Exact computation of emergy based on a mathematical reinterpretation of the rules of emergy algebra

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Exact Computation of Emergy Based on a Reformulation of the Rules of Emergy Algebra

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Abstract

The emergy algebra is based on four rules which use is sometimes confusing or reserved only to the experts of the domain. The emergy computation does not obey conservation logic (i.e. emergy computation does not obey Kirchoff-like circuit law). In this paper the authors propose to reformulate the emergy rules into four axioms which provide an exact algorithm to compute emergy within a system of interconnected processes at steady state modelized by an oriented graph named emergy graph.

Because emergy algebra follows a logic of memorization the evaluation principles deal with paths in emergy graph. The underlying algebraic structure is the the set of non-negative reals equipped with the maximum (max), the addition (+) and the multiplication (·). The maximum is associated with the co-product problem. The addition is linked with the split problem or more generaly with the independence of two emergy flows. And the

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multiplication is related to the logic of memorization. The axioms describe how to use the different operators max, + and · to combine flows without any confusion or ambiguity. The method is tested on five benchmark emergy examples.

Keywords: Track summing method, path, memory algebra, emergy algebra, exact emergy computation

1. Introduction

According to Odum [1] the emergy is defined as the total solar equivalent energy/exergy of one form that was used up directely or indirectely in the work of making a product or a service. In Emergy point of view the comparison of interconnected processes/components can be based on the same fundamentals and provide reliable sustainability developement dimensionless numbers. The idea of the emergy is based on the maximum power principle stated by the biologist Lotka [2].

However, as mentioned in e.g. Hau and Bakshi [3] even if the idea of emergy is attractive only Odum and a small circle of co-workers have developped the notion of emergy and emergy analysis since the 1980's. Even if there are attractive features it is mentioned in Hau and Bakshi [3, Section 1 and subsection 3.2] that emergy analysis received many criticisms. Most of these criticisms could be applied to other popular methods which try to analyze in the same framework environmental and industrial/human systems. As mentioned in Hau and Bakshi [3, Section 5] emergy analysis of large and complex systems is one of the main challenges of emergy approach. A system is large when it possesses a large number of components. A system is complex when there are splits and co-products within the same system. Roughly speaking, an emergy system (see the precise definition in Section 2) is represented by an oriented graph. Each node represents a process/component. The emergy circulates on the branches of the graph (or diagram associated with the system) and is assigned at the nodes of the system. Because emergy can be considered as the memory of all solar used during a process (see e.g. Bastianoni et al. [4]) the notion of pathway from a source is the fundamental notion to manipulate for emergy analysis. A pathway from a source of emergy (e.g.: sun, wind, fuel, ...) on the graph represents the sequel of assignations of the emergy source. According to Odum [1, Chap. 6, p. 90] in a split branching a pathway of the emergy system divides into several branches of the same kind e.g. as in hydraulic systems. In a co-product branching, the flow in each branch is of a different kind e.g. as in combined heat and power plants (described in e.g. Horlock [5]). The complexity comes from the fact that the flow circulating on a branch is in fact a combination of splits and co-products coming upstream this branch. And the emergy upstream flows cannot be counted more than once.

The way to combine the emergy upstream flows is described and explained in e.g. [1, Chap. 6]. It is summarized in e.g. Sciubba and Ulgiati [6, pp. 1965-1966] as follows under the name *emergy algebra*.

R1 : When only one product is obtained from a process (i.e. a process with only one output), all source-emergy is assigned to it.

Concerning processes with more than one output we have.

R2 : When a flow (of emergy) splits the total emergy splits accordingly, based on the exergy/energy flowing through each pathway.

R3 : When two or more co-products are generated in a process, the total source-emergy is assigned to each of them.

Finally, a fourth rule describes how emergy is assigned within a system of interconnected processes.

R4 : Emergy cannot be counted twice within a system.

R4.1 : Emergy in feedbacks cannot be double counted.

R4.2 : Co-products, when reunited, cannot be summed. Only the emergy of the largest co-product flow is accounted for.

The general method of emergy analysis consists in propagating these rules from emergy sources to the outputs of the system of interconnected processes. Difficulty occurs for large and complex systems. Moreover, the use of these rules are not easy and seems to be confusing. E.g. concerning the application of rule R4.2 it is clearly noticed in Lazzaretto [7, p.2201]: "As observed by one of the reviewers the rule counting the largest emergy value [arriving at a node] is a rather "crude way" of avoiding double counting". This approximation is made in e.g. Li et al. [8, (2) p. 415] when authors studied the output emergy at node G of the emergy graph (see their figures 8 and 9).

To bypass these difficulties several numerical methods have been proposed. Most of them are approximation methods based on linear algebra (It means that they do not use the operator maximum). Some of them are based on pre-analysis of the system which is not well-suited for an emergy computation. For more details on such approaches see e.g. Li et al. [8, subsection 1.3 and references therein]. Few simulation solutions have been proposed (see Odum and Peterson [9], Maud [10] and references therein). All these solutions have no mathematical framework and it is difficult to validate their results. To the best knowledge of the authors only two mathematical framework have been proposed in the literature. The first is Giannantoni [11] who proposed another approach based on (non)-linear differential equations and on a variant of fractional derivatives concept. The second is the approach of Bastianoni et al. [4] based on (commutative) free monoids.

Contributions of the paper are as follows.

To respect the logic of memorization of the emergy algebra a new path-oriented method is proposed. A path-oriented method is a method which manipulates paths in a graph. In this paper the proposed method is based on the *Track summing method* developed by Tennenbaum [12]. The Track summing method is a path-oriented method which is exact and has been implemented for emergy systems with splits and without co-product. More precisely authors start from the expression given in Tennenbaum [12, p. viii] for acyclic source requirements and extend the Track summing method to interconnected systems with splits and co-products.

It is noticed that the Tennenbaum's Track summing method can be divided into two different parts. The first part is a path-finding problem. The second part is a computational problem. The path-finding problem can be solved by method based on a slight modification of methods to enumerate elementary paths in a graph which have been developed by e.g. Kaufmann and Malgrange [13], Kaufmann [14], Benzaken [15], Backhouse and Carré [16], also mentioned in e.g. Gondran and Minoux [17]. And this is clearly

Figure 1: List of Emergy Symbols

not the purpose of this paper. It is the subject of a companion paper which is in preparation.

Thus, assuming that all emergy paths ending by a given arc of the emergy graph are known the major contributions are the following ones.

- The paper describes how to compute the exact value of emergy flowing on this arc (see the recursive algorithm subsection 3.1).
- To proceed an axiomatic basis is proposed as a reformulation of the rules R1-R4 to avoid confusing applications of the rules and decide whether or not emergy flows are independent. As an example the abovementioned problem with co-product (i.e the application of rule R4.2) noticed in e.g. Lazzaretto [7, p.2201] is solved (see the illustrative example of subsection 4.4).

2. Example and important definitions

The way by which emergy circulates in a multicomponent system is modelized by an oriented graph. The graph has input nodes called sources, intermediate nodes and output (or final) nodes. Each node is represented by an integer (i.e. an element of $\mathbb{N}$).

The drawing conventions for the emergy graph are depicted in Figure 1. A source is represented by the symbol Fig 1.A, an intermediate node on the emergy graph is represented by Fig 1.B, an output node is represented by

Figure 2: Emergy graph G_1

Fig 1.C. Splits are modelized by Fig 1.D and co-products are modelized by Fig 1.E.

Let us consider a system with its associated emergy graph (or diagram) G_1 described by the Figure 2.

According to convention notations of Figure 1 the set of sources is $\{1, 2\}$. For numerical application authors assume that the emergy of 1 is 400 seJ and the emergy of node 2 is 100 seJ. The set of intermediate nodes is $\{3, 4, 5, 6\}$ and the set of output nodes is $\{7\}$. Finally, the set of all nodes is $\{1, 2, 3, 4, 5, 6, 7\}$.

The set of the arcs is:

$$\mathcal{A}_1 = \{[1; 3], [2; 4], [3; 4], [3; 5], [4; 6], [5; 6], [6; 5], [6; 7]\}.$$

The weight (i.e. the fraction of emergy which is assumed to be given in this paper) of the arcs $[1; 3]$, $[2; 4]$, $[3; 4]$, $[3; 5]$, $[4; 6]$, $[5; 6]$ is 1. The weight of the arc $[6; 7]$ is $4/5$ and the weight of the arc $[6; 5]$ is $1/5$.

There is a split at node 6 and a co-product at node 3.

All this information is encoded using the following 8-tuple:

$$G_1 = (\mathcal{L}^s_1, \mathcal{L}^i_1, \mathcal{L}^o_1, \mathcal{A}_1, \boldsymbol{R}_{G_1}, \boldsymbol{\Omega}_{G_1}, \boldsymbol{\epsilon}_{G_1}, \mathcal{E}_1) \tag{1}$$

Where:

- $\mathcal{L}^s_1 = \{1, 2\}, \mathcal{L}^i_1 = \{3, 4, 5, 6\}, \mathcal{L}^o_1 = \{7\}$.

- The relations between the arcs are stored in the array $\boldsymbol{R}_{G_1}$:

	[1; 3]	[2; 4]	[3; 4]	[3; 5]	[4; 6]	[5; 6]	[6; 5]	[6; 7]
[1; 3]	id	$\perp$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
[2; 4]	$\perp$	id	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
[3; 4]	$\emptyset$	$\emptyset$	id	$\parallel$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
[3; 5]	$\emptyset$	$\emptyset$	$\parallel$	id	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
[4; 6]	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	id	$\emptyset$	$\emptyset$	$\emptyset$
[5; 6]	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	id	$\emptyset$	$\emptyset$
[6; 5]	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	id	$\perp$
[6; 7]	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\perp$	id

  The relation id denotes the identity relation (i.e. equality of the arcs). The relation $\emptyset$ means that there is no relation between the arcs. The relation $\parallel$ means that there is a co-product. In the example there is a co-product at node 3, thus we have: $[3;4] \parallel [3;5]$ (or equivalently $\boldsymbol{R}_{G_1}([3;4],[3;5]) =\parallel$) and $[3;5] \parallel [3;4]$. To indicate that flows which circulate on arcs are independent we use the symbol relation $\perp$. There are two cases of independence. The first case is for a split. In the example there is a split at node 6, thus $[6;5] \perp [6;7]$ and $[6;7] \perp [6;5]$ (or equivalently $\boldsymbol{R}_{G_1}([6;7],[6;5]) =\perp$). The second case is for the sources. In the example, the arcs $[1;3]$ and $[2;4]$ satisfy this condition, thus $[1;3] \perp [2;4]$ and $[2;4] \perp [1;3]$.

- The matrix of the weights of the graph, $\boldsymbol{\Omega}_{G_1}$, is as follows:

$$\boldsymbol{\Omega}_{G_1} = \begin{array}{|c|ccccccc|}
\hline
\curvearrowright & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
6 & 0 & 0 & 0 & 0 & 1/5 & 0 & 4/5 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}$$

- The vector of assigned emergy sources is:

$$\boldsymbol{\epsilon}_{G_1} = (400, 100, 0, 0, 0, 0, 0).$$

On this graph we define different notion of paths. A *path* π has the form $\pi = \underline{1}$ (unit path, i.e. a path with no arc) or e.g. $\pi = [3;4][4;6][6;5][5;6]$ which is a path from first node 3 to last node 6 in G_1. A *path from a source* is a path such that its first node is a source. E.g. $[1;3][3;4][4;6][6;5][5;6][6;5]$ is a path from a source (1) to node 5. A *simple path* is a path such that all its nodes are different. E.g. $[4;6][6;5]$ is a simple path from node 4 to node 5. A *simple path from a source* is a simple path such that its first node is a source. E.g. $[2;4][4;6][6;5]$ is a simple path from a source (2) to node 5. Finally, an *emergy path* of n ($n \geq 1$) arcs is a path such that the path with the $n-1$ first arcs is a simple path from a source. E.g. $[2;4][4;6][6;5]$, $[2;4][4;6][6;5][5;6]$ are emergy paths. But the path $[2;4][4;6][6;5][5;6][6;7]$ is not an emergy path because the path $[2;4][4;6][6;5][5;6]$ is not a simple path from a source.

- The set of all emergy paths of G_1 is assumed to be given in this paper is as follows:

$$\mathcal{E}_1 = \left\{ \begin{array}{c} [1;3], [1;3][3;4], [1;3][3;5], [1;3][3;5][5;6][6;5], \\ [1;3][3;4][4;6][6;5], \\ [1;3][3;4][4;6], [1;3][3;5][5;6], [1;3][3;4][4;6][6;5][5;6], \\ [1;3][3;4][4;6]][6;7], [1;3][3;5][5;6][6;7], [2;4], [2;4][4;6][6;5], \\ [2;4][4;6], [2;4][4;6][6;5][5;6], [2;4][4;6][6;7] \end{array} \right\}.$$

Recall that the computation of emergy paths is a further work.

Concatenation.

We define the concatenation of paths by analogy with the concatenation of letters to form words. For example the concatenation of the path $[1;3][3;4][4;6][6;5]$ with the path $[5;6][6;5]$ gives the path $[1;3][3;4][4;6][6;5][5;6][6;5]$. For pure mathematical reasons we add that the path $\underline{1}$ concatened with any other path π gives π (i.e. $\underline{1}\pi = \pi\underline{1} = \pi$). That is why $\underline{1}$ is called the unit path.

If $\mathcal{U}$ denotes a set of paths. Then for any path π the set

$$\pi\mathcal{U}$$

denotes the union of the paths obtained by the concatenation of π with the paths of $\mathcal{U}$. Following the logic of memorization of the emergy algebra the path π can be interpreted as the *past* of the paths of $\pi\mathcal{U}$.

