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# MAXIMAL ACCRETIVE EXTENSIONS OF SCHRÖDINGER OPERATORS ON VECTOR BUNDLES OVER INFINITE GRAPHS

OGNJEN MILATOVIĆ AND FRANÇOISE TRUC

ABSTRACT. Given a Hermitian vector bundle over an infinite weighted graph, we define the Laplacian associated to a unitary connection on this bundle and study a perturbation of this Laplacian by an operator-valued potential. We give a sufficient condition for the resulting Schrödinger operator to serve as the generator of a strongly continuous contraction semigroup in the corresponding  $\ell^p$ -space. Additionally, in the context of  $\ell^2$ -space, we study the essential self-adjointness of the corresponding Schrödinger operator.

## 1. INTRODUCTION

In recent years, there has been quite a bit of interest in the study of the Laplacian in  $\ell^p$ -spaces on infinite graphs. More precisely, let  $(X, b, m)$  be a weighted graph as described in section 2.1 below, and let us define a form  $Q^{(c)}$  on (complex-valued) finitely supported functions on  $X$  by

$$\begin{aligned} Q^{(c)}(u, v) &:= \frac{1}{2} \sum_{x, y \in X} b(x, y)(u(x) - u(y))\overline{(v(x) - v(y))} \\ &\quad + \sum_{x \in X} w(x)u(x)\overline{v(x)}, \end{aligned} \tag{1}$$

where  $w: X \rightarrow [0, \infty)$ . We denote by  $\ell_m^p(X)$  the space of  $\ell^p$ -summable functions with weight  $m$ , by  $Q^{(D)}$  the closure of  $Q^{(c)}$  in  $\ell_m^2(X)$ , and by  $L$  the associated self-adjoint operator. Since  $Q^{(D)}$  is a Dirichlet form, the semigroup  $e^{-tL}$ ,  $t \geq 0$ , extends to a  $C_0$ -semigroup on  $\ell_m^p(X)$ , where  $p \in [1, \infty)$ . We denote by  $-L_p$  the generators of these semigroups. For the definition of a  $C_0$ -semigroup and its generator, see the Appendix. The following characterization of operators  $L_p$  is given in [19]:

*Assume that*

$$\sum_{n \in \mathbb{Z}_+} m(x_n) = \infty, \tag{A1}$$

*for any sequence  $\{x_n\}_{n \in \mathbb{Z}_+}$  of vertices such that  $x_n \sim x_{n+1}$  for all  $n \in \mathbb{Z}_+$ . Then for any  $p \in [1, \infty)$ , the operator  $L_p$  is the restriction of  $\tilde{L}$  to*

$$\text{Dom}(L_p) = \{u \in \ell_m^p(X) \cap \tilde{D}_s : \tilde{L}u \in \ell_m^p(X)\},$$

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where

$$\widetilde{D}_s := \{u: X \rightarrow \mathbb{C}: \sum_{y \in X} b(x, y)|u(y)| < \infty, \forall x \in X\},$$

$\widetilde{L} := \Delta_{b, m} + w/m$ , and

$$(\Delta_{b, m}u)(x) := \frac{1}{m(x)} \sum_{y \in X} b(x, y)(u(x) - u(y)). \quad (2)$$

Actually, (A1) can be replaced when  $w = 0$  by the existence of a compatible intrinsic metric (see [13]), or if moreover  $p = 2$ , by the existence of an intrinsic metric so that  $\frac{1}{m(x)} \sum_{y \in X} b(x, y)$  is bounded on the combinatorial neighborhood of each distance ball (see [15]).

In the case of Schrödinger operators on a Riemannian manifold  $M$ , it is natural to study maximal accretivity or self-adjointness properties of operators acting on sections of vector bundles over  $M$ . But the notion of vector bundle is also relevant on graphs; see for example [1], [11], [20], and [27]. The aim of this paper is precisely to study such properties in the setup of a vector bundle over an infinite weighted graph. In particular, we give sufficient conditions for the equality of the operator  $H_{p, \max}$  (vector-bundle analogue of  $L_p$ ) and the closure in  $\Gamma_{\ell_m^p}(X, F)$  (the corresponding  $\ell^p$ -space of sections of the bundle  $F \rightarrow X$ ) of the restriction of  $\widetilde{H}_{W, \Phi}$  (vector-bundle analogue of  $\widetilde{L}$ ) to the set of finitely supported sections.

The paper is organized as follows. In sections 2.1, 2.2 and 2.3 we describe the setting: discrete sets, Hermitian vector bundle and connection, operators. The main results are presented in section 2.4, with some comments. Section 3 contains preliminary results, such as Green's formula, Kato's inequality, and ground state transform. Sections 4, 5 and 6 are devoted to the proofs of the theorems. For readers' convenience, in the Appendix we review some concepts from the theory of semigroups of operators:  $C_0$ -semigroup, generator of a  $C_0$ -semigroup, and (maximal) accretivity. Additionally, the Appendix contains the statement of Hille–Yosida Theorem and a discussion of the connection between self-adjointness and maximal accretivity of operators in Hilbert spaces.

## 2. SETUP AND MAIN RESULTS

**2.1. Weighted Graph.** Let  $X$  be a countably infinite set, equipped with a measure  $m: X \rightarrow (0, \infty)$ . Let  $b: X \times X \rightarrow [0, \infty)$  be a function such that

- (i)  $b(x, y) = b(y, x)$ , for all  $x, y \in X$ ;
- (ii)  $b(x, x) = 0$ , for all  $x \in X$ ;
- (iii)  $\sum_{y \in X} b(x, y) < \infty$ , for all  $x \in X$ .

Vertices  $x, y \in X$  with  $b(x, y) > 0$  are called *neighbors*, and we denote this relationship by  $x \sim y$ . We call the triple  $(X, b, m)$  a *weighted graph*. We assume that  $(X, b, m)$  is connected, that is, for any  $x, y \in X$  there exists a path  $\gamma$  joining  $x$  and  $y$ . Here, a path  $\gamma$  is a sequence  $x_1, x_2, \dots, x_n \in X$  such that  $x = x_1, y = x_n$ , and  $x_j \sim x_{j+1}$  for all  $1 \leq j \leq n - 1$ .

**2.2. Hermitian Vector Bundles on Graphs and Connection.** A family of (finite-dimensional) complex linear spaces  $F = \bigsqcup_{x \in X} F_x$  is called a *complex vector bundle over  $X$*  and written  $F \rightarrow X$ , if any two  $F_x$  and  $F_y$  are isomorphic as complex vector spaces. Then the  $F_x$ 's are called the *fibers* of  $F \rightarrow X$ , and the complex linear space

$$\Gamma(X, F) := \prod_{x \in X} F_x = \{u \mid u: X \rightarrow F, u(x) \in F_x\}$$

is called the space of *sections in  $F \rightarrow X$* . We define the space of *finitely supported sections*  $\Gamma_c(X, F)$  of  $F \rightarrow X$  as the set of  $u \in \Gamma(X, F)$  such that  $u(x) = 0$  for all but finitely many  $x \in X$ .

