^{\frac{1}{2+\alpha}} R} \int_0^{\frac{\pi}{2}} g(\rho^{2+\alpha}) \rho^{1+\alpha} c^{\frac{2\alpha}{2+\alpha}} (cs)^{\frac{2}{2+\alpha}-1} d\theta d\rho \\ &= \int_0^{2^{\frac{1}{2+\alpha}} R} \int_0^{\frac{\pi}{2}} g(\rho^{2+\alpha}) \rho^{1+\alpha} c^{\frac{\alpha}{2+\alpha}} s^{\frac{-\alpha}{2+\alpha}} d\theta d\rho. \end{aligned}$$

The integral $\int_0^{\frac{\pi}{2}} c^{\frac{\alpha}{2+\alpha}} s^{\frac{-\alpha}{2+\alpha}} d\theta$ is finite since $\alpha \in]-1, +\infty[$. We have obtained that

$$CR \leq \frac{4}{2+\alpha} \int_0^{cR} g(\rho^{2+\alpha}) \rho^{1+\alpha} d\rho = F^{-1}((cR)^{2+\alpha}). \quad (33)$$

This implies that $F(R) \leq cteR^{2+\alpha}$, which contradicts the Keller Osserman condition. This completes the proof of Theorem 1.1.

5.4. Miscellaneous results. We complete this section with some results concerning boundary blow up solutions on a given annulus. Let us point out that these solutions are not strictly speaking BBUS since they do not blow up on the whole boundary, but on a part of the boundary, that is either the interior or the exterior of the annulus boundary. This result will be used in the sequel.

Proposition 5. *Assume that f satisfies the Keller Osserman condition. Assume that H satisfies (H1) (H2), (H3), (H4). There exists a minimal nonnegative solution in the annulus $R_1 < |x| < R_2$ with*

$$\begin{cases} H[u] = f(u) & \text{in } B(0, R_2) \setminus \overline{B(0, R_1)}, \\ u = 0 & \text{on } |x| = R_1, \\ u = +\infty & \text{on } |x| = R_2. \end{cases}$$

Furthermore u is radial and satisfies $u'(r) > 0$ for $r \in]R_1, R_2[$. Of course the symmetric result holds : There exists a minimal nonnegative solution in the annulus $R_1 < |x| < R_2$ with

$$\begin{cases} H[u] = f(u) & \text{in } B(0, R_2) \setminus \overline{B(0, R_1)}, \\ u = +\infty & \text{on } |x| = R_1, \\ u = 0 & \text{on } |x| = R_2. \end{cases}$$

with u radial and satisfies $u'(r) < 0$ for $r \in]R_1, R_2[$

Proof. The existence can be obtained by considering for $k \in \mathbb{N}$ the minimal solution relative to 0 of

$$\begin{cases} H[u] = f(u) & \text{in } B(0, R_2) \setminus \overline{B(0, R_1)}, \\ u = 0 & \text{on } |x| = R_1, \\ u = k & \text{on } |x| = R_2. \end{cases}$$

This solution u_k is radial and the sequence $(u_k)_k$ is nondecreasing by the minimality. Furthermore it is radial. We prove now that each u_k is increasing in r .

We begin to observe, using Hopf's principle and the arguments in the proof of Proposition 3 that $u'_k > 0$, Arguing as in the proof of Theorem 1.1, i. e. among other things using the existence of BBUS on balls, one observes that $(u_k)_k$ is locally uniformly bounded, and passing to the limit u_k tends to u on every compact set of $B(0, R_2) \setminus \overline{B(0, R_1)}$, finally using once more the arguments in Proposition 3 one

gets that u is increasing. Of course a symmetric observation permits to prove that in the second case the corresponding u is decreasing. \square

6. Some results about the blow-up rate of solutions.

6.1. Blow-up rate of radially symmetric solutions. In this section, we are concerned to provide the blow-up rate for solutions in an implicit form. We assume that $H[u] = |\nabla u|^\alpha \Delta u$. Actually, we focus on this operator since for general fully non-linear operators, we just can expect inequalities and not equivalents. We now state