For example,

$$[1;3]\{[3;4][4;6][6;7], [3;5][5;6][6;7]\} \quad = \{[1;3][3;4][4;6][6;7], [1;3][3;5][5;6][6;7]\}$$

3. Emergy evaluation principles

Let us recall that emergy algebra obeys a logic of memorization which implies that the definition of emergy is based on paths in emergy graph.

The general principle is that at a node of the emergy graph only emergy flows arriving at this node with the same past (or upstream flow) can be combined using the maximum, addition and multiplication operators. So, let us consider the set of nonnegative reals $\mathbb{R}_+$ equipped with the operations max (i.e. maximum), $+$ (the addition) and $\cdot$ (the multiplication). The max is associated with the co-product $\|$. The addition is associated with the independent relation $\perp$. And the multiplication is related to the logic of memorization of the emergy.

Based on previous preliminaries in this section, remarks in the summary and the introduction of the paper we propose the following definition for emergy.

Definition 3.1 (Emergy). *Let us consider the emergy graph*

$$G = (\mathcal{L}^s, \mathcal{L}^i, \mathcal{L}^o, \mathcal{A}, \boldsymbol{R}_G, \boldsymbol{\Omega}_G, \boldsymbol{\epsilon}_G, \mathcal{E}),$$

where $\mathcal{E}$ is assumed to be known in this paper. Then, the emergy flowing on arc $[l; l']$ with $l, l' \in \mathcal{L}$ is the function denoted $\mathsf{Em}(\mathcal{E}([l; l']))$, where $\mathcal{E}([l; l']) \subseteq \mathcal{E}$ denotes the set of all emergy paths ending by the arc $[l; l']$, which satisfies the following axioms which replace the rules R1-R4 of emergy algebra:

($\mathfrak{a}$.1) . *$\forall \pi$, $\forall k \geq 1$, $\forall a_1, \ldots, a_k \in \mathcal{A}$ s.t. $a_1 \top a_2 \cdots \top a_k$ with $\top \in \{\mathsf{id}, \perp, \|\}$, $\forall \mathcal{U}_1, \ldots, \mathcal{U}_k \subseteq \mathcal{E}([l; l'])$:*

($\mathfrak{a}$.1.1) . *In the case of one ouput (i.e. $\top = \mathsf{id}$) all emergy having the same past π is assigned to this output, that is:*
$\mathsf{Em}(\cup_{i=1}^k \pi a_i \mathcal{U}_i) = \mathsf{Em}(\pi a_1 (\cup_{i=1}^k \mathcal{U}_i))$.

($\mathfrak{a}$.1.2) . *If the arcs a_i are independent then the total emergy flowing on arc $[l;l']$, $\mathsf{Em}(\cup_{i=1}^{k}\pi a_i\mathcal{U}_i)$ is equal to the sum of the emergies flowing on arc $[l;l']$ of the system if there was only one arc a_i after the past π, $\mathsf{Em}(\pi a_i\mathcal{U}_i)$, $i=1,\ldots,k$, when reunited, i.e.:*
$\mathsf{Em}(\cup_{i=1}^{k}\pi a_i\mathcal{U}_i) = \sum_{i=1}^{k}\mathsf{Em}(\pi a_i\mathcal{U}_i)$, *if* $\top = \perp$.
(See the explanation in Appendix A).

($\mathfrak{a}$.1.3) . *If there is co-product just after π then the total emergy flowing on arc $[l;l']$, $\mathsf{Em}(\cup_{i=1}^{k}\pi a_i\mathcal{U}_i)$, is equal to the maximum of the emergies flowing on arc $[l;l']$ of the system if there was only one arc a_i after the past π, $\mathsf{Em}(\pi a_i\mathcal{U}_i)$, $i=1,\ldots,k$, when reunited, i.e.:*
$\mathsf{Em}(\cup_{i=1}^{k}\pi a_i\mathcal{U}_i) = \max_{i=1}^{k}\mathsf{Em}(\pi a_i\mathcal{U}_i)$, *if* $\top = \|$.
(See the explanation in Appendix B).

($\mathfrak{a}$.2) . *For all path π, for all $\mathcal{U}\subseteq\mathcal{E}$, $\mathsf{Em}(\pi\mathcal{U}) = \mathsf{Em}(\pi)\cdot\mathsf{Em}(\mathcal{U})$. It means that the computation of the emergy of emergy paths with the same past π can be divided into the computation of the past π and the computation of the downstream part of the emergy paths.*

($\mathfrak{a}$.3) . *For all path $[l_1;l_2]\ldots[l_{k-1};l_k]$,*

$$\mathsf{Em}([l_1;l_2]\ldots[l_{k-1};l_k]) = \begin{cases} \Pi_{i=1}^{k-1}\mathbf{\Omega}_G(l_i,l_{i+1}) & \textit{if } l_1\notin\mathcal{L}^s \\ \boldsymbol{\epsilon}_G(l_1)\cdot\Pi_{i=1}^{k-1}\mathbf{\Omega}_G(l_i,l_{i+1}) & \textit{if } l_1\in\mathcal{L}^s \end{cases}$$

In the case where $l_1\in\mathcal{L}^s$ $\mathsf{Em}([l_1;l_2]\ldots[l_{k-1};l_k])$ coincides with the emergy flowing on the path $[l_1;l_2]\ldots[l_{k-1};l_k]$ which is obtained as the fraction $\Pi_{i=1}^{k-1}\mathbf{\Omega}_G(l_i,l_{i+1})$ of the emergy of source l_1, $\boldsymbol{\epsilon}_G(l_1)$.

And for pure mathematical consideration we add:

($\mathfrak{a}$.4) . *For all path π:* $\mathsf{Em}(\{\pi\}) = \mathsf{Em}(\pi)$.

We call ($\mathfrak{a}$.1)-($\mathfrak{a}$.2) the *tree property*.

The emergy rules R1-R4 do not make clearly the difference between the qualitative analysis of the emergy (i.e. the enumeration problem of the emergy paths) and the quantitative analysis of the emergy (which is the focus of the paper). Nevertheless,

- the rule R1 has been expressed as a particular case of axioms ($\mathfrak{a}$.1.1) and ($\mathfrak{a}$.1.2, with $\forall i = 1, \ldots, k$: $a_i \in [\mathcal{L}^s; l]$, $\mathcal{U}_i = \{[l; l']\}$ for some $l, l' \in \mathcal{L} \setminus \mathcal{L}^s$), and the axioms ($\mathfrak{a}$.2)-($\mathfrak{a}$.4). This rule is illustrated in e.g. [18, Fig. 6.b p. 225]. However, let us remark that this rule seems not to be always written the same way in the literature (see e.g. [6] –also used in the Introduction of this paper–, [8], [19], [7], [20]).

  The basic case of n sources and one product, usually written under emergy tables, is completely treated in subsection 4.1 as an application of axioms ($\mathfrak{a}$.1.2) and ($\mathfrak{a}$.2)-($\mathfrak{a}$.4).

- The rule R2 concerning splits has been expressed by axioms ($\mathfrak{a}$.1.2) and ($\mathfrak{a}$.2)-($\mathfrak{a}$.4).

- The rule R3 is expressed as a particular case of the axiom ($\mathfrak{a}$.1.3) with $\mathcal{U}_i = \{\underline{1}\}$, $i = 1, \ldots, k$ and the application of ($\mathfrak{a}$.2)-($\mathfrak{a}$.4).

- The rule R4 concerning the double counting problem is expressed by the application of the axioms ($\mathfrak{a}$.1.3) and ($\mathfrak{a}$.2)-($\mathfrak{a}$.4) and the computation of the emergy paths $\mathcal{E}$.

3.1. Algorithm for emergy computation

In this Section we present a recursive algorithm to compute $\mathsf{Em}(\mathcal{E}([l; l']))$ which is as follows:

Figure 3: Emergy graph G_0 with n sources and one output

- Enter emergy graph $G = (\mathcal{L}^s, \mathcal{L}^i, \mathcal{L}^o, \mathcal{A}, \boldsymbol{R}_G, \boldsymbol{\Omega}_G, \boldsymbol{\epsilon}_G, \mathcal{E})$, l and l'

- $\mathcal{X} := \mathcal{E}([l; l'])$, where $\mathcal{E}([l; l'])$ is assumed to be known/given.

While $\mathcal{X} \neq \emptyset$ Do

1 Factorize $\mathcal{X}$ according to ($\mathfrak{a}$.1) using the same notations

2 Apply ($\mathfrak{a}$.1.1) if $\top = \mathsf{id}$ or ($\mathfrak{a}$.1.2) if $\top = \perp$ or ($\mathfrak{a}$.1.3) if $\top = \|$

3 Apply ($\mathfrak{a}$.2) to each $\pi a_i \mathcal{U}_i$, $i = 1, \ldots, k$ if $\top \in \{\perp, \|\}$ or Apply ($\mathfrak{a}$.2) to $\pi a_1 (\cup_{i=1}^k \mathcal{U}_i)$ if $\top = \mathsf{id}$

4 Evaluate by axiom ($\mathfrak{a}$.3) and store $\mathsf{Em}(\pi a_i)$, $i = 1, \ldots, k$

5 $\mathcal{X} := \mathcal{X} \setminus (\cup_{i=1}^k \{\pi a_i\})$

EndWhile

- Return $\mathsf{Em}(\mathcal{E}([l; l']))$

4. Numerical examples

4.1. n sources, one product

Let us consider the emergy graph G_0 with n sources and one output arc as depicted in Figure 3.

We have: $\mathcal{L}_0^s = \{1, 2, \ldots n\}$, $\mathcal{L}_0^i = \{n+1\}$ and $\mathcal{L}_0^o = \{n+2\}$.

$$\mathcal{A}_0 = \{[1; n+1], [2; n+1], \ldots, [n; n+1], [n+1; n+2]\}$$

The relations between the arcs are stored in the array $\boldsymbol{R}_{G_0}$:

	$[1;n+1]$	$[2;n+1]$	$\cdots$	$[n;n+1]$	$[n+1;n+2]]$
$[1;n+1]$	id	$\perp$	$\cdots$	$\perp$	$\emptyset$
$[2;n+1]$	$\perp$	id	$\perp$	$\cdots$	$\emptyset$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\vdots$
$[n;n+1]$	$\perp$	$\perp$	$\cdots$	id	$\emptyset$
$[n+1;n+2]$	$\emptyset$	$\emptyset$	$\cdots$	$\emptyset$	id

The matrix of the weights of the graph G_0, $\boldsymbol{\Omega}_{G_0}$, is as follows:

$$\boldsymbol{\Omega}_{G_0} = \begin{array}{|c|cccccc|}\hline \curvearrowright & 1 & 2 & \cdots & n & n+1 & n+2 \\ \hline 1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 2 & 0 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ n & 0 & 0 & \cdots & 0 & 1 & 0 \\ n+1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ n+2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline \end{array}$$

Finally, the vector of assigned emergy sources is:

$$\boldsymbol{\epsilon}_{G_0} = (\epsilon(1), \epsilon(2), \ldots, \epsilon(n), 0, 0).$$

The set of all emergy paths denoted $\mathcal{E}_0$ is:

$$\mathcal{E}_0 = \{[1;n{+}1], [2;n{+}1], \ldots, [n;n{+}1], [1;n{+}1][n{+}1;n{+}2], \ldots, [n;n{+}1][n{+}1;n{+}2]\}.$$

Let us compute the emergy flowing on arc $\mathsf{Em}(\mathcal{E}_0([n+1;n+2]))$. The direct application of the rule $R1$ gives that:

$$\mathsf{Em}(\mathcal{E}_0([n+1;n+2])) = \epsilon(1) + \ldots + \epsilon(n).$$

Now, let us compute $\mathsf{Em}(\mathcal{E}_0([n+1;n+2]))$ using our axiomatic basis.

First, let us assumed that the set of emergy paths ending by the arc $[n+1;n+2]$ is given. Thus:

$$\mathcal{E}_0([n{+}1;n{+}2]) = \{[1;n{+}1][n{+}1;n{+}2],[2;n{+}1][n{+}1;n{+}2],\ldots,[n;n{+}1][n{+}1;n{+}2]\}.$$

Rewrite $\mathcal{E}_0([n+1;n+2])$ as:

$$\mathcal{E}_0([n+1;n+2]) = \cup_{i=1}^{n} a_i \mathcal{U}_i,$$

with $a_i = [i,n+1], \mathcal{U}_i = \{[n+1;n+2]\}, \forall i = 1,\ldots,n$. And $a_1 \perp a_2 \perp \ldots \perp a_n$. Thus, applying ($\mathfrak{a}$.1.2) with $\pi = \underline{1}$ it comes:

$$\mathsf{Em}(\mathcal{E}_0([n+1;n+2])) = \sum_{i=1}^{n} \mathsf{Em}([i;n+1]\{[n+1;n+2]\}).$$

For all $i = 1,\ldots,n$ we apply ($\mathfrak{a}$.2), then:

$$\mathsf{Em}([i;n+1]\{[n+1;n+2]\}) = \mathsf{Em}([i;n+1]) \cdot \mathsf{Em}(\{[n+1;n+2]\})$$

By ($\mathfrak{a}$.3) because i is a source:

$$\mathsf{Em}([i;n+1]) = \epsilon(i) \cdot \mathbf{\Omega}_{G_0}(i,n+1).$$

Apply ($\mathfrak{a}$.4) we have:

$$\mathsf{Em}(\{[n+1;n+2]\}) = \mathsf{Em}([n+1;n+2]),$$

and by ($\mathfrak{a}$.3) noticing that $n+1$ is not a source:

$$\mathsf{Em}([n+1;n+2]) = \mathbf{\Omega}_{G_0}(n+1,n+2).$$

Thus, $\forall i = 1,\ldots,n$, by applying ($\mathfrak{a}$.2)-($\mathfrak{a}$.4) we have:

$$\mathsf{Em}([i;n+1]\{[n+1;n+2]\}) = \epsilon(i) \cdot \mathbf{\Omega}_{G_0}(i,n+1) \cdot \mathbf{\Omega}_{G_0}(n+1,n+2).$$

Finally we have:

$$\begin{aligned}\mathsf{Em}(\mathcal{E}_0([n+1;n+2])) &= \textstyle\sum_{i=1}^{n} \epsilon(i) \cdot \boldsymbol{\Omega}_{G_0}(i,n+1) \cdot \boldsymbol{\Omega}_{G_0}(n+1,n+2)\\ &= \textstyle\sum_{i=1}^{n} \epsilon(i) \cdot 1 \cdot 1\\ &= \textstyle\sum_{i=1}^{n} \epsilon(i).\end{aligned}$$

And the result obtained by application of the rule $R1$ is retrieved.