**Definition 2.1.** An assignment  $\Phi$  which associates to any  $x \sim y$  an isomorphism of complex vector spaces  $\Phi_{x,y}: F_x \rightarrow F_y$  is called a *connection on the complex vector bundle  $F \rightarrow X$*  if

$$\Phi_{y,x} = (\Phi_{x,y})^{-1} \quad \text{for all } x \sim y. \quad (3)$$

**Definition 2.2.** (i) A family of complex scalar products

$$\langle \cdot, \cdot \rangle_{F_x}: F_x \times F_x \rightarrow \mathbb{C}, \quad x \in X,$$

is called a *Hermitian structure on the complex vector bundle  $F \rightarrow X$* , and the pair given by  $F \rightarrow X$  and  $\langle \cdot, \cdot \rangle_{F_x}$  is called a *Hermitian vector bundle over  $X$* .

(ii) A connection  $\Phi$  on a complex vector bundle  $F \rightarrow X$  is called *unitary with respect to a Hermitian structure  $\langle \cdot, \cdot \rangle_{F_x}$*  if for all  $x \sim y$  one has

$$\Phi_{x,y}^* = \Phi_{x,y}^{-1},$$

where  $T^*$  denotes the Hermitian adjoint of an operator  $T: F_x \rightarrow F_y$  with respect to  $\langle \cdot, \cdot \rangle_{F_x}$  and  $\langle \cdot, \cdot \rangle_{F_y}$ .

**Definition 2.3.** The Laplacian  $\Delta_{b,m}^{F,\Phi}: \tilde{D} \rightarrow \Gamma(X, F)$  on a Hermitian vector bundle  $F \rightarrow X$  with a unitary connection  $\Phi$  is a linear operator with the domain

$$\tilde{D} := \{u \in \Gamma(X, F): \sum_{y \in X} b(x, y) |u(y)|_{F_y} < \infty, \text{ for all } x \in X\} \quad (4)$$

defined by the formula

$$(\Delta_{b,m}^{F,\Phi} u)(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y) (u(x) - \Phi_{y,x} u(y)). \quad (5)$$

**Remark 2.1.** The operator  $\Delta_{b,m}^{F,\Phi}$  is well-defined by the property (iii) of  $b(x, y)$ , definition (4), and unitarity of  $\Phi$ .

**Remark 2.2.** In the case  $F_x = \{x\} \times \mathbb{C}$  with the canonical Hermitian structure, the sections of the bundle  $F \rightarrow X$  can be canonically identified with complex-valued functions on  $X$ . Under this identification, any connection  $\Phi$  can be uniquely written as  $\Phi_{x,y} = e^{i\theta(y,x)}$ , where  $\theta: X \times X \rightarrow [-\pi, \pi]$  is a magnetic potential on  $(X, b)$ , which, due to (3), satisfies the property  $\theta(x, y) =$

$-\theta(y, x)$  for all  $x, y \in X$ . As a result, we get the magnetic Laplacian operator. In particular, if  $\theta \equiv 0$  we get the Laplacian operator (2).

**Remark 2.3.** If the property (iii) of  $b(x, y)$  is replaced by

$$\sharp \{y \in X : b(x, y) > 0\} < \infty, \text{ for all } x \in X,$$

where  $\sharp S$  denotes the number of elements in the set  $S$ , then the graph  $(X, b, m)$  is called locally finite. In this case, we have  $\tilde{D} = \Gamma(X, F)$ .

**2.3. Operators.** From now on we will always work in the setting of a Hermitian vector bundle  $F \rightarrow X$  over a connected weighted graph  $(X, b, m)$ , equipped with a unitary connection  $\Phi$ .

**Definition 2.4.** We define the Schrödinger-type operator  $\tilde{H}_{W, \Phi} : \tilde{D} \rightarrow \Gamma(X, F)$  by the formula

$$\tilde{H}_{W, \Phi} u := \Delta_{b, m}^{F, \Phi} u + W u, \quad (6)$$

where  $W(x) : F_x \rightarrow F_x$  is a linear operator for any  $x \in X$ , and  $\tilde{D}$  is as in (4).

**Definition 2.5.** (i) For any  $1 \leq p < \infty$  we denote by  $\Gamma_{\ell_m^p}(X, F)$  the space of sections  $u \in \Gamma(X, F)$  such that

$$\|u\|_p^p := \sum_{x \in X} m(x) |u(x)|_{F_x}^p < \infty,$$

where  $|\cdot|_{F_x}$  denotes the norm in  $F_x$  corresponding to the Hermitian product  $\langle \cdot, \cdot \rangle_{F_x}$ . The space of  $p$ -summable functions  $X \rightarrow \mathbb{C}$  with weight  $m$  will be denoted by  $\ell_m^p(X)$ .

(ii) By  $\Gamma_{\ell^\infty}(X, F)$  we denote the space of bounded sections of  $F$ , equipped with the norm

$$\|u\|_\infty := \sup_{x \in X} |u(x)|_{F_x}.$$

The space of bounded functions on  $X$  will be denoted by  $\ell^\infty(X)$ .

The space  $\Gamma_{\ell_m^2}(X, F)$  is a Hilbert space with the inner product

$$(u, v) := \sum_{x \in X} m(x) \langle u(x), v(x) \rangle_{F_x}$$

**Definition 2.6.** Let  $1 \leq p < +\infty$  and let  $\tilde{D}$  be as in (4). The maximal operator  $H_{p, \max}$  is given by the formula  $H_{p, \max} u = \tilde{H}_{W, \Phi} u$  with domain

$$\text{Dom}(H_{p, \max}) = \{u \in \Gamma_{\ell_m^p}(X, F) \cap \tilde{D} : \tilde{H}_{W, \Phi} u \in \Gamma_{\ell_m^p}(X, F)\}. \quad (7)$$

Moreover if

$$\tilde{H}_{W, \Phi}[\Gamma_c(X, F)] \subseteq \Gamma_{\ell_m^p}(X, F), \quad (8)$$

then we set  $H_{p, \min} := \tilde{H}_{W, \Phi}|_{\Gamma_c(X, F)}$ .

**Remark 2.4.** Note that under our assumptions on  $(X, b, m)$ , the inclusion (8) does not necessarily hold. It holds if we additionally assume that  $(X, b, m)$  is locally finite.

**2.4. Statement of the Results.** Let us denote by  $\overline{T}$  the closure of an operator  $T$ .

**Theorem 2.1.** *Let  $W(x): F_x \rightarrow F_x$  be a linear operator satisfying*

$$\operatorname{Re} \langle W(x)u(x), u(x) \rangle_{F_x} \geq 0, \quad \text{for all } x \in X. \quad (9)$$

*Then, the following properties hold:*

- (i) *Let  $1 < p < \infty$ , and assume that (8) and (A1) are satisfied. Then the operator  $-\overline{H_{p,\min}}$  generates a strongly continuous contraction semigroup on  $\Gamma_{\ell_m^p}(X, F)$ .*
- (ii) *Assume that (8) is satisfied for  $p = 1$ , and that  $(X, b, m)$  is stochastically complete. Then the operator  $-\overline{H_{1,\min}}$  generates a strongly continuous contraction semigroup on  $\Gamma_{\ell_m^1}(X, F)$ .*

**Remark 2.5.** By Definition 1.1 in [19], stochastic completeness of  $(X, b, m)$  means that there is no non-trivial and non-negative  $w \in \ell^\infty(X)$  such that

$$(\Delta_{b,m} + \alpha)w \leq 0, \quad \alpha > 0,$$

where  $\Delta_{b,m}$  is as in (2).