Proposition 6. *We suppose that f satisfies the Keller Osserman condition and that φ is a radial BBUS in the ball B_1 . Then*

$$\int_{\varphi(r)}^{\infty} \frac{ds}{F(s)^{\frac{1}{2+\alpha}}} \sim_{r \rightarrow 1} (2 + \alpha)^{\frac{1}{2+\alpha}} (1 - r). \quad (34)$$

Remark 11. *The results here enclosed can be extended to the case where $H = H^\pm$ and f is nondecreasing near infinity, with some obvious changes. Indeed, in that case we have seen that φ'' is > 0 for r close to the boundary and then the equation in that set is either $a|\nabla u|^\alpha \Delta u = f(u)$ for H^- or $A|\nabla u|^\alpha \Delta u = f(u)$ for H^+ . As a consequence in the case of H^+ one has*

$$\int_{\varphi(r)}^{\infty} \frac{ds}{F(s)^{\frac{1}{2+\alpha}}} \sim_{r \rightarrow 1} \left(\frac{2 + \alpha}{A} \right)^{\frac{1}{2+\alpha}} (1 - r). \quad (35)$$

and in the case of H^-

$$\int_{\varphi(r)}^{\infty} \frac{ds}{F(s)^{\frac{1}{2+\alpha}}} \sim_{r \rightarrow 1} \left(\frac{2 + \alpha}{a} \right)^{\frac{1}{2+\alpha}} (1 - r). \quad (36)$$

It is worth to emphasize that we assume here f nondecreasing near infinity to ensure that $\varphi'' > 0$ near to the boundary, and then to allow us to perform computations on a well-defined equality.

Remark 12. *One can also consider the case of the operators of the form*

$$H(p, M) = |p|^\alpha \left(q_1 \operatorname{tr}(M) + q_2 \left\langle M \frac{p}{|p|}, \frac{p}{|p|} \right\rangle \right)$$

with $q_1 > 0$, $q_1 + q_2 > 0$. This class of operators includes the case of the p Laplacian for which $q_1 = 1$ and $q_2 = p - 2$, but also some operators which cannot be written in divergence form. Even though these operators do not satisfy (H3), for radial solutions one has

$$H[\varphi] = |\varphi'|^\alpha \left((q_1 + q_2) \varphi'' + q_2 \frac{(N-1)}{r} \varphi' \right)$$

and it is clear that the arguments used for the regularity of the radial solution for the operators satisfying (H1), (H2), (H3) are still valid. As a consequence the solutions are C^2 for $r > 0$. One can note in addition that in the case $\alpha \leq 0$ it is proved in [12] that any solution of the equations $F(\nabla u, D^2 u) = f(u)$ (not only radial) are C^2 .

In that case the rate of convergence is

$$\int_{\varphi(r)}^{\infty} \frac{ds}{F(s)^{\frac{1}{2+\alpha}}} \sim_{r \rightarrow 1} \left(\frac{2 + \alpha}{p_1 + p_2} \right)^{\frac{1}{2+\alpha}} (1 - r).$$

and the proof is left to the reader.

We now move on the proof of Proposition 6.

Proof. Multiplying the equation satisfied by φ by $r^{(N-1)(2+\alpha)}\varphi'$ and integrating par parts the right hand side, one obtains

$$\begin{aligned} r^{(N-1)(2+\alpha)}|\varphi'|^{2+\alpha} &= (2+\alpha)[s^{(N-1)(2+\alpha)}F(\varphi(s))]_0^r \\ &\quad - (\alpha+2)^2(N-1)\int_0^r F(\varphi(s))s^{(N-1)(2+\alpha)-1}ds. \end{aligned}$$

