Regularity and uniqueness of the first eigenfunction for singular fully noninear equations

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1 Introduction

The concept of principal eigenvalue for boundary value problems of elliptic operators has been extended, in the last decades, to quasi-nonlinear and fully-nonlinear equations (somehow "abusing" the name of eigenvalue) see [1, 2, 28, 31, 24, 6, 25] etc.. In all the cases we know, two features of the operators are requested, *homogeneity* and *ellipticity*.

The meta-definition of these principal eigenvalues could be the following: Given a zero order and odd operator H with the same homogeneity than the second order elliptic operator F, and given a domain Ω,

λ is a principal eigenvalue if there exists a non trivial solution of constant sign of the problem

$$\begin{cases} F(x, \nabla u, D^2 u) + \lambda H(u) = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega; \end{cases}$$

not surprisingly, that function will be called the "eigenfunction".

Of course when F is a second order linear elliptic operator and $H(u) = u$, the principal eigenvalue is just the first eigenvalue in the "classical" sense, it is well known that it is both *simple* and *isolated*.

It is interesting to notice that when the operator is not "odd" with respect to the Hessian, it can be expected that there are two principal eigenvalues, one

corresponding to a positive eigenfunction and one corresponding to a negative eigenfunction, and this is the case for example when F is one of the Pucci operators e.g. for some $0 < a < A$

$$F(D^2u) := \mathcal{M}^+_{a,A}(D^2u) := a\sum_{e_i<0} e_i + A\sum_{e_i>0} e_i$$

where the e_i denote the eigenvalues of the matrix D^2u (see [10]). In these cases, or in general when the operator is uniformly elliptic and homogenous of degree 1, the principal eigenvalues have been proved to be simple and isolated (see [31, 24, 29]).

When the operator is "quasi linear" but in divergence form , for example in the case of the p-Laplacian, it is well known that the principal eigenvalue can be defined through the Rayleigh quotient and it was proved, independently by Anane [1] and by Otani and Teshima [28], that it is simple and isolated (see also [27]). The variational structure plays a key role there. On the other hand, for the ∞-Laplacian (see [25]), the question of the simplicity of the principal eigenvalue is still open.

The cases treated in this paper concern operators that have the "homogeneity" of the p-Laplacian, but are "fully-nonlinear" and hence are not variational. In previous works, we proved the existence of the principal eigenvalues for this large class of operators and many features related to them ([5],...,[8]). The inspiration for these definitions and results was the acclaimed work of Berestycki, Nirenberg and Varadhan [4] where the eigenvalue for linear elliptic operators in general bounded domains was defined through the maximum principle.

The main questions left open in our previous works were: are these eigenvalues "simple" ? are they "isolated"?

We shall now proceed to describe the results obtained in this note but for the sake of comprehension we shall do it for an operator that exemplifies well the cases treated here (the general conditions and hypothesis will be given in the next section). For some $0 \geq \alpha > -1$, and some Hölder continuous function h of exponent smaller than $1 + \alpha$, let:

$$F[u] := F(x, \nabla u, D^2u) = |\nabla u|^\alpha \mathcal{M}^+_{a,A}(D^2u) + h(x) \cdot \nabla u |\nabla u|^\alpha.$$

Suppose that Ω is a bounded, smooth domain of $\mathbb{R}^N$. Then we can define

$$\lambda^+ := \sup\{\lambda; \exists \phi > 0, \phi \in \mathcal{C}(\overline{\Omega}), \quad F[\phi] + \lambda\phi^{1+\alpha} \leq 0 \quad \text{in } \Omega\},$$

$$\lambda^- := \sup\{\lambda; \exists \phi < 0, \phi \in \mathcal{C}(\overline{\Omega}), \quad F[\phi] + \lambda\phi|\phi|^\alpha \geq 0 \quad \text{in } \Omega\}.$$

It is clear that the inequalities are meant in the "viscosity sense" adapted to these non smooth operators (see the next section for a precise definition).

For a much more general context and for any $\alpha > -1$ we proved in [6] that these are well defined; that for any $\lambda < \lambda^+$ the maximum principle holds i.e. considering the problem

$$\begin{cases} F[u] + \lambda u|u|^\alpha = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

if u is a viscosity subsolution of (1.1) then $u \leq 0$ in Ω and for any $\lambda \leq \min\{\lambda^+, \lambda^-\}$ and any continuous f there exists a solution of

$$\begin{cases} F[u] + \lambda u|u|^\alpha = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

which is Lipschitz continuous.

Of course we also proved that there exists ϕ^+ and ϕ^- respectively positive and negative eigenfunctions in the sense that e.g. for $\lambda = \lambda^+$ there exists $\phi^+ > 0$ viscosity solution of (1.1).

One of the question we raise here is , if $\phi > 0$ is another solution of (1.1) with $\lambda = \lambda^+$, is it true that there exists $t > 0$ such that $\phi^+ = t\phi$? The answer is yes for $\alpha \in (-1, 0]$, for any domain Ω such that $\partial\Omega$ has only one connected component. When $\partial\Omega$ has two connected components we can prove the result when $N = 2$.

Let us recall that for any $\alpha > -1$ and any N the simplicity of the eigenvalue has been proved in [8] in the case of radial solutions.

It is clear that these results are somehow equivalent to a "strong comparison principle" i.e. it is equivalent to know that if two solutions are one above the other in some open set $\mathcal{O}$ and they "touch" at some point of $\mathcal{O}$ then they coincide in $\mathcal{O}$. Now in Proposition 4.3, we prove such a result when at least one of the solutions has the gradient away from zero in $\mathcal{O}$.

This restriction implies that in order to apply Proposition 4.3 we need to know that there is some subset $\mathcal{O}$ of Ω where this condition is satisfied. Naturally the Hopf's lemma together with a C^1 regularity is the convenient ingredient since it guarantees that the gradient is away from zero in a neighborhood of $\partial\Omega$. This explains why we start by proving a $W^{2,p}$ regularity result, which is interesting in itself but furthermore is essential in the proof of the simplicity of the principal eigenvalue.

This is done through a fixed point theorem. Let us mention that in general the tools to prove regularity are the Alexandrov-Bakelman-Pucci inequality and some "sub-linearity" of the operator. In a recent paper Davila, Felmer and Quaas have proved ABP [16] for singular fullynonlinear operators, but in this case it does not seem to be useful to prove $W^{2,p}$ regularity because the difference of a sub and super solution may not be a sub-solution of an elliptic equation. It is interesting to remark that Imbert [23] used ABP to prove a local Hölder regularity.

The dimensional restriction is due to the fact that when there are two connected components of the boundary of Ω, we use Sard's Theorem. A famous counterexample of Whitney shows that Sard's theorem doesn't hold if the functions are only C^1. It seems that the least regularity that can be asked is $W^{N,p}$ (see e.g. [21]), and since the solutions are in $W^{2,p}$ we have to take $N = 2$.

On the other hand the restriction on α negative is only due to the fact that we can prove $C^{1,\beta}$ regularity for those α.

Other important results concerning eigenvalues are given as a consequence of simplicity. In particular we prove that there are no eigenfunctions that change sign for $\lambda = \lambda^{\pm}$. Further results include the strict monotonicity of the eigenvalue with respect to the inclusion of domains. And finally that the eigenvalues are isolated.

The paper is organized as follows: In the next section we give the precise hypothesis concerning the operator and we recall the known results concerning singular operators. In section three we prove the $C^{1,\beta}$ regularity of the solutions. Section four is devoted to the strict comparison principle and simplicity of the principal eigenvalues. We end the paper with other properties of the eigenvalues.

2 Assumptions and known results

Let Ω be a bounded regular domain of $\mathbb{R}^N$.

Let us recall what we mean by *viscosity solutions*, adapted to our context.

Definition 2.1 *Let Ω be a bounded domain in $\mathbb{R}^N$, let g be a continuous function on $\Omega \times \mathbb{R}$, then v, continuous on $\overline{\Omega}$ is called a viscosity super-solution (respectively sub-solution) of $F(x, \nabla u, D^2u) = g(x, u)$ if for all $x_0 \in \Omega$,*

-Either there exists an open ball $B(x_0, \delta)$, $\delta > 0$ in Ω on which $v = cte = c$ and $0 \leq g(x, c)$, for all $x \in B(x_0, \delta)$ (respectively $0 \geq g(x, c)$ for all $x \in B(x_0, \delta)$).

-Or $\forall \varphi \in \mathcal{C}^2(\Omega)$, such that $v - \varphi$ has a local minimum (respectively local maximum) at x_0 and $\nabla\varphi(x_0) \neq 0$, one has

$$F(x_0, \nabla\varphi(x_0), D^2\varphi(x_0)) \leq g(x_0, v(x_0)).$$

(respectively

$$F(x_0, \nabla\varphi(x_0), D^2\varphi(x_0)) \geq g(x_0, v(x_0)).)$$

A viscosity solution is a function which is both a super-solution and a sub-solution.

Remark 2.2 *When F is continuous in p, and $F(x, 0, 0) = 0$, this definition is equivalent to the classical definition of viscosity solutions, as in the User's guide [15].*

We now state the assumptions satisfied by the operator F. Let S be the set of $N \times N$ symmetric matrices, and let $\alpha \in]-1, 0[$. Then F defined on $\Omega \times \mathbb{R}^N \setminus \{0\} \times S$ satisfies

$$F(x, p, M) = |p|^\alpha (\tilde{F}(x, M) + h(x) \cdot p). \tag{2.1}$$

on $\tilde{F}$ we suppose

(F) $\tilde{F}(x, tM) = t\tilde{F}(x, M)$ for any $t \in \mathbb{R}^+$, and there exist $A \geq a > 0$ such that for any $M \in S$, and any $N \in S$ such that $N \geq 0$

$$atr(N) \leq \tilde{F}(x, M + N) - \tilde{F}(x, M) \leq Atr(N). \tag{2.2}$$

Furthermore $(x, M) \mapsto \tilde{F}(x, M)$ is continuous.

