

# OPEN PROBLEMS IN GAUSSIAN FLUID QUEUEING THEORY

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**ABSTRACT.** We present three challenging open problems that originate from the analysis of the asymptotic behavior of Gaussian fluid queueing models. In particular, we address the problem of characterizing the *correlation structure* of the stationary buffer content process, the *speed of convergence to stationarity*, and analysis of an *asymptotic constant* associated with the stationary buffer content distribution (the so-called *Pickands constant*).

**Key words:** Buffer content process — Fractional Brownian motion — Gaussian processes — Queues

**AMS 2010 Subject Classification:** Primary 60G15, Secondary 60G70.

## 1. INTRODUCTION

Over the last years a substantial effort has been devoted to the analysis of queueing fluid systems driven by Gaussian processes; see the monograph [16] and references therein. On the one hand, the interest in such models stems from both the flexibility and richness of the family of Gaussian processes; more specifically, Gaussian processes cover a broad spectrum of correlation structures, which for instance cover phenomena as *long-range dependance* and *self-similarity*. On the other hand, both empirical and theoretical considerations legitimate the use of Gaussian processes as models for traffic streams in modern communication networks. The empirical evidence consists of a variety of measurement studies that statistically assess the properties of network traffic. As a theoretical back-up we mention in particular [9, 15], which proved that in a heavy traffic environment parameterization, large numbers of i.i.d. on-off sources may be approximated by a Gaussian process (with the same covariance structure as the on-off process). Additionally, contributions by Taqqu *et al.* [26] and Mikosch *et al.* [20] gave a formal argument for the use of a specific Gaussian process, *viz. fractional Brownian motion*. In [6, 8] it was proved that these central limit theorem type of results carry over to the buffer content process level. This formally justified the analysis of buffer content processes for queues fed by Gaussian input.

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Let  $\{Q(t) : t \geq 0\}$  be the stationary buffer content process for a fluid queue fed by a centered Gaussian stochastic process  $\{X(t) : t \in \mathbb{R}\}$  with stationary increments, continuous sample paths a.s. and variance function  $\sigma_X^2(t) = \text{Var}(X(t))$ . We assume that the system is emptied with a constant output rate  $c > 0$ . Due to Reich [23], the following representation for  $Q(t)$  holds on the process level:

$$(1) \quad \{Q(t) : t \geq 0\} =_d \left\{ \sup_{s \leq t} (X(t) - X(s) - c(t - s)) : t \geq 0 \right\},$$

with the interpretation that  $A(s, t) = X(t) - X(s)$  is the amount of input that entered the system in time interval  $[s, t)$ ,  $s < t$ . For notational convenience we write  $Q$  instead of  $Q(t)$ , if one dimensional properties of the queueing process are analyzed (in stationarity).

The following special cases of  $X(\cdot)$  play a crucial role in the literature:

- The case of fractional Brownian motion (FBM):  $X(t) = B_H(t)$ , where  $B_H(\cdot)$  is a fractional Brownian motion with Hurst parameter  $H \in [1/2, 1)$ ; that is,  $X(\cdot)$  is a centered Gaussian process with stationary increments, continuous sample paths,  $X(0) = 0$  a.s., and variance function  $\sigma_X^2(t) = t^{2H}$ ; see, e.g., [6, 13, 16, 21].
- The case of Integrated Gaussian (IG) processes:  $X(t) = \int_0^t Z(s) ds$ , where  $Z(\cdot)$  is a centered stationary Gaussian process with continuous covariance function  $R(t) = \text{Cov}(Z(s+t), Z(s))$ ; see, e.g., [8, 15, 16].

Both classes of inputs possess the property that  $\sigma_X^2(t)$  is regularly varying at  $\infty$  with index  $\alpha_\infty \in [1, 2)$ . We refer to the monograph [16] for a complete survey on Gaussian queueing models.

In this paper we present three problems in Gaussian fluid model theory. Despite the substantial research efforts devoted, these problems are still open, as far as we are aware of.

## 2. CORRELATION STRUCTURE OF GAUSSIAN QUEUE

For Gaussian queues, so far the focus was on the characterizing the steady-state distribution of  $\{Q(t) : t \geq 0\}$ . Much less is known about the dependance structure of the queueing process, represented by the correlation function:

$$\rho(t) := \text{Corr}(Q(t), Q(0)) = \text{Cov}(Q(t), Q(0)) / \text{Var}(Q(0)).$$

Properties of  $\rho(t)$  have been studied for several other queueing systems, see [12, 24] and references therein. It is the dependence structure of the Gaussian input process that makes standard techniques not applicable. An additional difficulty in analyzing the asymptotics of  $\rho(t)$  is that for a general Gaussian input process  $X(\cdot)$ , there are no explicit expressions available for the (stationary) distribution of  $Q$ , let alone of the transient  $Q(t)$ .