Dividing by $r^{(N-1)(2+\alpha)}$ one gets

$$\begin{aligned} |\varphi'|^{2+\alpha} &= (2+\alpha)F(\varphi(r)) \\ &\quad - (N-1)\frac{(2+\alpha)^2}{r}\int_0^r F(\varphi(s))\left(\frac{s}{r}\right)^{(N-1)(2+\alpha)-1}ds \quad (37) \\ &= (2+\alpha)(F(\varphi(r)) - G_\varphi(r)), \end{aligned}$$

with $G_\varphi(r) = (N-1)\frac{(2+\alpha)}{r}\int_0^r F(\varphi(s))\left(\frac{s}{r}\right)^{(N-1)(2+\alpha)-1}ds$. Observe that $G_\varphi(r) = o(F(\varphi(r)))$ when r goes to 1. Indeed let $r < 1$ and ϵ such that $1 - \epsilon < r$. One cuts the integral using the fact that F is non decreasing and get

$$\begin{aligned} \frac{G_\varphi(r)}{F(\varphi(r))} &= (N-1)\frac{(2+\alpha)}{r}\int_0^{1-\epsilon}\left(\frac{s}{r}\right)^{(N-1)(2+\alpha)-1}\frac{F(\varphi(s))}{F(\varphi(r))}ds \\ &\quad + (N-1)\frac{(2+\alpha)}{r}\int_{1-\epsilon}^r\left(\frac{s}{r}\right)^{(N-1)(2+\alpha)-1}\frac{F(\varphi(s))}{F(\varphi(r))}ds \\ &\leq \frac{F(\varphi(1-\epsilon))}{F(\varphi(r))} + C\epsilon. \end{aligned}$$

Since $\varphi(r) \rightarrow +\infty$ when r goes to 1, let r be large enough, in order that $F(\varphi(r)) > \frac{F(\varphi(1-\epsilon))}{\epsilon}$, then letting ϵ go to zero we get the result. We have obtained that

$$\frac{\varphi'}{((2+\alpha)F(\varphi(r)))^{\frac{1}{2+\alpha}}} = \left(1 - \frac{G_\varphi(r)}{F(\varphi(r))}\right)^{\frac{1}{2+\alpha}} \sim_{r \rightarrow 1} 1. \quad (38)$$

Hence integrating between r and 1 one gets,

$$\int_r^1 \frac{\varphi'(r)dr}{((2+\alpha)F(\varphi(r)))^{\frac{1}{2+\alpha}}} \sim (1-r).$$

We have obtained the result since the integral on the left can be written after changing variable $\int_{\varphi(r)}^\infty \frac{ds}{((2+\alpha)F(s))^{\frac{1}{2+\alpha}}}$. \square

Remark 13. *It is contained in the proof of the previous proposition that if f is non decreasing near infinity, there exists some positive constant C such that for r close to 1,*

$$\frac{F(\varphi)}{C} \leq (\varphi')^{2+\alpha} \leq CF(\varphi).$$

This inequality is a mere consequence of (38). This will be used in the last section about uniqueness of radial solutions when f is non decreasing.

The following statement provides inequalities for radial solutions to general fully nonlinear equations.

Corollary 1. *Suppose that f satisfies the Keller Osserman condition, is non decreasing and that H is an operator which satisfies (H1) and (H2). Let φ be the minimal blow up solution relative to 0 of $H[u] = f(u)$ in the ball $B(0, 1)$. Then*

$$\left(\frac{2+\alpha}{a}\right)^{\frac{1}{2+\alpha}}(1-r) + o(1-r) \geq \int_{\varphi(r)}^{\infty} \frac{ds}{F(s)^{\frac{1}{2+\alpha}}} \geq \left(\frac{2+\alpha}{A}\right)^{\frac{1}{2+\alpha}}(1-r) + o(1-r).$$

Proof. Let us recall that if φ^+ is a blow up solution for H^+ in the ball, it is a blow up supersolution for $H[u] = f(u)$ and according to Theorem 5.1 $0 \leq \varphi \leq \varphi^+$. As a consequence one gets the right hand side inequality.

For the left hand side, let us note that φ is a boundary blow up super-solution for the equation $H^-[u] = f(u)$, and then there exists φ^- a boundary blowup solution for $H^-[u] = f(u)$, relative to zero with $\varphi^- \leq \varphi \leq \varphi^+$. Then the inequality on the left can easily be derived from the estimate for φ^- proved before. \square

We complete this section by a precision concerning the rate of convergence for solutions which blow up on part of the boundary of an annulus.