(J) There exists a continuous function ω with $\omega(0) = 0$, such that if $(X, Y) \in S^2$ and $\zeta \in \mathbb{R}^+$ satisfy

$$-\zeta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 4\zeta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and I is the identity matrix in $\mathbb{R}^N$, then for all $(x, y) \in \mathbb{R}^N$, $x \neq y$

$$\tilde{F}(x, X) - \tilde{F}(y, -Y) \leq \omega(\zeta|x - y|^2).$$

On h we suppose that :

(H) h is Hölder continuous of exponent $1 + \alpha$.

Remark 2.3 *Since $\alpha < 0$, if $\tilde{F}$ satisfies condition (J), then so does F i.e. for X, Y and ζ as above*

$$F(x, \zeta(x-y), X) - F(y, \zeta(x-y), -Y) \leq \omega(\zeta|x-y|^2).$$

Example 2.4 *We suppose that h satisfies (H).*

1) Let $0 < a < A$ and $\mathcal{M}^+_{a,A}(M)$ be the Pucci's operator $\mathcal{M}^+_{a,A}(M) = Atr(M^+) - atr(M^-)$ where $M^\pm$ are the positive and negative part of M, and $\mathcal{M}^-_{a,A}(M) = -\mathcal{M}^+_{a,A}(-M)$. Then F defined as

$$F(x, p, M) = |p|^\alpha(\mathcal{M}^\pm_{a,A}(M) + h(x) \cdot p)$$

satisfies the assumptions.

2) More generally let $B(x)$ be some matrix with Lipschitz coefficients which is invertible for all $x \in \Omega$. Let us consider $A(x) = B^\star B(x)$ and the operator

$F(x, p, M) = |p|^\alpha(tr(A(x)(M)) + h(x) \cdot p)$, then $\tilde{F}$ satisfies (F) and (J), arguing as in [5], example 2.4.

We begin to recall some of the results obtained in [6] which will be needed in this article.

Theorem 2.5 *Suppose that F, h are as above and c is continuous and bounded and satisfies $c \leq 0$.*

Suppose that f_1 and f_2 are continuous and bounded and that u and v satisfy

$$\begin{array}{rcl} F(x, \nabla u, D^2u) + c(x)|u|^\alpha u & \geq & f_1 \quad in \quad \Omega \\ F(x, \nabla v, D^2v) + c(x)|v|^\alpha v & \leq & f_2 \quad in \quad \Omega \\ u \leq v & & on \quad \partial\Omega. \end{array}$$

Suppose that $f_2 < f_1$, then $u \leq v$ in Ω. Moreover if $c < 0$ in Ω, and $f_2 \leq f_1$ the result still holds.

This comparison theorem allows to prove, using the existence of sub- and supersolutions constructed with the aid of the distance function to $\partial\Omega$, together with Perron's method adapted to our context, the following existence's result :

Theorem 2.6 *Suppose that F, h and c are as above and that $c \leq 0$. Suppose that f is continuous and bounded, then there exists a continuous solution to*

$$\begin{cases} F(x, \nabla u, D^2 u) + c(x)|u|^\alpha u = f & in \quad \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

If furthermore $f \leq 0$, $u \geq 0$, and if $f \geq 0$, $u \leq 0$. Moreover if $c < 0$ in Ω, the solution is unique.

Remark 2.7 *The Hopf principle asserts that if $\mathcal{O}$ is a smooth bounded domain, and u is a solution of*

$$F(x, \nabla u, D^2 u) \leq 0 \tag{2.4}$$

in $\mathcal{O}$, such that $u > c$ inside $\mathcal{O}$ and $u(\bar{x}) = c$ at some boundary point of $\mathcal{O}$ then "$\frac{\partial u}{\partial n}(\bar{x})$" < 0.

In particular this implies that a non constant solution of (2.4) in a domain Ω has no interior minimum.

We also recall some regularity results

Proposition 2.8 *Suppose that $\tilde{F}$ satisfies (F), (J) and h is continuous. Let f be some continuous function on $\overline{\Omega}$. Let u be a viscosity non-negative bounded solution of*

$$\begin{cases} F(x, \nabla u, D^2 u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Then, for any $\gamma \leq 1$ there exists some constant C which depends only on $|f|_\infty$, $|h|_\infty$ and $|u|_\infty$ such that :

$$|u(x) - u(y)| \leq C|x - y|^\gamma$$

for any $(x, y) \in \overline{\Omega}^2$.

This Proposition implies some compactness for bounded sequences of solutions, this will be used in section 3.

We shall also need in the proof of Theorem 4.1 the following comparison principle, proved in [7].

Theorem 2.9 *Suppose that $\tilde{F}$ satisfies (F), (J), and h satisfies (H), that c is continuous and bounded. Suppose that $c + \lambda$ is positive on $\overline{\Omega}$ and that u and v are respectively positive continuous super and sub solutions of*

$$|\nabla u|^\alpha \left(\tilde{F}(x, D^2 u) + h(x).\nabla u \right) + (c(x) + \lambda) u^{1+\alpha} = 0$$

1) If $u \geq v > 0$ on $\partial\Omega$ then $u \geq v$ in Ω.
2) If $u > v$ on $\partial\Omega$ then $u > v$ on $\overline{\Omega}$.

3 Regularity

In this section for $\alpha \leq 0$ we establish that the solutions of (2.5) are $W^{2,p}$ for all $p < \infty$, and consequently also $C^{1,\beta}$; this will be derived from the known regularity results in the case $\alpha = 0$.

We expect that some $C^{1,\beta}$ regularity of the solutions be true for more general operators i.e. operators that are only homogeneous and singularly elliptic but not necessarily of the form given in (2.1). As an example, in the second subsection we illustrate this for a class of operators which does not satisfy (2.1) and is somehow close to the ∞-Laplacian, though not as degenerate.

3.1 Regularity result for $\alpha \leq 0$

To prove the regularity results announced (which will be stated precisely in Corollary 3.3) , we remark that the solutions of the Dirichlet problem can be obtained as a fixed point of some operator acting in $\mathcal{C}_o(\Omega) \cap W^{1,\infty}$.

We define $\mathcal{C}_o(\Omega)$ as the space of continuous function on $\overline{\Omega}$ which are zero on the boundary, and $\mathcal{L}(\Omega) := \{u \in \mathcal{C}_o(\Omega);\ u \text{ is Lipschitz }\}$ with the norm $|u|_{\mathcal{L}} := \sup_{x,y\in\Omega, x\neq y} \dfrac{|u(x)-u(y)|}{|x-y|}$.

And we suppose here that $g : \mathbb{R}^+ \to (\mathbb{R}^+)^\star$ is continuous and decreasing, while $l \mapsto lg(|l|)$ is increasing on $\mathbb{R}$.

Proposition 3.1 *Suppose that $\tilde{F}$, h satisfy respectively (F), (J) and (H). Let $f \in L^\infty(\Omega)$, let T_ϵ be the operator $\mathcal{L} \mapsto \mathcal{L}$ such that $T_\epsilon u = v$, where v is the unique solution of*

$$\begin{cases} \tilde{F}(x, D^2 v) + h(x)\cdot\nabla v = (f + \epsilon g(|u|)u)\frac{1}{g(|\nabla u|)} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

then there exists ϵ_o small enough in order that for $\epsilon < \epsilon_o$, T_ϵ has a non trivial fixed point in $\mathcal{L}$.

Remark 3.2 *The solution is taken in the sense of L^p viscosity solutions,see [13] and [34].*

Before giving the proof of the Proposition 3.1 we shall prove the main result of this section i.e. the following

Corollary 3.3 *Suppose that $\tilde{F}$, h satisfy respectively (F) (J) and (H) and that $f \in \mathcal{C}(\overline{\Omega})$. Let u be a solution in the sense of Definition 2.1 of*

$$\begin{cases} |\nabla u|^\alpha \left(\tilde{F}(x, D^2u) + h(x)\cdot\nabla u\right) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Then $u \in W^{2,p}(\Omega)$ for all $p < \infty$, hence it is $\mathcal{C}^{1,\beta}(\overline{\Omega})$ for all $\beta \in (0,1)$.

Proof of Corollary 3.3. We prove first the announced regularity result for the equation

$$\begin{cases} |\nabla u|^\alpha \left(\tilde{F}(x, D^2u) + h(x)\cdot\nabla u\right) - \epsilon|u|^\alpha u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for ϵ small enough.

Let $\delta > 0$ and suppose that v_δ is a fixed point for T_ϵ with $g(l) = (|l|^2 + \delta^2)^{\frac{\alpha}{2}}$ which exists by Proposition 3.1. We shall prove that it is the unique solution, in the sense of Definition 2.1, of

$$\begin{cases} (|\nabla u|^2 + \delta^2)^{\frac{\alpha}{2}} \left(\tilde{F}(x, D^2u) + h(x)\cdot\nabla u\right) - \epsilon(\delta^2 + |u|^2)^{\frac{\alpha}{2}} u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Observe first that since $\tilde{F}$ satisfies (F), by standard regularity results $v_\delta \in W^{2,p}(\Omega)$ for all $p < \infty$ (see Evans [18], Caffarelli- Cabré [14] and Winter [34]).

