Essential self-adjointness of Schrödinger operators on vector bundles over infinite graphs

Ognjen Milatovic, Francoise Truc

ESSENTIAL SELF-ADJOINTNESS OF SCHRÖDINGER OPERATORS ON VECTOR BUNDLES OVER INFINITE GRAPHS

OGNJEN MILATOVIC AND FRANÇOISE TRUC

Abstract. Given a Hermitian vector bundle over an infinite weighted graph, we define the Laplacian associated to a unitary connection on this bundle and study the essential self-adjointness of a perturbation of this Laplacian by an operator-valued potential. Additionally, we give a sufficient condition for the resulting Schrödinger operator to serve as the generator of a strongly continuous contraction semigroup in the corresponding ℓ^p-space.

1. Introduction

In recent years, there has been quite a bit of interest in the study of self-adjoint extensions of the adjacency operator, (magnetic) Laplacians and Schrödinger-type operators on infinite graphs, (see, for instance, [1, 5, 6, 15, 16, 17, 19, 22, 30, 31, 32, 35, 36, 39, 42, 49, 50, 51, 52]) and also in the study of Laplacian in ℓ^p-spaces on infinite graphs. To describe the context of some of those studies, let (X, b, m) be a weighted graph with vertex set X, edge weights b (satisfying the conditions described in section 2.1 below), and vertex weights $m\colon X \to (0, \infty)$. Let $Q^{(c)}$ be the form defined on (complex-valued) finitely supported functions on X,

$$Q^{(c)}(u, v) := \frac{1}{2} \sum_{x,y\in X} b(x, y)(u(x) - u(y))(\overline{v(x) - v(y)}) + \sum_{x\in X} w(x)u(x)\overline{v(x)}, \tag{1}$$

where $w\colon X \to [0, \infty)$. Let $\ell^p_m(X)$ be the space of ℓ^p-summable functions with weight m as in Definition 2.5 below, let $Q^{(D)}$ be the closure of $Q^{(c)}$ in $\ell^2_m(X)$, and let L be the associated self-adjoint operator. Since $Q^{(D)}$ is a Dirichlet form, the semigroup e^{-tL}, $t \geq 0$, extends to a C_0-semigroup on $\ell^p_m(X)$, where $p \in [1, \infty)$. Let L_p denote the generators of these semigroups. The following assumption on (X, b, m) plays an important role in the description of operators L_p:

Assumption (A1) For any sequence $\{x_n\}_{n\in\mathbb{Z}_+}$ of vertices such that $x_n \sim x_{n+1}$ for all $n \in \mathbb{Z}_+$, the following equality holds:

$$\sum_{n\in\mathbb{Z}_+} m(x_n) = \infty.$$

Before describing the operators L_p, we introduce some notations. Let

$$\widetilde{D_s} := \{u\colon X \to \mathbb{C}\colon \sum_{y\in X} b(x, y)|u(y)| < \infty, \text{ for all } x \in X\},$$

2000 *Mathematics Subject Classification.* 35J10, 39A12, 47B25.

let $\Delta_{b,m}$ be the Laplacian operator acting on functions $u \in \widetilde{D_s}$,

$$(\Delta_{b,m}u)(x) := \frac{1}{m(x)} \sum_{y \in X} b(x,y)(u(x) - u(y)), \tag{2}$$

and let $\widetilde{L} := \Delta_{b,m} + w/m$ with domain $\widetilde{D_s}$.

Under the assumption (A1), the following characterization of operators L_p is given in [36, Theorem 5]: for any $p \in [1, \infty)$ the operator L_p is the restriction of $\widetilde{L}$ to

$$\mathrm{Dom}(L_p) = \{u \in \ell^p_m(X) \cap \widetilde{D_s} : \widetilde{L}u \in \ell^p_m(X)\}.$$

Actually, assumption (A1) can be replaced (in the case $w = 0$) by the existence of an intrinsic metric with finite jump size and such that the restriction of the weighted degree

$$\mathrm{Deg}_m(x) := \frac{1}{m(x)} \sum_{y \in X} b(x,y), \qquad x \in X,$$

to every distance ball is bounded, see [30] (for $p = 2$) and [26] (for any p). In analogy with Karp's theorem for Riemannian manifolds, the authors of [26] additionally prove an ℓ^p-Liouville type theorem, which, in the case $p \in (1, \infty)$, generalizes earlier results of [24], [25], [39], and [45]. In the setting of a graph that has uniform subexponential growth with respect to an intrinsic metric with finite jump size, the paper [2] establishes p-independence of the spectrum of the operator L_p described above.

As in the continuous case (Schrödinger operators on Riemannian manifolds), it is natural to extend self-adjointness and spectral problems to operators acting on vector-valued functions or, more generally, sections of vector bundles over graphs. In the last two decades, the notion of vector bundles and connections on graphs has attracted quite a bit of attention, with fruitful applications, such as analysis of large data sets [48] and the theory of molecular bonds [4]. In the case of finite graphs, the author of [38] extends the classical matrix-tree theorem (which relates the determinant of the combinatorial Laplacian to the number of spanning trees) to the context of one- and two-dimensional vector-bundle Laplacians. The vector-bundle Laplacians and Schrödinger-type operators considered in the present paper (see definitions 2.3 and 2.4) are a generalization to infinite graphs of the operator from [38], and our goal is to investigate conditions which ensure maximal accretivity or essential self-adjointness. Let us also mention the paper [20], which establishes a Feynman–Kac-type formula for Schrödinger operators on Hermitian vector bundles over arbitrary weighted graphs.

We close this section with an outline of the organization of our paper. In sections 2.1 and 2.2 we describe the setting of discrete sets and the notion of (Hermitian) vector bundle and connection. The operator theoretic preliminaries are explained in section 2.3. The main results are presented in section 2.4, with some comments. Section 3 contains preliminary results, such as Green's formula, Kato's inequality, and ground state transform. The last part of the paper is devoted to the proofs of the theorems.

2. Setup and main results

2.1. **Weighted graph.** Let X be a countably infinite set. We assume that X is equipped with a measure $m\colon X \to (0,\infty)$. Let $b\colon X \times X \to [0,\infty)$ be a function such that

(i) $b(x,y) = b(y,x)$, for all $x,\, y \in X$;

(ii) $b(x,x) = 0$, for all $x \in X$;

(iii) $\mathrm{Deg}_1(x) := \sum_{y\in V} b(x,y) < \infty$, for all $x \in X$.

Vertices $x,\, y \in X$ with $b(x,y) > 0$ are called *neighbors*, and we denote this relationship by $x \sim y$. We call the triple (X,b,m) a *weighted graph.* We assume that (X,b,m) is connected, that is, for any $x,\, y \in X$ there exists a path γ joining x and y. Here, γ is a sequence $x_1,\, x_2,\, \ldots, x_n \in X$ such that $x = x_1$, $y = x_n$, and $x_j \sim x_{j+1}$ for all $1 \le j \le n-1$.

2.2. **Hermitian vector bundles on graphs and connection.** Following the discussion of [38, Section 3], which concerned finite graphs, we define a Hermitian vector bundle over the graph (X,b,m) described in section 2.1. A family of (finite-dimensional) complex linear spaces $F = (F_x)_{x\in X}$ is called a *complex vector bundle over* X and written $F \to X$, if any two F_x and F_y are isomorphic as complex vector spaces. Then the F_x's are called the *fibers* of $F \to X$, and the complex linear space

$$\Gamma(X,F) := \prod_{x\in X} F_x = \{u|\, u\colon X \to F,\, u(x) \in F_x\}$$

is called the space of *sections in* $F \to X$. We define the space of *finitely supported sections* $\Gamma_c(X,F)$ of the bundle $F \to X$ to be the set of $u \in \Gamma(X,F)$ such that $u(x) = 0$ for all but finitely many $x \in X$.

Definition 2.1. *Let $F \to X$ be a complex vector bundle. An assignment Φ which associates to any $x \sim y$ an isomorphism of complex vector spaces $\Phi_{x,y}\colon F_x \to F_y$ is called a connection on $F \to X$ if*

$$\Phi_{y,x} = (\Phi_{x,y})^{-1} \qquad \textit{for all } x \sim y. \tag{3}$$

We now define Hermitian vector bundles and the corresponding unitary connections on discrete sets:

Definition 2.2. *(i) A family of complex scalar products*

$$\langle\cdot,\cdot\rangle_{F_x}\colon F_x \times F_x \to \mathbb{C}, \quad x \in X,$$

is called a Hermitian structure on the complex vector bundle $F \to X$, and the pair given by $F \to X$ and $\langle\cdot,\cdot\rangle_{F_x}$ is called a Hermitian vector bundle over X.

(ii) Let $F \to X$ be a complex vector bundle with a connection Φ defined on it. Then Φ is called unitary with respect to a Hermitian structure $\langle\cdot,\cdot\rangle_{F_x}$ on $F \to X$ if for all $x \sim y$ one has

$$\Phi^*_{x,y} = \Phi^{-1}_{x,y}, \tag{4}$$

where T^* denotes the Hermitian adjoint of an operator T with respect to $\langle\cdot,\cdot\rangle_{F_x}$.

Definition 2.3. *The* Laplacian $\Delta^{F,\Phi}_{b,m}\colon \widetilde{D}\to\Gamma(X,F)$ *on a Hermitian vector bundle $F\to X$ with a unitary connection Φ is a linear operator with the domain*

$$\widetilde{D} := \{u\in\Gamma(X,F)\colon \sum_{y\in X} b(x,y)|u(y)|_{F_y}<\infty,\ \textit{for all } x\in X\} \tag{5}$$

defined by the formula

$$(\Delta^{F,\Phi}_{b,m}u)(x) = \frac{1}{m(x)}\sum_{y\in X} b(x,y)(u(x)-\Phi_{y,x}u(y)). \tag{6}$$

Remark 2.1. *The operator $\Delta^{F,\Phi}_{b,m}$ is well-defined by the property (iii) of $b(x,y)$, definition (5), and unitarity of Φ.*

Remark 2.2. *In the case $F_x=\{x\}\times\mathbb{C}$ with the canonical Hermitian structure, the sections of the bundle $F\to X$ can be canonically identified with complex-valued functions on X. Under this identification, any connection Φ can be uniquely written as $\Phi_{x,y}=e^{i\theta(y,x)}$, where $\theta\colon X\times X\to[-\pi,\pi]$ is a magnetic potential on (X,b), which, due to (3), satisfies the property $\theta(x,y)=-\theta(y,x)$ for all x, $y\in X$. As a result, we get the magnetic Laplacian operator. In particular, if $\theta\equiv 0$ we get the Laplacian operator (2).*

Remark 2.3. *If the property (iii) of $b(x,y)$ is replaced by*

$$\sharp\{y\in X\colon b(x,y)>0\}<\infty,\ \textit{for all } x\in X,$$

where $\sharp S$ denotes the number of elements in the set S, then the graph (X,b,m) is called locally finite. In this case, we have $\widetilde{D}=\Gamma(X,F)$.

2.3. **Operators and quadratic forms.** Let (X,b,m) be a connected weighted graph and let $F\to X$ be a Hermitian vector bundle with a unitary connection Φ.

Definition 2.4. *We define the Schrödinger-type operator $\widetilde{H}_{W,\Phi}\colon\widetilde{D}\to\Gamma(X,F)$ by the formula*

$$\widetilde{H}_{W,\Phi}u := \Delta^{F,\Phi}_{b,m}u + Wu, \tag{7}$$

where $W(x)\colon F_x\to F_x$ is a linear operator for any $x\in X$, and $\widetilde{D}$ is as in (5).

We give now a description of ℓ^p-spaces.

Definition 2.5. *(i) For any $1\le p<\infty$ we denote by $\Gamma_{\ell^p_m}(X,F)$ the space of sections $u\in\Gamma(X,F)$ such that*

$$\|u\|_p^p := \sum_{x\in X} m(x)|u(x)|^p_{F_x}<\infty, \tag{8}$$

where $|\cdot|_{F_x}$ denotes the norm in F_x corresponding to the Hermitian product $\langle\cdot,\cdot\rangle_{F_x}$. The space of p-summable functions $X\to\mathbb{C}$ with weight m will be denoted by $\ell^p_m(X)$.
(ii) By $\Gamma_{\ell^\infty}(X,F)$ we denote the space of bounded sections of F, equipped with the norm

$$\|u\|_\infty := \sup_{x\in X}|u(x)|_{F_x}. \tag{9}$$

The space of bounded functions on X will be denoted by $\ell^\infty(X)$.