Proposition 7. *We assume that $H[u] = |\nabla u|^\alpha \Delta u$. We suppose that f satisfies the Keller Osserman condition and that φ is a radial minimal solution on the annulus $R_1 < |x| < R_2$ of*

$$\begin{cases} H[u] = f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } |x| = R_1, \\ u = 0 & \text{on } |x| = R_2. \end{cases}$$

Then

$$\int_{\varphi(r)}^{\infty} \frac{ds}{F(s)^{\frac{1}{2+\alpha}}} \sim_{r \rightarrow R_1} (2+\alpha)^{\frac{1}{2+\alpha}}(r - R_1). \quad (39)$$

Proof.

We act as before by observing that in place of φ' one must deal with $|\varphi'| = -\varphi'$. So we get

$$\frac{-\varphi'}{((2+\alpha)F(\varphi(r)))^{\frac{1}{2+\alpha}}} = \left(1 - \frac{G_\varphi(r)}{F(\varphi(r))}\right)^{\frac{1}{2+\alpha}} \sim_{r \rightarrow R_1} 1. \quad (40)$$

Hence integrating between R_1 and r one gets

$$\int_{\varphi(r)}^{\infty} \frac{ds}{((2+\alpha)F(s))^{\frac{1}{2+\alpha}}} \sim_{r \rightarrow R_1} (r - R_1). \quad (41)$$

\square

Remark 14. *The previous result is still valid for fully nonlinear operator satisfying (H1), (H2) and (H3), if we assume moreover that f is nondecreasing near infinity, once again to enforce $\varphi'' > 0$.*

6.2. Blow up rates of boundary blow up solutions in arbitrary domains.

We still assume in this section that $H[u] = |\nabla u|^\alpha \Delta u$. We want to extend the estimate near the boundary for radial solutions in balls to blow up solutions in arbitrary domains.

Theorem 6.1. *Assume that Ω satisfies the uniform interior and exterior sphere condition. Assume that f satisfies the Keller Osserman condition and let u be a blow up solution in Ω . Then for all $x_0 \in \partial\Omega$, for all x that converges towards x_0 in such a way that the open ball centered at x and of radius $|x - x_0|$ is included in Ω*

$$\lim_{x \rightarrow x_0} \frac{\int_{u(x)}^{\infty} \frac{dt}{((2+\alpha)F(t))^{\frac{1}{2+\alpha}}}}{d(x, \partial\Omega)} = 1.$$

Proof. We argue as in [25]. Let x close to Ω and $B(x, |x_0 - x|)$ be a ball tangent to $\partial\Omega$ on x_0 . Let η be < 1 . Let $v(y) = \sup_{O \in O(\mathbb{R}^N)} u(O(y - x) + x)$. Then v is a sub-solution in $B(x, \eta|x - x_0|)$, it is radial and $u \leq v$. Consider the minimal blow up solution φ on $B(x, \eta|x - x_0|)$ relative to v . Then

$$\int_{u(x)}^{\infty} \frac{dt}{(F(t))^{\frac{1}{2+\alpha}}} \geq \int_{\varphi(x)}^{\infty} \frac{dt}{(F(t))^{\frac{1}{2+\alpha}}} \sim (2 + \alpha)^{\frac{1}{2+\alpha}} \eta |x - x_0|. \quad (42)$$

This leads to, letting η going to 1,

$$\lim_{x \rightarrow x_0} \frac{\int_{u(x)}^{\infty} \frac{dt}{((2+\alpha)F(t))^{\frac{1}{2+\alpha}}}}{d(x, \partial\Omega)} \geq 1.$$

We now move to the reverse inequality. We consider some ball $B(x_1, |x_1 - x_0|)$ included in $\mathbb{R}^N \setminus \Omega$ with $\overline{B(x_1, |x_1 - x_0|)} \cap \overline{\Omega} = \{x_0\}$. Let $B(x_1, R')$ with R' large enough which contains Ω . Let η be < 1 . Consider the annulus $B(x_1, R') \setminus B(x_1, \eta|x_1 - x_0|)$. Let v be the minimal solution in the annulus of

$$\begin{cases} H[v] = f(v) & \text{in } B(x_1, R') \setminus B(x_1, \eta|x_1 - x_0|) \\ v = +\infty & \text{on } |y - x_1| = \eta|x_1 - x_0| \\ v = 0 & \text{on } |y - x_1| = R' \end{cases}$$