We prove only the super-solution case. Let $\bar{x} \in \Omega$ and φ be in $\mathcal{C}^2$ such that $(v_\delta - \varphi)(x) \geq (v_\delta - \varphi)(\bar{x}) = 0$, with $\nabla\varphi(\bar{x}) \neq 0$. Since v_δ is $\mathcal{C}^{1,\beta}$, $\nabla\varphi(\bar{x}) = \nabla v_\delta(\bar{x})$ and since v_δ is a super-solution of the fixed point equation:

$$\begin{aligned} \tilde{F}(\bar{x}, D^2\varphi(\bar{x})) + h(\bar{x})\cdot\nabla\varphi(\bar{x}) &\leq (f(\bar{x}) + \epsilon(|v_\delta(\bar{x})|^2 + \delta^2)^{\frac{\alpha}{2}} v_\delta(\bar{x}))(|\nabla v_\delta|^2(\bar{x}) + \delta^2)^{-\frac{\alpha}{2}} \\ &= (f + \epsilon(|\varphi(\bar{x})|^2 + \delta^2)^{\frac{\alpha}{2}}\varphi(\bar{x}))(|\nabla\varphi|^2(\bar{x}) + \delta^2)^{\frac{-\alpha}{2}} \end{aligned}$$

i.e.

$$(|\nabla\varphi|^2(\bar{x}) + \delta^2)^{\frac{\alpha}{2}} \left(\tilde{F}(\bar{x}, D^2\varphi(\bar{x})) + h(\bar{x})\cdot\nabla\varphi(\bar{x})\right) - \epsilon(|\varphi(\bar{x})|^2 + \delta^2)^{\frac{\alpha}{2}}\varphi(\bar{x}) \leq f.$$

and then v_δ is a supersolution of (3.1).

We suppose now that for some c_o, $v_\delta(x) = c_o$ for x in a neighborhood of $\bar{x}$. From the equation we get that $-\epsilon(\delta^2 + c_o^2)^{\frac{\alpha}{2}} c_o \leq f$ in a neighborhood of $\bar{x}$ which is the same condition required in the Definition 2.1.

Proceeding similarly for the subsolution we have obtained that v_δ is a solution of (3.1) and there exists C depending only on the structural constants of $\tilde{F}$ and h such that

$$\begin{aligned} \|v_\delta\|_{W^{2,p}(\Omega)} &\leq C(|f|_\infty + \epsilon |v_\delta|_\infty^{1+\alpha})(|\nabla v_\delta|_\infty^2 + \delta^2)^{-\frac{\alpha}{2}} \\ &\leq C(|f|_\infty + \epsilon |v_\delta|_\infty^{1+\alpha})(|\nabla v_\delta|_\infty^2 + 1)^{-\frac{\alpha}{2}}. \end{aligned}$$

Arguing as in ([6]), one can prove that v_δ is uniformly Lipschitz; this implies both that the bounds on the $W^{2,p}$ norm don't depend on δ and that we can pass to the limit in the equation (3.1); hence we obtain the same estimate for $\delta = 0$. Stability of viscosity solutions implies that v_δ converges to the unique solution of

$$\begin{cases} |\nabla u|^\alpha \left(\tilde{F}(x, D^2 u) + h(x) \cdot \nabla u \right) - \epsilon |u|^\alpha u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We now consider the case $\epsilon = 0$. We write the equation in the following way

$$|\nabla u|^\alpha \left(\tilde{F}(x, D^2 u) + h(x) \cdot \nabla u \right) - \epsilon |u|^\alpha u = f - \epsilon |u|^\alpha u.$$

Replacing f by $f - \epsilon |u|^\alpha u$ since $u \in L^\infty$, it gives the $W^{2,p}$ regularity of u and ends the proof.

Remark 3.4 *Under the hypothesis of Corollary 3.3 if $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded, then any solution of*

$$\begin{cases} |\nabla u|^\alpha \left(\tilde{F}(x, D^2 u) + h(x) \cdot \nabla u \right) + g(x, u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is in $W^{2,p}(\Omega)$ for any $p < \infty$.

Proof of Proposition 3.1. Clearly $\mathcal{L}(\Omega)$, equipped with the norm $|.|_{\mathcal{L}}$ is a Banach space. It is well known that for any $k \in L^\infty$ there exists a unique solution v of

$$\begin{cases} \tilde{F}(x, D^2 v) + h(x) \cdot \nabla v = k & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore v is Lipschitz continuous and there exists some constant c which depends only on the structural data such that

$$|v|_{\mathcal{L}} \leq c|k|_{\infty}.$$

Let ϵ be small enough in order that $\epsilon c\ d_{\Omega} < 1$ where d_{Ω} is the diameter of Ω.

Let G be the reciprocal function of $l \mapsto lg(l)$. Suppose first that $d_{\Omega} \leq 1$. Then we choose $f_o = \frac{c|f|_{\infty}}{1-\epsilon c}$ and

$$\mathcal{B} = \{u \in \mathcal{L}(\Omega), \text{such that } |u|_{\mathcal{L}} \leq G\left(f_o\right)\}.$$

$\mathcal{B}$ is a convex and compact set of $\mathcal{L}$. We use the fact that $l \mapsto lg(|l|)$ and $\frac{1}{g}$ are increasing to obtain that, if $u \in \mathcal{B}$,

$$\begin{aligned} |T_{\epsilon}(u)|_{\mathcal{L}} &\leq c\left(|f|_{\infty} + \epsilon\ g(d_{\Omega}G(f_o))d_{\Omega}G(f_o)\right)\frac{1}{g\left(G(f_o)\right)} \\ &\leq c\left(|f|_{\infty} + \epsilon\ f_o\right)\frac{G(f_o)}{f_o} = G(f_o). \end{aligned}$$

On the other hand if $d_{\Omega} \geq 1$, then we choose $f_o = \frac{c|f|_{\infty}}{1-\epsilon d_{\Omega} c}$ and $\mathcal{B}$ as before.

Hence if $u \in \mathcal{B}$, using that g is decreasing:

$$\begin{aligned} |T_{\epsilon}(u)|_{\mathcal{L}} &\leq c\left(|f|_{\infty} + \epsilon\ g(d_{\Omega}G(f_o))d_{\Omega}G(f_o)\right)\frac{1}{g\left(G(f_o)\right)} \\ &\leq c\left(|f|_{\infty} + \epsilon\ d_{\Omega}\ g(G(f_o))G(f_o)\right)\frac{1}{g\left(G(f_o)\right)} \\ &\leq c\left(|f|_{\infty} + \epsilon d_{\Omega}\ f_o\right)\frac{G(f_o)}{f_o} = G(f_o). \end{aligned}$$

In both cases $T_{\epsilon}(B) \subset B$ and furthermore T_{ϵ} is continuous and compact, then Schauder's fixed point theorem implies the result.

3.2 Other operators

We now present an example to which the results of the previous section can be extended but which does not satisfy the previous assumptions.

Let $q \geq 0$, and $\alpha \leq 0$,

$$\mathcal{F}(p, M) = |p|^{\alpha}\left(trM + q\langle\frac{Mp}{|p|}, \frac{p}{|p|}\rangle\right) := |p|^{\alpha}\tilde{F}(M, p).$$

Proposition 3.5 *Suppose that u is a viscosity solution of*

$$\begin{cases} \mathcal{F}(\nabla u, D^2 u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

with $f \in \mathcal{C}(\overline{\Omega})$. Then there exists $\beta = \beta(q, N) \in (0,1)$ such that u is in $C^{1,\beta}(\overline{\Omega})$.

Proof

Recall that the $(q+2)$-Laplacian is defined by

$$\Delta_{q+2} u = |\nabla u|^q \left(\Delta u + q \left(D^2 u(\frac{\nabla u}{|\nabla u|}), \frac{\nabla u}{|\nabla u|} \right) \right).$$

We first prove a fixed point property. For $\epsilon > 0$ and $\delta > 0$ let T_{lap} be the map $u \mapsto v$ where v is the solution of

$$\begin{cases} \Delta_{q+2}(v) = (f + \epsilon(|u|^2 + \delta^2)^{\frac{\alpha}{2}} u)(|\nabla u|^2 + \delta^2)^{\frac{q-\alpha}{2}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Recall that by regularity results ([19, 17, 33, 26]) of the $q+2$-Laplacian, there exist $\beta \in (0,1)$ and C_{lap} such that

$$|T_{lap}(u)|_{\mathcal{C}^{1,\beta}} \leq C_{lap}(|f|_\infty + \epsilon(|u|_\infty^2 + \delta^2)^{\frac{\alpha}{2}} |u|_\infty)^{\frac{1}{q+1}} (|\nabla u|_\infty^2 + \delta^2)^{\frac{q-\alpha}{2(q+1)}}. \tag{3.3}$$

For ϵ small enough, let l_ϵ be a solution of

$$l_\epsilon^{q+1} = (C_{lap}(|f|_\infty + \epsilon((d_\Omega l_\epsilon)^2 + \delta^2)^{\frac{\alpha}{2}} d_\Omega l_\epsilon)(l_\epsilon^2 + \delta^2)^{\frac{q-\alpha}{2}},$$

and define the closed convex compact set in $\mathcal{L}(\Omega)$

$$\mathcal{B} = \{u \in \mathcal{L}(\Omega),\ |u|_{\mathcal{L}} \leq l_\epsilon\}.$$