The space $\Gamma_{\ell^2_m}(X,F)$ is a Hilbert space with the inner product

$$(u,v) := \sum_{x\in X} m(x)\langle u(x), v(x)\rangle_{F_x} \tag{10}$$

In what follows, when there is no ambiguity, we will write $\|\cdot\|$ instead of $\|\cdot\|_2$.

Definition 2.6. *Let $1 \le p < +\infty$ and let $\widetilde{D}$ be as in (5). The maximal operator $H_{p,\max}$ is given by the formula $H_{p,\max}u = \widetilde{H}_{W,\Phi}u$ with domain*

$$\mathrm{Dom}(H_{p,\max}) = \{u \in \Gamma_{\ell^p_m}(X,F) \cap \widetilde{D} : \widetilde{H}_{W,\Phi}u \in \Gamma_{\ell^p_m}(X,F)\}. \tag{11}$$

Moreover if

$$\widetilde{H}_{W,\Phi}[\Gamma_c(X,F)] \subseteq \Gamma_{\ell^p_m}(X,F), \tag{12}$$

then we set $H_{p,\min} := \widetilde{H}_{W,\Phi}|_{\Gamma_c(X,F)}$.

Remark 2.4. *Note that under our assumptions on (X,b,m), the inclusion (12) does not necessary hold. It holds if we additionally assume that (X,b,m) is locally finite.*

We now define two quadratic forms and their associated operators, which will be used in the statement of Theorem 2.3. Let $W(x)\colon F_x \to F_x$ be a linear operator satisfying

$$\langle W(x)u(x), v(x)\rangle_{F_x} = \langle u(x), W(x)v(x)\rangle_{F_x} \tag{13}$$

and

$$\langle W(x)u(x), u(x)\rangle_{F_x} \ge 0, \qquad \text{for all } x \in X. \tag{14}$$

We first define a symmetric sesqui-linear form $Q^{(c)}_{W,\Phi}$ in $\Gamma_{\ell^2_m}(X,F)$ with the domain $\mathrm{Dom}(Q^{(c)}_{W,\Phi}) = \Gamma_c(X,F)$:

$$\begin{aligned} Q^{(c)}_{W,\Phi}(u,v) := \frac{1}{2}\sum_{x,y\in X} b(x,y)\langle u(x) - \Phi_{y,x}u(y), v(x) - \Phi_{y,x}v(y)\rangle_{F_x} \\ + \sum_{x\in X} m(x)\langle W(x)u(x), v(x)\rangle_{F_x}. \end{aligned} \tag{15}$$

If W satisfies (14), then $Q^{(c)}_{W,\Phi}$ is closable, and we denote its closure in $\Gamma_{\ell^2_m}(X,F)$ by $Q^{(D)}_{W,\Phi}$. Let $H^{(D)}_{W,\Phi}$ be the self-adjoint operator associated with $Q^{(D)}_{W,\Phi}$. We may think of $H^{(D)}_{W,\Phi}$ as a Schrödinger operator with Dirichlet-type boundary conditions. We define the form $Q^{(N)}_{W,\Phi}$ by the same formula as in (15), with the domain

$$\mathrm{Dom}(Q^{(N)}_{W,\Phi}) := \{u \in \Gamma_{\ell^2_m}(X,F)\colon Q^{(N)}_{W,\Phi}(u,u) < \infty\}.$$

The form $Q^{(N)}_{W,\Phi}$ is symmetric, non-negative and closed. The associated self-adjoint operator in $\Gamma_{\ell^2_m}(X,F)$ will be denoted by $H^{(N)}_{W,\Phi}$ and may be thought of as a Schrödinger operator with Neumann-type boundary conditions.

2.4. **Statement of the results.** In what follows, we often refer to Assumption (A1) specified in section 1, and we denote by $\overline{T}$ the closure of an operator T. We are now ready to state the first result.

Theorem 2.1. *Assume that (X, b, m) is a connected weighted graph, and let $F \to X$ be a Hermitian vector bundle with a unitary connection Φ. Let $W(x)\colon F_x \to F_x$ be a linear operator satisfying*

$$Re\,\langle W(x)u(x), u(x)\rangle_{F_x} \geq 0, \qquad \text{for all } x \in X. \tag{16}$$

Then, the following properties hold:

(i) *Assume that (A1) is satisfied. Let $1 < p < \infty$, and assume that (12) is satisfied. Then $\overline{H_{p,\min}}$ generates a strongly continuous contraction semigroup on $\Gamma_{\ell^p_m}(X, F)$.*

(ii) *Assume that (12) is satisfied for $p = 1$. Additionally, assume that (X, b, m) is stochastically complete. Then $\overline{H_{1,\min}}$ generates a strongly continuous contraction semigroup on $\Gamma_{\ell^1_m}(X, F)$.*

Remark 2.5. *By* [36, Definition 1.1]*, stochastic completeness of (X, b, m) means that there is no non-trivial and non-negative $w \in \ell^\infty(X)$ such that*

$$(\Delta_{b,m} + \alpha)w \leq 0, \qquad \alpha > 0,$$

where $\Delta_{b,m}$ is as in (2).

Remark 2.6. *By the remark preceding Theorem X.49 in* [44]*, the following property holds: if a linear operator T on a Banach space X generates a strongly continuous contraction semigroup, then T is a maximal accretive operator on X. For the definition of an accretive operator on a Banach space, see* [44, Section X.8]*. In particular, under the assumptions of Theorem 2.1, the operator $\overline{H_{p,\min}}$ is maximal accretive for all $1 \leq p < \infty$.*

In the next theorem, we make the following assumption, which is stronger than (12):

$$\widetilde{H}_{W,\Phi}[\Gamma_c(X, F)] \subseteq \Gamma_{\ell^p_m}(X, F) \cap \Gamma_{\ell^{p^*}_m}(X, F), \qquad \text{with} \qquad 1/p + 1/p^* = 1. \tag{17}$$

Remark 2.7. *If (X, b, m) is a locally finite graph then (17) is satisfied. If $\inf_{x\in X} m(x) > 0$ then (A1) and (17) are satisfied.*

Theorem 2.2. *Let (X, b, m) and Φ be as in Theorem 2.1. Assume that the hypotheses (A1) and (16) are satisfied. Then, the following properties hold:*

(i) *Let $1 < p < \infty$, and assume that (17) is satisfied. Then $\overline{H_{p,\min}} = H_{p,\max}$.*

(ii) *Assume that (17) is satisfied for $p = 1$. Additionally, assume that (X, b, m) is stochastically complete. Then $\overline{H_{1,\min}} = H_{1,\max}$.*

The following result is a vector bundle Laplacian analogue of [36, Theorem 6].

Theorem 2.3. *Let (X, b, m) and Φ be as in Theorem 2.1. Assume that W satisfies (13) and (14). Additionally, assume (A1) and $\widetilde{H}_{W,\Phi}[\Gamma_c(X, F)] \subseteq \Gamma_{\ell^2_m}(X, F)$. Then, the following properties hold:*

(i) *the operator* $H_{2,\min}$ *is essentially self-adjoint, and* $\overline{H_{2,\min}} = H_{2,\max} = H^{(D)}_{W,\Phi} = H^{(N)}_{W,\Phi}$*;*

(ii) $Q^{(D)}_{W,\Phi} = Q^{(N)}_{W,\Phi}$.

Before stating the next result, we introduce some notations. In what follows, $d_c(x,y)$ denotes the combinatorial distance between x, $y \in X$, defined as the number of edges in the shortest path connecting the vertices x and y. Fix a vertex $x_0 \in X$ and define $r_c(x) := d_c(x_0, x)$, for all $x \in X$. The n-neighborhood $B^{(c)}_n(x_0)$ of $x_0 \in X$ with respect to d_c is defined as

$$\{x \in X : r_c(x) \le n\} \cup \{\{x,y\} \in E : r_c(x) \le n \text{ and } r_c(y) \le n\}, \tag{18}$$

where $n \in \mathbb{Z}_+$, and E stands for the set of (unoriented) edges of the graph (X, b, m).

In the next result, we assume that (X, b, m) is a locally finite weighted graph satisfying the following condition:

Assumption (A2) Assume that

$$\lim_{n\to\infty} \frac{\delta_n b_n}{n^2} = 0, \tag{19}$$

where

$$\delta_n := \max_{x \in B^{(c)}_n(x_0)} (\deg(x)) \qquad \text{and} \qquad b_n := \max_{x \in B^{(c)}_n(x_0)} \left(\max_{y \sim x} \left(\frac{b(x,y)}{m(x)} \right) \right). \tag{20}$$

Here, $B^{(c)}_n(x_0)$ as in (18), and $\deg(x)$ stands for the degree of $x \in X$, that is, the number of neighbors of $x \in X$.

Theorem 2.4. *Assume that* (X, b, m) *is a locally finite, weighted and connected graph, which satisfies the condition (A2). Let* $F \to X$ *be a Hermitian vector bundle with a unitary connection* Φ*, and let* $W(x) \colon F_x \to F_x$ *be a linear operator satisfying (13). Additionally, assume that there exists a constant* $C \in \mathbb{R}$ *such that*

$$(\widetilde{H}_{W,\Phi} u, u) \ge -C\|u\|^2 \qquad \text{for all } u \in \Gamma_c(X, F), \tag{21}$$

where $(\cdot,\cdot)$ *is as in (10) and* $\|\cdot\|$ *is the corresponding norm. Then* $\widetilde{H}_{W,\Phi}$ *is essentially self-adjoint on* $\Gamma_c(X, F)$*.*

In the remaining results we will use the notion of intrinsic metric.

Definition 2.7. *A pseudo metric (see* [30]*) is a map* $d \colon X \times X \to [0, \infty)$ *such that* $d(x,y) = d(y,x)$*, for all* x, $y \in X$*;* $d(x,x) = 0$*, for all* $x \in X$*; and* $d(x,y)$ *satisfies the triangle inequality. A pseudo metric* $d = d_\sigma$ *is called a path pseudo metric if there exists a map* $\sigma \colon X \times X \to [0, \infty)$ *such that* $\sigma(x,y) = \sigma(y,x)$*, for all* x, $y \in X$*;* $\sigma(x,y) > 0$ *if and only if* $x \sim y$*; and*

$$d_\sigma(x,y) = \inf\{l_\sigma(\gamma) \colon \gamma \text{ path connecting } x \text{ and } y\},$$

where the length l_σ *of the path* $\gamma = (x_0, x_1, \ldots, x_n)$ *is given by*

$$l_\sigma(\gamma) = \sum_{i=0}^{n-1} \sigma(x_i, x_{i+1}). \tag{22}$$

It is known that on a locally finite graph a path pseudo metric is a metric; see [30, Lemma A.3(a)]. The following definitions come from [30]:

Definition 2.8. *(i) A pseudo metric d on (X, b, m) is called intrinsic if*

$$\frac{1}{m(x)} \sum_{y \in X} b(x,y)(d(x,y))^2 \leq 1, \qquad \text{for all } x \in X.$$

(ii) An intrinsic path pseudo metric $d = d_\sigma$ on (X, b, m) is called strongly intrinsic if

$$\frac{1}{m(x)} \sum_{y \in X} b(x,y)(\sigma(x,y))^2 \leq 1, \qquad \text{for all } x \in X.$$

Remark 2.8. *On a locally finite graph (X, b, m), the formula*

$$\sigma_1(x,y) = b(x,y)^{-1/2} \min\left\{ \frac{m(x)}{Deg_1(x)}, \frac{m(y)}{Deg_1(y)} \right\}^{1/2}, \qquad \text{with } x \sim y, \tag{23}$$

where $Deg_1(x)$ is as in property (iii) of $b(x,y)$, defines a strongly intrinsic path metric; see [30, Example 2.1].

Our next result states that Theorem 2.4 remains true without condition (A2) provided (X, d) is metrically complete with respect to an intrinsic metric.

Theorem 2.5. *Assume that (X, b, m) is a locally finite, weighted, and connected graph. Let $F \to X$ be a Hermitian vector bundle with a unitary connection Φ. Let $d = d_\sigma$ be an intrinsic path metric on X such that (X, d) is metrically complete. Let $W(x)\colon F_x \to F_x$ be a linear operator satisfying (13). Additionally, assume that there exists a constant $C \in \mathbb{R}$ such that (21) is satisfied. Then $\widetilde{H}_{W,\Phi}$ is essentially self-adjoint on $\Gamma_c(X, F)$.*

Next results concern operators that are not necessarily semi-bounded from below.