The relevance of insight into properties of  $\rho(t)$  (as  $t \rightarrow \infty$ ) is evident, not only in view of engineering purposes. From a more general standpoint, an interesting and important fundamental question can be stated: *is the short/long range dependance*

property on the level of input process  $X(\cdot)$  inherited by the workload process? Apart from the case of the Brownian queue (also referred to as reflected Brownian motion), for which  $\rho(t)$  decays exponentially fast to 0 (see [1]), the answer to this question is open. Interestingly, it is anticipated that for  $\{X(t)\}$  being long-range dependent, the asymptotics of  $\rho(t)$  are decaying polynomially, equally fast as the asymptotics of  $\text{Cov}(X(1), X(t+1) - X(t))$  [18]. More precisely, we expect the following.

**Conjecture 2.1.** (i) If  $\alpha_\infty = 1$ , then for some constant  $\gamma_1 \in (0, \infty)$ , as  $t \rightarrow \infty$ ,

$$(2) \quad \lim_{t \rightarrow \infty} \frac{\log \rho(t)}{t} = -\gamma_1.$$

(ii) If  $\alpha_\infty \in (1, 2)$ , then for some constant  $\gamma_{\alpha_\infty} \in (0, \infty)$ , as  $t \rightarrow \infty$ ,

$$(3) \quad \lim_{t \rightarrow \infty} \frac{\rho(t)}{\sigma_X^2(t)/t^2} = \gamma_{\alpha_\infty}.$$

The above conjectures are to some extent supported by findings in [4, 11], where the asymptotics of  $\mathbb{P}(Q(0) > u, Q(t_u) > u)$ , as  $u \rightarrow \infty$  were derived for various classes of functions  $t_u$ .

### 3. SPEED OF CONVERGENCE TO STATIONARITY

Assuming that at time 0 the system is empty, the *transient* buffer content  $Q_{\text{tr}}(t)$  at time  $t > 0$  obeys  $Q_{\text{tr}}(t) =_d \sup_{s \in [0, t]} (X(s) - cs)$ . Knowledge of the *speed of convergence* of  $Q_{\text{tr}}(t)$  to  $Q$  is intimately related to many aspects of the analyzed queueing system. For instance, it helps to determine how long one should simulate  $X(t) - ct$  in order to get the assumed accuracy in estimation of  $\mathbb{P}(Q > u)$ .

Let

$$\gamma(u, t) := \mathbb{P}(Q > u) - \mathbb{P}(Q_{\text{tr}}(t) > u).$$

Notice that

$$\gamma(u, t) = \mathbb{P} \left( \sup_{s \in [0, t]} (X(s) - cs) \leq u; \sup_{s \in (t, \infty)} (X(s) - cs) > u \right).$$

Apart from the case of Brownian input, for which

$$\gamma(u, t) = e^{-2uc} \mathbb{P}(\mathcal{N} > (ct - u)/\sqrt{t}) - \mathbb{P}(\mathcal{N} > (ct + u)/\sqrt{t}),$$

where  $\mathcal{N}$  is the standard normal random variable, little is known about the behavior of  $\gamma(u, t)$  for general Gaussian inputs.

**Open Problem 3.1.** Find the asymptotics of  $\gamma(u, t)$ , as  $t \rightarrow \infty$ , for fixed  $u$ .

This setting is intimately related with the notion of *relaxation time*, which was intensively investigated in the classical, non-Gaussian queueing context.

Recent progress in understanding the behavior of  $\gamma(u, t)$  was made in [18], where

$$\mathbb{D}_1(t) := \sup_{u>0} \gamma(u, t), \quad \text{and} \quad \mathbb{D}_2(t) := \int_0^\infty \gamma(u, t) du$$

were studied for queues driven by fractional Brownian motion. Interestingly, in [18] it was proved that

$$(4) \quad \lim_{t \rightarrow \infty} \frac{\log \mathbb{D}_1(t)}{t^{2-2H}} = \lim_{t \rightarrow \infty} \frac{\log \mathbb{D}_2(t)}{t^{2-2H}} = \lim_{t \rightarrow \infty} \frac{\log \mathbb{P}(K > t)}{t^{2-2H}},$$

where

$$K := \inf\{s \geq 0 : Q(s) = 0\} - \sup\{s \leq 0 : Q(s) = 0\}$$

denotes the *ongoing* busy period at time 0; the rightmost decay rate in (4) was computed in [17]. We expect that this result carries over to the class of Gaussian inputs with regularly varying variance function at  $\infty$ . In view of (4), the following challenging question arises.

**Open Problem 3.2.** *Find the exact asymptotics of metrics  $\mathbb{D}_1(t), \mathbb{D}_2(t)$ , as well as those of  $\mathbb{P}(K > t)$ , as  $t \rightarrow \infty$ .*

#### 4. ESTIMATES AND SIMULATION OF THE ASYMPTOTIC CONSTANT

Over the past decade, the asymptotic behavior of  $\mathbb{P}(Q > u)$ , as  $u \rightarrow \infty$ , was an important research theme, both for FBM and IG driven queues; see [3, 10, 13, 14]. The structural form of these asymptotics is known by now, and captured by the following general formula:

$$(5) \quad \mathbb{P}(Q > u) = \mathcal{C} u^\beta \Psi(m(u))(1 + o(1)),$$

as  $u \rightarrow \infty$ , with known  $\beta > 0$ ,

$$m(u) := \min_{t \geq 0} \frac{u + ct}{\sigma_X(t)},$$

and  $\Psi(u) := \mathbb{P}(\mathcal{N} > u)$ . The *asymptotic constant*  $\mathcal{C}$  in (5) can be expressed in terms of the so-called *generalized Pickands constant*  $\mathcal{H}_\eta$ , associated with a Gaussian process  $\eta(t)$  that directly relates to our input process on  $X(t)$ . Generally, neither an explicit formula nor an accurate approximation for  $\mathcal{C}$  is known.