Then $T_{lap} B \subset B$ and using the Schauder fixed point theorem one gets that there exists $u_\delta \in \mathcal{C}^{1,\beta}$ which solves

$$\begin{cases} \Delta_{q+2}(u) = (f + \epsilon(|u|^2 + \delta^2)^{\frac{\alpha}{2}} u)(|\nabla u|^2 + \delta^2)^{\frac{q-\alpha}{2}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

As before, we can show that in fact u_δ is a solution in the sense of definition 2.1 of

$$\begin{cases} (|\nabla u|^2 + \delta^2)^{\frac{\alpha-q}{2}} \Delta_{q+2}(u) - \epsilon(|u|^2 + \delta^2)^{\frac{\alpha}{2}} u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Arguing as in the proof of Hölder's and Lipschitz's regularity in [6], one can prove some uniform Lipschitz estimates on v_δ. Then it is clear from the estimates in (3.3) that the $C^{1,\beta}$ norm of v_δ is independent of δ, we can then let δ go to zero and obtain that u_δ converges to a solution u of

$$\begin{cases} |\nabla u|^\alpha \left(\Delta u + q(D^2 u(\frac{\nabla u}{|\nabla u|}), \frac{\nabla u}{|\nabla u|}) \right) - \epsilon |u|^\alpha u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We then obtain the regularity when $\epsilon = 0$ by writing the equation

$$\begin{cases} |\nabla u|^\alpha \left(\Delta u + q(D^2 u(\frac{\nabla u}{|\nabla u|}), \frac{\nabla u}{|\nabla u|}) \right) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

when $f \in L^\infty$, under the form

$$\begin{cases} |\nabla u|^\alpha \left(\Delta u + q(D^2 u(\frac{\nabla u}{|\nabla u|}), \frac{\nabla u}{|\nabla u|}) \right) - \epsilon |u|^\alpha u = f - \epsilon |u|^\alpha u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Using the fact that u is in L^∞ one gets the regularity result.

4 Strict comparison principle and uniqueness of the first eigenfunction

We now assume that $c(x)$ is some continuous bounded function, and we consider the principal eigenvalues λ^+, λ^- for the operator

$$|\nabla u|^\alpha \left(\tilde{F}(x, D^2 u) + h(x) \cdot \nabla u \right) + c(x) |u|^\alpha u$$

as defined in the introduction. We suppose that conditions (F) and (J) are satisfied by $\tilde{F}$, and h satisfies (H).

We prove the uniqueness result in the case λ^+, the changes to bring for λ^- being obvious.

Theorem 4.1 *Suppose that Ω is a bounded regular domain such that either $\partial\Omega$ is connected or $N = 2$ and $\partial\Omega$ has at most two connected components. Suppose that c satisfies $c(x) + \lambda^+ > 0$ in Ω.*

If ψ and φ are two positive eigenfunctions for the eigenvalue λ^+ there exists $t > 0$ such that $\psi = t\varphi$.

Remark 4.2 *The condition $N = 2$ is used only in order to apply Sard's theorem. Indeed in [21], Sard's theorem is proved to hold for functions that are in $W^{N,p}(\mathbb{R}^N)$ for $p > N$ but we only know that u is in $W^{2,p}$ for all $p < \infty$.*

In order to prove Theorem 4.1 we shall need a few results concerning comparison principle and applications of Hopf's principle. We begin by a strong comparison principle inside Ω for sub and supersolutions u and v which coincide on one point $\bar{x}$ inside Ω, where $|\nabla u|(\bar{x}) \neq 0$, or $|\nabla v|(\bar{x}) \neq 0$:

Proposition 4.3 *Suppose that u and v are respectively nonnegative C^1 solutions of*

$$\begin{aligned} F(x, \nabla u, D^2 u) &\leq f \\ F(x, \nabla v, D^2 v) &\geq g \end{aligned}$$

with $f \leq g$. Suppose that $\mathcal{O}$ is an open connected subset of Ω, such that
1) $u \geq v$, in $\mathcal{O}$,
2) $\exists\ x_o \in \mathcal{O}$ such that $u(x_o) > v(x_o)$,
3) either $|\nabla u(x)| \neq 0$ or $|\nabla v(x)| \neq 0$ in $\mathcal{O}$,
Then $u > v$ in $\mathcal{O}$.

Proof of Proposition 4.3. Using the connectedness of $\mathcal{O}$ it is sufficient to prove the result in some ball containing x_o instead of $\mathcal{O}$. in Suppose by contradiction that there exists some point x_1 close to x_o, such that $u(x_1) = v(x_1)$ and in the ball $B(x_o, R)$, for $R = |x_1 - x_o|$, x_1 is the only point in the closure of that ball on which u and v coincide. One can also assume that $B(x_o, \frac{3R}{2}) \subset \Omega$.

We shall prove that there exists some constant $c > 0$ and some $\delta > 0$ such that

$$u \geq v + \delta(e^{-c|x-x_o|} - e^{\frac{-3cR}{2}}) \equiv v + w$$

in the annulus $\frac{R}{2} \leq |x - x_o| = r \leq \frac{3R}{2}$. This will contradict the fact that $u = v$ on x_1.

Let $\delta = \min_{|x-x_o|=\frac{R}{2}} (u - v)$, so that

$$u \geq v + w \quad \text{on} \quad \partial\left(B(x_o, \frac{3R}{2}) \setminus \overline{B(x_o, \frac{R}{2})}\right).$$

Let φ be some test function for v from above, a simple calculation on w implies that, if $c \geq \frac{1}{a}(\frac{2(2A(N-1)}{R} + 2|h|_\infty)$ then

$$
\begin{aligned}
|\nabla\varphi+\nabla w|^\alpha \qquad & \left(\tilde F(x,D^2\varphi+D^2w)+h(x)\cdot(\nabla\varphi+\nabla w)\right)\\
\geq\ & |\nabla\varphi+\nabla w|^\alpha\left(\tilde F(x,D^2\varphi)+h(x)\cdot\nabla\varphi\right)\\
& +|\nabla\varphi+\nabla w|^\alpha\left(\mathcal{M}^-(D^2w)+h(x)\cdot\nabla w\right)\\
\geq\ & |\nabla\varphi+\nabla w|^\alpha\frac{g}{|\nabla\varphi|^\alpha}+\\
& +|\nabla\varphi+\nabla w|^\alpha\left(ac^2-Ac(\frac{N-1}{r})-|h|_\infty c\right)\delta e^{-cr}\\
\geq\ & |\nabla\varphi+\nabla w|^\alpha\frac{g}{|\nabla\varphi|^\alpha}+|\nabla\varphi+\nabla w|^\alpha\frac{ac^2}{2}\delta e^{-cr}.
\end{aligned}
$$

We also impose that $\delta<\frac{RL_1e}{16}$, which implies in particular that $|\nabla w|\leq\frac{|\nabla\varphi]}{8}$
We now use the inequalities

$$
||\nabla\varphi+\nabla w|^\alpha-|\nabla\varphi|^\alpha|\leq|\alpha||\nabla w||\nabla\varphi|^{\alpha-1}\left(\frac12\right)^{\alpha-1}\leq\frac{|\nabla\varphi|^\alpha}{2}
$$

to get

$$
\begin{aligned}
|\nabla\varphi+\nabla w|^\alpha\left(\tilde F(x,D^2\varphi+D^2w)\right. \quad &+\quad h(x)\cdot(\nabla\varphi+\nabla w))\\
&\geq\ g-|g|_\infty|\nabla\varphi|^{-1}|\alpha|2^{1-\alpha}c\delta e^{-cr}+L_2^\alpha\frac{ac^2}{4}\delta e^{-cr}.
\end{aligned}
$$

It is now enough to choose

$$
c=\sup\left\{\frac{2(\frac{2A(N-1)}{R}+|h|_\infty)}{a},\frac{2^{4-\alpha}|g|_\infty}{aL_1L_2^\alpha}\right\}
$$

to finally obtain

$$
\begin{aligned}
|\nabla\varphi+\nabla w|^\alpha\left(\tilde F(x,D^2\varphi+D^2w)\right. \quad &+\quad h(x)\cdot(\nabla\varphi+\nabla w))\\
&\geq\ f+\frac{ac^2\delta L_2^\alpha e^{-cr}}{8}.
\end{aligned}
$$

i.e.

$$
F(x,\nabla(v+w),D^2(v+w))>F(x,\nabla u,D^2u).
$$

We now are in a position to use the comparison principle in theorem 2.5 and get that

$$u \geq v + w$$

in the annulus $B(x_o, \frac{3R}{2}) \setminus \overline{B(x_o, \frac{R}{2})}$, the desired contradiction. This ends the proof of Proposition 4.3.