Theorem 2.6. *Assume that (X, b, m) is a locally finite, weighted and connected graph, and let $F \to X$ be a Hermitian vector bundle with a unitary connection Φ. Let d_σ be a strongly intrinsic path metric on X. Let $q\colon X \to [1, \infty)$ be a function satisfying*

$$|q^{-1/2}(x) - q^{-1/2}(y)| \leq K\sigma(x,y), \tag{24}$$

for all $x, y \in X$ such that $x \sim y$, where K is a constant. Let $W(x)\colon F_x \to F_x$ be a linear operator satisfying (13). Additionally, assume that there exists $\varepsilon \in [0, 1)$ such that

$$\varepsilon(\Delta_{b,m}^{F,\Phi} u, u) + (Wu, u) \geq -(qu, u), \quad \text{for all } u \in \Gamma_c(X, F), \tag{25}$$

where $(\cdot, \cdot)$ is as in (10). Let

$$\sigma_q(x,y) = \min\{q^{-1/2}(x), q^{-1/2}(y)\} \cdot \sigma(x,y) \tag{26}$$

and let d_{σ_q} be the path metric corresponding to σ_q. Assume that (X, d_{σ_q}) is metrically complete. Then $\widetilde{H}_{W,\Phi}$ is essentially self-adjoint on $\Gamma_c(X, F)$.

In the next (and final) result, we will use the notion of a *regular graph* introduced in [7]. Let us first recall the definition of the boundary of a given set $A \subseteq X$:

$$\partial A := \{x \in A \colon \text{there exists } y \in X \backslash A \text{ such that } y \sim x\}.$$

In the sequel, we denote by $(\widehat{X}, \widehat{d})$ the metric completion of (X, d), and we define the *Cauchy boundary* X_∞ as follows: $X_\infty := \widehat{X} \backslash X$. Note that (X, d) is metrically complete if and only if X_∞ is empty. For a path metric $d = d_\sigma$ on X and $x \in X$, we set

$$D(x) := \inf_{z \in X_\infty} \widehat{d}_\sigma(x, z). \tag{27}$$

Definition 2.9. *Let (X, b, m) be a graph with a path metric d_σ. Let $\varepsilon > 0$ be given and let*

$$X_\varepsilon := \{x \in X \colon D(x) \geq \varepsilon\}, \tag{28}$$

where $D(x)$ is as in (27). We say that a graph (X, b, m) with a path metric d_σ is regular *if for any sufficiently small ε, any bounded subset of ∂X_ε (for the metric d_σ) is finite.*

Remark 2.9. *Definition 2.9 covers a broad class of graphs. For instance, weighted graphs whose first Betti number is finite are regular; see* [7, Corollary 4.1]. *In particular, any weighted tree is regular; see* [7, Proposition 4.2].

We are ready to state our last theorem.

Theorem 2.7. *Assume that (X, b, m) is a locally finite, weighted, and connected graph. Let $d = d_\sigma$ be an intrinsic path metric on X such that (X, d) is not metrically complete. Assume that (X, b, m) is a regular graph in the sense of Definition 2.9. Let $F \to X$ be a Hermitian vector bundle with a unitary connection Φ and let $W(x) \colon F_x \to F_x$ be a linear operator satisfying (13). Additionally, assume that there exists a constant C such that*

$$\langle W(x)u(x), u(x)\rangle_{F_x} \geq \left(\frac{1}{2(D(x))^2} - C\right) |u(x)|^2_{F_x}, \tag{29}$$

for all $x \in X$ and all $u \in \Gamma_c(X, F)$, where $D(x)$ is as in (27). Then $\widetilde{H}_{W,\Phi}$ is essentially self-adjoint on $\Gamma_c(X, F)$.

2.5. **Comments on the existing literature.** In relation to Theorems 2.1 and 2.2, we should mention that Kato [34, Section A] gave sufficient conditions under which the closure of the minimal operator corresponding to $-\Delta + W$, where Δ is the standard Laplacian on $\mathbb{R}^n$ and $0 \leq W \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, generates a strongly continuous contractive semigroup in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Furthermore, under the conditions of [34, Section A], one has the equality of the minimal and maximal operators corresponding to $-\Delta + W$. The paper [41] contains an extension of the results in [34, Section A] to the context of magnetic Schrödinger operators in $L^p(M)$, $1 < p < \infty$, where M is a manifold of bounded geometry. With regard to Theorem 2.4, we should point out that the condition (A2) comes from the paper [39], which studied the essential self-adjointness of the Laplacian on 1-forms in the context of a locally finite graph. The authors of [1] generalize the self-adjointness result of [39] by replacing the hypothesis (A2) with weaker requirements of "completeness" and "homogeneity" of the graph; see [1, Section

3.3.1]. Theorem 2.5 is an extension of [19, Theorem 2.10(b)], which concerned the essential self-adjointness of a semibounded from below magnetic Schrödinger operator on a locally finite graph with a complete intrinsic metric. Theorem 2.6 generalizes [42, Theorem 1.10], which was proven for the magnetic Schrödinger operator satisfying the condition (25) with $\varepsilon = 0$. We should note that the condition (25) goes back to [46]. Theorem 2.7 is an extension of [42, Theorem 1.5], which was proven for the magnetic Schrödinger operator on a regular graph in the sense of Definition 2.9. In the context of the Schrödinger operator on a regular graph of bounded degree, a result of this kind was proven in [7, Theorem 4.2]. Let us point out that Theorem 1.8 of [42] could be extended in a similar way.

Recently, several researchers have come up with the concept of an intrinsic metric on a graph. The definition used in the present paper goes back to [14]. For applications of intrinsic metrics in different settings, see, for instance, [2, 3, 11, 12, 13, 18, 19, 23, 26, 28, 29, 30, 40]. For a proof of a Hopf–Rinow-type theorem for locally finite weighted graphs with a path metric, see [30]. For a proof of Feynman–Kac–Itô formula for magnetic Schrödinger operators with a general class of potentials on arbitrary weighted graphs see [19].

3. Preliminary lemmas

3.1. **Green's formula.** We begin with a variant of Green's formula, which is analogous to [19, Lemma 2.1] and [21, Lemma 4.7] concerning Schrödinger operators with magnetic potential and without magnetic potential, respectively.

Notation 3.1. *Assume that (X, b, m) is a connected weighted graph, and let $F \to X$ be a Hermitian vector bundle with a unitary connection Φ. Let $W(x)\colon F_x \to F_x$ be a linear operator. We denote by W^* the Hermitian adjoint of W, that is, $(W(x))^*$ is the Hermitian adjoint of $W(x)$ with respect to $\langle\cdot,\cdot\rangle_{F_x}$.*

Lemma 3.1. *Let (X, b, m), Φ, F, and W be defined as in Notation 3.1, and let $\widetilde{H}_{W,\Phi}$ be as in (7). Then, the following properties hold:*

(i) *if $\widetilde{H}_{W,\Phi}[\Gamma_c(X,F)] \subseteq \Gamma_{\ell^p_m}(X,F)$ for some $1 \le p \le \infty$, then any $u \in \Gamma_{\ell^{p^*}_m}(X,F)$ with $1/p + 1/p^* = 1$ belongs to the set $\widetilde{D}$ defined by (5);*

(ii) *for all $u \in \widetilde{D}$ and all $v \in \Gamma_c(X,F)$, the sums*

$$\sum_{x\in X} m(x)\langle \widetilde{H}_{W,\Phi}u, v\rangle_{F_x}, \qquad \sum_{x\in X} m(x)\langle u, \widetilde{H}_{W^*,\Phi}v\rangle_{F_x},$$

and the expression

$$\frac{1}{2}\sum_{x,y\in X} b(x,y)\langle u(x) - \Phi_{y,x}u(y), v(x) - \Phi_{y,x}v(y)\rangle_{F_x} + \sum_{x\in V} m(x)\langle W(x)u(x), v(x)\rangle_{F_x} \tag{30}$$

converge absolutely and agree;

(iii) *if $\widetilde{H}_{W,\Phi}[\Gamma_c(X,F)] \subseteq \Gamma_{\ell^2_m}(X,F)$ and W satisfies (13), then*

$$Q^{(c)}_{W,\Phi}(u,v) = (\widetilde{H}_{W,\Phi}u, v) = (u, \widetilde{H}_{W,\Phi}v), \qquad \text{for all } u,\, v \in \Gamma_c(X,F), \tag{31}$$

where $Q^{(c)}_{W,\Phi}$ is as in (15) and $(\cdot,\cdot)$ is as in (10).

Proof. To make the notations simpler, throughout the proof we suppress F_x in $|\cdot|_{F_x}$. From the assumption $\widetilde{H}_{W,\Phi}[\Gamma_c(X,F)] \subseteq \Gamma_{\ell^p_m}(X,F)$, it is easily seen that the function $y \mapsto b(x,y)/m(y)$ belongs to $\ell^p_m(X)$, for all $x \in X$. In the case $1 < p^* < \infty$, for all $u \in \Gamma_{\ell^{p^*}_m}(X,F)$, by Hölder's inequality with $1/p + 1/p^* = 1$ we have

$$\sum_{y\in X} b(x,y)|u(y)| \leq \left(\sum_{y\in X}\left(\frac{b(x,y)}{m(y)}\right)^p m(y)\right)^{1/p}\left(\sum_{y\in X}|u(y)|^{p^*}m(y)\right)^{1/p^*}. \tag{32}$$

In the case $p^* = 1$, for all $u \in \Gamma_{\ell^1_m}(X,F)$, by Hölder's inequality with $p = \infty$ and $p^* = 1$ we have

$$\sum_{y\in X} b(x,y)|u(y)| \leq \sup_{y\in X}\left(\frac{b(x,y)}{m(y)}\right)\left(\sum_{y\in X}|u(y)|m(y)\right). \tag{33}$$

In the case $p^* = \infty$, for all $u \in \Gamma_{\ell^\infty}(X,F)$, by Hölder's inequality with $p = 1$ and $p^* = \infty$ we have

$$\sum_{y\in X} b(x,y)|u(y)| \leq \sup_{y\in X}(|u(y)|)\left(\sum_{y\in X} b(x,y)\right). \tag{34}$$

This concludes the proof of property (i). Let us prove property (ii). Since $v \in \Gamma_c(X,F)$, the first two sums are performed over finitely many $x \in X$. We now turn to the sum (30). By Cauchy–Schwarz inequality and unitarity of $\Phi_{y,x}$ we get

$$\begin{aligned}\sum_{x,y\in X}|b(x,y)\langle u(x),\Phi_{y,x}v(y)\rangle_{F_x}| &\leq \sum_{x,y\in X} b(x,y)|u(x)||v(y)| \\ = \sum_{y\in X}|v(y)|\left(\sum_{x\in X} b(x,y)|u(x)|\right) &< \infty,\end{aligned}$$

where the convergence follows from the fact that $u \in \widetilde{D}$ and $v \in \Gamma_c(X,F)$.

Similarly,

$$\begin{aligned}\sum_{x,y\in X}|b(x,y)\langle u(x),v(x)\rangle_{F_x}| &\leq \sum_{x,y\in X} b(x,y)|u(x)||v(x)| \\ = \sum_{x\in X}|u(x)||v(x)|\left(\sum_{y\in X} b(x,y)\right) &< \infty,\end{aligned}$$

where the convergence follows by property (iii) of $b(x,y)$ and since $v \in \Gamma_c(X,F)$. Hence, the three sums converge absolutely. The equality of sums follows directly from Fubini's theorem. This shows property (ii). Finally, (31) follows from the equality of three sums and the definition of $Q^{(c)}_{W,\Phi}$. □

In the next Lemma, T^* denotes the adjoint of operator T.

Lemma 3.2. *Let (X, b, m), Φ and F be defined as in Notation 3.1. Assume that $W(x)\colon F_x \to F_x$ satisfies (13), and that $\widetilde{H}_{W,\Phi}[\Gamma_c(X,F)] \subseteq \Gamma_{\ell^2_m}(X,F)$. Let $H_{2,\min}$ and $H_{2,\max}$ be as in Definition 2.6 with $p = 2$. Then, $H_{2,\min}$ is a symmetric operator in $\Gamma_{\ell^2_m}(X,F)$ and $(H_{2,\min})^* = H_{2,\max}$.*

Proof. The symmetry of $H_{2,\min}$ follows by Lemma 3.1(iii). The inclusion $\Gamma_{\ell^2_m}(X,F) \subseteq \widetilde{D}$, where $\widetilde{D}$ is as in (5), follows by Lemma 3.1(i). Thus, by (11) with $p = 2$ we have

$$\mathrm{Dom}(H_{2,\max}) = \{u \in \Gamma_{\ell^2_m}(X,F)\colon \widetilde{H}_{W,\Phi}u \in \Gamma_{\ell^2_m}(X,F)\}.$$

Using Lemma 3.1(ii) we get

$$(u, \widetilde{H}_{W,\Phi}v) = (\widetilde{H}_{W,\Phi}u, v), \qquad \text{for all } u \in \mathrm{Dom}(H_{2,\max}) \text{ and } v \in \Gamma_c(X,F),$$

from which $(H_{2,\min})^* = H_{2,\max}$ follows easily. □

3.2. **Kato's inequality.** The version of Kato's inequality used in the present paper is analogous to that of [9], which was proven in the case of magnetic Laplacian (acting on functions) with $m(x) \equiv 1$ and $b(x,y) \equiv 1$ whenever $x \sim y$.