4.2. Tennenbaum-like example

Let us consider the emergy graph G_2 corresponding to the Figure 4.

Figure 4: Emergy graph G_2 Tennenbaum-like net

We have:

$$\mathcal{L}_2^s = \{1,2\}, \mathcal{L}_2^i = \{3,4\}, \mathcal{L}_2^o = \{5\}.$$

$$\mathcal{A}_2 = \{[1;3],[2;4],[3;4],[4;3],[4;5]\}.$$

The relations between the arcs are stored in the array $\boldsymbol{R}_{G_2}$:

	$[1;3]$	$[2;4]$	$[3;4]$	$[4;3]$	$[4;5]$
$[1;3]$	id	$\perp$	$\emptyset$	$\emptyset$	$\emptyset$
$[2;4]$	$\perp$	id	$\emptyset$	$\emptyset$	$\emptyset$
$[3;4]$	$\emptyset$	$\emptyset$	id	$\emptyset$	$\emptyset$
$[4;3]$	$\emptyset$	$\emptyset$	$\emptyset$	id	$\perp$
$[4;5]$	$\emptyset$	$\emptyset$	$\emptyset$	$\perp$	id

The matrix of the weights of the graph G_2, $\boldsymbol{\Omega}_{G_2}$, is as follows:

$$\mathbf{\Omega}_{G_2} = \begin{array}{|c|ccccc|}
\hline
\curvearrowright & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 0 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 \\
4 & 0 & 0 & 3/5 & 0 & 2/5 \\
5 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}$$

And the vector of assigned emergy is:

$$\boldsymbol{\epsilon}_{G_2} = (\epsilon(1), \epsilon(2), 0, 0, 0).$$

The set of all emergy paths denoted $\mathcal{E}_2$ is:

$$\begin{aligned}
\mathcal{E}_2 \;\; &= \{[1;3], [1;3][3;4][4;3], [1;3][3;4], [1;3][3;4][4;5], \\
&[2;4][4;3], [2;4], [2;4][4;3][3;4], [2;4][4;5]\}
\end{aligned}$$

For example let us give the close formula for emergy circulating on arc $[4;5]$.

The set of all emergy paths ending by the arc $[4;5]$ is:

$$\mathcal{E}_2([4;5]) = \{[1;3][3;4][4;5], [2;4][4;5]\},$$

and we compute the emergy $\mathsf{Em}(\mathcal{E}_2([4;5]))$ as follows.

- 1 and 2 are emergy sources, thus by definition of $\perp$ we have $[1;3] \perp [2;4]$ (i.e. $\boldsymbol{R}_{G_2}([1;3], [2;4]) = \perp$).

- Rewrite $\mathsf{Em}(\mathcal{E}_2([4;5]))$ as:

  $$\mathsf{Em}(\mathcal{E}_2([4;5])) = \mathsf{Em}([1;3]\{[3;4][4;5]\} \cup [2;4]\{[4;5]\}),$$

  with $[1;3] \perp [2;4]$.

- Apply ($\mathfrak{a}$.1.2) with $\pi = \underline{1}$, $k = 2$, $a_1 = [1;3]$, $\mathcal{U}_1 = \{[3;4][4;5]\}$, $a_2 = [2;4]$ and $\mathcal{U}_2 = \{[4;5]\}$. Then,

$$\mathsf{Em}(\mathcal{E}_2([4;5])) = \mathsf{Em}([1;3]\{[3;4][4;5]\}) + \mathsf{Em}([2;4]\{[4;5]\})$$

- Compute $\mathsf{Em}([2;4]\{[4;5]\})$ as follows:

$$\begin{aligned} \mathsf{Em}([2;4]\{[4;5]\}) &= \mathsf{Em}([2;4]) \cdot \mathsf{Em}(\{[4;5]\}) \\ &\quad \text{by } (\mathfrak{a}.2,\ \pi = [2;4],\ \mathcal{U} = \{[4;5]\}) \\ &= \epsilon(2) \cdot \mathbf{\Omega}_{G_2}(2,4) \cdot \mathsf{Em}(\{[4;5]\}) \\ &\quad \text{by } (\mathfrak{a}.3,\ l = 2,\ l' = 4) \\ &= \epsilon(2) \cdot \mathbf{\Omega}_{G_2}(2,4) \cdot \mathsf{Em}([4;5]) \\ &\quad \text{by } (\mathfrak{a}.4) \\ &= \epsilon(2) \cdot \mathbf{\Omega}_{G_2}(2,4) \cdot \mathbf{\Omega}_{G_2}(4,5) \\ &\quad \text{by } (\mathfrak{a}.3, \text{ noticing that } 4,5 \notin \mathcal{L}_2^s). \end{aligned}$$

- Compute $\mathsf{Em}([1;3]\{[3;4][4;5]\}$ as follows.

$$\begin{aligned} \mathsf{Em}([1;3]\{[3;4][4;5]\} &= \mathsf{Em}([1;3]) \cdot \mathsf{Em}(\{[3;4][4;5]\}) \\ &\quad \text{by } (\mathfrak{a}.2) \\ &= \mathsf{Em}([1;3]) \cdot \mathsf{Em}([3;4][4;5]) \\ &\quad \text{by } (\mathfrak{a}.4) \end{aligned} .$$

  Then, applying ($\mathfrak{a}$.3) to $\mathsf{Em}([1;3])$, $\mathsf{Em}([3;4][4;5])$, we have:

$$\begin{aligned} \mathsf{Em}([1;3]) &= \epsilon(1) \cdot \mathbf{\Omega}_{G_2}(1,3) \\ \mathsf{Em}([3;4][4;5]) &= \mathbf{\Omega}_{G_2}(3,4) \cdot \mathbf{\Omega}_{G_2}(4,5). \end{aligned}$$

Finally, we have:

$$\mathsf{Em}(\mathcal{E}_2([4;5])) = \epsilon(1) \cdot \mathbf{\Omega}_{G_2}(1,3) \cdot \mathbf{\Omega}_{G_2}(3,4) \cdot \mathbf{\Omega}_{G_2}(4,5) + \epsilon(2) \cdot \mathbf{\Omega}_{G_2}(2,4) \cdot \mathbf{\Omega}_{G_2}(4,5)$$

Numerical application.

$\epsilon(1) = 400$ seJ and $\epsilon(2) = 100$ seJ.

Thus,

$$\mathsf{Em}(\mathcal{E}_2([4;5])) = 400 \cdot 1 \cdot 1 \cdot \frac{2}{5} + 100 \cdot 1 \cdot 1 \cdot \frac{2}{5} = 160 + 40 = 200 \text{ seJ},$$

which is the value obtained at the output of the graph [18, Fig 8.b p. 226].

Remark 4.1. *The emergy computed corresponds to the entry* $(4,5)$ *of the matrix FRM in the Tennenbaum's program (see Tennenbaum [12, pp. 122-126]).*

4.3. Example of Section 2 continued

Let us recall that the set of all emergy paths $\mathcal{E}_1$ is:

$$\mathcal{E}_1 = \left\{ \begin{array}{c} [1;3], [1;3][3;4], [1;3][3;5], [1;3][3;5][5;6][6;5], \\ [1;3][3;4][4;6][6;5], \\ [1;3][3;4][4;6], [1;3][3;5][5;6], [1;3][3;4][4;6][6;5][5;6], \\ [1;3][3;4][4;6]][6;7], [1;3][3;5][5;6][6;7], [2;4], [2;4][4;6][6;5], \\ [2;4][4;6], [2;4][4;6][6;5][5;6], [2;4][4;6][6;7] \end{array} \right\}$$

As an illustrative example, let us compute the emergy flowing on the arc $[6;5]$, i.e. $\mathsf{Em}(\mathcal{E}([6;5])$ with $\mathcal{E}_1([6;5]) = \{[1;3][3;5][5;6][6;5], [1;3][3;4][4;6][6;5], [2;4][4;6][6;5]\}$.

Because $1, 2 \in \mathcal{L}_1^s$ we have: $[1;3] \perp [2;4]$, by definition of $\perp$. Thus, we express $\mathcal{E}_1([6;5])$ as follows:

$$\mathcal{E}_1([6;5]) = [1;3]\mathcal{U}_1 \cup [2;4]\mathcal{U}_2,$$

with $\mathcal{U}_1 = \{[3;5][5;6][6;5],[3;4][4;6][6;5]\}$ and $\mathcal{U}_2 = \{[4;6][6;5]\}$. And we obtain:

$$\begin{aligned}\mathsf{Em}(\mathcal{E}_1([6;5])) &= \mathsf{Em}([1;3]\mathcal{U}_1 \cup [2;4]\mathcal{U}_2)\\ &= \mathsf{Em}([1;3]\mathcal{U}_1) + \mathsf{Em}([2;4]\mathcal{U}_2) \quad \text{by } (\mathfrak{a}.1.2).\end{aligned}$$

By an easy computation we have:

$$\begin{aligned}\mathsf{Em}([2;4]\mathcal{U}_2) &= \mathsf{Em}([2;4]\{[4;6][6;5]\})\\ &= \mathsf{Em}([2;4]) \cdot \mathsf{Em}(\{[4;6][6;5]\}), \text{ by } (\mathfrak{a}.2)\\ &= \mathsf{Em}([2;4]) \cdot \mathsf{Em}([4;6][6;5]), \text{ by } (\mathfrak{a}.4)\\ &= \epsilon(2) \cdot \mathbf{\Omega}_{G_1}(2,4) \cdot \mathbf{\Omega}_{G_1}(4,6) \cdot \mathbf{\Omega}_{G_1}(6,5), \text{ by } (\mathfrak{a}.3).\end{aligned}$$

Let us detail the computation of $\mathsf{Em}([1;3]\mathcal{U}_1)$. It comes:

$$\begin{aligned}\mathsf{Em}([1;3]\mathcal{U}_1) &= \mathsf{Em}([1;3]) \cdot \mathsf{Em}(\mathcal{U}_1) && \text{by } (\mathfrak{a}.2)\\ &= \epsilon(1) \cdot \mathbf{\Omega}_{G_1}(1,3) \cdot \mathsf{Em}(\mathcal{U}_1) && \text{by } (\mathfrak{a}.3).\end{aligned}$$

Now, we just have to compute $\mathsf{Em}(\mathcal{U}_1)$. We remark that:

$$\mathcal{U}_1 = [3;4]\{[4;6][6;5]\} \cup [3;5]\{[5;6][6;5]\},$$

with $[3;4] \parallel [3;5]$ because there is a co-product at node 3. Then, by applying $(\mathfrak{a}.1.3)$ we have:

$\mathsf{Em}(\mathcal{U}_1) = \max(\mathbf{\Omega}_{G_1}(3,4) \cdot \mathsf{Em}(\{[4;6][6;5]\}), \mathbf{\Omega}_{G_1}(3,5) \cdot \mathsf{Em}(\{[5;6][6;5]\}))$.

Using $(\mathfrak{a}.2)$ and $(\mathfrak{a}.3)$ we have:

$\mathsf{Em}([3;4]\{[4;6][6;5]\}) = \mathbf{\Omega}_{G_1}(3,4) \cdot \mathbf{\Omega}_{G_1}(4,6) \cdot \mathbf{\Omega}_{G_1}(6,5)$

and

$\mathsf{Em}([3;5]\{[5;6][6;5]\}) = \mathbf{\Omega}_{G_1}(3,5) \cdot \mathbf{\Omega}_{G_1}(5,6) \cdot \mathbf{\Omega}_{G_1}(6,5)$.

Finally, we obtain:

$$\begin{aligned}\mathsf{Em}(\mathcal{E}_1([6;5])) &= \epsilon(2)\cdot\mathbf{\Omega}_{G_1}(2,4)\cdot\mathbf{\Omega}_{G_1}(4,6)\cdot\mathbf{\Omega}_{G_1}(6,5)\\ &+\epsilon(1)\cdot\mathbf{\Omega}_{G_1}(1,3)\cdot\max(\mathbf{\Omega}_{G_1}(3,4)\cdot\mathbf{\Omega}_{G_1}(4,6)\cdot\mathbf{\Omega}_{G_1}(6,5),\\ &\mathbf{\Omega}_{G_1}(3,5)\cdot\mathbf{\Omega}_{G_1}(5,6)\cdot\mathbf{\Omega}_{G_1}(6,5)).\end{aligned}$$

Numerical application.

$\epsilon(1) = 400$ seJ, $\epsilon(2) = 100$ seJ.