**Remark 2.6.** The notions of generator of a strongly continuous semigroup and (maximal) accretivity are reviewed in the Appendix. In particular, under the assumptions of Theorem 2.1, the operator  $\overline{H_{p,\min}}$  is maximal accretive for all  $1 \leq p < \infty$ .

In the next theorem, we make the following assumption, which is stronger than (8):

$$\tilde{H}_{W,\Phi}[\Gamma_c(X, F)] \subseteq \Gamma_{\ell_m^p}(X, F) \cap \Gamma_{\ell_m^{p^*}}(X, F), \quad (10)$$

with  $1/p + 1/p^* = 1$ .

**Remark 2.7.** If  $(X, b, m)$  is a locally finite graph then (10) is satisfied. If  $\inf_{x \in X} m(x) > 0$  then (A1) and (10) are satisfied.

**Theorem 2.2.** *Assume that the hypotheses (A1) and (9) are satisfied. Then, the following properties hold:*

- (i) *Let  $1 < p < \infty$ , and assume that (10) is satisfied. Then  $\overline{H_{p,\min}} = H_{p,\max}$ .*
- (ii) *Assume that (10) is satisfied for  $p = 1$ , and that  $(X, b, m)$  is stochastically complete. Then  $\overline{H_{1,\min}} = H_{1,\max}$ .*

Regarding self-adjointness problems, let us point out that the results of [3, 4, 21, 24, 25] and Theorem 5 in [18] can be extended to the vector-bundle setting. As an illustration, we state and prove an extension of Theorem 1.5 from [25]. Before doing this, we recall the notion of intrinsic metric.

**Definition 2.7.** *A pseudo metric is a map  $d: X \times X \rightarrow [0, \infty)$  such that  $d(x, y) = d(y, x)$ , for all  $x, y \in X$ ;  $d(x, x) = 0$ , for all  $x \in X$ ; and  $d(x, y)$  satisfies the triangle inequality.*

*A pseudo metric  $d = d_\sigma$  is called a path pseudo metric if there exists a map  $\sigma: X \times X \rightarrow [0, \infty)$  such that  $\sigma(x, y) = \sigma(y, x)$ , for all  $x, y \in X$ ;  $\sigma(x, y) > 0$  if and only if  $x \sim y$ ; and  $d_\sigma(x, y) =$*

$\inf\{l_\sigma(\gamma) : \gamma \text{ path connecting } x \text{ and } y\}$ , where the length  $l_\sigma$  of the path  $\gamma = (x_0, x_1, \dots, x_n)$  is given by

$$l_\sigma(\gamma) = \sum_{i=0}^{n-1} \sigma(x_i, x_{i+1}).$$

On a locally finite graph a path pseudo metric is a metric; see [15].

**Definition 2.8.** A pseudo metric  $d$  on  $(X, b, m)$  is called *intrinsic* if

$$\frac{1}{m(x)} \sum_{y \in X} b(x, y) (d(x, y))^2 \leq 1, \quad \text{for all } x \in X.$$

**Remark 2.8.** The concept of intrinsic pseudo metric goes back to [9] which discusses a more general situation. For graphs it has been discussed in [14] and [8]. Related earlier material can be found in [22].

We will also use the notion of a *regular graph* introduced in [3], which is a (not yet published) revised version of [2]. Let us first recall the definition of the boundary of a given set  $A \subseteq X$ :

$$\partial A := \{x \in A : \text{there exists } y \in X \setminus A \text{ such that } y \sim x\}.$$

In the sequel, we denote by  $(\widehat{X}, \widehat{d})$  the metric completion of  $(X, d)$ , and we define the *Cauchy boundary*  $X_\infty$  as follows:  $X_\infty := \widehat{X} \setminus X$ . Note that  $(X, d)$  is metrically complete if and only if  $X_\infty$  is empty. For a path metric  $d = d_\sigma$  on  $X$  and  $x \in X$ , we set

$$D(x) := \inf_{z \in X_\infty} \widehat{d}_\sigma(x, z). \quad (11)$$

**Definition 2.9.** Let  $(X, b, m)$  be a graph with a path metric  $d_\sigma$ . Let  $\varepsilon > 0$  be given and let

$$X_\varepsilon := \{x \in X : D(x) \geq \varepsilon\}. \quad (12)$$

We say that  $(X, b, m)$  is *regular* if for any sufficiently small  $\varepsilon$ , any bounded subset of  $\partial X_\varepsilon$  (for the metric  $d_\sigma$ ) is finite.

**Remark 2.9.** Metrically complete graphs  $(X, d)$  are regular since  $D(x) = \infty$  for any  $x \in X$ , which implies that  $X_\varepsilon = X$ , so that  $\partial X_\varepsilon = \emptyset$ .

**Remark 2.10.** Definition 2.9 covers also a broad class of metrically non-complete graphs. For instance, weighted graphs whose first Betti number is finite are regular. In particular, any weighted tree is regular; see [3].

**Theorem 2.3.** Let  $(X, b, m)$  be a locally finite graph with an intrinsic path metric  $d = d_\sigma$ . Assume that  $(X, b, m)$  is regular. Let  $W(x) : F_x \rightarrow F_x$  be a linear self-adjoint operator such that there exists a constant  $C$  satisfying

$$\langle W(x)u(x), u(x) \rangle_{F_x} \geq \left( \frac{1}{2(D(x))^2} - C \right) |u(x)|_{F_x}^2, \quad (13)$$

for all  $x \in X$  and all  $u \in \Gamma_c(X, F)$ , where  $D(x)$  is as in (11). Then  $\widetilde{H}_{W, \Phi}$  is essentially self-adjoint on  $\Gamma_c(X, F)$ .

### 3. PRELIMINARY LEMMAS

**3.1. Green's Formula.** We now give a variant of Green's formula, which is analogous to Lemma 2.1 in [10] and Lemma 4.7 in [12].

**Notation 3.1.** Let  $W(x): F_x \rightarrow F_x$  be a linear operator. We denote by  $W^*$  the Hermitian adjoint of  $W$ , that is,  $(W(x))^*$  is the Hermitian adjoint of  $W(x)$  with respect to  $\langle \cdot, \cdot \rangle_{F_x}$ .

**Lemma 3.1.** Let  $\tilde{H}_{W,\Phi}$  be as in (6). The following properties hold:

- (i) if  $\tilde{H}_{W,\Phi}[\Gamma_c(X, F)] \subseteq \Gamma_{\ell_m^p}(X, F)$  for some  $1 \leq p \leq \infty$ , then any  $u \in \Gamma_{\ell_m^{p^*}}(X, F)$  with  $1/p + 1/p^* = 1$  belongs to the set  $\tilde{D}$  defined by (4);
- (ii) for all  $u \in \tilde{D}$  and all  $v \in \Gamma_c(X, F)$ , the sums

$$\sum_{x \in X} m(x) \langle \tilde{H}_{W,\Phi} u, v \rangle_{F_x}, \quad \sum_{x \in X} m(x) \langle u, \tilde{H}_{W^*,\Phi} v \rangle_{F_x},$$

and the expression

$$\begin{aligned} & \frac{1}{2} \sum_{x,y \in X} b(x,y) \langle u(x) - \Phi_{y,x} u(y), v(x) - \Phi_{y,x} v(y) \rangle_{F_x} \\ & + \sum_{x \in X} m(x) \langle W(x)u(x), v(x) \rangle_{F_x} \end{aligned} \tag{14}$$

converge absolutely and agree.