Since $v < +\infty$ on $\partial\Omega$, then $u \geq v$ on Ω . Then for $x \in \Omega$ such that $x_0 - x = x_1 - x_0$

$$\begin{aligned} \int_{u(x)}^{\infty} \frac{dt}{(F(t))^{\frac{1}{2+\alpha}}} &\leq \int_{v(x)}^{\infty} \frac{dt}{(F(t))^{\frac{1}{2+\alpha}}} \\ &\sim (2 + \alpha)^{\frac{1}{2+\alpha}} ((2 - \eta)(|x - x_0|)) \\ &\sim (2 + \alpha)^{\frac{1}{2+\alpha}} (2 - \eta) d(x, \partial\Omega), \end{aligned}$$

using the estimate in Proposition 7. We let $\eta \rightarrow 1$ to obtain the result. \square

Remark 15. *Of course arguing as in the proof of Corollary 1, if H is any operator satisfying (H1) (H2), and if f is non decreasing, any BBUS for H in Ω satisfies the inequalities*

$$\begin{aligned} \frac{1}{A^{\frac{1}{2+\alpha}}} &\leq \liminf_{x \rightarrow x_0} \frac{\int_{u(x)}^{\infty} \frac{dt}{((2+\alpha)F(t))^{\frac{1}{2+\alpha}}}}{d(x, \partial\Omega)} \leq \\ \limsup_{x \rightarrow x_0} \frac{\int_{u(x)}^{\infty} \frac{dt}{((2+\alpha)F(t))^{\frac{1}{2+\alpha}}}}{d(x, \partial\Omega)} &\leq \frac{1}{a^{\frac{1}{2+\alpha}}}. \end{aligned} \quad (43)$$

7. The uniqueness of radial BBUS in the case where f is nondecreasing.

In this section we assume that f satisfies the Keller Osserman condition, is nondecreasing, and we prove the uniqueness of radial BBUS for the operators H^\pm . We also prove the uniqueness of BBUS on balls.

Proposition 8. *Suppose that f is non decreasing. The radial solutions of*

$$H^\pm[\varphi] = f(\varphi) \quad (44)$$

in the ball $B(0, R)$ are ordered.

Proof. Since f is nondecreasing we know that the equation satisfied by radial solutions is either $A|\nabla u|^\alpha \Delta u = f(u)$ or $a|\nabla u|^\alpha \Delta u = f(u)$ so, up to changing f by $\frac{f}{a}$ or $\frac{f}{A}$, we can assume that $a = A = 1$.

Suppose that $u_1(0) = u_2(0)$, by $u_1'(0) = 0 = u_2'(0)$, then Proposition 3 gives that $u_1 \equiv u_2$.

Suppose that $u_1(0) > u_2(0)$, and suppose by contradiction that r_0 is such that for $r_0 > r > 0$, $u_1 > u_2$ and $u_1(r_0) = u_2(r_0)$. Then by the mean value theorem there exists $r_1 \in]0, r_0[$, with $(u_1 - u_2)'(r_1) < 0$. Now we observe that for all $r \in]0, r_0[$, $r^{(N-1)(1+\alpha)}(u_1')^{1+\alpha}(r) - r^{(N-1)(1+\alpha)}(u_2')^{1+\alpha}(r) = \int_0^r s^{(N-1)(1+\alpha)}(f(u_1(s)) - f(u_2(s)))ds \geq 0$. This implies that $(u_1 - u_2)'(r) \geq 0$ for $r \in]0, r_0[$, a contradiction. \square

Let us observe that this Proposition has also the following consequence :

If u and v are two radial solutions in $B(0, R)$ and $u < v$ on $\partial B(0, R)$, $u < v$ in Ω .

This will be used in step 1 in Theorem 7.1.

Theorem 7.1. *Suppose that f satisfies the Keller Osserman condition, that f is a nondecreasing and C^1 function such that $f(0) = 0$. Consider the fully nonlinear equation*

$$H^\pm[u] = f(u).$$

There exists a unique boundary blow up solution for this equation on the unit ball.

Proof. As in the previous proof one can assume that $a = A = 1$.