Thus,

$$\mathsf{Em}(\mathcal{E}_1([6;5])) = 100\cdot 1\cdot 1\cdot \frac{1}{5} + 400\cdot 1\cdot \max(1\cdot 1\cdot \frac{1}{5}, 1\cdot 1\cdot \frac{1}{5}) = 100 \text{ seJ}.$$

4.4. Emergy graph with splits and one co-product

Let us consider the emergy graph G_3 of Figure 5 borrowed from Li et al. [8, Fig 8 and 9]. There are splits at nodes 3, 5, 6, 7 and 10, and a co-product at node 4. The set of sources is $\mathcal{L}_3^s = \{1, 2\}$, the set of internal nodes is $\mathcal{L}_3^i = \{3, 4, 5, 6, 7, 8, 9, 10\}$ and the set of the output nodes is $\mathcal{L}_3^o = \{11, 12, 13, 14\}$. Because 1 and 2 are sources we have: $[1;3] \perp [2;10]$. Because 3, 5, 6, 7 and 10 are splitted we have: $[3;4] \perp [3;5]$, $[6;8] \perp [6;9]$, $[7;9] \perp [7;10]$ and $[10;4] \perp [10;11]$. Because of the co-product at node 4 we have: $[4;6] \parallel [4;7]$.

Figure 5: Net with splits and one co-product at node 4

Let us give the main steps of the computation of the emergy circulating on the arc $[9;13]$ denoted $\mathsf{Em}(\mathcal{E}_3([9;13]))$, recalling that $\mathcal{E}_3([9;13])$ is the set of all emergy paths ending by $[9;13]$.

We assume that the computation of the set $\mathcal{E}_3([9;13])$ has already been made and:

$$\mathcal{E}_3([9;13]) = \left\{ \begin{array}{l} [1;3][3;5][5;7][7;10][10;4][4;6][6;9][9;13], \\ [1;3][3;5][5;7][7;9][9;13], \\ [1;3][3;4][4;6][6;9][9;13], [1;3][3;4][4;7][7;9][9;13], \\ [2;10][10;4][4;6][6;9][9;13], [2;10][10;4][4;7][7;9][9;13] \end{array} \right\}.$$

Because 1 and 2 are sources: $[1;3] \perp [2;10]$, by definition of $\perp$. Then, $\mathcal{E}_3([9;13])$ is decomposed as follows:

$$\mathcal{E}_3([9;13]) = [1;3]\mathcal{U}_1 \cup [2;10]\mathcal{U}_2,$$

and applying ($\mathfrak{a}$.1.2) we have:

$$\mathsf{Em}(\mathcal{E}_3([9;13]) = \mathsf{Em}([1;3]\mathcal{U}_1) + \mathsf{Em}([2;10]\mathcal{U}_2)$$

with:
$\mathcal{U}_1 = \left\{ \begin{array}{c} [3;5][5;7][7;10][10;4][4;6][6;9][9;13], [3;5][5;7][7;9][9;13], \\ [3;4][4;6][6;9][9;13], [3;4][4;7][7;9][9;13] \end{array} \right\}$ and
$\mathcal{U}_2 = \{[10;4][4;6][6;9][9;13], [10;4][4;7][7;9][9;13]\}$.

Applying ($\mathfrak{a}$.2) and ($\mathfrak{a}$.3) to $\mathsf{Em}([1;3]\mathcal{U}_1)$ we have:

$$\mathsf{Em}([1;3]\mathcal{U}_1) = \epsilon(1) \cdot \mathbf{\Omega}_{G_3}(1,3) \cdot \mathsf{Em}(\mathcal{U}_1)$$

and

$$\mathsf{Em}([2;10]\mathcal{U}_2) = \epsilon(2) \cdot \mathbf{\Omega}_{G_3}(2,10) \cdot \mathsf{Em}(\mathcal{U}_2).$$

Computation of $\mathsf{Em}(\mathcal{U}_1)$. There is a split at node 3 thus $[3;4] \perp [3;5]$ and $\mathcal{U}_1$ is decomposed as follows:

$$\mathcal{U}_1 = [3;4]\mathcal{U}_{11} \cup [3;5]\mathcal{U}_{12}.$$

Thus, applying ($\mathfrak{a}$.1.2), ($\mathfrak{a}$.2) and ($\mathfrak{a}$.3) we have:

$$\mathsf{Em}(\mathcal{U}_1) = \mathbf{\Omega}_{G_3}(3,4) \cdot \mathsf{Em}(\mathcal{U}_{11}) + \mathbf{\Omega}_{G_3}(3,5) \cdot \mathsf{Em}(\mathcal{U}_{12}),$$

with:

$\mathcal{U}_{11} = \{[4;6][6;9][9;13], [4;7][7;9][9;13]\}$ and

$\mathcal{U}_{12} = \{[5;7][7;10][10;4][4;6][6;9][9;13], [5;7][7;9][9;13]\}$.

There is a co-product at node 4 with $[4;6] \parallel [4;7]$, thus $\mathcal{U}_{11}$ is decomposed as follows:

$$\mathcal{U}_{11} = [4;6]\mathcal{U}_{111} \cup [4;7]\mathcal{U}_{112},$$

Hence, using ($\mathfrak{a}$.1.3), ($\mathfrak{a}$.2) and ($\mathfrak{a}$.3) we have:

$$\mathsf{Em}(\mathcal{U}_{11}) = \max(\mathbf{\Omega}_{G_3}(4,6) \cdot \mathsf{Em}(\mathcal{U}_{111}), \mathbf{\Omega}_{G_3}(4,7) \cdot \mathsf{Em}(\mathcal{U}_{112})),$$

with $\mathcal{U}_{111} = \{[6;9][9;13]\}$ and $\mathcal{U}_{112} = \{[7;9][9;13]\}$.

Noticing that $[7;9] \perp [7;10]$, $\mathcal{U}_{12}$ is decomposed as follows:

$$\mathcal{U}_{12} = [5;7][7;10]\mathcal{U}_{121} \cup [5;7][7;9]\mathcal{U}_{122},$$

by applying ($\mathfrak{a}$.1.2), ($\mathfrak{a}$.2) and ($\mathfrak{a}$.3) we have:

$$\mathsf{Em}(\mathcal{U}_{12}) = \mathbf{\Omega}_{G_3}(5,7) \cdot \mathbf{\Omega}_{G_3}(7,10) \cdot \mathsf{Em}(\mathcal{U}_{121}) + \mathbf{\Omega}_{G_3}(5,7) \cdot \mathbf{\Omega}_{G_3}(7,9) \cdot \mathsf{Em}(\mathcal{U}_{122}),$$

with: $\mathcal{U}_{121} = \{[10;4][4;6][6;9][9;13]\}$ and $\mathcal{U}_{122} = \{[9;13]\}$.

By applying ($\mathfrak{a}$.4), ($\mathfrak{a}$.2) and ($\mathfrak{a}$.3) we have:

$$\mathsf{Em}(\mathcal{U}_{121}) = \mathbf{\Omega}_{G_3}(10,4) \cdot \mathbf{\Omega}_{G_3}(4,6) \cdot \mathbf{\Omega}_{G_3}(6,9) \cdot \mathbf{\Omega}_{G_3}(9,13)$$

and

$$\mathsf{Em}(\mathcal{U}_{122}) = \mathbf{\Omega}_{G_3}(9,13).$$

Computation of $\mathsf{Em}(\mathcal{U}_2)$.

Noticing that $[4;6] \parallel [4;7]$, $\mathcal{U}_2$ is decomposed as follows:

$$\mathcal{U}_2 = [10;4][4;6]\mathcal{U}_{21} \cup [10;4][4;7]\mathcal{U}_{22},$$

by applying ($\mathfrak{a}$.1.3), ($\mathfrak{a}$.2) and ($\mathfrak{a}$.3) we have:

$$\mathsf{Em}(\mathcal{U}_2) = \max(\mathbf{\Omega}_{G_3}(10,4)\cdot\mathbf{\Omega}_{G_3}(4,6)\cdot\mathsf{Em}(\mathcal{U}_{21}), \mathbf{\Omega}_{G_3}(10,4)\cdot\mathbf{\Omega}_{G_3}(4,7)\cdot\mathsf{Em}(\mathcal{U}_{22})),$$

with $\mathcal{U}_{21} = \{[6;9][9;13]\}$ and $\mathcal{U}_{22} = \{[7;9][9;13]\}$.

By applying ($\mathfrak{a}$.4), ($\mathfrak{a}$.2) and ($\mathfrak{a}$.3) we have:

$$\mathsf{Em}(\mathcal{U}_{21}) = \mathbf{\Omega}_{G_3}(6,9) \cdot \mathbf{\Omega}_{G_3}(9,13)$$

and

$$\mathsf{Em}(\mathcal{U}_{22}) = \mathbf{\Omega}_{G_3}(7,9) \cdot \mathbf{\Omega}_{G_3}(9,13).$$

Finally, the following close formula for $\mathsf{Em}(\mathcal{E}_3([9;13]))$ is obtained:

$$\begin{aligned}
&\mathsf{Em}(\mathcal{E}_3([9;13])) = \epsilon(1) \cdot \mathbf{\Omega}_{G_3}(1,3) \cdot (\\
&\mathbf{\Omega}_{G_3}(3,5) \cdot \mathbf{\Omega}_{G_3}(5,7) \cdot (\\
&\mathbf{\Omega}_{G_3}(7,10) \cdot \mathbf{\Omega}_{G_3}(10,4) \cdot \mathbf{\Omega}_{G_3}(4,6) \cdot \mathbf{\Omega}_{G_3}(6,9) \cdot \mathbf{\Omega}_{G_3}(9,13) + \mathbf{\Omega}_{G_3}(7,9)\mathbf{\Omega}_{G_3}(9,13))\\
&+\mathbf{\Omega}_{G_3}(3,4) \cdot \max(\mathbf{\Omega}_{G_3}(4,6) \cdot \mathbf{\Omega}_{G_3}(6,9) \cdot \mathbf{\Omega}_{G_3}(9,13), \mathbf{\Omega}_{G_3}(4,7) \cdot \mathbf{\Omega}_{G_3}(7,9) \cdot \mathbf{\Omega}_{G_3}(9,13)))\\
&+\epsilon(2) \cdot \mathbf{\Omega}_{G_3}(2,10) \cdot \mathbf{\Omega}_{G_3}(10,4)\cdot\\
&\max(\mathbf{\Omega}_{G_3}(4,6) \cdot \mathbf{\Omega}_{G_3}(6,9) \cdot \mathbf{\Omega}_{G_3}(9,13), \mathbf{\Omega}_{G_3}(4,7) \cdot \mathbf{\Omega}_{G_3}(7,9) \cdot \mathbf{\Omega}_{G_3}(9,13)).
\end{aligned}$$

Numerical application.

$\epsilon(1) = 1000$ seJ, $\epsilon(2) = 500$ seJ.

$\mathbf{\Omega}_{G_3}(1,3) = \mathbf{\Omega}_{G_3}(2,10) = \mathbf{\Omega}_{G_3}(4,6) = \mathbf{\Omega}_{G_3}(4,7) = \mathbf{\Omega}_{G_3}(9,13) = 1.$

$\mathbf{\Omega}_{G_3}(3,4) = 5/8$, $\mathbf{\Omega}_{G_3}(3,5) = 3/8$, $\mathbf{\Omega}_{G_3}(5,7) = 4/5$, $\mathbf{\Omega}_{G_3}(6,9) = 1/5$, $\mathbf{\Omega}_{G_3}(7,9) = 2/3$, $\mathbf{\Omega}_{G_3}(7,10) = \mathbf{\Omega}_{G_3}(10,4) = 1/3$.

$$\begin{aligned}
\mathsf{Em}(\mathcal{E}_3([9;13])) &= 1000 \cdot 1 \cdot (\tfrac{3}{8} \cdot \tfrac{4}{5} \cdot (\tfrac{1}{3} \cdot \tfrac{1}{3} \cdot 1 \cdot \tfrac{1}{5} \cdot 1 + \tfrac{2}{3} \cdot 1) + \tfrac{5}{8} \cdot \max(1 \cdot \tfrac{1}{5} \cdot 1, \tfrac{2}{3} \cdot 1)) \\
&+500 \cdot 1 \cdot \tfrac{1}{3} \cdot \max(1 \cdot \tfrac{1}{5} \cdot 1, 1 \cdot \tfrac{2}{3} \cdot 1) \\
&= \tfrac{6610}{9} (\approx 734.44) \text{ seJ}.
\end{aligned}$$

Remark 4.2. *Let us remark that our formula avoid double counting of emergy flows with the same past. In Li et al. [8, p. 415, (2)] authors propose to compute the emergy flowing on arc* $[9;13]$ *as the maximum of emergy flowing on arc* $[6;9]$ *and the emergy flowing on arc* $[7;9]$. *From a numerical point of view this leads to the value of* 727.77 *(which is different than ours). But the problem is that emergy flowing on arc* $[6;9]$ *has not exactly the same past than the emergy flowing on arc* $[7;9]$. *In fact, the formula used in Li et al. [8] to compute emergy flowing on arc* $[9;13]$ *is:*

$$\begin{aligned}
&\max(\epsilon(1) \cdot \mathbf{\Omega}_{G_3}(1,3) \cdot \mathbf{\Omega}_{G_3}(3,5) \cdot \mathbf{\Omega}_{G_3}(5,7) \cdot \mathbf{\Omega}_{G_3}(7,9) \\
&+\epsilon(1) \cdot \mathbf{\Omega}_{G_3}(1,3) \cdot \mathbf{\Omega}_{G_3}(3,4) \cdot \mathbf{\Omega}_{G_3}(4,7) \cdot \mathbf{\Omega}_{G_3}(7,9) \\
&+\epsilon(2) \cdot \mathbf{\Omega}_{G_3}(2,10) \cdot \mathbf{\Omega}_{G_3}(10,4) \cdot \mathbf{\Omega}_{G_3}(4,7) \cdot \mathbf{\Omega}_{G_3}(7,9), \\
&\epsilon(1) \cdot \mathbf{\Omega}_{G_3}(1,3) \cdot \mathbf{\Omega}_{G_3}(3,5) \cdot \mathbf{\Omega}_{G_3}(5,7) \cdot \mathbf{\Omega}_{G_3}(7,10) \cdot \mathbf{\Omega}_{G_3}(10,4) \cdot \mathbf{\Omega}_{G_3}(4,6) \cdot \mathbf{\Omega}_{G_3}(6,9) \\
&+\epsilon(1) \cdot \mathbf{\Omega}_{G_3}(1,3) \cdot \mathbf{\Omega}_{G_3}(3,4) \cdot \mathbf{\Omega}_{G_3}(4,6) \cdot \mathbf{\Omega}_{G_3}(6,9) \\
&+\epsilon(2) \cdot \mathbf{\Omega}_{G_3}(2,10) \cdot \mathbf{\Omega}_{G_3}(10,4) \cdot \mathbf{\Omega}_{G_3}(4,6) \cdot \mathbf{\Omega}_{G_3}(6;9)).
\end{aligned}$$

This example illustrates the remark of one of the reviewers in Lazzaretto [7, p.2201]: "As observed by one of the reviewers the rule counting the largest emergy value [arriving at a node] is a rather "crude way" of avoiding double counting".