Another way of formulating this proposition is the following

Corollary 4.4 *Suppose that u and v are as in Proposition 4.3 but instead of condition 2) we have that there exists $\bar{x} \in \mathcal{O}$ such that $u(\bar{x}) = v(\bar{x})$, then $u \equiv v$ in $\mathcal{O}$.*

Another consequence of Proposition 4.3 is the following : Let $\partial_\nu u$ be the normal derivative $\nabla u \cdot \vec{\nu}$ where $\vec{\nu}$ is the unit outer normal to $\partial\Omega$. Then

Proposition 4.5 *Suppose that Ω is a smooth bounded domain of $\mathbb{R}^N$. Suppose that u and v are respectively nonnegative $\mathcal{C}^1(\overline{\Omega})$ solutions of*

$$F(x, \nabla u, D^2 u) \leq f$$
$$F(x, \nabla v, D^2 v) \geq g$$

with $f \leq g$, and $u = v = 0$ on $\partial\Omega$
1) $u \geq v$ in Ω,
2) there exists $\bar{x} \in \partial\Omega$ such that $\partial_\nu u(\bar{x}) = \partial_\nu v(\bar{x})$
then there exists $\epsilon > 0$ such that

$$u \equiv v \text{ in } \Omega \setminus \overline{\Omega}_\epsilon$$

where Ω_ϵ is the set of points of Ω whose distance to the connected component of the boundary which contains $\bar{x}$ is greater than ϵ.

Proof. First with Hopf Principle $|\nabla u| > 0$ and $|\nabla v| > 0$ on the boundary and by the regularity results there exist $\delta > 0$, L_1 and L_2 such that $L_1 \leq |\nabla u|, |\nabla v| \leq L_2$ in $\Omega \setminus \overline{\Omega}_\epsilon$.

If there exists a point x_1 of $\Omega \setminus \overline{\Omega}_\epsilon$ such that $u(x_1) = v(x_1)$ by the previous Corollary we have nothing to prove. So we can suppose by contradiction that there exists a ball $B \subset \Omega$ which is tangent to $\partial\Omega$ in $\bar{x}$ where $u > v$. Let x_2 be the center of that ball. Let $R = |\bar{x} - x_2|$.

Let $w = \delta(e^{-c|x-x_2|} - e^{-c|x_2-\bar{x}|})$ where δ is chosen such that $\delta \leq \inf_{|x-x_2|=\frac{R}{2}}(u-v)$.

Reasoning as in Proposition 4.3 we get that there exists c such that

$$u \geq v + w,$$

in the annulus $B(x_2, R) \setminus B(x_2, \frac{R}{2})$

This implies in particular that

$$|\partial_\nu u(\bar{x})| \geq |(\partial_\nu v + \partial_\nu w)(\bar{x})|.$$

Since $|(\partial_\nu v + \partial_\nu w)(\bar{x})| > |\partial_\nu v(\bar{x})|$ this leads to a contradiction. This ends the proof of Proposition 4.5.

In the sequel we shall need the following well known result :

Lemma 4.6 *Suppose that O is an open bounded set. There exists some point on ∂O where ∂O satisfies the interior sphere condition.*

See [11] for more complete results on that property.

Proof of Theorem 4.1.

We suppose first that $\partial\Omega$ is connected. Let $d(x)$ denote the distance to the boundary of Ω. Suppose that ψ and φ are two positive eigenfunctions and let $\Gamma = \sup \frac{\psi}{\varphi}$. This extremum is well defined because, using Hopf lemma and the comparison principle, there exist c_1 and c_2 such that in a neighborhood of $\partial\Omega$:

$$c_1 d(x) \leq \psi(x), \varphi(x) \leq c_2 d(x),$$

(see [6] for the details). Moreover this supremum is achieved on the boundary in the sense that e.g. there exists a sequence $(x_n)_n$ which goes to the boundary such that $\frac{\psi}{\varphi}(x_n) \to \Gamma$:

Indeed, suppose not, then, there would exist an open set Ω', $\Omega' \subset\subset \Omega$ such that on $\Omega \setminus \overline{\Omega'}$

$$\frac{\psi}{\varphi} \leq \Gamma - \epsilon$$

for some $\epsilon > 0$. Using the comparison Theorem 2.9 in Ω', and remarking that $\overline{\lambda}(\Omega) < \overline{\lambda}(\Omega')$ one would get that $\psi \leq (\Gamma - \epsilon)\varphi$ on Ω' and finally on all Ω, a contradiction.

We are in a position to apply Corollary 4.4 and Proposition 4.5 and obtain that $\psi = \Gamma\phi$ in a neighborhood of $\partial\Omega$. Considering the infimum of the ratio

one gets also that if $\gamma = \inf \frac{\psi}{\varphi}$ then $\frac{\psi}{\varphi} = \gamma$ on a neighborhood of the boundary, therefore $\gamma = \Gamma$ and the conclusion follows.

We now assume that $N = 2$ and since $\partial\Omega$ has at most two connected components there exist Ω_1 and Ω_2 simply connected and smooth such that $\Omega_1 \subset\subset \Omega_2$ and $\Omega = \Omega_2 \setminus \overline{\Omega_1}$.

We define as before $\Gamma = \sup \frac{\psi}{\varphi}$ and $\gamma = \inf \frac{\psi}{\varphi}$, and arguing as previously there exists some sequence x_n and $i \in \{1, 2\}$ such that $x_n \to \partial\Omega_i$ and $\frac{\psi}{\varphi}(x_n) \to \Gamma$. By Corollary 4.4 and Proposition 4.5 with $f := -(c(x) + \overline{\lambda})\psi^{1+\alpha} \leq -(c(x) + \overline{\lambda})\varphi^{1+\alpha} := g$,

$$\psi \equiv \Gamma\varphi \text{ in a neighborhood of } \partial\Omega_i.$$

In the same manner we prove that the infimum $\gamma = \inf \frac{\psi}{\varphi}$ is achieved on the other connected component, and also that $\psi = \gamma\varphi$ in a neighborhood of this part of the frontier.

We have obtained that

$$\psi \equiv \Gamma_1\varphi \text{ in a neighborhood of } \partial\Omega_1$$

and

$$\psi \equiv \Gamma_2\varphi \text{ in a neighborhood of } \partial\Omega_2.$$

with $\{\Gamma_1, \Gamma_2\} = \{\Gamma, \gamma\}$.

For $i = 1, 2$, let

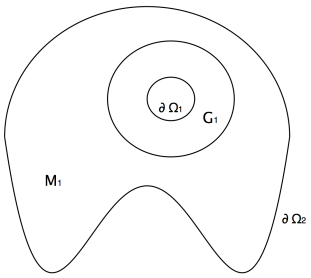
A_i be the connected component of $\{x, \psi(x) = \Gamma_i\varphi(x),\ \nabla\psi(x) \neq 0 \text{ or } \nabla\varphi(x) \neq 0\}$

whose boundary contains $\partial\Omega_i$.

A_i is open. Indeed if $x_1 \in A_1$ then $\psi(x_1) = \Gamma_1\varphi(x_1)$ and there is N_{x_1} a neighborhood of x_1 where either $\nabla\psi \neq 0$ or $\nabla\varphi \neq 0$ and then using Corollary 4.4, in N_{x_1}, $\psi = \Gamma_1\varphi$ and then $N_{x_1} \subset A_1$.

Observe that if $\partial A_1 \cap \partial\Omega_2 \neq \emptyset$ or $\partial A_2 \cap \partial\Omega_1 \neq \emptyset$, this ends the proof of Theorem 4.1 since it would imply that $\Gamma = \gamma$. Hence we suppose that these intersections are empty.

Let $G_i := \overline{A_i}$; $K_i := \partial G_i \cap \Omega$, and $M_i := \Omega \setminus G_i$.



Let us note that K_i satisfies that, for all $x \in K_i$, $\psi(x) = \Gamma_i\varphi(x)$ and $\nabla\psi(x) = 0$ and $\nabla\varphi(x) = 0$. To see this it is sufficient to observe that $\partial G_i \subset \partial A_i$ since G_i is closed, and then it is sufficient to prove that these inequalities are true on the boundary of A_i:

The first equality is true by continuity and the others two because otherwise $\psi = \Gamma_i\varphi$ in an open neighborhood of that point with $\nabla\psi(x) \neq 0$ in that neighborhood , or $\nabla\varphi(x) \neq 0$ which contradicts the notion of boundary.

Moreover $\partial M_1 = \partial\Omega_2 \cup K_1$ (and $\partial M_2 = \partial\Omega_1 \cup K_2$).

Indeed note that $\overline{M_1} \cap \partial\Omega_1 = \emptyset$ since A_1 contains an open neighborhood of $\partial\Omega_1$, then $\partial M_1 \subset (\overline{\Omega} \cap \partial G_1) \cup (\overline{C(G_1)} \cap \partial\Omega) \subset (\Omega \cap \partial G_1) \cup \partial\Omega_1 \cup \partial\Omega_2$ and since $\partial M_1 \cap \partial\Omega_1 = \emptyset$, hence $\partial M_1 \subset \partial\Omega_2 \cup K_1$. To prove the reverse inclusion, let $x \in K_1$, then for all $r > 0$, $B(x,r) \cap (\mathbb{R}^N \setminus G_1) \neq \emptyset$. Taking r such that $B(x,r) \subset \Omega$ one gets $x \in \overline{M_1}$. On the other hand $x \notin M_1$, since if $x \in M_1$ there exists some ball $B(x,\epsilon)$ included in M_1, and $x \in K_1$ implies $B(x,\epsilon) \cap G_1 \neq \emptyset$, a contradiction.

It is clear that $\partial\Omega_2$ is included in $\overline{M_1}$, and it has no point of M_1 since M_1 is an open subset of Ω.

Let us admit for a while these three simple claims,

Claim 1 M_i is connected for $i = 1, 2$.

Claim 2 ∂G_i has at most two connected components i.e. K_i is connected.

Claim 3 $\partial(M_1 \cap M_2) = K_1 \cup K_2$,

and let us finish the proof. Using the fact that both K_i are connected, since $\psi \in W^{2,p}$ for all $p < \infty$ we can apply Sard's theorem in Sobolev spaces [21], to conclude that there exists two constants c_i such that $\psi_{|K_i} = c_i$ for $i = 1$ and $i = 2$, .