Step 1: To begin with we observe that any boundary blow up solution u can be sandwiched between two radial boundary blow up solutions. By below by the minimal boundary blow up solution, that is radial. By above, as follows. Consider a smaller ball of radius $R < 1$ than the unit ball and the minimal BBUS u_R corresponding to this ball. Then for $\varepsilon < R$ small enough, $u < u_R$ on $B(0, R) \setminus \overline{B(0, R - \varepsilon)}$. Hence by the maximum principle $u < u_R$ everywhere on the ball of radius $R - \varepsilon$, and then on the ball of radius R . Since f is non decreasing, using the argument in the end of the proof of Proposition 8 the sequence $R \mapsto u_R$ is decreasing and converges to a BBUS which is radial and larger than u .

Step 2: Consider u_1 and u_2 two radial boundary BBUS. Using Proposition 8 we can assume that $u_1(r) < u_2(r)$ for any r . Let us recall that any radial BBUS satisfies

$$|u'|^\alpha(u'' + \frac{N-1}{r}u') = f(u). \quad (45)$$

We use the change of variable introduced by Costin and Dupaigne, [19]. Instead of considering u as a function of r , we consider r as a function of u , so we define $g(u(r)) = r$, that we differentiate to obtain $1 = g'(u(r))u'(r)$. Let $V(u) = u'(g(u))$.

Differentiating one obtains $V'(u) = u''(g(u))g'(u)$ and then $u''(g(u)) = V'(u)V(u)$. In the following we will denote for simplicity g as $r(u)$.

The previous equation reads then

$$V^\alpha \left(V \frac{dV}{du} + \frac{N-1}{r} V \right) = f(u). \quad (46)$$

One has $r'(u) = \frac{1}{V(u)}$ and then $r'(u)u'(r) = \frac{u'(r)}{V(u(r))}$. Let us integrate between r and 1 and use the change of variable $u(r)$ to get $g(u(1)) - g(u(r)) = \int_r^1 \frac{u'(r)}{V(u(r))} = \int_u^\infty \frac{1}{V(s)} ds$ which is nothing else but

$$1 - r = \int_u^{+\infty} \frac{ds}{V(s)}. \quad (47)$$

Step 3: To u_1 (respectively u_2) we associate r_1, V_1 (respectively r_2, V_2). We claim and prove that $r_2 < r_1$. Actually, $r_1(u)$ is the time r when the trajectory $u_1(r)$ reaches the level u . Since $u_1 < u_2$, the trajectory u_2 reaches the level u before u_1 .

Step 4: We state and prove: $V_2 < V_1$ for u large enough. Assume that this inequality is not valid. Then two cases may occur. Either $V_1 < V_2$ for u large enough, or there exists a sequence $u_k \rightarrow +\infty$ such that $(V_1(u_k) - V_2(u_k))(-1)^k > 0$ and $\frac{d}{du}((V_1 - V_2)(u_k)) = 0$. In the former case, we have

$$1 - r_2 = \int_u^{+\infty} \frac{1}{V_2} \leq \int_u^{+\infty} \frac{1}{V_1} = 1 - r_1, \quad (48)$$

and then a contradiction with Step 3. In the latter case, we write

$$\frac{d}{du}(V_2 - V_1) + (N-1)\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = f(u)\left(\frac{1}{V_2^{\alpha+1}} - \frac{1}{V_1^{\alpha+1}}\right). \quad (49)$$

Computing this at $u = u_{2k}$ leads to a contradiction.

Step 5: We now prove that $V_1 - V_2$ converges to 0 while u goes to $+\infty$. In this step we prove an inequality. Introduce now the function

$$w(u) = (r_1^{(\alpha+2)(N-1)} V_1^{\alpha+2} - r_2^{(\alpha+2)(N-1)} V_2^{\alpha+2})(u).$$

We infer from (46) that

$$\frac{dw}{du} = (\alpha+2)f(u)(r_1^{(\alpha+2)(N-1)} - r_2^{(\alpha+2)(N-1)}). \quad (50)$$

Therefore, due to the previous step, the function w is non decreasing. We plan to integrate this identity between $u_0 = u_2(\frac{1}{2})$ and u . Before this, we make an observation on the choice of u_0 .