4.5. Odum example

Let us consider the graph G_4 of Figure 6 borrowed from Odum [1, p. 100]. And let us compute emergy flowing on arc d and emergy flowing on arc e.

Figure 6: Graph G_4 Odum diagram

Emergy flowing on arc d.

The set of the emergy paths endind by d is:

$$\mathcal{E}_4(d) = \{samd, sbcd, samegncd, sbcejlmd, sbceghklmd, flmd, flmegncd\}$$

Noticing that $s \perp f$, $\mathcal{E}_4(d)$ is decomposed as follows:

$$\mathcal{E}_4(d) = s\mathcal{U}_1 \cup f\mathcal{U}_2,$$

and applying ($\mathfrak{a}$.1.2) we have:

$$\mathsf{Em}(\mathcal{E}_4(d)) = \mathsf{Em}(s\mathcal{U}_1) + \mathsf{Em}(f\mathcal{U}_2)$$

with: $\mathcal{U}_1 = \{amd, bcd, amegncd, bcejlmd, bceghklmd\}$ and $\mathcal{U}_2 = \{lmd, lmegncd\}$.

Applying ($\mathfrak{a}$.2) and ($\mathfrak{a}$.3) we have:

$$\mathsf{Em}(s\mathcal{U}_1) = \epsilon(1) \cdot \mathbf{\Omega}_{G_4}(s) \cdot \mathsf{Em}(\mathcal{U}_1)$$

and

$$\mathsf{Em}(f\mathcal{U}_2) = \epsilon(2) \cdot \mathbf{\Omega}_{G_4}(f) \cdot \mathsf{Em}(\mathcal{U}_2).$$

Computation of $\mathsf{Em}(\mathcal{U}_1)$. We remark that $a \perp b$ thus $\mathcal{U}_1$ is decomposed as follows:

$$\mathcal{U}_1 = a\mathcal{U}_{11} \cup b\mathcal{U}_{12}.$$

Thus applying ($\mathfrak{a}$.1.2), ($\mathfrak{a}$.2) and ($\mathfrak{a}$.3) we have:

$$\mathsf{Em}(\mathcal{U}_1) = \mathbf{\Omega}_{G_4}(a) \cdot \mathsf{Em}(\mathcal{U}_{11}) + \mathbf{\Omega}_{G_4}(b) \cdot \mathsf{Em}(\mathcal{U}_{12}).$$

with: $\mathcal{U}_{11} = \{md, megncd\}$ and $\mathcal{U}_{12} = \{cd, cejlmd, ceghklmd\}$.

Noticing that $d \parallel e$, $\mathcal{U}_{11}$ is decomposed as follows:

$$\mathcal{U}_{11} = md\mathcal{U}_{111} \cup me\mathcal{U}_{112}.$$

Hence, using ($\mathfrak{a}$.1.3), ($\mathfrak{a}$.2) and ($\mathfrak{a}$.3) we have:

$$\mathsf{Em}(\mathcal{U}_{11}) = \max(\mathbf{\Omega}_{G_4}(m) \cdot \mathbf{\Omega}_{G_4}(d) \cdot \mathsf{Em}(\mathcal{U}_{111}), \mathbf{\Omega}_{G_4}(m) \cdot \mathbf{\Omega}_{G_4}(e) \cdot \mathsf{Em}(\mathcal{U}_{112})),$$

with: $\mathcal{U}_{111} = \{\underline{1}\}$ and $\mathcal{U}_{112} = \{gncd\}$.

Noticing that $\{\underline{1}\} = \underline{1}\{\underline{1}\}$ we have by ($\mathfrak{a}$.2) and ($\mathfrak{a}$.4): $\mathsf{Em}(\underline{1}) = \mathsf{Em}(\underline{1}) \cdot \mathsf{Em}(\underline{1})$ and because $\mathsf{Em}()$ is strictly positive we have: $\mathsf{Em}(\underline{1}) = 1 = \mathsf{Em}(\mathcal{U}_{111})$.

By ($\mathfrak{a}$.4) and ($\mathfrak{a}$.3) we have: $\mathsf{Em}(\mathcal{U}_{112}) = \mathbf{\Omega}_{G_4}(g) \cdot \mathbf{\Omega}_{G_4}(n) \cdot \mathbf{\Omega}_{G_4}(c) \cdot \mathbf{\Omega}_{G_4}(d)$.

Because $d \parallel e$, $\mathcal{U}_{12}$ is decomposed as follows:

$$\mathcal{U}_{12} = cd\mathcal{U}_{121} \cup ce\mathcal{U}_{122}$$

Hence, using ($\mathfrak{a}$.1.3), ($\mathfrak{a}$.2) and ($\mathfrak{a}$.3) we have:

$$\mathsf{Em}(U_{12}) = \max(\mathbf{\Omega}_{G_4}(c) \cdot \mathbf{\Omega}_{G_4}(d) \cdot \mathsf{Em}(U_{121}), \max(\mathbf{\Omega}_{G_4}(c) \cdot \mathbf{\Omega}_{G_4}(e) \cdot \mathsf{Em}(U_{122}))$$

with: $\mathcal{U}_{121} = \{\underline{1}\}$ (thus $\mathsf{Em}(U_{121}) = 1$) and $\mathcal{U}_{122} = \{jlmd, ghklmd\}$.

Noticing that $j \perp g$, $\mathcal{U}_{122}$ is decomposed as follows:

$$\mathcal{U}_{122} = j\{lmd\} \cup g\{hklmd\}$$

Using ($\mathfrak{a}$.1.2), ($\mathfrak{a}$.2), ($\mathfrak{a}$.3) and ($\mathfrak{a}$.4) we have:

$$\mathsf{Em}(\mathcal{U}_{122}) = \mathbf{\Omega}_{G_4}(j) \cdot \mathbf{\Omega}_{G_4}(l) \cdot \mathbf{\Omega}_{G_4}(m) \cdot \mathbf{\Omega}_{G_4}(d) + \mathbf{\Omega}_{G_4}(g) \cdot \mathbf{\Omega}_{G_4}(h) \cdot \mathbf{\Omega}_{G_4}(k) \cdot \mathbf{\Omega}_{G_4}(l) \cdot \mathbf{\Omega}_{G_4}(m) \cdot \mathbf{\Omega}_{G_4}(d).$$

Computation of $\mathsf{Em}(\mathcal{U}_2)$.

Noticing that $d \parallel e$, $\mathcal{U}_2$ is decomposed as follows:

$$\mathcal{U}_2 = lmd\mathcal{U}_{21} \cup lme\mathcal{U}_{22}$$

Hence, using ($\mathfrak{a}$.1.3), ($\mathfrak{a}$.2) and ($\mathfrak{a}$.3) we have:

$$\mathsf{Em}(\mathcal{U}_2) = \max(\mathbf{\Omega}_{G_4}(l)\cdot\mathbf{\Omega}_{G_4}(m)\cdot\mathbf{\Omega}_{G_4}(d)\cdot\mathsf{Em}(\mathcal{U}_{21}), \mathbf{\Omega}_{G_4}(l)\cdot\mathbf{\Omega}_{G_4}(m)\cdot\mathbf{\Omega}_{G_4}(e)\cdot\mathsf{Em}(\mathcal{U}_{22})$$

with: $\mathcal{U}_{21} = \{\underline{1}\}$ and $\mathcal{U}_{22} = \{gncd\}$.

As previously, $\mathsf{Em}(\mathcal{U}_{21}) = 1$ and by ($\mathfrak{a}$.4) and ($\mathfrak{a}$.3) we have: $\mathsf{Em}(\mathcal{U}_{22}) = \mathbf{\Omega}_{G_4}(g) \cdot \mathbf{\Omega}_{G_4}(n) \cdot \mathbf{\Omega}_{G_4}(c) \cdot \mathbf{\Omega}_{G_4}(d)$.

Finally,

$$\begin{aligned}
&\mathsf{Em}(\mathcal{E}_4(d)) = \epsilon(1) \cdot \mathbf{\Omega}_{G_4}(s) \cdot (\mathbf{\Omega}_{G_4}(a) \\
&\cdot \max(\mathbf{\Omega}_{G_4}(m) \cdot \mathbf{\Omega}_{G_4}(d) \cdot 1, \mathbf{\Omega}_{G_4}(m) \cdot \mathbf{\Omega}_{G_4}(e) \cdot \mathbf{\Omega}_{G_4}(g) \cdot \mathbf{\Omega}_{G_4}(n) \cdot \mathbf{\Omega}_{G_4}(c) \cdot \mathbf{\Omega}_{G_4}(d)) \\
&+\mathbf{\Omega}_{G_4}(b) \\
&\cdot \max(\mathbf{\Omega}_{G_4}(c) \cdot \mathbf{\Omega}_{G_4}(d) \cdot 1, \mathbf{\Omega}_{G_4}(c) \cdot \mathbf{\Omega}_{G_4}(e) \cdot (\mathbf{\Omega}_{G_4}(j) \cdot \mathbf{\Omega}_{G_4}(l) \cdot \mathbf{\Omega}_{G_4}(m) \cdot \mathbf{\Omega}_{G_4}(d) \\
&+\mathbf{\Omega}_{G_4}(g) \cdot \mathbf{\Omega}_{G_4}(h) \cdot \mathbf{\Omega}_{G_4}(k) \cdot \mathbf{\Omega}_{G_4}(l) \cdot \mathbf{\Omega}_{G_4}(m) \cdot \mathbf{\Omega}_{G_4}(d))) \\
&+\epsilon(2) \cdot \mathbf{\Omega}_{G_4}(f) \cdot \max(\mathbf{\Omega}_{G_4}(l) \cdot \mathbf{\Omega}_{G_4}(m) \cdot \mathbf{\Omega}_{G_4}(d) \cdot 1, \\
&\mathbf{\Omega}_{G_4}(l) \cdot \mathbf{\Omega}_{G_4}(m) \cdot \mathbf{\Omega}_{G_4}(e) \cdot \mathbf{\Omega}_{G_4}(g) \cdot \mathbf{\Omega}_{G_4}(n) \cdot \mathbf{\Omega}_{G_4}(c) \cdot \mathbf{\Omega}_{G_4}(d)).
\end{aligned}$$

Emergy flowing on arc e.

The set of emergy paths ending by e is:

$$\mathcal{E}_4(e) = \{same, sbce, flme\}$$

By decomposing $\mathcal{E}_4(e)$ as follows:

$$\mathcal{E}_4(e) = s(a\{me\} \cup sb\{ce\}) \cup f\{lme\},$$

and using ($\mathfrak{a}$.1.2), ($\mathfrak{a}$.2)-($\mathfrak{a}$.4) we obtain:

$$\mathsf{Em}(\mathcal{E}_4(e)) = \epsilon(1)\cdot\mathbf{\Omega}_{G_4}(s)\cdot(\mathbf{\Omega}_{G_4}(a)\cdot\mathbf{\Omega}_{G_4}(m)\cdot\mathbf{\Omega}_{G_4}(e)+\mathbf{\Omega}_{G_4}(b)\cdot\mathbf{\Omega}_{G_4}(c)\cdot\mathbf{\Omega}_{G_4}(e))$$
$$+\epsilon(2)\cdot\mathbf{\Omega}_{G_4}(f)\cdot\mathbf{\Omega}_{G_4}(l)\cdot\mathbf{\Omega}_{G_4}(m)\cdot\mathbf{\Omega}_{G_4}(e).$$

Numerical application.