*Proof.* To make the notations simpler, throughout the proof we suppress  $F_x$  in  $|\cdot|_{F_x}$ . From the assumption  $\tilde{H}_{W,\Phi}[\Gamma_c(X, F)] \subseteq \Gamma_{\ell_m^p}(X, F)$ , it is easily seen that the function  $y \mapsto b(x,y)/m(y)$  belongs to  $\ell_m^p(X)$ , for all  $x \in X$ . In the case  $1 < p^* < \infty$ , for all  $u \in \Gamma_{\ell_m^{p^*}}(X, F)$ , by Hölder's inequality with  $1/p + 1/p^* = 1$  we have

$$\sum_{y \in X} b(x,y) |u(y)| \leq \left( \sum_{y \in X} \left( \frac{b(x,y)}{m(y)} \right)^p m(y) \right)^{1/p} \left( \sum_{y \in X} |u(y)|^{p^*} m(y) \right)^{1/p^*}.$$

In the case  $p^* = 1$ , for all  $u \in \Gamma_{\ell_m^1}(X, F)$ , by Hölder's inequality with  $p = \infty$  and  $p^* = 1$  we have

$$\sum_{y \in X} b(x,y) |u(y)| \leq \sup_{y \in X} \left( \frac{b(x,y)}{m(y)} \right) \left( \sum_{y \in X} |u(y)| m(y) \right).$$

In the case  $p^* = \infty$ , for all  $u \in \Gamma_{\ell^\infty}(X, F)$ , by Hölder's inequality with  $p = 1$  and  $p^* = \infty$  we have

$$\sum_{y \in X} b(x,y) |u(y)| \leq \sup_{y \in X} (|u(y)|) \left( \sum_{y \in X} b(x,y) \right).$$

This concludes the proof of property (i). Let us prove property (ii). Since  $v \in \Gamma_c(X, F)$ , the first sum is performed over finitely many  $x \in X$ . Hence, this sum converges absolutely. The



proof of absolute convergence of the second sum and the expression (14) is based on the next two estimates. By Cauchy–Schwarz inequality and unitarity of  $\Phi_{y,x}$  we get

$$\sum_{x,y \in X} |b(x,y) \langle u(x), \Phi_{y,x} v(y) \rangle_{F_x}| \leq \sum_{y \in X} |v(y)| \left( \sum_{x \in X} b(x,y) |u(x)| \right) < \infty,$$

where the convergence follows from the fact that  $u \in \tilde{D}$  and  $v \in \Gamma_c(X, F)$ . Similarly,

$$\sum_{x,y \in X} |b(x,y) \langle u(x), v(x) \rangle_{F_x}| \leq \sum_{x \in X} |u(x)| |v(x)| \left( \sum_{y \in X} b(x,y) \right) < \infty,$$

where the convergence follows by property (iii) of  $b(x,y)$  and since  $v \in \Gamma_c(X, F)$ . The equality of the three sums follows directly from Fubini’s theorem. This shows property (ii).  $\square$

**3.2. Kato’s Inequality.** This version of Kato’s inequality extends that of [6].

**Lemma 3.2.** *Let  $\Delta_{b,m}$  and  $\Delta_{b,m}^{F,\Phi}$  be defined as in (2) and (5) respectively. Then, the following pointwise inequality holds for all  $u \in \tilde{D}$ :*

$$|u|(\Delta_{b,m}|u|) \leq \operatorname{Re} \langle \Delta_{b,m}^{F,\Phi} u, u \rangle_{F_x}, \quad (15)$$

where  $|\cdot|$  denotes the norm in  $F_x$ , and  $\operatorname{Re} z$  denotes the real part of a complex number  $z$ .

*Proof.* Using (2), (5), and the unitarity of  $\Phi_{y,x}$ , we obtain

$$\begin{aligned} & |u(x)|((\Delta_{b,m}|u|)(x)) - \operatorname{Re} \langle \Delta_{b,m}^{F,\Phi} u(x), u(x) \rangle_{F_x} \\ &= \frac{1}{m(x)} \sum_{y \in X} b(x,y) [\operatorname{Re} \langle \Phi_{y,x} u(y), u(x) \rangle_{F_x} - |u(x)||u(y)|] \leq 0. \end{aligned} \quad \square$$

**3.3. Ground State Transform.** Using the definition of  $\tilde{H}_{W,\Phi}$  and unitarity of  $\Phi_{y,x}$ , it is easy to prove the following vector-bundle analogue of “ground state transform” from [9], [10], and [12]. We omit the proof here.

**Lemma 3.3.** *Assume that  $W(x): F_x \rightarrow F_x$  is a self-adjoint operator. Assume that (8) is satisfied for  $p = 2$ . Let  $\lambda \in \mathbb{R}$ , and let  $u \in \tilde{D}$  so that*

$$(\tilde{H}_{W,\Phi} - \lambda)u = 0.$$

*Then, for all finitely supported functions  $g: X \rightarrow \mathbb{R}$ , we have*

$$((\tilde{H}_{W,\Phi} - \lambda)(gu), gu) = \frac{1}{2} \sum_{x,y \in X} b(x,y) (g(x) - g(y))^2 (\operatorname{Re} \langle u(x), \Phi_{y,x} u(y) \rangle_{F_x}).$$

#### 4. PROOF OF THEOREM 2.1

In Lemmas 4.1 and 4.3 below, we assume that the hypotheses of Theorem 2.1 are satisfied.

**Lemma 4.1.** *Let  $1 \leq p < \infty$ . Then, the operator  $H_{p,\min}$  satisfies the following inequality for all  $u \in \Gamma_c(X, F)$ :*

$$\operatorname{Re} \sum_{x \in X} m(x) \langle (H_{p,\min} u)(x), u(x) |u(x)|^{p-2} \rangle_{F_x} \geq 0. \quad (16)$$

*Proof.* Let  $u \in \Gamma_c(X, F)$  be arbitrary. By Lemma 3.1(ii) with  $W = 0$ ,  $u \in \Gamma_c(X, F)$  and  $v := u|u|^{p-2}$ , we have

$$\begin{aligned} & \operatorname{Re} \sum_{x \in X} m(x) \langle (\Delta_{b,m}^{F,\Phi} u)(x), u(x)|u(x)|^{p-2} \rangle_{F_x} = \frac{1}{2} \sum_{x,y \in X} b(x,y) [|u(x)|^p \\ & + |u(y)|^p - \operatorname{Re} \langle \Phi_{y,x} u(y), u(x)|u(x)|^{p-2} \rangle_{F_x} - \operatorname{Re} \langle \Phi_{x,y} u(x), u(y)|u(y)|^{p-2} \rangle_{F_y}] \\ & \geq \frac{1}{2} \sum_{x,y \in X} b(x,y) [|u(x)|^p + |u(y)|^p - |u(x)||u(y)|^{p-1} \\ & - |u(y)||u(x)|^{p-1}]. \end{aligned} \tag{17}$$

For  $p = 1$ , from (17) and the assumption (9) we easily get (16).