Lemma 3.3. *Let $\Delta_{b,m}$ and $\Delta^{F,\Phi}_{b,m}$ be defined as in (2) and (6) respectively. Then, the following pointwise inequality holds for all $u \in \widetilde{D}$:*

$$|u|(\Delta_{b,m}|u|) \le Re\,\langle \Delta^{F,\Phi}_{b,m}u, u\rangle_{F_x}, \tag{35}$$

where $|\cdot|$ denotes the norm in F_x, and $Re\,z$ denotes the real part of a complex number z.

Proof. Using (2), (6), and the unitarity of $\Phi_{y,x}$, we obtain

$$\begin{aligned}
&|u(x)|((\Delta_{b,m}|u|)(x)) - \mathrm{Re}\,\langle \Delta^{F,\Phi}_{b,m}u(x), u(x)\rangle_{F_x} \\
&= \frac{1}{m(x)}\sum_{y\in X} b(x,y)\,[\mathrm{Re}\,\langle \Phi_{y,x}u(y), u(x)\rangle_{F_x} - |u(x)||u(y)|] \le 0,
\end{aligned}$$

and the lemma is proven. □

3.3. **Ground state transform.** We will use an analogue of the"ground state transform" given in [19, Proposition 2.12] and [21, Proposition 3.2].

Lemma 3.4. *Let (X, b, m), Φ, and F be defined as in Notation 3.1. Assume that $W(x)\colon F_x \to F_x$ satisfies (13). Let $\lambda \in \mathbb{R}$, and let $u \in \widetilde{D}$ so that*

$$(\widetilde{H}_{W,\Phi} - \lambda)u = 0. \tag{36}$$

Then, for all finitely supported functions $g\colon X \to \mathbb{R}$, we have

$$\begin{aligned}
&Q^{(c)}_{W,\Phi}(gu, gu) = \lambda\|gu\|^2 \\
&+ \frac{1}{2}\sum_{x,y\in X} b(x,y)(g(x)-g(y))^2(Re\,\langle u(x), \Phi_{y,x}u(y)\rangle_{F_x}),
\end{aligned} \tag{37}$$

where $Q^{(c)}_{W,\Phi}(\cdot,\cdot)$ is as in (15).

Proof. By (36) and (7) we get

$$\lambda\|gu\|^2 = \lambda(u, g^2u) = \sum_{x\in X} m(x)\langle(\widetilde{H}_{W,\Phi}u)(x), g^2(x)u(x)\rangle_{F_x}$$
$$= \sum_{x,y\in X} b(x,y)\langle u(x) - \Phi_{y,x}u(y), g^2(x)u(x)\rangle_{F_x} + (Wgu, gu). \tag{38}$$

Note that the first quantity on the right hand side of the last equality in (38) is real-valued because $\|gu\|^2$ and (Wgu, gu) are real-valued. Let $Q^{(c)}_{0,\Phi}(\cdot,\cdot)$ be as in (15) with $W = 0$. Using the property (i) of $b(x, y)$, unitarity of $\Phi_{x,y}$, and (3) we have

$$\sum_{x,y\in X} b(x,y)\langle u(x) - \Phi_{y,x}u(y), g^2(x)u(x)\rangle_{F_x}$$
$$= \frac{1}{2}\sum_{x,y\in X} b(x,y)\langle u(x) - \Phi_{y,x}u(y), g^2(x)u(x)\rangle_{F_x}$$
$$+ \frac{1}{2}\sum_{x,y\in X} b(x,y)\langle u(y) - \Phi_{x,y}u(x), g^2(y)u(y)\rangle_{F_y}$$
$$= \frac{1}{2}\sum_{x,y\in X} b(x,y)\langle u(x) - \Phi_{y,x}u(y), g^2(x)u(x) - \Phi_{y,x}(g^2(y)u(y))\rangle_{F_x}$$
$$= Q^{(c)}_{0,\Phi}(gu, gu) + \frac{1}{2}\sum_{x,y\in X} b(x,y)g(x)(g(y) - g(x))\langle\Phi_{y,x}u(y), u(x)\rangle_{F_x}$$
$$+ \frac{1}{2}\sum_{x,y\in X} b(x,y)g(y)(g(x) - g(y))\langle u(x), \Phi_{y,x}u(y)\rangle_{F_x}, \tag{39}$$

where the last equality follows by direct computation. Taking the real parts in (39) and using the observation that the leftmost side of (39) is real-valued, and noting that

$$\mathrm{Re}\,\langle u(x), \Phi_{y,x}u(y)\rangle_{F_x} = \mathrm{Re}\,\langle\Phi_{y,x}u(y), u(x)\rangle_{F_x},$$

we get

$$\sum_{x,y\in X} b(x,y)\langle u(x) - \Phi_{y,x}u(y), g^2(x)u(x)\rangle_{F_x}$$
$$= Q^{(c)}_{0,\Phi}(gu, gu) - \frac{1}{2}\sum_{x,y\in X} b(x,y)(g(x) - g(y))^2(\mathrm{Re}\,\langle u(x), \Phi_{y,x}u(y)\rangle_{F_x}). \tag{40}$$

Combining (38) and (40) we get (37). □

3.4. **Cut-off functions.** In this section, we define cut-off functions used in several proofs below. Assume that (X, b, m) is locally finite, and let d_σ be a path metric. Fix $x_0 \in X$ and define

$$\chi_n(x) := \left(\left(\frac{2n - d_\sigma(x_0, x)}{n}\right) \vee 0\right) \wedge 1, \qquad x \in X, \quad n \in \mathbb{Z}_+. \tag{41}$$

Denote

$$B^\sigma_n(x_0) := \{x \in X : d_\sigma(x_0, x) \le n\}. \tag{42}$$

The sequence $\{\chi_n\}_{n\in\mathbb{Z}_+}$ satisfies the following properties, which are checked easily: (i) $0 \le \chi_n(x) \le 1$, for all $x \in X$; (ii) $\chi_n(x) = 1$ for $x \in B_n^\sigma(x_0)$ and $\chi_n(x) = 0$ for $x \notin B_{2n}^\sigma(x_0)$; (iii) for all $x \in X$, we have $\lim_{n\to\infty} \chi_n(x) = 1$; and (iv) the functions χ_n satisfy the inequality

$$|\chi_n(x) - \chi_n(y)| \le \frac{d_\sigma(x,y)}{n}, \qquad \text{for all } x \sim y. \tag{43}$$

4. Proof of Theorem 2.1

We begin by recalling an abstract fact about generators of strongly continuous contraction semigroups. Let T be a linear operator on a Banach space $\mathcal{X}$ and let $\rho(T)$ denote the resolvent set of T. By Hille–Yosida Theorem (see [10, Theorem II.3.5]), the operator T generates a strongly continuous contraction semigroup on $\mathcal{X}$ if and only if the following three conditions are satisfied:

(C1) T is densely defined and closed;
(C2) $(-\infty, 0) \subset \rho(T)$;
(C3) $\|(\gamma + T)^{-1}\| \le \gamma^{-1}$, for all $\gamma > 0$,

where $\|\cdot\|$ denotes the operator norm (of a bounded linear operator $\mathcal{X} \to \mathcal{X}$).

In Lemmas 4.1 and 4.3 below, we assume that the hypotheses of Theorem 2.1 are satisfied.

Lemma 4.1. *Let $1 \le p < \infty$. Then, the operator $H_{p,\min}$ satisfies the following inequality:*

$$\mathrm{Re} \sum_{x\in X} m(x)\langle (H_{p,\min}u)(x), u(x)|u(x)|^{p-2}\rangle_{F_x} \ge 0, \qquad \text{for all } u \in \Gamma_c(X,F). \tag{44}$$

Proof. By Lemma 3.1, for all u, $v \in \Gamma_c(X,F)$ we have

$$\sum_{x\in X} m(x)\langle (\Delta_{b,m}^{F,\Phi}u)(x), v(x)\rangle_{F_x} = Q_{0,\Phi}^{(c)}(u,v), \tag{45}$$

where $Q_{0,\Phi}^{(c)}$ is as in (15) with $W = 0$. Using (45) with $u \in \Gamma_c(X,F)$ and $v := u|u|^{p-2}$, we have

$$\begin{aligned}
&\mathrm{Re} \sum_{x\in X} m(x)\langle (\Delta_{b,m}^{F,\Phi}u)(x), u(x)|u(x)|^{p-2}\rangle_{F_x} = \frac{1}{2}\sum_{x,y\in X} b(x,y)\,[|u(x)|^p + |u(y)|^p \\
&-\mathrm{Re}\,\langle \Phi_{y,x}u(y), u(x)|u(x)|^{p-2}\rangle_{F_x} - \mathrm{Re}\,\langle \Phi_{x,y}u(x), u(y)|u(y)|^{p-2}\rangle_{F_y}] \\
&\ge \frac{1}{2}\sum_{x,y\in X} b(x,y)\left[|u(x)|^p + |u(y)|^p - |u(x)||u(y)|^{p-1} - |u(y)||u(x)|^{p-1}\right].
\end{aligned} \tag{46}$$

For $p = 1$, from (46) and the assumption (16) we easily get (44).

Let $1 < p < \infty$ and let p^* satisfy $1/p + 1/p^* = 1$. By Young's inequality we have

$$|u(x)||u(y)|^{p-1} \le \frac{|u(x)|^p}{p} + \frac{(|u(y)|^{p-1})^{p^*}}{p^*} = \frac{|u(x)|^p}{p} + \frac{(p-1)|u(y)|^p}{p} \tag{47}$$

and, likewise,

$$|u(y)||u(x)|^{p-1} \le \frac{|u(y)|^p}{p} + \frac{(p-1)|u(x)|^p}{p}. \tag{48}$$

From (47) and (48) we get

$$-|u(x)||u(y)|^{p-1} - |u(y)||u(x)|^{p-1} \geq -|u(x)|^p - |u(y)|^p. \tag{49}$$

Using (49), (46), and the assumption (16), we obtain (44). □

The following lemma is a special case of [35, Proposition 8]:

Lemma 4.2. *Assume (A1). Let $\alpha > 0$ and $1 \leq p < \infty$. Let $\Delta_{b,m}$ be as in (2). Assume that $u \in \ell^p_m(X)$ is a real-valued function satisfying the inequality $(\Delta_{b,m} + \alpha)u \geq 0$. Then $u \geq 0$.*

Remark 4.1. *The case $p = \infty$ is more complicated and involves the notion of stochastic completeness; see, for instance,* [12], [27], [35], [36], [37], [51], *and* [52].

In the remainder of this section and in section 5, we will use certain arguments of [34, Part A] in our setting. In the sequel, Ran T denotes the range of an operator T.