Let us recall (see Remark 13) that there exists some constant C_* such that for any blow up solution u and for some u'_0 large enough and $u \geq u'_0$

$$\frac{1}{C_*} F(u) \leq V^{\alpha+2} \leq C_* F(u), \quad (51)$$

Define then $c(N, \alpha) = (N-1)2^{(2\alpha+3)(N-1)-1} C_*^{\alpha+3}$ and take u'_0 large enough such that for $u_0 \geq u'_0$, due to Keller-Osserman hypothesis

$$c(N, \alpha) \int_{u_0}^{+\infty} \frac{1}{F(v)^{\frac{1}{\alpha+2}}} dv \leq \frac{1}{2}. \quad (52)$$

If this is not valid for our choice $u'_0 = u_2(\frac{1}{2})$, we set $u'_0 = u_2(r_0)$ for r_0 close enough to 1. We set $u_0 = u_2(\frac{1}{2})$ to fix ideas.

On that set $r_2, r_1 \geq \frac{1}{2}$, so we get

$$\begin{aligned} & [(r_1^{(N-1)}V_1)^{\alpha+2} - (r_2^{(N-1)}V_2)^{\alpha+2}](u) \\ & - [(r_1^{(N-1)}V_1)^{\alpha+2} - (r_2^{(N-1)}V_2)^{\alpha+2}](u_2(\frac{1}{2})) \\ & = \int_{u_2(\frac{1}{2})}^u f(v)(r_1^{(\alpha+2)(N-1)} - r_2^{(\alpha+2)(N-1)})(v)dv \\ & \leq (\alpha+2)(N-1)2^{(\alpha+2)(N-1)-1} \int_{u_2(\frac{1}{2})}^u f(v)(r_1(v) - r_2(v))dv. \end{aligned}$$

Introducing C which denotes a generic constant that depends on u_1, u_2 and that may vary from one line to one another, the previous inequality reads also

$$0 \leq w(u) \leq C + (\alpha+2)(N-1)2^{(\alpha+2)(N-1)-1} \int_{u_0}^u f(v)(r_1(v) - r_2(v))dv. \quad (53)$$

Going back to the very definition of $r = 1 - \int_u^{+\infty} \frac{1}{V}$, we then obtain

$$0 \leq w(u) \leq C + (\alpha+2)(N-1)2^{(\alpha+2)(N-1)-1} \int_{u_0}^u f(v) \left(\int_v^{+\infty} \frac{V_1(s) - V_2(s)}{V_1(s)V_2(s)} ds \right) dv. \quad (54)$$

Since $V_1 - V_2 \leq \frac{1}{\alpha+2} \frac{V_1^{\alpha+2} - V_2^{\alpha+2}}{\inf(V_1^{\alpha+1}, V_2^{\alpha+1})}$ and we obtain using (51)

$$\frac{V_1(s) - V_2(s)}{V_1(s)V_2(s)} \leq \frac{C_*^{\frac{\alpha+3}{\alpha+2}} (V_1^{\alpha+2} - V_2^{\alpha+2})}{(\alpha+2)(F(s))^{\frac{\alpha+3}{\alpha+2}}}. \quad (55)$$

On the other hand, since $r_1 \geq r_2 \geq \frac{1}{2}$, then

$$V_1^{\alpha+2} - V_2^{\alpha+2} \leq 2^{(N-1)(1+\alpha)} (r_1^{(N-1)(\alpha+2)} (V_1^{\alpha+2} - V_2^{\alpha+2})) \leq 2^{(N-1)(1+\alpha)} w(u). \quad (56)$$

Gathering all these inequalities together we obtain that

$$0 \leq w(u) \leq C + c(N, \alpha) \int_{u_0}^u f(v) \left(\int_v^{+\infty} \frac{w(s)}{F(s)^{\frac{\alpha+3}{\alpha+2}}} ds \right) dv. \quad (57)$$

We now integrate by parts to obtain, recalling that $F' = f$,

$$0 \leq w(u) \leq C + c(N, \alpha) \left(\int_{u_0}^u \frac{w(v)}{F(v)^{\frac{1}{\alpha+2}}} dv + F(u) \int_u^{+\infty} \frac{w(s)}{F(s)^{\frac{\alpha+3}{\alpha+2}}} ds \right). \quad (58)$$