Let  $1 < p < \infty$  and let  $p^*$  satisfy  $1/p + 1/p^* = 1$ . By Young's inequality we have

$$|u(x)||u(y)|^{p-1} \leq \frac{|u(x)|^p}{p} + \frac{(|u(y)|^{p-1})^{p^*}}{p^*} = \frac{|u(x)|^p}{p} + \frac{(p-1)|u(y)|^p}{p}$$

and, likewise,

$$|u(y)||u(x)|^{p-1} \leq \frac{|u(y)|^p}{p} + \frac{(p-1)|u(x)|^p}{p}.$$

From the last two inequalities we get

$$-|u(x)||u(y)|^{p-1} - |u(y)||u(x)|^{p-1} \geq -|u(x)|^p - |u(y)|^p. \tag{18}$$

Using (18), (17), and the assumption (9), we obtain (16).  $\square$

The following lemma is a special case of Proposition 8 in [18]:

**Lemma 4.2.** *Assume (A1). Let  $\alpha > 0$  and  $1 \leq p < \infty$ . Let  $\Delta_{b,m}$  be as in (2). Assume that  $u \in \ell_m^p(X)$  is a real-valued function satisfying the inequality  $(\Delta_{b,m} + \alpha)u \geq 0$ . Then  $u \geq 0$ .*

**Remark 4.1.** The case  $p = \infty$  is more complicated and involves the notion of stochastic completeness; see, for instance, [14], [18], [19].

In the remainder of this section and in section 5, we will use certain arguments of Section A in [17] and [23] in our setting. In the sequel,  $\operatorname{Ran} T$  denotes the range of an operator  $T$ .

**Lemma 4.3.** *Let  $1 < p < \infty$  and let  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda > 0$ . Then,  $\operatorname{Ran} (H_{p,\min} + \lambda)$  is dense in  $\ell_m^p(X)$ .*

*Proof.* Let  $u \in (\Gamma_{\ell_m^p}(X, F))^* = \Gamma_{\ell_m^{p^*}}(X, F)$ , be a continuous linear functional that annihilates  $(\lambda + H_{p,\min})\Gamma_c(X, F)$ :

$$\sum_{x \in X} m(x) \langle (\lambda + H_{p,\min})v(x), u(x) \rangle_{F_x} = 0, \quad \text{for all } v \in \Gamma_c(X, F). \tag{19}$$

By assumption (8) we know that  $\tilde{H}_{W,\Phi} v \in \Gamma_{\ell_m^p}(X, F)$ . Since  $u \in \Gamma_{\ell_m^{p^*}}(X, F)$ , by Lemma 3.1(i) we have  $u \in \tilde{D}$ . Now using Lemma 3.1(ii) in (19), we get

$$\sum_{x \in X} m(x) \langle v(x), (\bar{\lambda} + \tilde{H}_{W^*,\Phi})u(x) \rangle_{F_x} = 0, \quad \text{for all } v \in \Gamma_c(X, F),$$

where  $\bar{\lambda}$  is the complex conjugate of  $\lambda$ . The last equality leads to

$$(\bar{\lambda} + \Delta_{b,m}^{F,\Phi} + W^*)u = 0. \quad (20)$$

Using Kato's inequality (15), assumption (9), and (20) we have

$$\begin{aligned} |u|(\Delta_{b,m}|u|) &\leq \operatorname{Re} \langle \Delta_{b,m}^{F,\Phi} u, u \rangle_{F_x} \\ &= -(\operatorname{Re} \lambda)|u|^2 - \operatorname{Re} \langle W^* u, u \rangle_{F_x} \leq -(\operatorname{Re} \lambda)|u|^2, \end{aligned}$$

where  $|u| \in \ell_m^{p^*}(X)$  with  $1 < p^* < \infty$ . Rewriting the last inequality, we obtain

$$|u|(\Delta_{b,m}|u| + (\operatorname{Re} \lambda)|u|) \leq 0.$$

For all  $x \in X$  such that  $u(x) \neq 0$ , we may divide both sides of the last inequality by  $|u(x)|$  to get

$$(\Delta_{b,m} + \operatorname{Re} \lambda)|u| \leq 0. \quad (21)$$

Note that the inequality (21) also holds for those  $x \in X$  such that  $u(x) = 0$ ; in this case, the left hand side of (21) is non-positive by (2). Thus, the inequality (21) holds for all  $x \in X$ . By Lemma 4.2, from (21) we get  $|u| \leq 0$ . Hence,  $u = 0$ .  $\square$

**End of the Proof of Theorem 2.1(i).** The inequality (16) means that  $H_{p,\min}$  is accretive in  $\Gamma_{\ell_m^p}(X, F)$ ; see (R1) in the Appendix with  $j(u) = u|u|^{p-2}$ . Hence,  $H_{p,\min}$  is closable and  $\overline{H_{p,\min}}$  is accretive in  $\Gamma_{\ell_m^p}(X, F)$ ; see the Appendix. Therefore, for all  $u \in \operatorname{Dom}(\overline{H_{p,\min}})$  the following inequality holds:

$$\operatorname{Re} \sum_{x \in X} m(x) \langle (H_{p,\min} u)(x), u(x)|u(x)|^{p-2} \rangle_{F_x} \geq 0. \quad (22)$$

Let  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda > 0$ . Using Hölder's inequality, from (22) we get

$$(\operatorname{Re} \lambda)\|u\|_p \leq \|(\lambda + \overline{H_{p,\min}})u\|_p, \quad (23)$$

for all  $u \in \operatorname{Dom}(\overline{H_{p,\min}})$ . By Lemma 4.3 we know that  $\operatorname{Ran} (H_{p,\min} + \lambda)$  is dense in  $\Gamma_{\ell_m^p}(X, F)$ . This, together with (23), shows that  $\operatorname{Ran} (\overline{H_{p,\min}} + \lambda) = \Gamma_{\ell_m^p}(X, F)$ . Hence, from (23) we get

$$\|(\xi + \overline{H_{p,\min}})^{-1}\| \leq \frac{1}{\xi}, \quad \text{for all } \xi > 0,$$

where  $\|\cdot\|$  is the operator norm  $\Gamma_{\ell_m^p}(X, F) \rightarrow \Gamma_{\ell_m^p}(X, F)$ . Thus,  $-\overline{H_{p,\min}}$  satisfies the conditions (C1), (C2) and (C3) of Hille–Yosida Theorem; see the Appendix. Hence,  $-\overline{H_{p,\min}}$  is the generator of a strongly continuous contraction semigroup on  $\Gamma_{\ell_m^p}(X, F)$ .  $\square$

**Proof of Theorem 2.1(ii).** Repeating the proof of Lemma 4.3 in the case  $p = 1$  and using Remark 2.5, from (21) with  $u \in \Gamma_{\ell^\infty}(X, F)$  we obtain  $|u| = 0$ . Therefore, for all  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda > 0$ , the set  $\operatorname{Ran} (H_{1,\min} + \lambda)$  is dense in  $\Gamma_{\ell_m^1}(X, F)$ . From here on, we may repeat the proof of Theorem 2.1(i).  $\square$

## 5. PROOF OF THEOREM 2.2

We begin with the following lemma.