Lemma 4.3. *Let $1 < p < \infty$ and let $\lambda \in \mathbb{C}$ with Re $\lambda > 0$. Then, Ran $(H_{p,\min} + \lambda)$ is dense in $\ell^p_m(X)$.*

Proof. Let $u \in (\Gamma_{\ell^p_m}(X, F))^* = \Gamma_{\ell^{p^*}_m}(X, F)$, be a continuous linear functional that annihilates $(\lambda + H_{p,\min})\Gamma_c(X, F)$:

$$\sum_{x \in X} m(x)\langle(\lambda + H_{p,\min})v(x), u(x)\rangle_{F_x} = 0, \qquad \text{for all } v \in \Gamma_c(X, F). \tag{50}$$

By assumption (12) we know that $\widetilde{H}_{W,\Phi}v \in \Gamma_{\ell^p_m}(X, F)$. Since $u \in \Gamma_{\ell^{p^*}_m}(X, F)$, we may use Lemma 3.1 (iii) in (50). As a result, we get

$$\sum_{x \in X} m(x)\langle v(x), (\overline{\lambda} + \widetilde{H}_{W^*,\Phi})u(x)\rangle_{F_x} = 0, \qquad \text{for all } v \in \Gamma_c(X, F),$$

where $\overline{\lambda}$ is the complex conjugate of λ. The last equality leads to

$$(\bar{\lambda} + \Delta^{F,\Phi}_{b,m} + W^*)u = 0. \tag{51}$$

Using Kato's inequality (35), assumption (16), and (51) we have

$$\begin{aligned} |u|(\Delta_{b,m}|u|) &\leq \mathrm{Re}\,\langle \Delta^{F,\Phi}_{b,m}u, u\rangle_{F_x} \\ &= -(\mathrm{Re}\ \lambda)|u|^2 - \mathrm{Re}\,\langle W^*u, u\rangle_{F_x} \leq -(\mathrm{Re}\ \lambda)|u|^2, \end{aligned}$$

where $|u| \in \ell^{p^*}_m(X)$ with $1 < p^* < \infty$. Rewriting the last inequality, we obtain

$$|u|(\Delta_{b,m}|u| + (\mathrm{Re}\,\lambda)|u|) \leq 0.$$

For all $x \in X$ such that $u(x) \neq 0$, we may divide both sides of the last inequality by $|u(x)|$ to get

$$(\Delta_{b,m} + \mathrm{Re}\ \lambda)|u| \leq 0. \tag{52}$$

Note that the inequality (52) also holds for those $x \in X$ such that $u(x) = 0$; in this case, the left hand side of (52) is non-positive by (2). Thus, the inequality (52) holds for all $x \in X$. By Lemma 4.2, from (52) we get $|u| \leq 0$. Hence, $u = 0$. □

End of the proof of Theorem 2.1(i). By the definition of accretivity for an operator in Banach space (see, for instance, [44, Section X.8]), the inequality (44) means that the operator $H_{p,\min}$ is accretive in $\Gamma_{\ell^p_m}(X,F)$. By an abstract fact (see the remark preceding Theorem X.48 in [44]), the operator $H_{p,\min}$ is closable and $\overline{H_{p,\min}}$ is accretive in $\Gamma_{\ell^p_m}(X,F)$. Thus, the following inequality holds:

$$\mathrm{Re}\sum_{x\in X} m(x)\langle (H_{p,\min}u)(x), u(x)|u(x)|^{p-2}\rangle_{F_x} \geq 0 \qquad \text{for all } u\in \mathrm{Dom}(\overline{H_{p,\min}}). \tag{53}$$

Let $\lambda\in\mathbb{C}$ with $\mathrm{Re}\,\lambda>0$. Using Hölder's inequality, from (53) we get

$$(\mathrm{Re}\,\lambda)\|u\|_p \leq \|(\lambda+\overline{H_{p,\min}})u\|_p, \qquad \text{for all } u\in \mathrm{Dom}(\overline{H_{p,\min}}). \tag{54}$$

By Lemma 4.3 we know that Ran $(H_{p,\min}+\lambda)$ is dense in $\Gamma_{\ell^p_m}(X,F)$. This, together with (54) shows that Ran $(\overline{H_{p,\min}}+\lambda)=\Gamma_{\ell^p_m}(X,F)$. Hence, from (54) we get

$$\|(\gamma+\overline{H_{p,\min}})^{-1}\| \leq \frac{1}{\gamma}, \qquad \text{for all } \gamma>0,$$

where $\|\cdot\|$ is the operator norm $\Gamma_{\ell^p_m}(X,F)\to\Gamma_{\ell^p_m}(X,F)$. Thus, conditions (C1), (C2) and (C3) of Hille–Yosida Theorem are satisfied. Hence, the operator $\overline{H_{p,\min}}$ is the generator of a contraction semigroup on $\Gamma_{\ell^p_m}(X,F)$. □

Proof of Theorem 2.1(ii). Repeating the proof of Lemma 4.3 in the case $p=1$ and using Remark 2.5, from (52) with $u\in\Gamma_{\ell^\infty}(X,F)$ we obtain $|u|=0$. Therefore, for all $\lambda\in\mathbb{C}$ with $\mathrm{Re}\,\lambda>0$, the set Ran $(H_{1,\min}+\lambda)$ is dense in $\Gamma_{\ell^1_m}(X,F)$. From here on, we may repeat the proof of Theorem 2.1(i). □

5. Proof of Theorem 2.2

We begin with the following lemma.

Lemma 5.1. *Let $1\leq p<\infty$ and $1/p+1/p^*=1$. Assume that (17) is satisfied. Then $H_{p,\max}$ is a closed operator.*

Proof. Let u_k be a sequence in $\mathrm{Dom}(H_{p,\max})$ such that $u_k\to u$ and $H_{p,\max}u_k\to f$, as $k\to\infty$, using the norm convergence in $\Gamma_{\ell^p_m}(X,F)$. We need to show that $u\in\mathrm{Dom}(H_{p,\max})$ and $f=H_{p,\max}u$. Let $v\in\Gamma_c(X,F)$ be arbitrary, and consider the sum

$$\sum_{x\in X} m(x)\langle (H_{p,\max}u_k)(x), v(x)\rangle_{F_x} = \sum_{x\in X} m(x)\langle (\widetilde{H}_{W,\Phi}u_k)(x), v(x)\rangle_{F_x}.$$

Since $\widetilde{H}_{W,\Phi}[\Gamma_c(X,F)]\subseteq\Gamma_{\ell^{p^*}_m}(X,F)$, we may use Lemma 3.1 to get

$$\sum_{x\in X} m(x)\langle (\widetilde{H}_{W,\Phi}u_k)(x), v(x)\rangle_{F_x} = \sum_{x\in X} m(x)\langle u_k(x), (\widetilde{H}_{W^*,\Phi}v)(x)\rangle_{F_x}. \tag{55}$$

Using the norm convergence $u_k \to u$ in $\Gamma_{\ell^p_m}(X,F)$ and the assumption $\widetilde{H}_{W,\Phi}v \in \Gamma_{\ell^{p^*}_m}(X,F)$ with $1/p + 1/p^* = 1$, by Hölder's inequality we get

$$\sum_{x\in X} m(x)\langle u_k(x), (\widetilde{H}_{W^*,\Phi}v)(x)\rangle_{F_x} \to \sum_{x\in X} m(x)\langle u(x), (\widetilde{H}_{W^*,\Phi}v)(x)\rangle_{F_x}.$$

Using the norm convergence $\widetilde{H}_{W,\Phi}u_k \to f$ in $\Gamma_{\ell^p_m}(X,F)$, by Hölder's inequality we get

$$\sum_{x\in X} m(x)\langle (\widetilde{H}_{W,\Phi}u_k)(x), v(x)\rangle_{F_x} \to \sum_{x\in X} m(x)\langle f(x), v(x)\rangle_{F_x}.$$

Therefore, taking the limit as $k \to \infty$ on both sides of (55), we obtain

$$\sum_{x\in X} m(x)\langle u(x), (\widetilde{H}_{W^*,\Phi}v)(x)\rangle_{F_x} = \sum_{x\in X} m(x)\langle f(x), v(x)\rangle_{F_x}. \tag{56}$$

Since $u \in \Gamma_{\ell^p_m}(X,F)$ and since $\widetilde{H}_{W,\Phi}[\Gamma_c(X,F)] \subseteq \Gamma_{\ell^{p^*}_m}(X,F)$, we may use Lemma 3.1 to rewrite the left-hand side of (56) as follows:

$$\sum_{x\in X} m(x)\langle u(x), (\widetilde{H}_{W^*,\Phi}v)(x)\rangle_{F_x} = \sum_{x\in X} m(x)\langle (\widetilde{H}_{W,\Phi}u)(x), v(x)\rangle_{F_x}. \tag{57}$$

Since $v \in \Gamma_c(X,F)$ is arbitrary, by (56) and (57) we get $\widetilde{H}_{W,\Phi}u = f$. Additionally, since $u \in \Gamma_{\ell^p_m}(X,F)$, by Lemma 3.1 we get $u \in \widetilde{D}$, where $\widetilde{D}$ is as in (5). Thus, $u \in \mathrm{Dom}(H_{p,\max})$ and $H_{p,\max}u = f$. Therefore, $H_{p,\max}$ is a closed operator. □

Maximal operator associated with $\Delta_{b,m}$. Let $1 \le p < \infty$ and let $\Delta_{b,m}$ be as in (2). We define the maximal operator $L_{p,\max}$ in $\ell^p_m(X)$ by the formula $L_{p,\max}u = \Delta_{b,m}u$ with the domain

$$\mathrm{Dom}(L_{p,\max}) = \{u \in \ell^p_m(X) \cap \widetilde{D} : \Delta_{b,m}u \in \ell^p_m(X)\}, \tag{58}$$

where $\widetilde{D}$ is as in (5) and sections are replaced by functions $X \to \mathbb{C}$.

Under the assumption (A1), it is known that $L_{p,\max}$ generates a contraction semigroup on $\ell^p_m(X)$ for all $1 \le p < \infty$; see [36, Theorem 5]. Thus, by Hille–Yosida Theorem we have

$$(-\infty, 0) \subset \rho(L_{p,\max}) \qquad \text{and} \qquad \|(\gamma + L_{p,\max})^{-1}\| \le \frac{1}{\gamma}, \qquad \text{for all } \gamma > 0, \tag{59}$$

where $\|\cdot\|$ denotes the operator norm (for a bounded linear operator $\ell^p_m(X) \to \ell^p_m(X)$) and $\rho(T)$ denotes the resolvent set of an operator T.

Lemma 5.2. *Let $1 \le p < \infty$ and let $\lambda \in \mathbb{C}$ with $Re\ \lambda > 0$. Assume that the hypotheses (A1), (12), and (16) are satisfied. Then, the following properties hold:*

(i) *for all $u \in \mathrm{Dom}(H_{p,\max})$, we have*

$$(Re\ \lambda)\|u\|_p \le \|(\lambda + H_{p,\max})u\|_p; \tag{60}$$

(ii) *the operator $\lambda + H_{p,\max}\colon \mathrm{Dom}(H_{p,\max}) \subset \ell^p_m(X) \to \ell^p_m(X)$ is injective.*

Proof. Let $u \in \mathrm{Dom}(H_{p,\max})$ and $f := (\lambda + H_{p,\max})u$. By the definition of $\mathrm{Dom}(H_{p,\max})$, we have $f \in \Gamma_{\ell^p_m}(X, F)$, where $1 < p < +\infty$. Using (35) and (16) we get

$$\begin{aligned} |u|((\mathrm{Re}\,\lambda + \Delta_{b,m})|u|) &\le \mathrm{Re}\,\langle(\lambda + \Delta^{F,\Phi}_{b,m})u, u\rangle_{F_x} \\ &\le \mathrm{Re}\,\langle(\lambda + \Delta^{F,\Phi}_{b,m} + W)u, u\rangle_{F_x} = \mathrm{Re}\,\langle f, u\rangle_{F_x} \le |f||u|. \end{aligned}$$

In what follows, we denote $\gamma := \mathrm{Re}\,\lambda$. For all $x \in X$ such that $u(x) \neq 0$, we may divide both sides of the last inequality by $|u(x)|$ to get

$$(\gamma + \Delta_{b,m})|u| \ \le\ |f|. \tag{61}$$

Note that the inequality (61) also holds for those $x \in X$ such that $u(x) = 0$; in this case, the left hand side of (61) is non-positive by (2). Thus, the inequality (61) holds for all $x \in X$.

According to (59) the linear operator

$$(\gamma + L_{p,\max})^{-1} \colon \ell^p_m(X) \to \ell^p_m(X)$$

is bounded. Hence, we can rewrite (61) as

$$(\gamma + \Delta_{b,m})[(\gamma + L_{p,\max})^{-1}|f| - |u|] \ge 0. \tag{62}$$

Since

$$(\gamma + L_{p,\max})^{-1}|f| \in \ell^p_m(X) \qquad \text{and} \qquad |u| \in \ell^p_m(X),$$

it follows that $((\gamma + L_{p,\max})^{-1}|f| - |u|) \in \ell^p_m(X)$. Hence, applying Lemma 4.2 to (62) we get

$$|u| \le (\gamma + L_{p,\max})^{-1}|f|. \tag{63}$$

Taking the l^p norms on both sides and using (59) we get

$$\|u\|_p \le \|(\gamma + L_{p,\max})^{-1}|f|\|_p \le \frac{1}{\gamma}\|f\|_p,$$

and (60) is proven. We now turn to property (ii). Assume that $u \in \mathrm{Dom}(H_{p,\max})$ and $(\lambda + H_{p,\max})u = 0$. Using (60) with $f = 0$, we get $\|u\|_p = 0$, and hence $u = 0$. This shows that $\lambda + H_{p,\max}$ is injective. □

End of the proof of Theorem 2.2. We will consider the cases $1 < p < \infty$ and $p = 1$ simultaneously, keeping in mind the stochastic completeness assumption on (X, b, m) when $p = 1$. Since $H_{p,\min} \subset H_{p,\max}$ and since $H_{p,\max}$ is closed (see Lemma 5.1), it follows that $\overline{H_{p,\min}} \subset H_{p,\max}$. To prove the equality $\overline{H_{p,\min}} = H_{p,\max}$, it is enough to show that $\mathrm{Dom}(H_{p,\max}) \subset \mathrm{Dom}(\overline{H_{p,\min}})$. Let $\gamma > 0$, let $u \in \mathrm{Dom}(H_{p,\max})$, and consider

$$v := (\overline{H_{p,\min}} + \gamma)^{-1}(H_{p,\max} + \gamma)u. \tag{64}$$

By Theorem 2.1, the element v is well-defined, and $v \in \mathrm{Dom}(\overline{H_{p,\min}})$.