$\epsilon(1) = 10,000$ seJ and $\epsilon(2) = 20,000$ seJ.

$$\begin{aligned}\mathsf{Em}(\mathcal{E}_4(d)) &= 10,000\cdot 1\cdot(\tfrac{3}{10}\cdot\max(1\cdot 1\cdot 1, 1\cdot 1\cdot\tfrac{1}{2}\cdot 1\cdot 1\cdot 1)\\ &+\tfrac{7}{10}\cdot\max(1\cdot 1\cdot 1, 1\cdot 1\cdot(\tfrac{1}{2}\cdot 1\cdot 1\cdot 1+\tfrac{1}{2}\cdot 1\cdot\tfrac{1}{2}\cdot 1\cdot 1\cdot 1)))\\ &+20,000\cdot 1\cdot\max(1\cdot 1\cdot 1\cdot 1, 1\cdot 1\cdot 1\cdot\tfrac{1}{2}\cdot 1\cdot 1\cdot 1)\\ &= 30,000 \text{ seJ.}\end{aligned}$$

$$\begin{aligned}\mathsf{Em}(\mathcal{E}_4(e)) &= 10,000\cdot 1\cdot(\tfrac{3}{10}\cdot 1\cdot 1+\tfrac{7}{10}\cdot 1\cdot 1)+20,000\cdot 1\cdot 1\cdot 1\cdot 1\\ &= 30,000 \text{ seJ.}\end{aligned}$$

Remark 4.3. *We can compute every emergy flowing on each arc of the graph. As a further example the numerical expression of the emergy flowing on arc m is:*

$$\begin{aligned}\mathsf{Em}(\mathcal{E}_4(m)) &= 10,000\cdot 1\cdot(\tfrac{3}{10}\cdot 1+\tfrac{7}{10}\cdot 1\cdot 1(\tfrac{1}{2}\cdot 1\cdot 1+\tfrac{1}{2}\cdot 1\cdot\tfrac{1}{2}\cdot 1\cdot 1))\\ &+20,000\cdot 1\cdot 1\\ &= 28,250 \ seJ,\end{aligned}$$

with

$$\mathcal{E}_4(m) = \{sam, sbcejlm, sbceghklm, flm\}.$$

5. Conclusion

In this paper the Tennenbaum's Track Summing method has been extended to the case of emergy networks with both splits and co-products. To obtain this extension the emergy rules R1-R4 (see the Introduction) have been reformulated into the axiomatic basis ($\mathfrak{a}$.1)-($\mathfrak{a}$.4).

The data processing implementation of this axiomatic basis is carried out by a recursive method (see Section 3.1). And it does not require an expert on emergy algebra. Even if we cannot formally prove that our axiomatic basis is logically equivalent to the rules R1-R4 apply on emergy flows with the same past (in the sense of graph theory) this method has been tested on benchmark emergy examples and gives the same results.

Last but not least. Let us note that our method is not only a computational method. It also provides a rigourous framework based on an axiomatic basis to do the emergy evaluation of an emergy graph.

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Appendix A. Explanation of axiom ($\mathfrak{a}$.1.2)

Let us consider the emergy graph of Figure A.7 such that $\epsilon(1) = 300$ seJ. Let us compute $\mathsf{Em}(\mathcal{E}([6;5]))$. We have $\mathcal{E}([6;5]) = \{[1;3][3;4][4;6][6;5], [1;3][3;5][5;6][6;5]\}$. Because there is a split at node 3: $[3;4] \perp [3;5]$, thus the set $\mathcal{E}([6;5])$ is decomposed as follows:

$$\mathcal{E}([6;5]) = [1;3][3;4]\mathcal{U}_1 \cup [1;3][3;5]\mathcal{U}_2$$

with: $\mathcal{U}_1 = \{[4;6][6;5]\}$ and $\mathcal{U}_2 = \{[5;6][6;5]\}$.

Figure A.7: Emergy graph with split

The graph of Figure A.8 explains how to compute the emergy flowing on arc $[6;5]$ of the system where there is only the arc $[3;4]$ after the past $[1;3]$, that is $\mathsf{Em}([1;3][3;4]\mathcal{U}_1)$.

Figure A.8: First pathway from 1 to 5

The graph of Figure A.9 explains how to compute the emergy flowing on arc $[6;5]$ of the system where there is only the arc $[3;5]$ after the past $[1;3]$, that is $\mathsf{Em}([1;3][3;5]\mathcal{U}_2)$.

Figure A.9: Second pathway from 1 to 5

Finally, when reunited the graph of Figure A.10 explains how to compute the whole emergy flowing on arc $[6;5]$ and illustrates the formula:

$$\mathsf{Em}([1;3][3;4]\mathcal{U}_1 \cup [1;3][3;5]\mathcal{U}_2) = \mathsf{Em}([1;3][3;4]\mathcal{U}_1) + \mathsf{Em}([1;3][3;5]\mathcal{U}_2).$$

In the general case we have:

$$\mathsf{Em}(\cup_{i=1}^{k} \pi a_i \mathcal{U}_i) = \sum_{i=1}^{k} \mathsf{Em}(\pi a_i \mathcal{U}_i),$$

and the addition is well associated with independent relation $\perp$.

Figure A.10: Total emergy flowing on arc $[6;5]$

Appendix B. Explanation of axiom ($\mathfrak{a}$.1.3)

Let us consider the emergy graph of Figure B.11 such that $\epsilon(1) = 500$ seJ. Let us compute $\mathsf{Em}(\mathcal{E}([6;5])$. We have $\mathcal{E}([6;5]) = \{[1;3][3;4][4;6][6;5], [1;3][3;5][5;6][6;5]\}$. Because there is a co-product at node 3: $[3;4] \parallel [3;5]$, thus the set $\mathcal{E}([6;5])$ is decomposed as follows:

$$\mathcal{E}([6;5]) = [1;3][3;4]\mathcal{U}_1 \cup [1;3][3;5]\mathcal{U}_2$$

with: $\mathcal{U}_1 = \{[4;6][6;5]\}$ and $\mathcal{U}_2 = \{[5;6][6;5]\}$.

Figure B.11: Emergy graph with co-product

The graph of Figure B.12 explains how to compute the emergy flowing on arc $[6;5]$ of the system where there is only the arc $[3;4]$ after the past $[1;3]$, that is $\mathsf{Em}([1;3][3;4]\mathcal{U}_1)$.

Figure B.12: Emergy on the first pathway from 1 to 5

The graph of Figure B.13 explains how to compute the emergy flowing on arc $[6;5]$ of the system where there is only the arc $[3;5]$ after the past $[1;3]$, that is $\mathsf{Em}([1;3][3;5]\mathcal{U}_2)$.

Figure B.13: Emergy on the second pathway from 1 to 5

Finally, when reunited the graph of Figure B.14 explains how to compute the whole emergy flowing on arc $[6;5]$ and illustrates the formula:

$$\mathsf{Em}([1;3][3;4]\mathcal{U}_1 \cup [1;3][3;5]\mathcal{U}_2) = \max(\mathsf{Em}([1;3][3;4]\mathcal{U}_1), \mathsf{Em}([1;3][3;5]\mathcal{U}_2)).$$

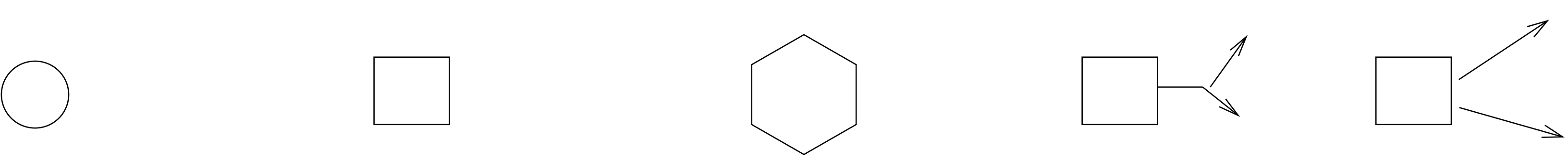
In the general case we have:

$$\mathsf{Em}(\cup_{i=1}^{k} \pi a_i \mathcal{U}_i) = \max_{i=1}^{k} \mathsf{Em}(\pi a_i \mathcal{U}_i),$$

and the maximum is well associated with the co-product $\|$.

Figure B.14: Total emergy flowing on arc $[6;5]$

Figure 1



(A) (B) (C) (D) (E)