**Lemma 5.1.** *Let  $1 \leq p < \infty$  and  $1/p + 1/p^* = 1$ . Assume that (10) is satisfied. Then  $H_{p,\max}$  is a closed operator.*

*Proof.* Let  $u_k$  be a sequence of elements in  $\text{Dom}(H_{p,\max})$  such that  $u_k \rightarrow u$  and  $H_{p,\max}u_k \rightarrow f$ , as  $k \rightarrow \infty$ , using the norm convergence in  $\Gamma_{\ell_m^p}(X, F)$ . We need to show that  $u \in \text{Dom}(H_{p,\max})$  and  $f = H_{p,\max}u$ . Let  $v \in \Gamma_c(X, F)$  be arbitrary, and consider the sum

$$\sum_{x \in X} m(x) \langle (H_{p,\max}u_k)(x), v(x) \rangle_{F_x} = \sum_{x \in X} m(x) \langle (\tilde{H}_{W,\Phi}u_k)(x), v(x) \rangle_{F_x}.$$

By Lemma 3.1(ii) we have

$$\sum_{x \in X} m(x) \langle (\tilde{H}_{W,\Phi}u_k)(x), v(x) \rangle_{F_x} = \sum_{x \in X} m(x) \langle u_k(x), (\tilde{H}_{W^*,\Phi}v)(x) \rangle_{F_x}. \quad (24)$$

Using the norm convergence  $u_k \rightarrow u$  in  $\Gamma_{\ell_m^p}(X, F)$  and the assumption  $\tilde{H}_{W,\Phi}v \in \Gamma_{\ell_m^{p^*}}(X, F)$  with  $1/p + 1/p^* = 1$ , by Hölder's inequality we get

$$\sum_{x \in X} m(x) \langle u_k(x), (\tilde{H}_{W^*,\Phi}v)(x) \rangle_{F_x} \rightarrow \sum_{x \in X} m(x) \langle u(x), (\tilde{H}_{W^*,\Phi}v)(x) \rangle_{F_x}.$$

Using the norm convergence  $\tilde{H}_{W,\Phi}u_k \rightarrow f$  in  $\Gamma_{\ell_m^p}(X, F)$ , by Hölder's inequality we get

$$\sum_{x \in X} m(x) \langle (\tilde{H}_{W,\Phi}u_k)(x), v(x) \rangle_{F_x} \rightarrow \sum_{x \in X} m(x) \langle f(x), v(x) \rangle_{F_x}.$$

Therefore, taking the limit as  $k \rightarrow \infty$  on both sides of (24), we obtain

$$\sum_{x \in X} m(x) \langle u(x), (\tilde{H}_{W^*,\Phi}v)(x) \rangle_{F_x} = \sum_{x \in X} m(x) \langle f(x), v(x) \rangle_{F_x}. \quad (25)$$

Since  $u \in \Gamma_{\ell_m^p}(X, F)$  and since  $\tilde{H}_{W,\Phi}[\Gamma_c(X, F)] \subseteq \Gamma_{\ell_m^{p^*}}(X, F)$ , we may use Lemma 3.1(i) to conclude  $u \in \tilde{D}$ . Using Lemma 3.1(ii), we rewrite the left-hand side of (25) as follows:

$$\sum_{x \in X} m(x) \langle u(x), (\tilde{H}_{W^*,\Phi}v)(x) \rangle_{F_x} = \sum_{x \in X} m(x) \langle (\tilde{H}_{W,\Phi}u)(x), v(x) \rangle_{F_x}. \quad (26)$$

Since  $v \in \Gamma_c(X, F)$  is arbitrary, by (25) and (26) we get  $\tilde{H}_{W,\Phi}u = f$ . Thus,  $u \in \text{Dom}(H_{p,\max})$  and  $H_{p,\max}u = f$ . Therefore,  $H_{p,\max}$  is closed.  $\square$

**Maximal Operator Associated with  $\Delta_{b,m}$ .** Let  $1 \leq p < \infty$  and let  $\Delta_{b,m}$  be as in (2). We define the maximal operator  $L_{p,\max}$  in  $\ell_m^p(X)$  by the formula  $L_{p,\max}u = \Delta_{b,m}u$  with the domain

$$\text{Dom}(L_{p,\max}) = \{u \in \ell_m^p(X) \cap \tilde{D} : \Delta_{b,m}u \in \ell_m^p(X)\},$$

where  $\tilde{D}$  is as in (4) and sections are replaced by functions  $X \rightarrow \mathbb{C}$ .

Under the assumption (A1), it is known that  $-L_{p,\max}$  generates a strongly continuous contraction semigroup on  $\ell_m^p(X)$  for all  $1 \leq p < \infty$ ; see Theorem 5 in [19]. Thus, by Hille–Yosida Theorem (see the Appendix), we have

$$(0, \infty) \subset \rho(-L_{p,\max}) \quad \text{and} \quad \|(\xi + L_{p,\max})^{-1}\| \leq \frac{1}{\xi}, \quad (27)$$

for all  $\xi > 0$ , where  $\rho(T)$  denotes the resolvent set of an operator  $T$ .

**Lemma 5.2.** *Let  $1 \leq p < \infty$  and let  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda > 0$ . Assume that the hypotheses (A1) and (9) are satisfied. Then, the following properties hold:*

(i) *for all  $u \in \operatorname{Dom}(H_{p,\max})$ , we have*

$$(\operatorname{Re} \lambda)\|u\|_p \leq \|(\lambda + H_{p,\max})u\|_p; \quad (28)$$

(ii) *the operator  $\lambda + H_{p,\max}: \operatorname{Dom}(H_{p,\max}) \subset \Gamma_{\ell_m^p}(X, F) \rightarrow \Gamma_{\ell_m^p}(X, F)$  is injective.*

*Proof.* Let  $u \in \operatorname{Dom}(H_{p,\max})$  and  $f := (\lambda + H_{p,\max})u$ . By the definition of  $\operatorname{Dom}(H_{p,\max})$ , we have  $f \in \Gamma_{\ell_m^p}(X, F)$ , where  $1 < p < +\infty$ . Using (15) and (9) we get

$$\begin{aligned} |u|((\operatorname{Re} \lambda + \Delta_{b,m})|u|) &\leq \operatorname{Re} \langle (\lambda + \Delta_{b,m}^{F,\Phi})u, u \rangle_{F_x} \\ &\leq \operatorname{Re} \langle (\lambda + \Delta_{b,m}^{F,\Phi} + W)u, u \rangle_{F_x} = \operatorname{Re} \langle f, u \rangle_{F_x} \leq |f||u|. \end{aligned}$$

In what follows, we denote  $\xi := \operatorname{Re} \lambda$ . For all  $x \in X$  such that  $u(x) \neq 0$ , we may divide both sides of the last inequality by  $|u(x)|$  to get

$$(\xi + \Delta_{b,m})|u| \leq |f|. \quad (29)$$

Note that the inequality (29) also holds for those  $x \in X$  such that  $u(x) = 0$ ; in this case, the left hand side of (29) is non-positive by (2). Thus, the inequality (29) holds for all  $x \in X$ .