Since $\overline{H_{p,\min}} \subset H_{p,\max}$, from (64) we get

$$(H_{p,\max} + \gamma)(v - u) = 0.$$

Since $H_{p,\max} + \gamma$ is an injective operator (see Lemma 5.2), we get $v = u$. Therefore, $u \in \mathrm{Dom}(\overline{H_{p,\min}})$. □

6. Proof of Theorem 2.3

The equality $\overline{H_{2,\min}} = H_{2,\max}$ follows directly from Theorem 2.2 in the case $p = 2$. Note that the operator $\overline{H_{2,\min}}$ is symmetric (as the closure of a symmetric operator). Furthermore, by Theorem 2.1 and Remark 2.6 the operator $\overline{H_{2,\min}}$ is maximal accretive. Hence, by [33, Problem V.3.32] the operator $\overline{H_{2,\min}}$ is self-adjoint. Note that the equality (31) extends to

$$(u, H_{2,\min} v) = Q^{(D)}_{W,\Phi}(u, v), \qquad \text{for all } u \in \mathrm{Dom}(Q^{(D)}_{W,\Phi}) \text{ and } v \in \Gamma_c(X, F).$$

Hence, we have the operator relation $H_{2,\min} \subseteq H^{(D)}_{W,\Phi}$, which leads to $\overline{H_{2,\min}} \subseteq H^{(D)}_{W,\Phi}$. Since $H^{(D)}_{W,\Phi}$ and $\overline{H_{2,\min}}$ are self-adjoint, it follows that $\overline{H_{2,\min}} = H^{(D)}_{W,\Phi}$. Again, by Lemma 3.1, for all $u \in \mathrm{Dom}(H^{(N)}_{W,\Phi})$ and $v \in \Gamma_c(X, F)$ we have

$$\sum_{x \in X} m(x) \langle \widetilde{H}_{W,\Phi} u, v \rangle_{F_x} = Q^{(D)}_{W,\Phi}(u, v) = (H^{(N)}_{W,\Phi} u, v).$$

This shows that $H^{(N)}_{W,\Phi} u = \widetilde{H}_{W,\Phi} u$ for all $u \in \mathrm{Dom}(H^{(N)}_{W,\Phi})$. Hence, we have the operator relation $H^{(N)}_{W,\Phi} \subseteq H_{2,\max}$. Since $H^{(N)}_{W,\Phi}$ and $H_{2,\max}$ are self-adjoint, it follows that $H_{2,\max} = H^{(N)}_{W,\Phi}$. This concludes the proof of property (i). Now property (ii) follows directly from the equality $H^{(D)}_{W,\Phi} = H^{(N)}_{W,\Phi}$. □

7. Proof of Theorem 2.4

Throughout this section we assume that the hypotheses of Theorem 2.4 are satisfied. Let $H_{2,\min}$ and $H_{2,\max}$ be as in Definition 2.6 and let $\mathrm{Id}(x)$ denote the identity endomorphism of F_x. Using (21), we may add, without loss of generality, the operator $k\,\mathrm{Id}(x)$ to $\widetilde{H}_{W,\Phi}$, where $k \in \mathbb{R}$ is a sufficiently large constant, to get

$$(H_{2,\min} v, v) \geq \|v\|^2, \qquad \text{for all } v \in \Gamma_c(X, F). \tag{65}$$

Since $H_{2,\min}$ is a symmetric operator satisfying (65) and since $(H_{2,\min})^* = H_{2,\max}$ (as seen in Lemma 3.2), the essential self-adjointness of $H_{2,\min}$ is equivalent to the following statement: $\ker(H_{2,\max}) = \{0\}$; see [44, Theorem X.26].

Let $u \in \mathrm{Dom}(H_{2,\max})$ satisfy $H_{2,\max} u = 0$. Let d_c be the combinatorial distance as in (18) and let $\{\chi_n\}_{n \in \mathbb{Z}_+}$ be as in (41) with $d_\sigma = d_c$. Note that the sequence $\{\chi_n\}_{n \in \mathbb{Z}_+}$ satisfies the properties (i)–(iii) of section 3.4, with $B^\sigma_n(x_0)$ replaced by $B^{(c)}_n(x_0)$. Additionally, note that (43) becomes $|\chi_n(x) - \chi_n(y)| \leq 1/n$. Also note that by the definition of d_c, the functions χ_n are finitely supported. Since X is locally finite, it follows that $\widetilde{D} = \Gamma(X, F)$. Thus, using (65) with $v = \chi_n u$, Lemma 3.4 with $g = \chi_n$ and $\lambda = 0$, and (31) we get

$$\begin{aligned} \sum_{x \in B^{(c)}_n(x_0)} m(x) |u(x)|^2 &\leq \|\chi_n u\|^2 \leq (H_{2,\min}(\chi_n u), \chi_n u) = Q^{(c)}_{W,\Phi}(\chi_n u, \chi_n u) \\ &= \frac{1}{2} \sum_{x,y \in X} b(x, y) (\chi_n(x) - \chi_n(y))^2 (\mathrm{Re}\, \langle u(x), \Phi_{y,x} u(y) \rangle_{F_x}). \end{aligned} \tag{66}$$

Using (66), properties of χ_n, the definition (20), and the inequality

$$|\mathrm{Re}\,\langle u(x), \Phi_{y,x}u(y)\rangle_{F_x}| \le |u(x)||\Phi_{y,x}u(y)| = |u(x)||u(y)| \le \frac{|u(x)|^2}{2} + \frac{|u(y)|^2}{2}, \tag{67}$$

we obtain

$$\sum_{x\in B_n^{(c)}(x_0)} m(x)|u(x)|^2 \le \frac{\delta_n b_n}{n^2}\|u\|^2.$$

We now let $n\to\infty$ and use (19) to get $\|u\|=0$, which gives $u=0$. □

8. Proof of Theorem 2.5

Throughout this section we assume that the hypotheses of Theorem 2.5 are satisfied. As in the proof of Theorem 2.4, we may assume (65), and the argument reduces to showing that if $u\in \mathrm{Dom}(H_{2,\max})$ satisfies $H_{2,\max}u=0$, then $u=0$.

Let $\{\chi_n\}_{n\in\mathbb{Z}_+}$ be as in (41). By hypothesis, we know that (X,d_σ) is a complete metric space and, thus, balls with respect to d_σ are finite; see, for instance, [30, Theorem A.1]. Thus, the functions χ_n are finitely supported. Since X is locally finite, it follows that $\widetilde{D}=\Gamma(X,F)$. Thus, using (65) with $v=\chi_n u$, Lemma 3.4 with $g=\chi_n$ and $\lambda=0$, and (31), we get (66) with $B_n^{(c)}(x_0)$ replaced by $B_n^{\sigma}(x_0)$. Using (66), properties of χ_n, Definition 2.8 (i), and the inequality (67), we obtain

$$\sum_{x\in B_n^{\sigma}(x_0)} m(x)|u(x)|^2 \le \frac{1}{n^2}\|u\|^2.$$

Letting $n\to\infty$ we get $\|u\|=0$; hence, $u=0$. □

9. Proof of Theorem 2.6

The arguments used in this section are based on the method of [47] in the setting of Riemannian manifolds. Throughout the section we assume that the hypotheses of Theorem 2.6 are satisfied. Let $H_{2,\min}$ and $H_{2,\max}$ be as in Definition 2.6. We begin with a generalization of [42, Proposition 4.1], which was proven there in the context of magnetic Schrödinger operators and under the assumption (25) with $\varepsilon=0$.

Proposition 9.1. *If $u\in\mathrm{Dom}(H_{2,\max})$, then*

$$\begin{aligned}&\sum_{x,y\in X} b(x,y)\min\{q^{-1}(x),q^{-1}(y)\}|u(x)-\Phi_{y,x}u(y)|^2\\ &\le \frac{4}{1-\varepsilon}\left(\|\widetilde{H}_{W,\Phi}u\|\|u\| + (K^2+1)\|u\|^2\right),\end{aligned} \tag{68}$$

where $\widetilde{H}_{W,\Phi}$ is as in (7), ε is as in (25), and K is as in (24).

Proof. Let $u\in\mathrm{Dom}(H_{2,\max})$ and let $g\colon X\to\mathbb{R}$ be a finitely supported function. Define

$$I := \left(\sum_{x,y\in X} b(x,y)|u(x)-\Phi_{y,x}u(y)|^2(g^2(x)+g^2(y))\right)^{1/2}. \tag{69}$$

A direct computation, which uses the properties $b(x,y)=b(y,x)$, (3) and (4), shows that

$$I^2 = 4\mathrm{Re}\,(g^2\Delta_{b,m}^{F,\Phi}u,u) + \mathrm{Re}\,Z, \tag{70}$$

where

$$Z := \sum_{x,y\in X} b(x,y)(g^2(x)-g^2(y))\langle \Phi_{y,x}u(y)-u(x), \Phi_{y,x}u(y)+u(x)\rangle_{F_x}.$$

By (70) we have

$$\begin{aligned}\mathrm{Re}\,(g^2\Delta_{b,m}^{F,\Phi}u,u) &= \frac{1}{4}(I^2-\mathrm{Re}\,Z)\\ &\geq (\Delta_{b,m}^{F,\Phi}(gu),(gu)) - \frac{1}{4}S^2,\end{aligned} \tag{71}$$

where

$$S^2 := \sum_{x,y\in X} b(x,y)|u(x)+\Phi_{y,x}u(y)|^2(g(x)-g(y))^2.$$

The inequality in (71) can be verified by using (31) to rewrite the term $(\Delta_{b,m}^{F,\Phi}(gu),(gu))$ and writing out the sums on the left-hand and right-hand side of the inequality.

Starting from (70) and using the factorization

$$g^2(x)-g^2(y) = (g(x)-g(y))(g(x)+g(y)),$$

Cauchy–Schwarz inequality, and

$$(g(x)+g(y))^2 \leq 2(g^2(x)+g^2(y)),$$

we obtain

$$I^2 \leq 4\mathrm{Re}\,(g^2\Delta_{b,m}^{F,\Phi}u,u) + \sqrt{2}IS,$$

where S is as in (71). For $\varepsilon\in[0,1)$ as in (25), the last inequality gives

$$\begin{aligned}(1-\varepsilon)I^2 &\leq 4(1-\varepsilon)\mathrm{Re}\,(g^2\Delta_{b,m}^{F,\Phi}u,u) + \sqrt{2}(1-\varepsilon)IS\\ &= 4\mathrm{Re}\,(g^2\widetilde{H}_{W,\Phi}u,u) - 4(g^2Wu,u) - 4\varepsilon\mathrm{Re}\,(g^2\Delta_{b,m}^{F,\Phi}u,u) + \sqrt{2}(1-\varepsilon)IS\\ &\leq 4\mathrm{Re}\,(g^2\widetilde{H}_{W,\Phi}u,u) - 4(g^2Wu,u) - 4\varepsilon(\Delta_{b,m}^{F,\Phi}(gu),gu) + \varepsilon S^2 + \sqrt{2}(1-\varepsilon)IS\\ &\leq 4\mathrm{Re}\,(g^2\widetilde{H}_{W,\Phi}u,u) + 4(g^2qu,u) + \varepsilon S^2 + \sqrt{2}(1-\varepsilon)IS,\end{aligned} \tag{72}$$

where in the first inequality we used (71) and in the second inequality we used (25).

Using the inequality $ab \leq \frac{a^2}{4}+b^2$ with $a=\sqrt{2(1-\varepsilon)}\cdot I$ in the term $\sqrt{2}(1-\varepsilon)IS$ and rearranging, we get

$$I^2 \leq \frac{2}{1-\varepsilon}\left(4\mathrm{Re}\,(g^2\widetilde{H}_{W,\Phi}u,u) + 4(g^2qu,u) + S^2\right) \tag{73}$$

Let $\{\chi_n\}_{n\in\mathbb{Z}_+}$ be as in (41). Since (X,d_{σ_q}) is a complete metric space, by [30, Theorem A.1] it follows that the balls with respect to d_{σ_q} are finite. Let $B_{2n}^{\sigma_q}(x_0)$ be as in (42) with d_σ replaced by d_{σ_q}. Since $q\geq 1$ it follows that $B_{2n}^{\sigma}(x_0)\subseteq B_{2n}^{\sigma_q}(x_0)$. Thus, the functions χ_n are finitely supported.