Figure 2

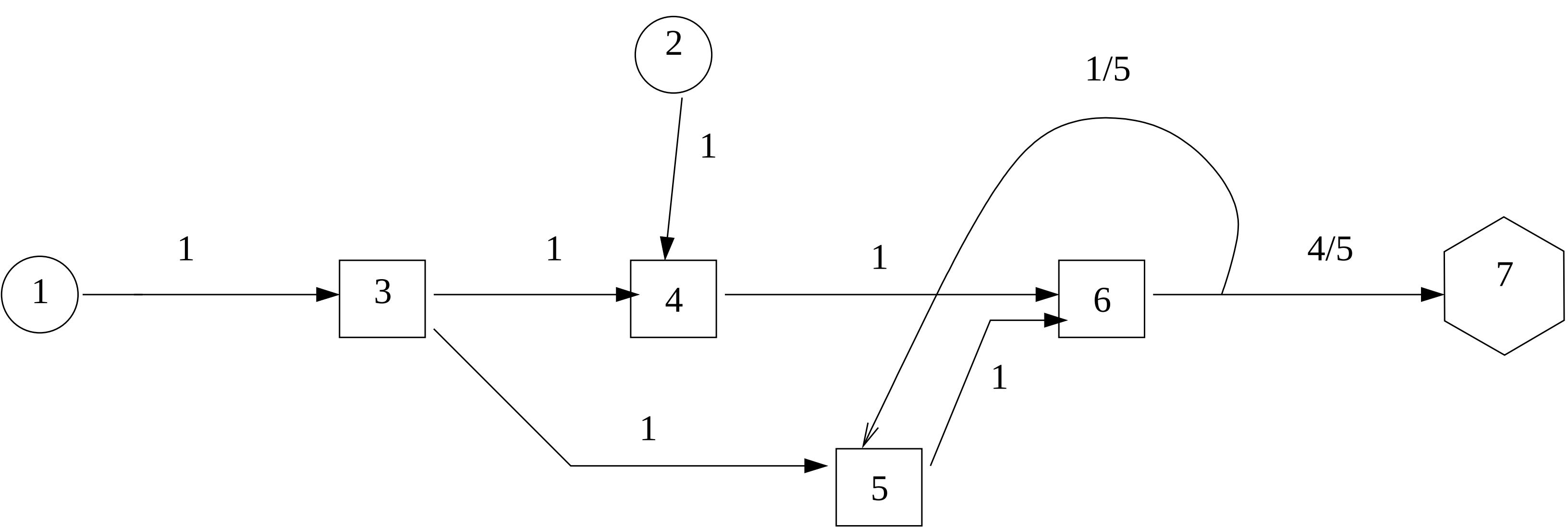


Figure 3

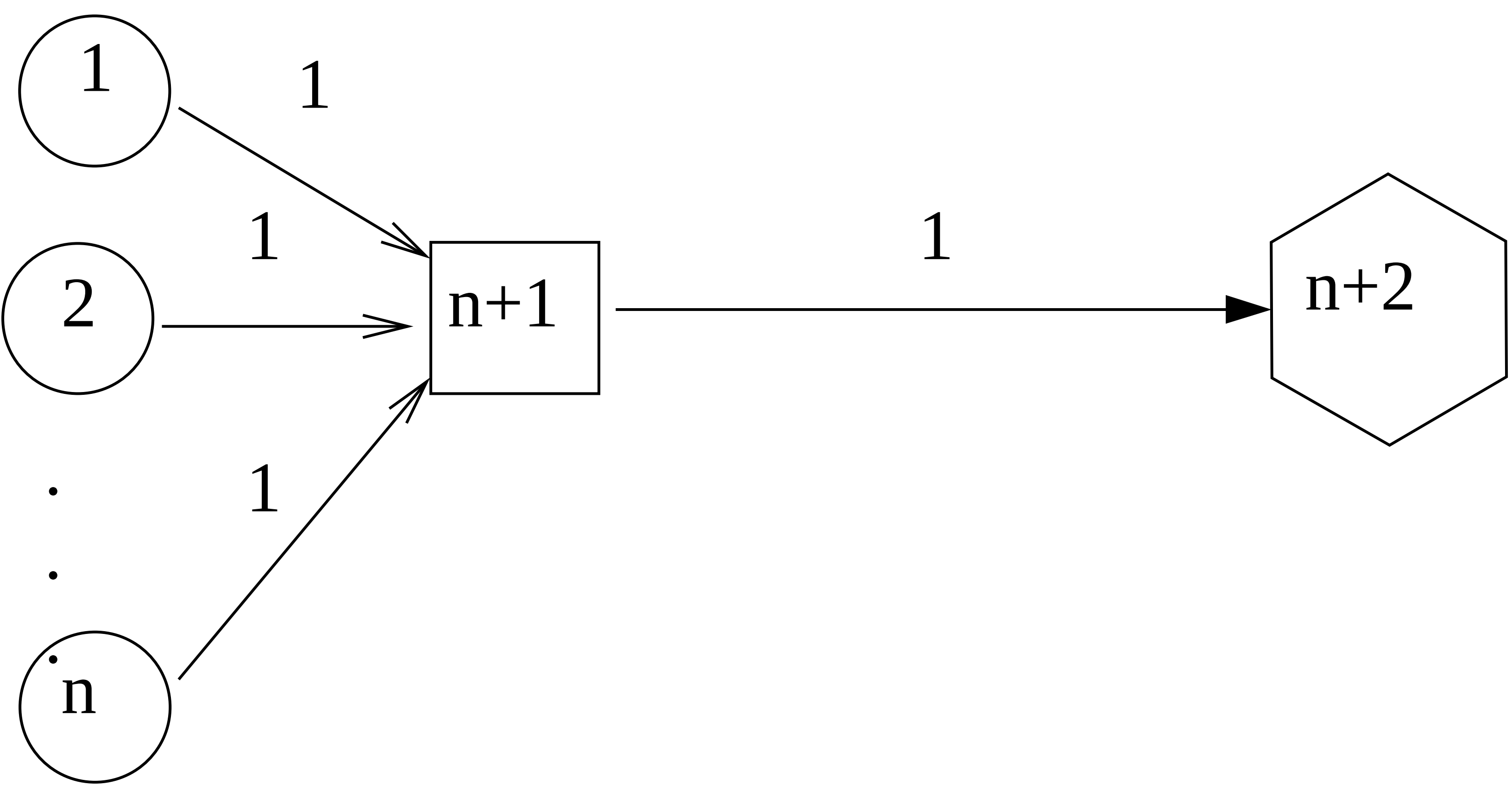


Figure 4

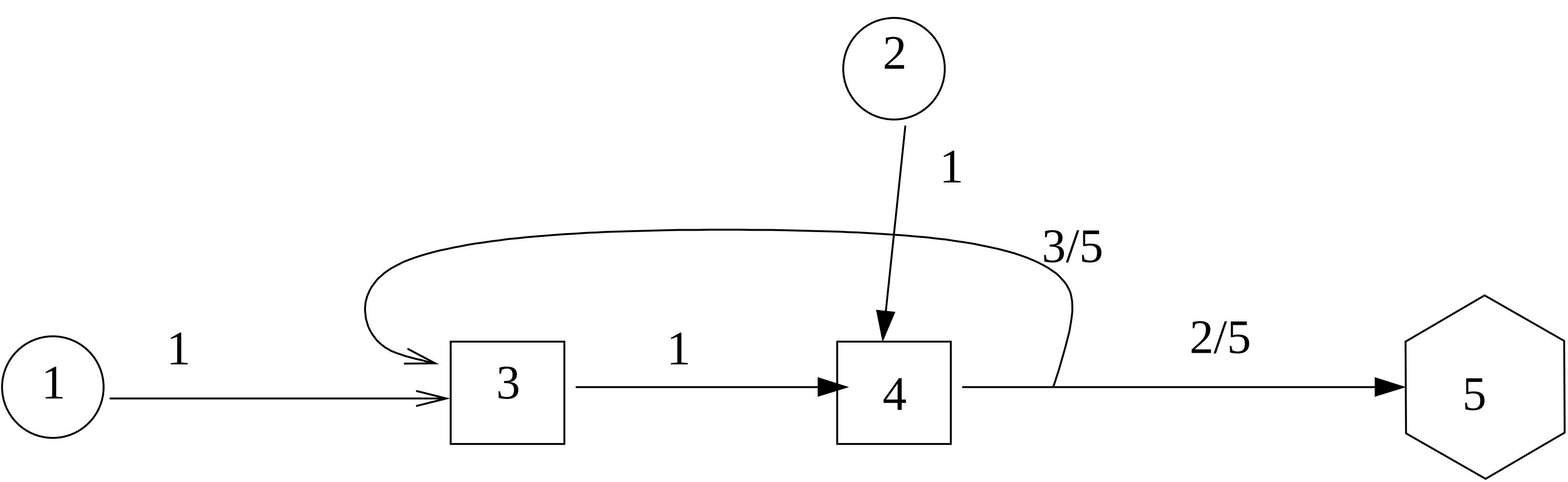


Figure 5

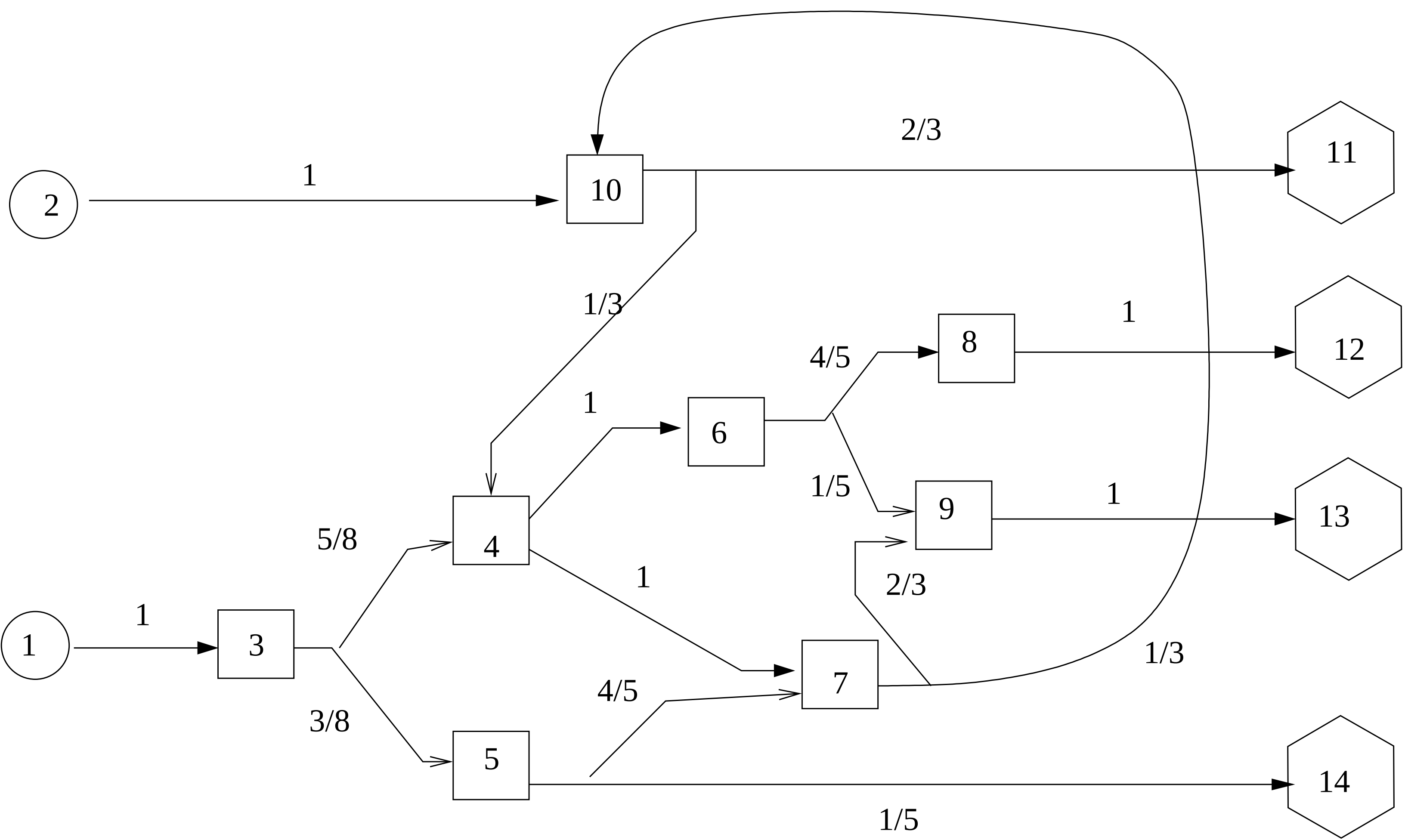


Figure 6

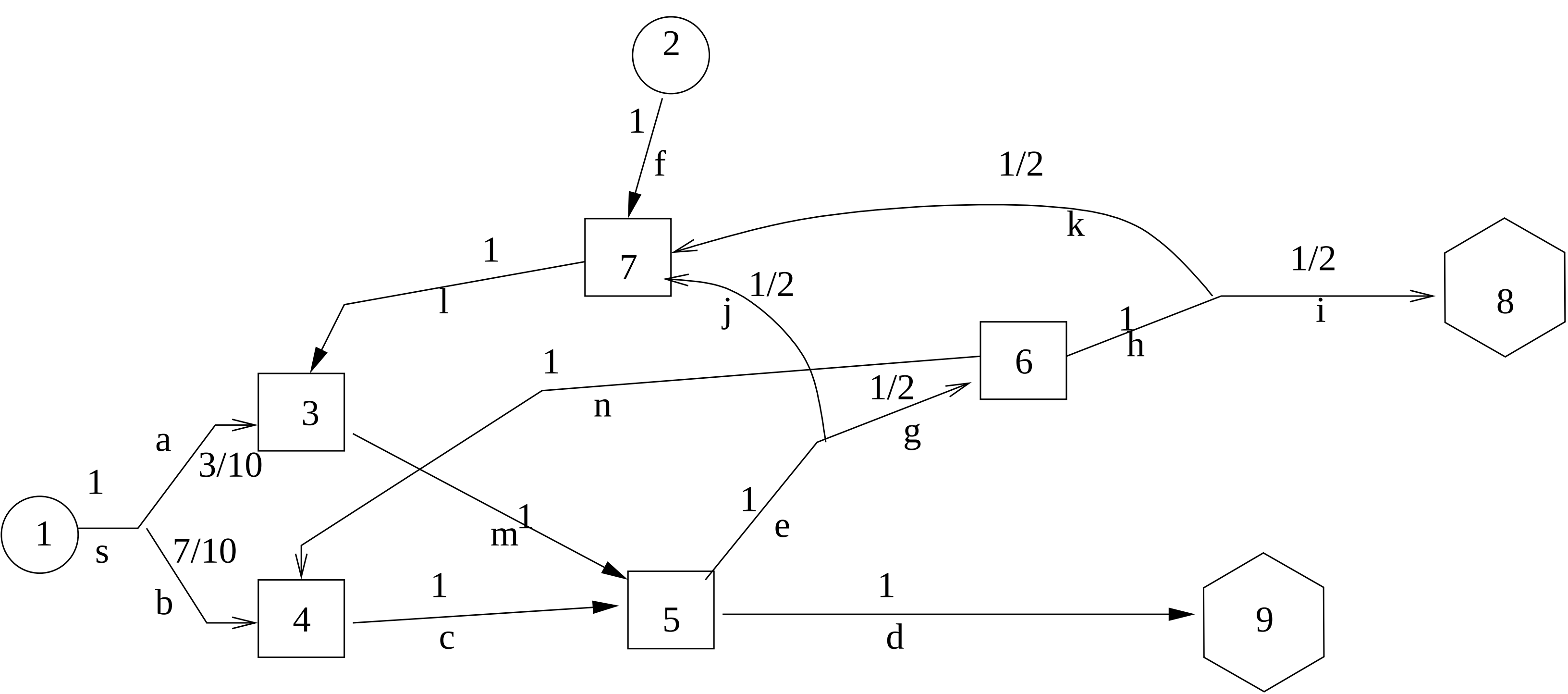


Figure 7

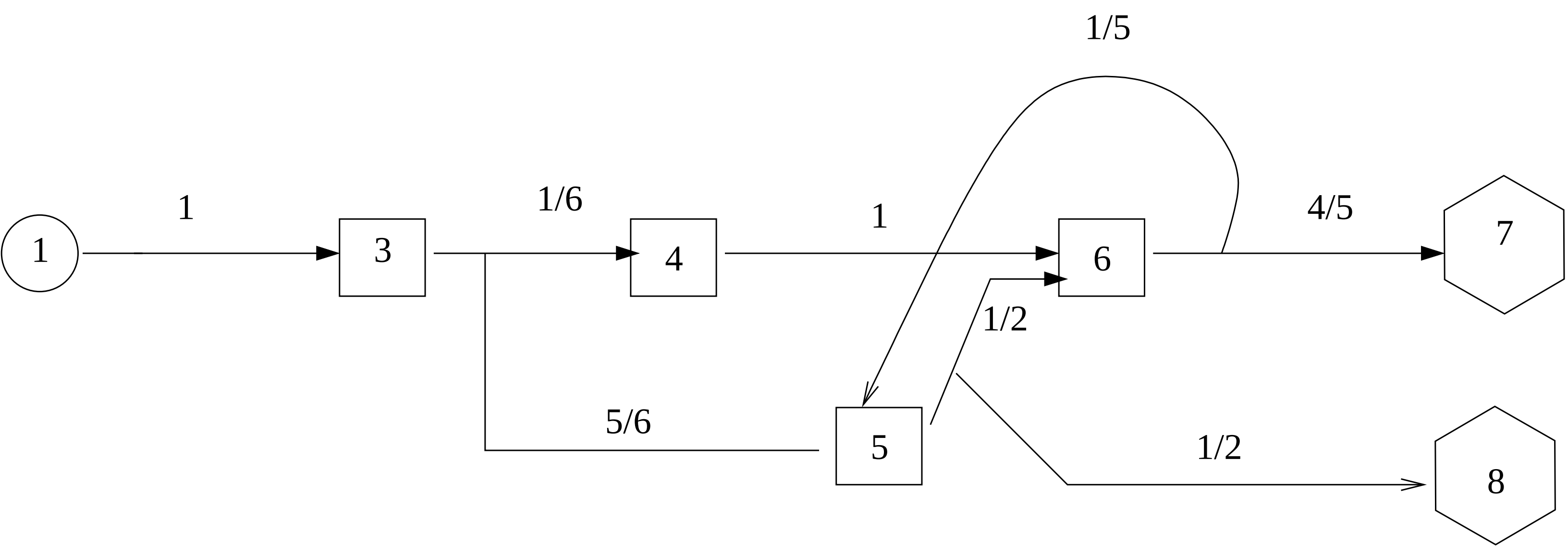


Figure 8