According to (27) the linear operator

$$(\xi + L_{p,\max})^{-1}: \ell_m^p(X) \rightarrow \ell_m^p(X)$$

is bounded. Hence, we can rewrite (29) as

$$(\xi + \Delta_{b,m})[(\xi + L_{p,\max})^{-1}|f| - |u|] \geq 0. \quad (30)$$

Since

$$(\xi + L_{p,\max})^{-1}|f| \in \ell_m^p(X) \quad \text{and} \quad |u| \in \ell_m^p(X),$$

it follows that  $((\xi + L_{p,\max})^{-1}|f| - |u|) \in \ell_m^p(X)$ . Hence, applying Lemma 4.2 to (30) we get

$$|u| \leq (\xi + L_{p,\max})^{-1}|f|.$$

Taking the  $\ell^p$ -norms on both sides and using (27) we get

$$\|u\|_p \leq \|(\xi + L_{p,\max})^{-1}|f|\|_p \leq \frac{1}{\xi}\|f\|_p,$$

and (28) is proven. We turn to property (ii). Assume that  $u \in \operatorname{Dom}(H_{p,\max})$  and  $(\lambda + H_{p,\max})u = 0$ . Using (28) we get  $\|u\|_p = 0$ , and hence  $u = 0$ . This shows that  $\lambda + H_{p,\max}$  is injective.  $\square$

**End of the Proof of Theorem 2.2.** We will consider the cases  $1 < p < \infty$  and  $p = 1$  simultaneously, keeping in mind the stochastic completeness assumption on  $(X, b, m)$  when  $p = 1$ . Since  $H_{p,\min} \subset H_{p,\max}$  and since  $H_{p,\max}$  is closed (see Lemma 5.1), it follows that  $\overline{H_{p,\min}} \subset H_{p,\max}$ . To prove the equality  $\overline{H_{p,\min}} = H_{p,\max}$ , it is enough to show that  $\text{Dom}(H_{p,\max}) \subset \text{Dom}(\overline{H_{p,\min}})$ . Let  $\xi > 0$ , let  $u \in \text{Dom}(H_{p,\max})$ , and consider

$$v := (\overline{H_{p,\min}} + \xi)^{-1}(H_{p,\max} + \xi)u. \quad (31)$$

By Theorem 2.1, the element  $v$  is well-defined, and  $v \in \text{Dom}(\overline{H_{p,\min}})$ .

Since  $\overline{H_{p,\min}} \subset H_{p,\max}$ , from (31) we get

$$(H_{p,\max} + \xi)(v - u) = 0.$$

Since  $H_{p,\max} + \xi$  is an injective operator (see Lemma 5.2), we get  $v = u$ . Therefore,  $u \in \text{Dom}(\overline{H_{p,\min}})$ .  $\square$

## 6. PROOF OF THEOREM 2.3

The following lemma, whose proof is given in Proposition 4.1 of [3], describes an important property of regular graphs. For the case of metrically complete graphs, see [15].

**Lemma 6.1.** *Assume that  $(X, b, m)$  is a locally finite graph with a path metric  $d_\sigma$ . Additionally, assume that  $(X, b, m)$  is regular in the sense of Definition 2.9. Let  $X_\varepsilon$  be as in (12). Then, closed and bounded subsets of  $X_\varepsilon$  are finite.*

By Remark 2.4 and Lemma 3.1(ii),  $\tilde{H}_{W,\Phi}|_{\Gamma_c(X,F)}$  is a symmetric operator in  $\Gamma_{\ell_m^2}(X, F)$ . To prove Theorem 2.3 we follow the method of Theorem 1.5 in [25], which goes back to [5] in the continuous setting. The main ingredient is the following Agmon-type estimate:

**Lemma 6.2.** *Let  $\lambda \in \mathbb{R}$  and let  $v \in \Gamma_{\ell_m^2}(X, F)$  be a weak solution of  $(\tilde{H}_{W,\Phi} - \lambda)v = 0$ . Assume that there exists a constant  $c_1 > 0$  such that, for all  $u \in \Gamma_c(X, F)$*

$$(u, (\tilde{H}_{W,\Phi} - \lambda)u) \geq \frac{1}{2} \sum_{x \in X} \max\left(\frac{1}{D(x)^2}, 1\right) m(x)|u(x)|_{F_x}^2 + c_1 \|u\|^2, \quad (32)$$

where  $D(x)$  is as in (11). Then  $v \equiv 0$ .

*Proof.* Let  $\rho$  be a number such that  $0 < \rho < 1/2$ . For any  $\varepsilon > 0$ , we define  $f_\varepsilon: X \rightarrow \mathbb{R}$  by  $f_\varepsilon(x) = F_\varepsilon(D(x))$ , where  $D(x)$  is as in (11) and  $F_\varepsilon: \mathbb{R}^+ \rightarrow \mathbb{R}$  is given by  $F_\varepsilon(s) = 0$  for  $s \leq \varepsilon$ ;  $F_\varepsilon(s) = (s - \varepsilon)/(\rho - \varepsilon)$  for  $\varepsilon \leq s \leq \rho$ ;  $F_\varepsilon(s) = s$  for  $\rho \leq s \leq 1$ ;  $F_\varepsilon(s) = 1$  for  $s \geq 1$ .

Let us fix a vertex  $x_0$ . For any  $\alpha > 0$ , we define  $g_\alpha: X \rightarrow \mathbb{R}$  by  $g_\alpha(x) = G_\alpha(d_\sigma(x_0, x))$ , where  $G_\alpha: \mathbb{R}^+ \rightarrow \mathbb{R}$  is given by  $G_\alpha(s) = 1$  for  $s \leq 1/\alpha$ ;  $G_\alpha(s) = -\alpha s + 2$  for  $1/\alpha \leq s \leq 2/\alpha$ ;  $G_\alpha(s) = 0$  for  $s \geq 2/\alpha$ . We also define

$$E_{\varepsilon,\alpha} := \{x \in X : \varepsilon \leq D(x) \text{ and } d_\sigma(x_0, x) \leq 2/\alpha\}.$$

By Lemma 6.1 the set  $E_{\varepsilon,\alpha}$  is finite because  $E_{\varepsilon,\alpha}$  is a closed and bounded subset of  $X_\varepsilon$ , where  $X_\varepsilon$  is as in (12). Since the support of  $f_\varepsilon g_\alpha$  is contained in  $E_{\varepsilon,\alpha}$ , it follows that  $f_\varepsilon g_\alpha$  is finitely supported. Using Lemma 4.1 in [2] it is easy to see that  $f_\varepsilon g_\alpha$  is a  $\beta$ -Lipschitz function with

respect to  $d_\sigma$ , where  $\beta = \rho/(\rho - \varepsilon) + \alpha$ . By Lemma 3.3 with  $g$  replaced by  $f_\varepsilon g_\alpha$ , unitarity of  $\tilde{\Phi}_{y,x}$ ,  $\beta$ -Lipschitz property of  $f_\varepsilon g_\alpha$ , and Definition 2.8, we have

$$(f_\varepsilon g_\alpha v, (\tilde{H}_{W,\Phi} - \lambda)(f_\varepsilon g_\alpha v)) \leq \frac{1}{2} \left( \frac{\rho}{\rho - \varepsilon} + \alpha \right)^2 \sum_{x \in X} m(x) |v(x)|_{F_x}^2. \quad (33)$$

On the other hand, by the definitions of  $f_\varepsilon$  and  $g_\alpha$  and the assumption (32) we have

$$(f_\varepsilon g_\alpha v, (\tilde{H}_{W,\Phi} - \lambda)(f_\varepsilon g_\alpha v)) \geq \frac{1}{2} \sum_{x \in S_{\rho,\alpha}} m(x) |v(x)|_{F_x}^2 + c_1 \|f_\varepsilon g_\alpha v\|^2, \quad (34)$$

where

$$S_{\rho,\alpha} := \{x \in X : \rho \leq D(x) \text{ and } d_\sigma(x_0, x) \leq 1/\alpha\}.$$