Recall that by *generalized Pickands constant*  $\mathcal{H}_\eta$  we understand the limit [3]

$$(6) \quad \lim_{T \rightarrow \infty} \frac{\mathcal{H}_\eta(T)}{T} = \mathcal{H}_\eta,$$

where  $\mathcal{H}_\eta(T) := \mathbb{E} \exp(\max_{t \in [0, T]} (\sqrt{2}\eta(t) - \sigma_\eta^2(t)))$ . In order to ensure that  $\mathcal{H}_\eta$  is well defined, it is assumed that  $\eta(t)$  is a centered Gaussian process with stationary increments, a.s. continuous sample paths,  $\eta(0) = 0$  and such that the variance function  $\sigma_\eta^2(t)$  satisfies

- C1**  $\sigma_\eta^2(t) \in C^1([0, \infty))$  is strictly increasing and there exists  $\epsilon > 0$  such that  $\limsup_{t \rightarrow \infty} t \dot{\sigma}_\eta^2(t) / \sigma_\eta^2(t) \leq \epsilon$ ;
- C2**  $\sigma_\eta^2(t)$  is regularly varying at 0 with index  $\alpha_0 \in (0, 2]$  and  $\sigma_\eta^2(t)$  is regularly varying at  $\infty$  with index  $\alpha_\infty \in (0, 2)$ .

We note that, for models with  $\{X(t) : t \in \mathbb{R}\}$  having a regularly varying variance function at  $\infty$  with  $\alpha_\infty > 1$ , the asymptotic constant  $\mathcal{C}$  reduces to the *classical* Pickands constant  $\mathcal{H}_{B_H}$  with  $H = \alpha_\infty/2$ ; see [10, 13].

The constant  $\mathcal{H}_{B_H}$  has been widely studied, but just a few partial results have been obtained so far. In particular, the exact value of  $\mathcal{H}_{B_H}$  is known only for  $\mathcal{H}_{B_{1/2}} = 1$  and  $\mathcal{H}_{B_1} = 1/\sqrt{\pi}$ ; see [22]. Some estimates for  $\mathcal{H}_{B_H}$  are given in [3, 5, 7, 25], but the gap between the lower and upper bounds is still quite substantial. For example, bounds for  $\mathcal{H}_{B_H}$  are precise only in the neighborhood of  $H = 1/2$  and  $H = 1$ , see [5]. This makes the following open problem to be particularly important.

**Open Problem 4.1.** *Find further characterizations of  $\mathcal{H}_\eta$  that lead to more precise estimates of (6).*

The following conjecture should perhaps be seen as mathematical folklore, as it lacks any firm heuristic support. However, due to the lack of precision of currently known estimates, it has not been falsified so far.

**Conjecture 4.1.**  $\mathcal{H}_{B_H} = 1/\Gamma\left(\frac{1}{2H}\right)$ .

Obtaining exact values for  $\mathcal{H}_\eta$  being prohibitively hard, it is of great importance to develop stable algorithms for estimating Pickands constants by simulation. It is evident that methods based on the definition (6) cannot produce any efficient algorithms. This is not only due to the fact that

$$\exp\left(\max_{t \in [0, T]} \left(\sqrt{2}\eta(t) - \sigma_\eta^2(t)\right)\right)$$

is asymptotically lognormal, but also a consequence of the fact that for each  $\epsilon > 0$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \exp\left(\max_{t \in [0, T]} \left(\sqrt{2}\eta(t) - (1 + \epsilon)\sigma_\eta^2(t)\right)\right) = 0,$$

while

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \exp\left(\max_{t \in [0, T]} \left(\sqrt{2}\eta(t) - (1 - \epsilon)\sigma_\eta^2(t)\right)\right) = \infty.$$

This explains that the methods needed cannot be straightforward, crude Monte Carlo type of procedures. There is some hope to develop a reliable simulation method based on the so-called *change of measure* technique. For the very special case of  $\eta(t) = \int_0^t Z(s)ds$ , where  $Z(t)$  is an Ornstein-Uhlenbeck process, this approach produced a stable algorithm, see [7]. However, applying a somewhat similar technique to estimate  $\mathcal{H}_{B_H}$ , resulted in two completely different estimates: compare [2] with [19].

**Open Problem 4.2.** *Develop a reliable simulation algorithm for efficient estimation of  $\mathcal{H}_\eta$ .*

Finding an efficient technique to estimate  $\mathcal{H}_\eta$  by simulation is a challenging task which is important not only from the perspective of the theory Gaussian fluid queues, but also in light of the theory of extreme values of stochastic processes in a more general sense.

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