Let q be as in (25), and define

$$\psi_n(x) := \chi_n(x) q^{-1/2}(x). \tag{74}$$

Clearly, functions ψ_n have finite support. Furthermore, by property (i) of χ_n and since $q \geq 1$, we have

$$0 \leq \psi_n(x) \leq q^{-1/2}(x) \leq 1, \qquad \text{for all } x \in X, \tag{75}$$

and by property (iii) of χ_n we have

$$\lim_{n\to\infty} \psi_n(x) = q^{-1/2}(x), \qquad \text{for all } x \in X. \tag{76}$$

Finally, by (24), properties (i) and (43) of χ_n, and the inequality $q \geq 1$, we have

$$|\psi_n(x) - \psi_n(y)| \leq \left(\frac{1}{n} + K\right) \sigma(x,y), \qquad \text{for all } x \sim y, \tag{77}$$

where K is as in (24). In what follows, we define I_n and S_n by the same formulas as I in (69) and S in (71), respectively, with $g = \psi_n$. Using (77), the inequality

$$|\Phi_{y,x} u(y) + u(x)|^2 \leq 2(|u(x)|^2 + |u(y)|^2)$$

and Definition 2.8(ii), we get

$$S_n^2 \leq 4 \left(\frac{1}{n} + K\right)^2 \|u\|^2,$$

which, together with (73) and (75), gives

$$\begin{aligned} I_n^2 &\leq \frac{2}{1-\varepsilon} \left(4\mathrm{Re}\,(\psi_n^2 \widetilde{H}_{W,\Phi} u, u) + 4(\psi_n^2 q u, u) + S_n^2\right) \\ &\leq \frac{2}{1-\varepsilon} \left(4\|\widetilde{H}_{W,\Phi} u\| \|u\| + 4\|u\|^2 + 4\left(n^{-1} + K\right)^2 \|u\|^2\right). \end{aligned} \tag{78}$$

Letting $n \to \infty$ in (78) and using (76), Fatou's lemma, and the inequality

$$2\min\{q^{-1}(x), q^{-1}(y)\} \leq q^{-1}(x) + q^{-1}(y), \qquad \text{for all } x,\, y \in X,$$

we get (68). □

End of the proof of Theorem 2.6. From now on the proof proceeds as in [42, Theorem 1.9]. For completeness, we include the argument. By [33, Problem V.3.10] the operator $H_{2,\min}$ is essentially self-adjoint if and only if

$$(H_{2,\max} u, v) = (u, H_{2,\max} v), \qquad \text{for all } u\,, v \in \mathrm{Dom}(H_{2,\max}). \tag{79}$$

Let d_{σ_q} be as in the hypothesis of Theorem 2.6. Fix $x_0 \in X$ and define

$$P(x) := d_{\sigma_q}(x_0, x), \qquad x \in X. \tag{80}$$

For a function $f\colon X \to \mathbb{R}$, define $f^+(x) := \max\{f(x), 0\}$. Let $u\,, v \in \mathrm{Dom}(H_{2,\max})$, let $s > 0$, and define

$$J_s := \sum_{x\in X} m(x) \left(1 - \frac{P(x)}{s}\right)^+ \left(\langle (\widetilde{H}_{W,\Phi} u)(x), v(x)\rangle_{F_x} - \langle u(x), (\widetilde{H}_{W,\Phi} v)(x)\rangle_{F_x}\right). \tag{81}$$

As indicated in the proof of Proposition 9.1, the set $\{x \in X \colon P(x) \leq s\}$ is finite. Thus, for all $s > 0$, the summation in (81) is carried out over finitely many vertices. From the definition of J_s and dominated convergence theorem it is easy to see that

$$\lim_{s\to+\infty} J_s = (H_{2,\max}u, v) - (u, H_{2,\max}v), \qquad \text{for all } u,\, v \in \mathrm{Dom}(H_{2,\max}). \tag{82}$$

Using the definition of J_s, properties $b(x,y) = b(y,x)$, (3), and (4), we get

$$\begin{aligned} 2J_s = \sum_{x,y\in X} \big[\big((1-P(x)/s)^+ - (1-P(y)/s)^+ \big)\, b(x,y) \\ \cdot \left(\langle u(x), \Phi_{y,x}v(y) - v(x)\rangle_{F_x} - \langle \Phi_{y,x}u(y) - u(x), v(y)\rangle_{F_x} \right) \big], \end{aligned}$$

which, together with the triangle inequality and property

$$|f_1^+(x) - f_2^+(x)| \leq |f_1(x) - f_2(x)|,$$

leads to

$$\begin{aligned} 2|J_s| \leq & \frac{1}{s} \sum_{x,y\in X} b(x,y)|P(x)-P(y)| \left(|\Phi_{y,x}v(y) - v(x)||u(x)| \right. \\ & \left. + |\Phi_{y,x}u(y) - u(x)||v(x)| \right). \end{aligned} \tag{83}$$

Additionally, by (80) and (26), for all $x \sim y$ we have

$$\begin{aligned} |P(x) - P(y)| &\leq d_{\sigma_q}(x,y) \leq \sigma_q(x,y) \\ &= \min\{q^{-1/2}(x), q^{-1/2}(y)\} \cdot \sigma(x,y). \end{aligned} \tag{84}$$

We now combine (83) and (84) and use Cauchy–Schwarz inequality together with Definition 2.8(ii). As a result, we obtain

$$|J_s| \leq \frac{1}{2s}(\|v\|T_u + \|u\|T_v), \qquad \text{for all } u,\, v \in \mathrm{Dom}(H_{2,\max}), \tag{85}$$

where

$$T_u := \left(\sum_{x,y\in X} b(x,y) \min\{q^{-1}(x), q^{-1}(y)\} |u(x) - \Phi_{y,x}u(y)|^2 \right)^{1/2}.$$

Since $u \in \mathrm{Dom}(H_{2,\max})$ and $v \in \mathrm{Dom}(H_{2,\max})$, by Lemma 3.2 it follows that $\widetilde{H}_{W,\Phi}u \in \Gamma_{\ell^2_m}(X,F)$ and $\widetilde{H}_{W,\Phi}v \in \Gamma_{\ell^2_m}(X,F)$. Thus, by Proposition 9.1, the expressions T_u and T_v are finite. We now let $s \to +\infty$ in (85) to get $J_s \to 0$, which, together with (82), shows (79). □

10. Proof of Theorem 2.7

The following lemma, whose proof is given in [7, Proposition 4.1], describes an important property of regular graphs.

Lemma 10.1. *Assume that (X,b,m) is a locally finite graph with a path metric d_σ. Additionally, assume that (X,b,m) is regular in the sense of Definition 2.9. Let X_ε be as in (28). Then, closed and bounded subsets of X_ε are finite.*

To prove Theorem 2.7 we follow the method of [42, Theorem 1.5] (see also [7]) . For completeness, all details are given below. The main idea is the Agmon-type estimate given in the next lemma. This kind of estimate has been first used in [43] for Schrödinger operators on an open set with compact boundary in $\mathbb{R}^n$, and refined in [8] for magnetic Laplacians.

Lemma 10.2. *Let $\lambda \in \mathbb{R}$ and let $v \in \Gamma_{\ell^2_m}(X, F)$ be a weak solution of $(\widetilde{H}_{W,\Phi} - \lambda)v = 0$. Assume that there exists a constant $c_1 > 0$ such that, for all $u \in \Gamma_c(X, F)$*

$$(u,\, (\widetilde{H}_{W,\Phi} - \lambda)u) \geq \frac{1}{2} \sum_{x \in X} \max\left(\frac{1}{D(x)^2}, 1\right) m(x)|u(x)|^2_{F_x} + c_1\|u\|^2, \tag{86}$$

where $D(x)$ is as in (27).Then $v \equiv 0$.

Proof. Let ρ be a number such that $0 < \rho < 1/2$. For any $\varepsilon > 0$, we define $f_\varepsilon \colon X \to \mathbb{R}$ by $f_\varepsilon(x) = F_\varepsilon(D(x))$, where $D(x)$ is as in (27) and $F_\varepsilon \colon \mathbb{R}^+ \to \mathbb{R}$ is given by

$$F_\varepsilon(s) = \begin{cases} 0 \text{ for } s \leq \varepsilon \\ \rho(s-\varepsilon)/(\rho-\varepsilon) \text{ for } \varepsilon \leq s \leq \rho \\ s \text{ for } \rho \leq s \leq 1 \\ 1 \text{ for } 1 \leq s. \end{cases}$$

Let us fix a vertex x_0. For any $\alpha > 0$, we define $g_\alpha \colon X \to \mathbb{R}$ by $g_\alpha(x) = G_\alpha(d_\sigma(x_0, x))$, where $G_\alpha \colon \mathbb{R}^+ \to \mathbb{R}$ is given by

$$G_\alpha(s) = \begin{cases} 1 \text{ for } s \leq 1/\alpha \\ -\alpha s + 2 \text{ for } 1/\alpha \leq s \leq 2/\alpha \\ 0 \text{ for } s \geq 2/\alpha. \end{cases}$$

We also define

$$E_{\varepsilon,\alpha} := \{x \in X \colon \varepsilon \leq D(x) \text{ and } d_\sigma(x_0, x) \leq 2/\alpha\}. \tag{87}$$

By Lemma 10.1 the set $E_{\varepsilon,\alpha}$ is finite because $E_{\varepsilon,\alpha}$ is a closed and bounded subset of X_ε, where X_ε is as in (28). Since the support of $f_\varepsilon g_\alpha$ is contained in $E_{\varepsilon,\alpha}$, it follows that $f_\varepsilon g_\alpha$ is finitely supported. Additionally, note that

$$\begin{aligned} |f_\varepsilon(x)g_\alpha(x) - f_\varepsilon(y)g_\alpha(y)| &\leq |f_\varepsilon(x)||g_\alpha(x) - g_\alpha(y)| + |g_\alpha(y)||f_\varepsilon(x) - f_\varepsilon(y)| \\ &\leq \frac{\rho}{\rho - \varepsilon}|D(x) - D(y)| + \alpha|d_\sigma(x_0, x) - d_\sigma(x_0, y)|. \end{aligned}$$

Hence, by [5, Lemma 4.1] it follows that $f_\varepsilon g_\alpha$ is a β-Lipschitz function with respect to d_σ, where $\beta = \rho/(\rho - \varepsilon) + \alpha$.

By Lemma 3.4 with g replaced by $f_\varepsilon g_\alpha$, we have

$$\begin{aligned} &(f_\varepsilon g_\alpha v,\, (\widetilde{H}_{W,\Phi} - \lambda)(f_\varepsilon g_\alpha v)) \\ &= \frac{1}{2} \sum_{x,y \in X} b(x,y)(f_\varepsilon g_\alpha(x) - f_\varepsilon g_\alpha(y))^2 (\operatorname{Re} \langle u(x), \Phi_{y,x} u(y)\rangle_{F_x}). \end{aligned} \tag{88}$$

The unitarity of the operator $\Phi_{y,x}$ implies the inequality

$$\operatorname{Re}\langle u(x), \Phi_{y,x}u(y)\rangle_{F_x} \leq \frac{1}{2}(|v(x)|^2_{F_x} + |v(y)|^2_{F_y}),$$

so from the symmetry of the weight b we get that

$$\begin{aligned}(f_\varepsilon g_\alpha v,\, (\widetilde{H}_{W,\Phi} - \lambda)(f_\varepsilon g_\alpha v)) &\leq \frac{1}{2}\sum_{x\in X}\sum_{y\sim x} b(x,y)|v(x)|^2_{F_x}((f_\varepsilon g_\alpha)(x) - (f_\varepsilon g_\alpha)(y))^2 \\ &\leq \frac{1}{2}\left(\frac{\rho}{\rho-\varepsilon} + \alpha\right)^2 \sum_{x\in X}\sum_{y\sim x} |v(x)|^2_{F_x} b(x,y)(d_\sigma(x,y))^2 \\ &\leq \frac{1}{2}\left(\frac{\rho}{\rho-\varepsilon} + \alpha\right)^2 \sum_{x\in X} m(x)|v(x)|^2_{F_x}, \end{aligned} \tag{89}$$

where the second inequality uses the fact that $f_\varepsilon g_\alpha$ is a β-Lipschitz function with $\beta = \rho/(\rho-\varepsilon) + \alpha$, and the last one comes from Definition 2.8.