Suppose that there exists one point $\bar{x} \in M = M_1 \cap M_2$ such that $\psi(\bar{x}) < \min(c_1, c_2)$. Then ψ would have a local minimum inside M, a contradiction with Hopf principle, see Remark 2.7. Then the minimum is achieved on the boundary of M, suppose to fix the ideas that $c_1 \leq c_2$. Now take a ball in M

where $\psi > c_1$ that touches K_1 at some point x_1, by Hopf's principle $\nabla\psi(x_1) \neq 0$ which contradicts the fact that, for all $x \in K_i$, $\nabla\psi(x) = 0$.

Now since ψ cannot be locally constant we have proved that $M = \emptyset$ and then $\Gamma_1 = \Gamma_2$, hence $\Gamma = \gamma$. This ends the proof of Theorem 4.1 provided we prove the claims.

Proof of Claim 1

To fix the ideas we consider the case of $i = 1$.

Let us recall that $\partial M_1 = (\partial G_1 \cap \Omega) \cup \partial\Omega_2$. Suppose that M_1 has at least two connected components, $M_{1,1}$ and $M_{1,2}$ and one of them, say $M_{1,2}$, has a boundary which contains $\partial\Omega_2$. Then $\partial M_{1,1} \subset K_1 = \Omega \cap \partial G_1$. Then $M_{1,1}$ is a connected open set on the boundary of which ψ and ϕ have their gradient equal to zero.

We prove that $M_{1,1}$ is simply connected.

If not there exists some open regular domain O', $\overline{O'} \subset \Omega_2$, with $\partial O' \subset M_{1,1}$, O' is not included in $M_{1,1}$, and $\partial M_{1,1}$ is not included in O'. Let $x \in O' \setminus M_{1,1}$, then $x \in G_1$ and hence $G_1 \cap O' \neq \emptyset$.

Either $G_1 \subset O'$, then $\partial M_{1,1} \subset K_1 \subset O'$ and the contradiction is in the choice of O'.

Or $G_1 \cap C(O') \neq \emptyset$ which contradicts that $\partial O' \subset M_{1,1}$. We have obtained that $M_{1,1}$ is simply connected.

By Sard's theorem there exists a constant $c_{1,1}$ such that $\psi = c_{1,1}$ on $\partial M_{1,1}$. Then one gets a contradiction with Hopf's principle (because either there exists a minimum inside $M_{1,1}$ and this contradicts Remark 2.7, or $c_{1,1}$ is a minimum for ψ and taking some point on the boundary of $M_{1,1}$ which possesses the interior sphere condition, one gets once more a contradiction with Hopf's principle).

We have obtained that $M_{1,1} = \emptyset$ and M_1 is connected.

Proof of Claim 2.

We prove this claim by establishing that $G_1 \cup \Omega_1$ is simply connected. Let O be a simply connected open set , such that $\overline{O} \subset \Omega_2$, $\partial O \subset G_1 \cup \Omega_1$. We need to prove that $O \subset G_1 \cup \Omega_1$.

If $O \subset \Omega_1$ it is true. If not, $O \cap G_1 \neq \emptyset$ and if $O \subset G_1$ the result holds, so we assume that $O \cap G_1 \neq \emptyset$ and also $O \cap C(G_1) \neq \emptyset$. Then $O \cap M_1 \neq \emptyset$. Since M_1 is a connected set which meets O we claim that $\partial O \cap M_1 \neq \emptyset$. Indeed, if not $M_1 \subset O$. Then one would have $\overline{M_1} \subset \overline{O} \subset \Omega_2$, a contradiction since $\partial\Omega_2 \subset \overline{M_1}$. Then $\partial O \cap M_1 \neq \emptyset$ which implies that $M_1 \cap (G_1 \cup \Omega_1) \neq \emptyset$ once more a contradiction.

To prove that K_2 is a connected set, one proves proceeding as above that $M_2 \cup \Omega_1$ is simply connected.

Proof of Claim 3.

We already have

$$\begin{aligned}\partial(M_1 \cap M_2) &= \overline{M_1 \cap M_2} \setminus (M_1 \cap M_2) \\ &\subset \overline{M_1} \cap \overline{M_2} \cap ((\Omega \setminus M_1) \cup (\Omega \setminus M_2)) \\ &\subset (\partial M_1 \cap \overline{M_2}) \cup (\partial M_2 \cap \overline{M_1}) \\ &\subset K_1 \cup K_2.\end{aligned}$$

To prove the reverse inclusion, we recall that, by hypothesis, $G_1 \cap G_2 = \emptyset$.

Then $K_1 \subset G_1 \subset M_2$. Let $x \in K_1$ and $r > 0$. We need to prove that $B_r(x) \cap (M_1 \cap M_2) \neq \emptyset$. Since M_2 is open there exists a ball $B_\epsilon(x) \subset M_2$. One can assume that $\epsilon < r$. Since $x \in K_1 \subset \partial M_1$, $B_\epsilon(x) \cap \overline{M_1} \neq \emptyset$ and also $B_\epsilon(x) \cap M_1 \neq \emptyset$. Finally $B_\epsilon(x) \cap M_1 \cap M_2 \subset B_r(x) \cap M_1 \cap M_2$ hence $x \in \overline{M_1 \cap M_2}$. $x \in K_1$ implies $x \notin M_1$ hence $x \notin M_1 \cap M_2$ i.e. $K_1 \subset \partial(M_1 \cap M_2)$. In the same manner $K_2 \subset \partial(M_1 \cap M_2)$.

We have also obtained the following strong comparison principle

Theorem 4.7 *Suppose that $N = 2$ and that Ω and $\partial\Omega$ are connected. Suppose that u and v are $W^{2,p}(\Omega)$ for some $p > 2$ respectively super and sub solution of*

$$\begin{aligned}&F(x, \nabla u, D^2 u) = f \ \text{in } \Omega \\ &u = 0 \quad \text{on } \partial\Omega\end{aligned}$$

with $f \leq 0$, and f not identically zero. Suppose that $u \geq v$ in Ω.

Then either $u > v$ inside Ω or $u \equiv v$.

Proof :

First we remark that $u > 0$ since if not u is identically zero by the strict maximum principle, and then $f \equiv 0$.

Then by Hopf's principle $\partial_n u < 0$ on the boundary.

We begin to prove that there exists a neighborhood of the boundary where either $u \equiv v$ or $u > v$ and in that last case one can conclude by using the comparison principle in Theorem 2.9, and $u > v$ everywhere. We then consider the case where $u \equiv v$ on a neighborhood of $\partial\Omega$. As in the previous proof, let

A be the connected component of $\{x, u(x) = v(x),\ \nabla u(x) \neq 0,\ \text{or } \nabla v(x) \neq 0\}$

whose boundary contains $\partial\Omega$. Let $G := \overline{A}$, $K := \partial G \cap \Omega$, $M = \Omega \setminus G$.

We want to prove that $M = \emptyset$. Suppose not, proceeding as in the proof of Theorem 4.1, it can be proved that M and K are connected, hence $\partial M = K$. Furthermore, on K, $\nabla u = 0$ and $\nabla v = 0$.

Then, using Sard's theorem, $u = cte$ on K and, since M cannot contain a local minimum of u, the minimum of u is achieved on the boundary i.e. on K; this contradicts the Hopf principle on some point where the interior sphere condition is satisfied, since $\nabla u = 0$ on K. Finally $M = \emptyset$ and $u = v$ in Ω.

5 Further results about the principal eigenvalues

In all this section we suppose that F is given by (2.1) and $\tilde{F}$ satisfies (F) and (J) and h satisfies (H).

5.1 Properties concerning the dependance on the domain of the eigenvalues.

From the definition of λ^+ and λ^- it is clear that $\Omega_1 \subset \Omega_2$ implies $\lambda^+(\Omega_1) \geq \lambda^+(\Omega_2)$. The next two theorems concern the strict monotonicity of the principal eigenvalues with respect to the domain inclusion. We state them in the case of the eigenvalue $\lambda^+(\Omega)$, the symmetric results hold for $\lambda^-(\Omega)$ with obvious changes. In the sequel when no ambiguities arise we shall write λ^+ without writing the dependence on the domain.

Theorem 5.1 *Suppose that Ω is a smooth bounded domain in $\mathbb{R}^N$. Suppose that $c(x) + \lambda^+ > 0$ in Ω. If u is a positive solution of*

$$F(x, \nabla u, D^2 u) + (c(x) + \lambda^+) u^{1+\alpha} = 0 \text{ in } \Omega,$$

then there exists $(\partial\Omega)'$ a connected component of $\partial\Omega$ such that

$$u = 0 \quad on \quad (\partial\Omega)'.$$

Proof. To begin with, let us note that u must be zero somewhere on $\partial\Omega$. Suppose not, i.e. $u > 0$ on $\overline{\Omega}$. Using the hypothesis $c(x) + \lambda^+ > 0$, this implies that there exists $\varepsilon > 0$ and $\lambda' > \lambda^+$ such that $u_\varepsilon := u - \varepsilon > 0$ is a solution of

$$F(x, \nabla u_\varepsilon, D^2 u_\varepsilon) + (c(x) + \lambda') u_\varepsilon^{1+\alpha} \leq 0,$$

which contradicts the definition of λ^+.