Combining (34) and (33) we obtain

$$\frac{1}{2} \sum_{x \in S_{\rho,\alpha}} m(x) |v(x)|_{F_x}^2 + c_1 \|f_\varepsilon g_\alpha v\|^2 \leq \frac{1}{2} \left( \frac{\rho}{\rho - \varepsilon} + \alpha \right)^2 \sum_{x \in X} m(x) |v(x)|_{F_x}^2.$$

We fix  $\rho$  and  $\varepsilon$ , and let  $\alpha \rightarrow 0+$ . After that, we let  $\varepsilon \rightarrow 0+$ . Finally, we take the limit as  $\rho \rightarrow 0+$ . As a result, we get  $v \equiv 0$ .  $\square$

**End of the Proof of Theorem 2.3.** Since  $\Delta_{b,m}^{F,\Phi}|_{\Gamma_c(X,F)}$  is a non-negative operator, for all  $u \in \Gamma_c(X, F)$ , we have

$$(u, \tilde{H}_{W,\Phi} u) \geq \sum_{x \in X} m(x) \langle W(x)u(x), u(x) \rangle_{F_x}.$$

Therefore, using assumption (13) we obtain:

$$\begin{aligned} (u, (\tilde{H}_{W,\Phi} - \lambda)u) &\geq \frac{1}{2} \sum_{x \in X} \frac{1}{D(x)^2} m(x) |u(x)|_{F_x}^2 - (\lambda + C) \|u\|^2 \\ &\geq \frac{1}{2} \sum_{x \in X} \max\left(\frac{1}{D(x)^2}, 1\right) m(x) |u(x)|_{F_x}^2 - (\lambda + C + 1/2) \|u\|^2. \end{aligned} \quad (35)$$

Choosing, for example,  $\lambda = -C - 3/2$  in (35) we get the inequality (32) with  $c_1 = 1$ . Thus,  $(\tilde{H}_{W,\Phi} - \lambda)|_{\Gamma_c(X,F)}$  with  $\lambda = -C - 3/2$  is a symmetric operator satisfying  $(u, (\tilde{H}_{W,\Phi} - \lambda)u) \geq \|u\|^2$ , for all  $u \in \Gamma_c(X, F)$ . By Theorem X.26 in [26] we know that the essential self-adjointness of  $(\tilde{H}_{W,\Phi} - \lambda)|_{\Gamma_c(X,F)}$  is equivalent to the following statement: if  $v \in \Gamma_{\ell_m^2}(X, F)$  satisfies  $(\tilde{H}_{W,\Phi} - \lambda)v = 0$ , then  $v = 0$ . Thus, by Lemma 6.2, the operator  $(\tilde{H}_{W,\Phi} - \lambda)|_{\Gamma_c(X,F)}$  is essentially self-adjoint. Thus,  $\tilde{H}_{W,\Phi}|_{\Gamma_c(X,F)}$  is essentially self-adjoint.  $\square$

## APPENDIX

In this section we review some concepts from the theory of one-parameter semigroups of operators on Banach spaces. Our exposition follows Chapters I and II of [7]. A family of bounded linear operators  $(T(t))_{t \geq 0}$  on a Banach space  $\mathcal{X}$  is called a *strongly continuous semigroup* (or  *$C_0$ -semigroup*) if it satisfies the functional equation

$$T(t+s) = T(t)T(s), \quad \text{for all } t, s \geq 0, \quad T(0) = I,$$

and the maps  $t \mapsto T(t)u$  are continuous from  $\mathbb{R}_+$  to  $\mathcal{X}$  for all  $u \in \mathcal{X}$ . Here,  $I$  stands for the identity operator on  $\mathcal{X}$ .

The *generator*  $A: \text{Dom}(A) \subset \mathcal{X} \rightarrow \mathcal{X}$  of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $\mathcal{X}$  is the operator

$$Au := \lim_{h \rightarrow 0^+} \frac{T(h)u - u}{h}$$

defined for every  $u$  in its domain

$$\text{Dom}(A) := \{u \in \mathcal{X} : \lim_{h \rightarrow 0^+} h^{-1}(T(h)u - u) \text{ exists}\}.$$

By Theorem II.1.4 in [7], the generator of a strongly continuous semigroup is a closed and densely defined operator that determines the semigroup uniquely.

A linear operator  $A$  on a Banach space  $\mathcal{X}$  with norm  $\|\cdot\|$  is called *accretive* if

$$\|(\xi + A)u\| \geq \xi\|u\|,$$

for all  $\xi > 0$  and all  $u \in \text{Dom}(A)$ . In the literature on semigroups of operators, the term *dissipative* is used when referring to an operator  $A$  such that  $-A$  is accretive. If  $A$  is a densely defined accretive operator, then  $A$  is closable and its closure  $\overline{A}$  is also accretive; see Proposition II.3.14 in [7].

We now give another description of accretivity. Let  $\mathcal{X}^*$  be the dual space of  $\mathcal{X}$ . By the Hahn-Banach theorem, for every  $u \in \mathcal{X}$  there exists  $u^* \in \mathcal{X}^*$  such that  $\langle u, u^* \rangle = \|u\|^2 = \|u^*\|^2$ , where  $\langle u, u^* \rangle$  denotes the evaluation of the functional  $u^*$  at  $u$ . For every  $u \in \mathcal{X}$ , we define

$$\mathcal{J}(u) := \{u^* \in \mathcal{X}^* : \langle u, u^* \rangle = \|u\|^2 = \|u^*\|^2\}.$$

By Proposition II.3.23 of [7], an operator  $A$  is accretive if and only if for every  $u \in \text{Dom}(A)$  there exists  $j(u) \in \mathcal{J}(u)$  such that

$$\text{Re} \langle Au, j(u) \rangle \geq 0. \tag{R1}$$

An operator  $A$  on a Banach space  $\mathcal{X}$  is called *maximal accretive* if it is accretive and  $\xi + A$  is surjective for all  $\xi > 0$ . There is a connection between maximal accretivity and self-adjointness of operators on Hilbert spaces:  $A$  is a self-adjoint and non-negative operator if and only if  $A$  is symmetric, closed, and maximal accretive; see Problem V.3.32 in [16].

A contraction semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $\mathcal{X}$  is a semigroup such that  $\|T(t)\| \leq 1$  for all  $t \geq 0$ , where  $\|\cdot\|$  denotes the operator norm (of a bounded linear) operator  $\mathcal{X} \rightarrow \mathcal{X}$ .



Generators of strongly continuous contraction semigroups are characterized as follows (Theorem II.3.5 in [7]):

**Hille–Yosida Theorem.** *An operator  $A$  on a Banach space generates a strongly continuous contraction semigroup if and only if the following three conditions are satisfied:*

- (C1)  $A$  is densely defined and closed;
- (C2)  $(0, \infty) \subset \rho(A)$ , where  $\rho(A)$  is the resolvent set of  $A$ ;
- (C3)  $\|(\xi - A)^{-1}\| \leq \xi^{-1}$ , for all  $\xi > 0$ .

Finally, we note that if  $A$  generates a strongly continuous contraction semigroup, then  $-A$  is maximal accretive.

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