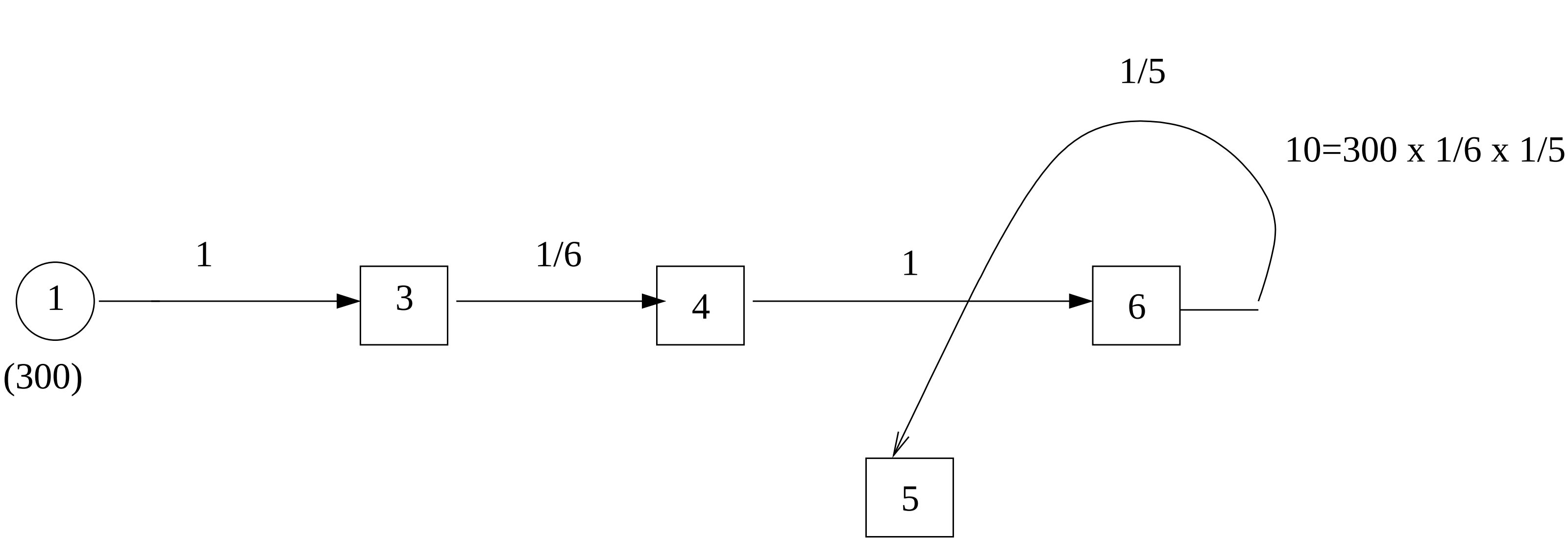


Figure 9

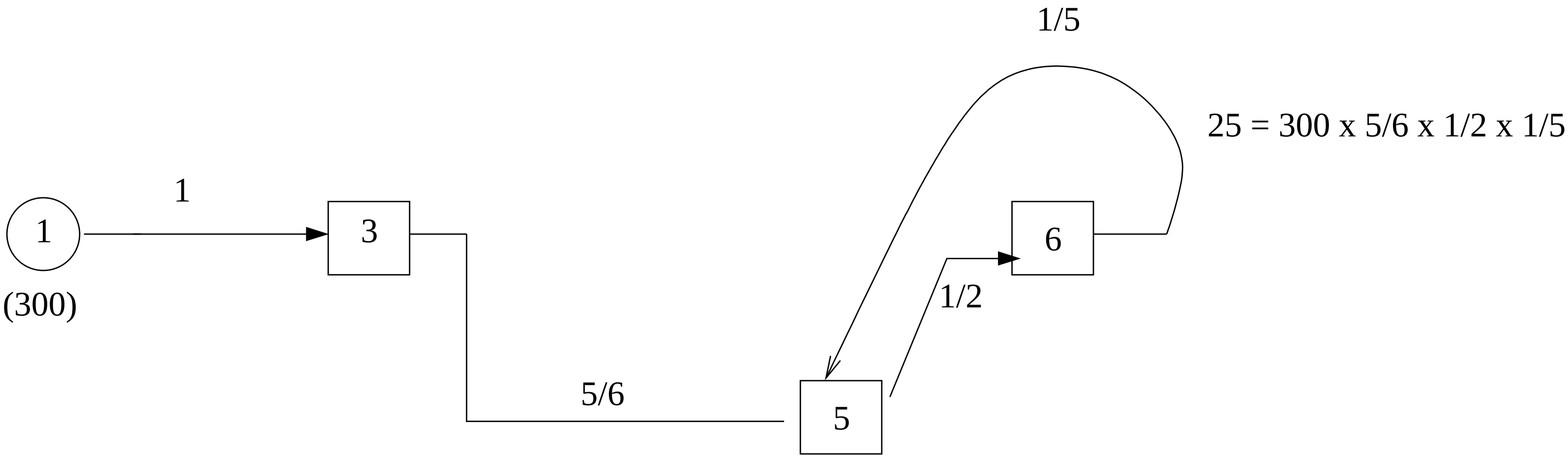


Figure 10

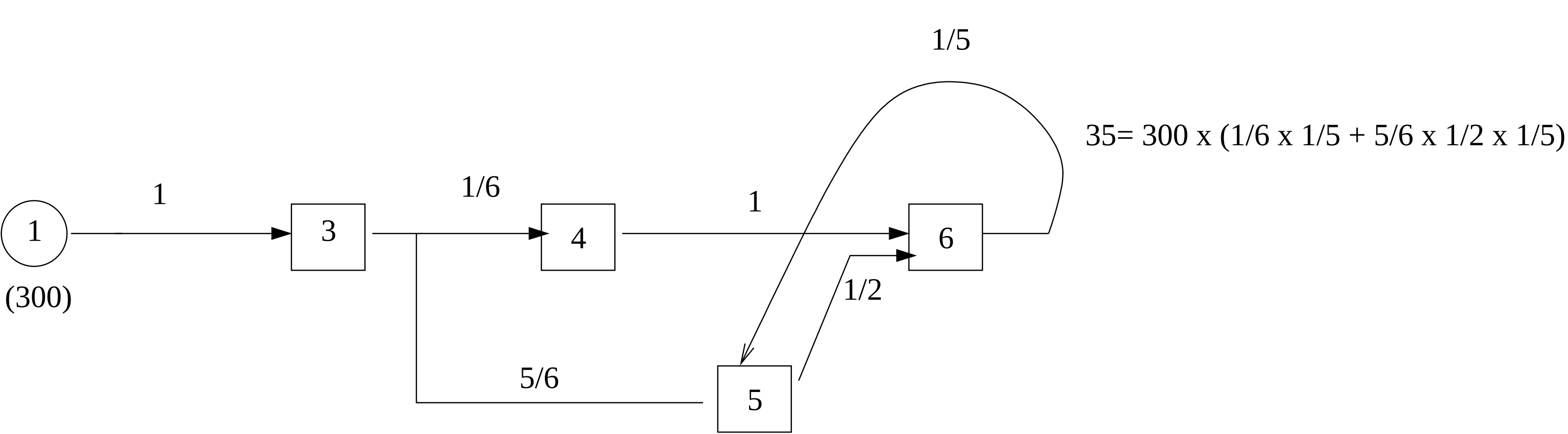


Figure 11

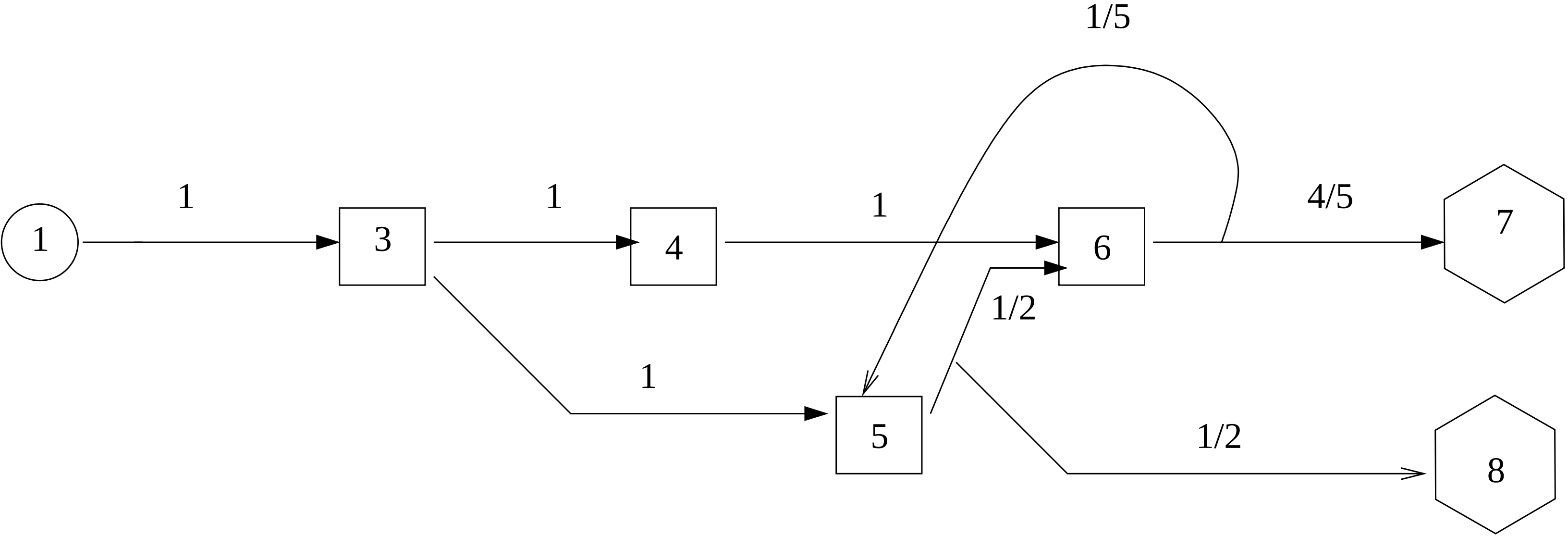


Figure 12

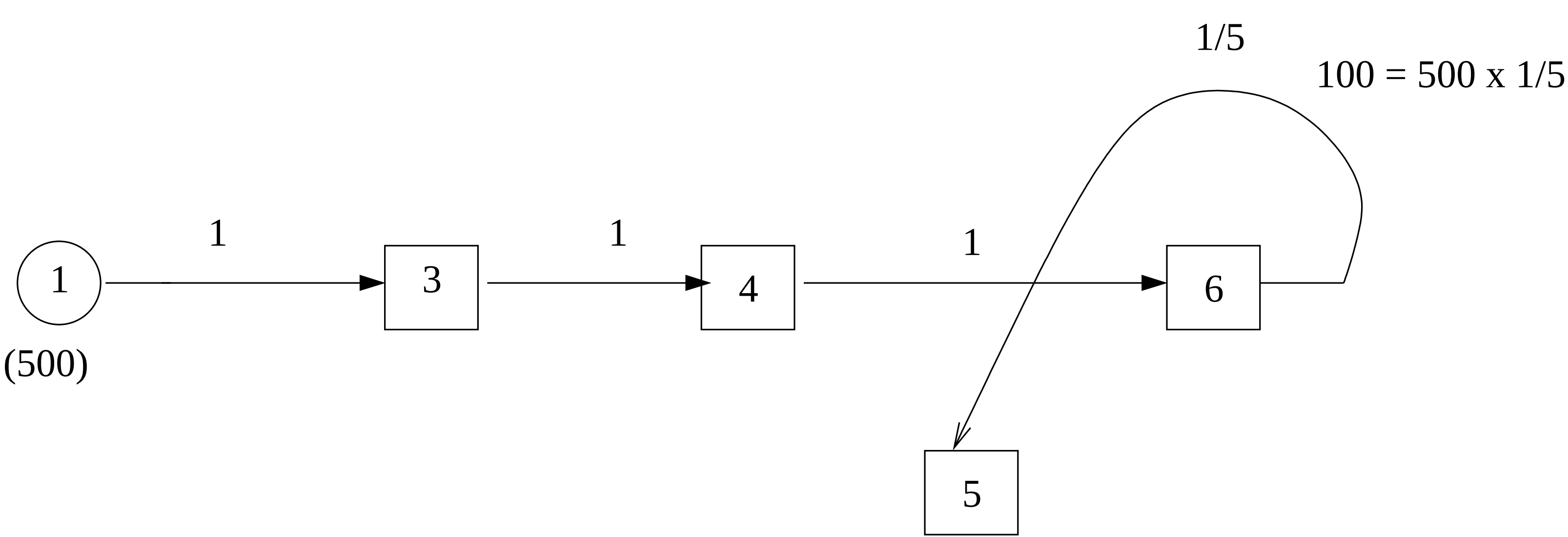


Figure 13

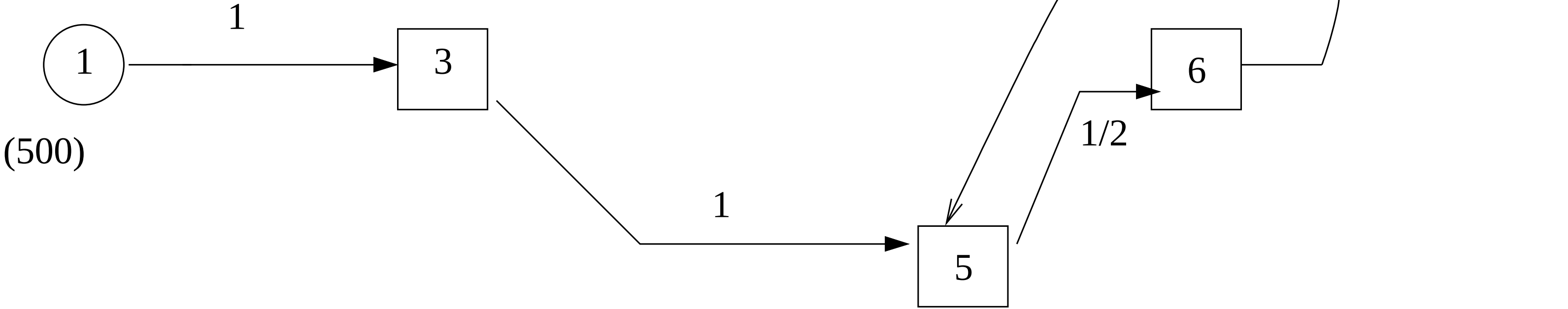


Figure 14