On the other hand, by the definitions of f_ε and g_α and the assumption (86) we have

$$(f_\varepsilon g_\alpha v,\, (\widetilde{H}_{W,\Phi} - \lambda)(f_\varepsilon g_\alpha v)) \geq \frac{1}{2}\sum_{x\in S_{\rho,\alpha}} m(x)|v(x)|^2_{F_x} + c_1\|f_\varepsilon g_\alpha v\|^2, \tag{90}$$

where

$$S_{\rho,\alpha} := \{x \in X \colon \rho \leq D(x) \text{ and } d_\sigma(x_0, x) \leq 1/\alpha\}.$$

Combining (90) and (89) we obtain

$$\frac{1}{2}\sum_{x\in S_{\rho,\alpha}} m(x)|v(x)|^2_{F_x} + c_1\|f_\varepsilon g_\alpha v\|^2 \leq \frac{1}{2}\left(\frac{\rho}{\rho-\varepsilon} + \alpha\right)^2 \sum_{x\in X} m(x)|v(x)|^2_{F_x}.$$

We fix ρ and ε, and let $\alpha \to 0+$. After that, we let $\epsilon \to 0+$. Finally, we take the limit as $\rho \to 0+$. As a result, we get $v \equiv 0$. □

End of the Proof of Theorem 2.7. Since $\Delta^{F,\Phi}_{b,m}|_{\Gamma_c(X,F)}$ is a non-negative operator, for all $u \in \Gamma_c(X,F)$, we have

$$(u,\, \widetilde{H}_{W,\Phi}u) \geq \sum_{x\in X} m(x)\langle W(x)u(x), u(x)\rangle_{F_x}.$$

Therefore, using assumption (29) we obtain:

$$\begin{aligned}(u,\, (\widetilde{H}_{W,\Phi} - \lambda)u) &\geq \frac{1}{2}\sum_{x\in X}\frac{1}{D(x)^2} m(x)|u(x)|^2_{F_x} - (\lambda + C)\|u\|^2 \\ &\geq \frac{1}{2}\sum_{x\in X}\max\left(\frac{1}{D(x)^2}, 1\right) m(x)|u(x)|^2_{F_x} - (\lambda + C + 1/2)\|u\|^2. \end{aligned} \tag{91}$$

Choosing, for example, $\lambda = -C - 3/2$ in (91) we get the inequality (86) with $c_1 = 1$.

Thus, $(\widetilde{H}_{W,\Phi} - \lambda)|_{\Gamma_c(X,F)}$ with $\lambda = -C - 3/2$ is a symmetric operator satisfying $(u,\, (\widetilde{H}_{W,\Phi} - \lambda)u) \geq \|u\|^2$, for all $u \in \Gamma_c(X,F)$. By [44, Theorem X.26] we know that the essential self-adjointness of $(\widetilde{H}_{W,\Phi} - \lambda)|_{\Gamma_c(X,F)}$ is equivalent to the following statement: if $v \in \Gamma_{\ell^2_m}(X,F)$

satisfies $(\widetilde{H}_{W,\Phi} - \lambda)v = 0$, then $v = 0$. Thus, by Lemma 10.2, the operator $(\widetilde{H}_{W,\Phi} - \lambda)|_{\Gamma_c(X,F)}$ is essentially self-adjoint. Thus, $\widetilde{H}_{W,\Phi}|_{\Gamma_c(X,F)}$ is essentially self-adjoint. □

References

[1] Anné, C., Torki-Hamza, N.: The Gauss–Bonnet operator of an infinite graph. arXiv:1301.0739v2

[2] Bauer, F., Hua, B., Keller, M.: On the l_p spectrum of Laplacians on graphs. arXiv:1211.6536v1

[3] Bauer, F., Keller, M., Wojciechowski, R. K.: Cheeger inequalities for unbounded graph Laplacians. arXiv:1209.4911v1. To appear in: J. Eur. Math. Soc. (JEMS)

[4] Chung, F. R. K., Sternberg, S.: Laplacian and vibrational spectra for homogeneous graphs. J. Graph Theory. **16**, 605–627 (1992)

[5] Colin de Verdière, Y., Torki-Hamza, N., Truc, F.: Essential self-adjointness for combinatorial Schrödinger operators II-Metrically non complete graphs. Math. Phys. Anal. and Geom. **14**, 21–38 (2011)

[6] Colin de Verdière, Y., Torki-Hamza, N., Truc, F.: Essential self-adjointness for combinatorial Schrödinger operators III-Magnetic fields. Ann. Fac. Sci. Toulouse Math. (6) **20**, 599–611 (2011)

[7] Colin de Verdière, Y., Torki-Hamza, N., Truc, F.: Essential self-adjointness for combinatorial Schrödinger operators II-Metrically non complete graphs. arXiv:1006.5778v3

[8] Colin de Verdière, Y., Truc, F: Confining quantum particles with a purely magnetic field. Ann. Inst. Fourier (Grenoble) **60** (7), 2333–2356 (2010)

[9] Dodziuk, J., Mathai, V.: Kato's inequality and asymptotic spectral properties for discrete magnetic Laplacians. In: Contemp. Math. vol. 398, pp. 69–81. Amer. Math. Soc., Providence (2006)

[10] Engel, K.-J, Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations. Graduate Texts in Mathematics 194, Springer (2000)

[11] Folz, M.: Gaussian upper bounds for heat kernels of continuous time simple random walks. Electron. J. Probab. **16**, 1693-1722 (2011)

[12] Folz, M.: Volume growth and stochastic completeness of graphs. arXiv:1201.5908v2. To appear in: Trans. Amer. Math. Soc.

[13] Folz, M.: Volume growth and spectrum for general graph Laplacians. arXiv:1204.4770v1

[14] Frank, R. L., Lenz, D., Wingert, D.: Intrinsic metrics for non-local symmetric Dirichlet forms and applications to spectral theory. arXiv:1012.5050v1. To appear in: J. Funct. Anal.

[15] Golénia, S.: Unboundedness of adjacency matrices of locally finite graphs. Lett. Math. Phys. **93** (2), 127–140 (2010)

[16] Golénia, S.: Hardy inequality and asymptotic eigenvalue distribution for discrete Laplacians. arXiv:1106.0658

[17] Golénia, S., Schumacher, C.: The problem of deficiency indices for discrete Schrödinger operators on locally finite graphs. J. Math. Phys. **52**, 063512 doi:10.1063/1.3596179 (2011)

[18] Grigor'yan, A., Huang, X., Masamune, J.: On stochastic completeness of jump processes. Math. Z. **271**, 1211–1239 (2012)

[19] Güneysu, B., Keller, M., Schmidt, M.: A Feynman–Kac–Itô formula for magnetic Schrödinger operators on graphs. arXiv:1301.1304v1

[20] Güneysu, B., Milatovic, O., Truc, F.: Generalized Schrödinger semigroups on infinite graphs. arXiv:1306.6062

[21] Haeseler, S., Keller, M.: Generalized solutions and spectrum for Dirichlet forms on graphs. In: Random Walks, Boundaries and Spectra. Progress in Probability, vol. 64, pp. 181–199. Birkhäuser, Basel (2011)

[22] Haeseler, S., Keller, M., Lenz, D., Wojciechowski, R. K.: Laplacians on infinite graphs: Dirichlet and Neumann boundary conditions. J. Spectr. Theory **2**, 397–432 (2012)

[23] Haeseler, S., Keller, M., Wojciechowski, R. K.: Volume growth and bounds for the essential spectrum for Dirichlet forms. arXiv:1205.4985v1. To appear in: J. Lond. Math. Soc. (2)

[24] Holopainen, I., Soardi, P. M.: A strong Liouville theorem for p-harmonic functions on graphs. Ann. Acad. Sci. Fenn. Math., **22**(1), 205–226 (1997)

[25] Hua, B., Jost, J.: l^q harmonic functions on graphs. arXiv:1301.3403v1
[26] Hua, B., Keller, M.: Harmonic functions of general graph Laplacians. arXiv:1303.7198v1
[27] Huang, X.: Stochastic incompleteness for graphs and weak Omori-Yau maximum principle. J. Math. Anal. Appl. **379**, 764–782 (2011)
[28] Huang, X.: On uniqueness class for a heat equation on graphs. J. Math. Anal. Appl. **393**, 377–388 (2012)
[29] Huang, X.: A note on the volume growth criterion for stochastic completeness of weighted graphs. arXiv:1209.2069v2
[30] Huang, X., Keller, M., Masamune, J., Wojciechowski, R. K.: A note on self-adjoint extensions of the Laplacian on weighted graphs. arXiv:1208.6358v1. To appear in: J. Funct. Anal.
[31] Jorgensen P.: Essential self-adjointness of the graph-Laplacian. J. Math. Phys. **49** (7), 073510 doi: 10.1063/1.2953684 (2008)
[32] Jorgensen, P., Pearse, E.: Spectral reciprocity and matrix representations of unbounded operators, J. Funct. Analysis **261** (3), 749–776 (2011)
[33] Kato, T.: Perturbation Theory for Linear Operators. Springer-Verlag, Berlin (1980)
[34] Kato, T.:L^p-theory of Schrödinger operators with a singular potential. In: Aspects of Positivity in Functional Analysis, R. Nagel, U. Schlotterbeck, M. P. H. Wolff (editors), pp. 63–78. North-Holland (1986)
[35] Keller, M., Lenz, D.: Unbounded Laplacians on graphs: basic spectral properties and the heat equation. Math. Model. Nat. Phenom. **5** (4), 198–224 (2010)
[36] Keller, M., Lenz, D.: Dirichlet forms and stochastic completneness of graphs and subgraphs. J. Reine Angew. Math. **666**, 189–223 (2012)
[37] Keller, M., Lenz, D., Wojciechowski, R. K.: Volume growth, spectrum and stochastic completeness of infinite graphs. arXiv:1105.0395. To appear in: Math. Z.
[38] Kenyon, R.: Spanning forests and the vector bundle Laplacian. Ann. Probab. **39**, 1983–2017 (2011)
[39] Masamune, J.: A Liouville property and its application to the Laplacian of an infinite graph. In: Contemporary Mathematics, vol. 484, pp. 103–115. American Mathematical Society, Providence (2009)
[40] Masamune, J., Uemura, T.: Conservation property of symmetric jump processes. Ann. Inst. Henri Poincaré Probab. Stat. **47**, 650–662 (2011)
[41] Milatovic, O.: On m-accretivity of perturbed Bochner Laplacian in L^p spaces on Riemannian manifolds. Integral Equations Operator Theory **68**, 243–254 (2010)
[42] Milatovic, O., Truc, F: Self-adjoint extensions of discrete magnetic Schrödinger operators. Ann. Henri Poincaré. doi: 10.1007/s00023-013-0261-9 (2013)
[43] Nenciu, G., Nenciu, I.: On confining potentials and essential self-adjointness for Schrödinger operators on bounded domains in $\mathbb{R}^n$. Ann. Henri Poincaré **10**, 377–394 (2009)
[44] Reed, M., Simon, B.: Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness. Academic Press, New York (1975)
[45] Rigoli, M., Salvatori, M., and Vignati, M.: Subharmonic functions on graphs. Israel J. Math. **99**, 1-27 (1997)
[46] Rofe-Beketov, F. S.: Conditions for the selfadjointness of the Schrödinger operator. Math. Notes **8**, 888–894 (1970)
[47] Shubin, M. A.: Essential self-adjointness for magnetic Schrödinger operators on non-compact manifolds. In: Séminaire Équations aux Dérivées Partielles (Polytechnique) (1998-1999), Exp. No. XV, pp. XV-1–XV-22. Palaiseau (1999)
[48] Singer, A., Wu, H.-T.: Vector diffusion maps and the connection Laplacian. Comm. Pure Appl. Math. **65**, 1067–1144 (2012).
[49] Torki-Hamza, N.: Laplaciens de graphes infinis I Graphes métriquement complets. Confluentes Math. **2**, 333–350 (2010)
[50] Weber, A.: Analysis of the physical Laplacian and the heat flow on a locally finite graph. J. Math. Anal. Appl. **370** (1), 146–158 (2010)

[51] Wojciechowski, R. K.: Stochastic completeness of graphs. Ph.D. Thesis, The Graduate Center of the City University of New York (2008)

[52] Wojciechowski, R. K.: Heat kernel and essential spectrum of infinite graphs. Indiana Univ. Math. J. **58** (3), 1419–1442 (2009)

Department of Mathematics and Statistics, University of North Florida, Jacksonville, FL 32224, USA

E-mail address: omilatov@unf.edu

Grenoble University, Institut Fourier, Unité mixte de recherche CNRS-UJF 5582, BP 74, 38402-Saint Martin d'Hères Cedex, France

E-mail address: francoise.truc@ujf-grenoble.fr