We now observe that since w is non decreasing

$$\int_{u_0}^u \frac{w(v)}{F(v)^{\frac{1}{\alpha+2}}} dv \leq w(u) \int_{u_0}^{+\infty} \frac{1}{F(v)^{\frac{1}{\alpha+2}}} dv. \quad (59)$$

we are led to

$$0 \leq w(u) \leq 2C + 2c(N, \alpha) F(u) \int_u^{+\infty} \frac{w(s)}{F(s)^{\frac{\alpha+3}{\alpha+2}}} ds. \quad (60)$$

Step 6: Introduce $G(u) = \int_u^{+\infty} \frac{w(s)}{F(s)^{\frac{\alpha+3}{\alpha+2}}} ds$. On the one hand, due to Keller-Osserman assumption, $\lim_{u \rightarrow +\infty} G(u) = 0$; this is valid since $w(s) \leq CF(s)$. On the other hand, due to (60), we have

$$0 \leq -G'(u)F(u)^{\frac{\alpha+3}{\alpha+2}} \leq 2C + 2c(N, \alpha)F(u)G(u). \quad (61)$$

Therefore, this implies

$$0 \leq \frac{\tilde{C}}{F(u)^{\frac{\alpha+3}{\alpha+2}}} + \frac{d}{du}(G(u) \exp(-2c(N, \alpha) \int_u^{+\infty} \frac{1}{F(s)^{\frac{1}{\alpha+2}}} ds)). \quad (62)$$

Integrating this between u and $+\infty$, and using Keller-Osserman assumption, we have that there exists a constant C such that

$$G(u) \leq C \int_u^{+\infty} \frac{1}{F(s)^{\frac{\alpha+3}{\alpha+2}}} ds \leq \frac{C}{F(u)}. \quad (63)$$

Going back to (60) we have that w is a bounded function of u . Therefore, using (56)

$$V_1^{\alpha+2}(u) - V_2^{\alpha+2}(u) \leq C, \quad (64)$$

Step 7: We prove that $u_2 - u_1 \rightarrow 0$ when r goes to 1.

We start from

$$1 - r = \int_{u_1}^{+\infty} \frac{1}{V_1} = \int_{u_2}^{+\infty} \frac{1}{V_2}. \quad (65)$$

This leads to

$$\int_{u_1}^{u_2} \frac{1}{V_1} = \int_{u_2}^{+\infty} \frac{V_1 - V_2}{V_1 V_2}. \quad (66)$$

Using (51), (66), (55) and (64) we then have

$$\frac{u_2 - u_1}{F(u_2)^{\frac{1}{\alpha+2}}} \leq C \int_{u_2}^{+\infty} \frac{V_1 - V_2}{V_2 V_1} \leq C \int_{u_2}^{+\infty} \frac{du}{(F(u))^{\frac{\alpha+3}{\alpha+2}}} \leq \frac{C}{(F(u_2)^{\frac{1}{\alpha+2}})} \int_{u_2}^{+\infty} \frac{du}{(F(u))} \quad (67)$$

and finally

$$u_2 - u_1 \leq C \int_{u_2}^{+\infty} \frac{1}{F}. \quad (68)$$

From this one derives that $\lim_{r \rightarrow 1} (u_2 - u_1) = 0$.

Step 8

Consider $u_1 < u_2$ two radial solutions. Then

$$\frac{d}{dr}(r^{(N-1)(1+\alpha)}(u_2 - u_1)) = (f(u_2) - f(u_1))r^{(N-1)(1+\alpha)} \geq 0. \quad (69)$$

Then $r \mapsto r^{(N-1)(1+\alpha)}(u_2 - u_1)$ is positive nondecreasing, bounded and goes to zero when r tends to 1. Then suppose that there exists $0 < r_0 < 1$ with $u_2(r_0) > u_1(r_0)$ and let $r > r_0$ so that by the asymptotic behaviour, $r^{(N-1)(1+\alpha)}(u_2 - u_1)(r) < \frac{r_0^{(N-1)(1+\alpha)}(u_2 - u_1)(r_0)}{2}$, then we get a contradiction. \square

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