Let v be an eigenfunction associated to λ^+, so that $v > 0$ in Ω and $v = 0$ on $\partial\Omega$. We now consider $\tau = \sup \frac{v}{u}$ which exists and is finite.

Indeed, it is sufficient to prove it near the boundary. As seen in the proof of Theorem 4.1 there exists some constant c_1 such that $v \leq c_1 d$; on the other hand by Hopf's principle there exists c_o such that $u \geq c_o d$; this implies that τ is finite.

Again reasoning as in Theorem 4.1, the supremum must be achieved at least on one point of the boundary in the sense that there exists a sequence $x_n \in \Omega$, $x_n \to \bar{x} \in \partial\Omega$ such that $\frac{v(x_n)}{u(x_n)} \to \tau$.

Let $(\partial\Omega)'$ be the connected component of $\partial\Omega$ that contains $\bar{x}$. Let $\Gamma^+ = \{x \in (\partial\Omega)', u(x) > 0\}$. We want to prove that $\Gamma^+ = \emptyset$.

For that aim we consider the set

$$A = \{x \in (\partial\Omega)',\ \limsup_{y \to x} \frac{v(y)}{u(y)} < \tau\}.$$

We shall prove that A is both closed and open in $(\partial\Omega)'$. If we suppose that Γ^+ is non empty, it will imply that A is not empty since $\Gamma^+ \subset A$ and therefore $A = (\partial\Omega)'$. We will have obtained that $\limsup_{y \to x} \frac{v(y)}{u(y)} < \tau$ everywhere on $(\partial\Omega)'$, contradicting the previous observation.

By Hopf principle and the regularity result one has $|\nabla v| \geq L_1 > 0$, on a neighborhood of the boundary. In the sequel for $x \in \partial\Omega$, we denote as $B_x(r_1)$ some ball such that $B_x(r_1) \subset \Omega$ and $\overline{B_x(r_1)} \cap \partial\Omega = \{x\}$. The existence of such ball for every point $x \in \partial\Omega$ is implied in particular by the assumption that $\partial\Omega$ is $\mathcal{C}^2$.

Claim : The complement of A is closed in $(\partial\Omega)'$.

We want to prove that if x_n is some sequence in $(\partial\Omega)'$ which converges to $x \in (\partial\Omega)'$ such that $\limsup_{y \to x_n} \frac{v(y)}{u(y)} = \tau$, then

$$\limsup_{y \to x} \frac{v(y)}{u(y)} = \tau.$$

In particular $u(x) = 0$. By the strict comparison principle Corollary 4.4 one has that there exists r_1 independent of n such that $v = \tau u$ in $B_{x_n}(r_1)$.

Since $B_{x_n}(r_1) \cap B_x(r_1) \neq \emptyset$ for n large enough, there exists $y_n \in B_x(r_1)$, $v(y_n) = \tau u(y_n)$, again using Corollary 4.4 we get that $v = \tau u$ in all the ball $B_x(r_1)$ and then x does not belong to A.

Claim : The set A is closed.

Let $x_n \in \partial\Omega$ such that for all n, $\limsup_{y\to x_n} \dfrac{v(y)}{u(y)} < \tau$ and x_n converges to $\bar{x} \in (\partial\Omega)'$, let us prove that $\bar{x} \in A$. Using Proposition 4.3, there exists r_1 such that $v < \tau u$ in $B_{x_n}(r_1)$, and since $B_{x_n}(r_1) \cap B_{\bar{x}}(r_1) \cap \Omega \neq \emptyset$, there exists $y \in B_{\bar{x}}(r_1)$ such that $v(y) < \tau u(y)$, which implies that $v < \tau u$ in all the ball $B_{\bar{x}}(r_1)$.

There are two cases either $v(\bar{x}) = 0 < u(\bar{x})$, then $\bar{x} \in A$ and we are done. Or $v(\bar{x}) = \tau u(\bar{x}) = 0$ and we need to prove that $\limsup_{y\to\bar{x}} \dfrac{v(y)}{u(y)} < \tau$. If not, acting as in the proof of Proposition 4.5, we obtain $v = \tau u$ in the whole ball $B_{\bar{x}}(r_1)$, a contradiction, so $\bar{x} \in A$. This ends the proof.

It is an immediate consequence of the definition, that if $\Omega' \subset \Omega$ then $\lambda^{\pm}(\Omega') \geq \lambda^{\pm}(\Omega)$. In the next Theorem we shall prove that the monotonicity is strict.

Theorem 5.2 *Let Ω be a domain with a connected boundary, and Ω' a subdomain of Ω.*

Suppose that $\partial\Omega$ is not included in $\partial\Omega'$ or, for $N = 2$, suppose only that $\overline{\Omega'} \neq \overline{\Omega}$. Then

$$\lambda^{\pm}(\Omega') > \lambda^{\pm}(\Omega).$$

Proof: Let us begin by the first case i.e. we suppose that there exists $x_0 \in \partial\Omega$ such that $x_0 \notin \partial\Omega'$ and hence, there exists $\delta > 0$, such that $B(x_0, \delta) \subset \mathbb{R}^N \setminus \overline{\Omega'}$.

Let Ω'' be a smooth domain whose boundary has only one connected component and such that

$$\Omega' \subset \Omega'' \subset \Omega \setminus \overline{B(x_0, \delta)}.$$

We know that

$$\lambda^{+}(\Omega') \geq \lambda^{+}(\Omega'') \geq \lambda^{+}(\Omega \setminus \overline{B(x_0, \delta)}) \geq \lambda^{+}(\Omega).$$

Suppose by contradiction that $\lambda^{+}(\Omega') = \lambda^{+}(\Omega)$; this implies that $\lambda^{+}(\Omega'') = \lambda^{+}(\Omega) := \lambda^{+}$.

Let v be an eigenfunction for Ω. So in particular it is positive in Ω. On the other hand it is also a solution of

$$F(x, \nabla\varphi, D^2\varphi) + (c(x) + \lambda^{+})\varphi^{1+\alpha} = 0,$$

in Ω'', then using Theorem 5.1, this would imply that $v = 0$ on $\partial\Omega''$, a contradiction with the fact that $v > 0$ in $\partial\Omega'' \cap \Omega$. This ends the first case.

We are left to prove the case $N = 2$ and $\partial\Omega \cap \partial\Omega' = \partial\Omega$. By contradiction we shall suppose that $\lambda^+(\Omega') = \lambda^+(\Omega)$. We can assume that Ω' is regular by replacing Ω' by some regular subset Ω'' such that $\partial\Omega'' \cap \partial\Omega = \partial\Omega$ which contains Ω' since it satisfies also $\lambda^+(\Omega'') = \lambda(\Omega)$. For simplicity we rename this set Ω' and we consider the respective eigenfunctions ϕ and ϕ'. By hypothesis they satisfy the same equation in Ω'.

Let

$$\tau = \sup_{\Omega'} \frac{\phi'}{\phi},$$

proceeding as in the proof of Theorem 4.1, it is possible to prove that τ is bounded and achieved on $\partial\Omega$.

Using Proposition 4.5 there exists G a closed connected neighborhood of $\partial\Omega$ where $\phi' = \tau\phi$ and such that

$$\nabla\phi = \nabla\phi' = 0 \text{ in } \ K := \partial G \cap \Omega.$$

Hence using Sard's theorem for functions in $W^{2,p}$, there exists a constant c such that

$$\phi = c, \ \phi' = \tau c, \ \text{in } K.$$

This of course leads to a contradiction, because either $\phi < c$ somewhere in $\Omega \setminus G$ and then ϕ would have a local minimum which contradicts Remark 2.7. Or $\phi \geq c$ in $\Omega \setminus G$ and then by Hopf's Lemma and Lemma 4.6 there is a point where $\nabla\phi \neq 0$ in K which is again a contradiction. This ends the proof.

To fix the ideas, in the next theorem we suppose $\lambda^+ < \lambda^-$, with obvious symmetric results in the other case.

Theorem 5.3 *Suppose that Ω is a bounded regular domain. Suppose that $f \leq 0$, then there exists no solution u for the equation*

$$\begin{cases} F(x, \nabla u, D^2u) + (c(x) + \lambda^+)u^{1+\alpha} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

such that $u(x_o) < 0$ for some $x_o \in \Omega$.

Suppose in addition that $\partial\Omega$ is connected, that $(c(x)+\lambda^-) > 0$, that $f \leq 0$ and that $f < 0$ somewhere near the boundary, then there is no solution to

$$\begin{cases} F(x, \nabla u, D^2u) + (c(x)+\lambda^-)|u|^\alpha u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

In the case where $N = 2$ the assumption $f < 0$ somewhere near the boundary can be removed, and the conclusion is that $f \equiv 0$ and u is an eigenfunction for λ^-.

Proof of Theorem 5.3: The first part is a mere application of the minimum principle : Since $f \leq 0$ and $\lambda^+ < \lambda^-$, and since $u = 0$ on the boundary, $u \geq 0$, which contradicts the assumption.

We now prove the second part.

We suppose first that $u \geq 0$. Then since $f \leq 0$ the strict maximum principle implies that either $u \equiv 0$ or $u > 0$, and this contradicts the definition of λ^+ and $\lambda^+ < \lambda^-$. We now then assume that $u(x_o) < 0$ for some x_o. Let $-\varphi^-$ be some normalized eigenfunction for λ^-. Let us note that $-u^-$ is a supersolution of the equation in Ω. Let us define $\Gamma = \inf\{t,\ u^- \leq t\varphi^-\}$. Then $\Gamma > 0$ and $u^- \leq \Gamma\varphi^-$. We prove that the supremum $\Gamma = \sup \frac{u^-}{\varphi^-}$ is achieved on the boundary. If not there exists some compact set K large enough in order that $\frac{u^-}{\varphi^-} \leq \Gamma'$ on $\Omega \setminus K$ and $\Gamma' < \Gamma$.

Since $-u^-$ is a supersolution and $-u^- \geq -\Gamma'\varphi^-$ on ∂K, then using the comparison Theorem 2.9 in its form for negative solutions, $-u^- \geq -\Gamma'\varphi^-$ in K, finally $-u^- \geq -\Gamma'\varphi^-$ in the whole of Ω, which yields a contradiction.

Then the supremum is "achieved" on the boundary and then the strict comparison principle in Proposition 4.3 implies that $-u^- = -\Gamma\varphi^-$ around the boundary. In particular one gets $f \equiv 0$ around the boundary, which is once more a contradiction.

Let $N = 2$, and f can be zero near the boundary. As in the previous argument if $u \geq 0$, one gets $u \equiv 0$. We now assume that there exists x_o such that $u(x_o) < 0$. By the previous conclusion $-u^- = -\Gamma\varphi^-$ on a neighborhood of the boundary, in fact since we have $-u^- \geq -\Gamma\varphi^-$ everywhere, and u^- satisfies

$$\begin{aligned} F(-\nabla u^-, D^2(-u^-)) &= f + (c(x)+\lambda^-)(u^-)^{1+\alpha} \leq (c(x)+\lambda^-)(\varphi^-)^{1+\alpha} \\ &\leq F(-\nabla\varphi^-, D^2(-\varphi^-)) \end{aligned}$$

using the strict comparison principle one gets that as long as $\nabla\varphi^- \neq 0$ one has $u^- = \Gamma\varphi^-$. Now applying Theorem 4.7 in its form with $f \geq 0$ and negative

solutions. one obtains that $u^- = \Gamma\varphi^-$ everywhere and since φ^- never takes the value 0 one gets that $u < 0$ and $f \equiv 0$ This ends the proof.

In recent papers, when $\alpha = 0$, Armstrong [3], Felmer, Quaas, Sirakov [32, 20] have studied some sort of "Fredholm" alternative in order to establish for which functions f there exists a solution of the Dirichlet problem when $\lambda^- \leq \lambda \leq \lambda^+$.

5.2 Further properties

We want to prove that we can recover some of the standard properties of eigenvalues for linear elliptic equations. We consider the Dirichlet problem

$$\begin{cases} F(x, \nabla u, D^2 u) + (c(x) + \lambda)u^{1+\alpha} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Non trivial solutions of (5.1) will be called eigenfunctions.

We now recall the following result, which is an easy consequence of the definitions of $\lambda^\pm$ and the maximum and the minimum principle (see [8],[9])

Theorem 5.4 *Suppose that Ω is a bounded regular domain,*

1) If $\lambda > \lambda_1 = \sup(\lambda^+(\Omega), \lambda^-(\Omega))$, then every non trivial solution of (5.1) changes sign in Ω.

2) For any λ between λ^+ and λ^- there are no nontrivial solutions of (5.1)

As a consequence of the previous section, we now have further results regarding the signs of the eigenfunctions.

Theorem 5.5 *Suppose that Ω is some bounded smooth domain, and suppose that one of the following holds*

1. *$\lambda^+ = \lambda^-$ and $\partial\Omega$ is connected*
2. *$N = 2$, $\lambda^+ = \lambda^-$, $\partial\Omega$ has at most two connected components*
3. *$N = 2$, $\partial\Omega$ is connected*

then any eigenfunction corresponding to $\lambda = \lambda^\pm$ is of constant sign.

An application of the previous result is the following

Corollary 5.6 *In the hypothesis of Theorem 5.5 the eigenvalues λ^+ or λ^- are isolated i.e. there exists $\delta > 0$ such that for any $\lambda \in]\lambda^{\pm}, \lambda^{\pm} + \delta[$, the solutions of (5.1) are trivial.*

Proof of Theorem 5.5. Let $\lambda_1 = \lambda^+ = \lambda^-$ and suppose that $\partial\Omega$ is connected. Suppose by contradiction that there exists a solution of (5.1) which changes sign. We define $\Omega^+ = \{x \in \Omega;\ u(x) > 0\}$ and $\Omega^- = \{x \in \Omega;\ u(x) < 0\}$. Then clearly, for any $\tilde{\Omega}^+$ (respectively $\tilde{\Omega}^-$) connected component of Ω^+ (respectively Ω^-):

$$\lambda^+(\tilde{\Omega}^+) = \lambda^-(\tilde{\Omega}^-) = \lambda_1.$$

$\partial\tilde{\Omega}^+ \cap \partial\Omega \neq \partial\Omega$ is not possible since it would imply, by Theorem 5.2 $\lambda^+(\tilde{\Omega}^+) > \lambda^+(\Omega)$. But, on the other hand, if $\partial\tilde{\Omega}^+ \cap \partial\Omega = \partial\Omega$, then $\partial\tilde{\Omega}^- \cap \partial\Omega \neq \partial\Omega$, and then the contradiction is given by the fact that it would imply $\lambda^-(\tilde{\Omega}^-) > \lambda^-(\Omega)$.

We now suppose to be in the second case. For $i = 1, 2$, Ω_i denote two open simply connected sets such that $\Omega_1 \subset\subset \Omega_2$ and $\Omega = \Omega_2 \setminus \overline{\Omega_1}$.

Let φ be a positive eigenfunction in Ω and v an eigenfunction which changes sign. Let $\Omega^{\pm}$ be the set where v is positive (respectively negative). If $\partial\Omega^+ \cap \partial\Omega_i \neq \emptyset$ then reasoning as in the proof of Theorem 5.1, $\partial\Omega^+ \cap \partial\Omega = \partial\Omega_i$. The same is true for Ω^-, hence one can assume without loss of generality that $\partial\Omega^+ \cap \partial\Omega = \partial\Omega_1$ and $\partial\Omega^- \cap \partial\Omega = \partial\Omega_2$, finally $\partial\Omega^+ \cap \Omega = \partial\Omega^- \cap \Omega \neq \emptyset$.

Since $\lambda^+ = \lambda^-$, $-\varphi$ is an eigenfunction corresponding to λ^-. Reasoning as in the proof of Theorem 4.1 we can define $\Gamma = \sup \frac{v}{\varphi}$ which exists according to the estimates on v near the boundary and it is "achieved" on $\partial\Omega_1$, and $\gamma = \sup \frac{-v}{\varphi}$ "achieved " on $\partial\Omega_2$, and prove that $v = \Gamma\varphi$ on a neighborhood of $\partial\Omega_1$ and $v = -\gamma\varphi$ on a neighborhood of $\partial\Omega_2$.

Defining as in the proof of Theorem 4.1 M_1 , K_1, M_2, K_2 with $\nabla v = \nabla\varphi = 0$ on K_1 and K_2.

Since $\varphi > 0$ in $M_1 \cap M_2$, using Sard's theorem $c_i = \varphi_{|_{K_i}}$. Again reasoning as in Theorem 4.1, φ must achieve its infimum on K_1 or K_2 where its gradient is zero, a contradiction with Hopf principle.

We consider the last case, i.e. $N = 2$, $\lambda^+ \neq \lambda^-$ and $\partial\Omega$ is connected. It is clear using the maximum principle that the result is true for the smallest of the two eigenvalues. We suppose to fix the ideas that $\lambda^- < \lambda^+$ and we prove that every eigenfunction corresponding to λ^+ is positive.

We denote by φ some positive eigenfunction for $\lambda^+(\Omega)$ and by ψ some eigenfunction which changes sign. We denote by Ω^+ and Ω^- the sets where ψ is

respectively positive and negative. This implies that $\lambda^+(\Omega^+) = \lambda^+(\Omega)$ and $\lambda^-(\Omega^-) = \lambda^-(\Omega)$; but, in dimension $N = 2$, this contradicts Theorem 5.2.

Proof of Corollary 5.6. The result needs to be proved for $\lambda_1 = \sup(\lambda^+, \lambda^-)$. Suppose by contradiction that there exists a sequence of eigenvalues λ_n, such that $\lambda_n \to \lambda_1$, $\lambda_n > \lambda_1$.

Let u_n be a sequence of solutions of (5.1) with $\lambda = \lambda_n$ such that $|u_n|_\infty = 1$. This implies that the Hölder's norm is uniformly bounded with respect to n (see Proposition 2.8).

Then $(u_n)_n$ is relatively compact and it converges up to a subsequence in $\mathcal{C}(\bar{\Omega})$ towards a solution u of (5.1) with $\lambda = \lambda_1$.

By Theorem 5.5, u must be either positive or negative, which implies that for n large enough, for any $K = \overline{\Omega}_1 \subset \Omega$, for n large enough, u_n has constant sign in K.

Without loss of generality we can suppose that $\lambda_1 = \lambda^+$ and $u > 0$ in Ω and hence u_n is positive in K.

We choose Ω_1 a regular subset in Ω such that $\lambda^-(\Omega \setminus \overline{\Omega_1}) > \lambda_n$.

By minimum principle, since

$$u_n \geq 0, \text{ in } \partial(\Omega \setminus \overline{\Omega_1})$$

it implies that $u_n \geq 0$ in $\Omega \setminus \overline{\Omega_1}$ and then u_n is positive in Ω which contradicts Theorem 5.4.

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