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► **To cite this version:**

Giada Basile, Stefano Olla. Energy Diffusion in Harmonic System with Conservative Noise. Journal of Statistical Physics, 2014, 155 (6), pp.1126-1142. 10.1007/s10955-013-0908-4 . hal-00839712

**HAL Id: hal-00839712**

**<https://hal.science/hal-00839712>**

Submitted on 29 Jun 2013

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# ENERGY DIFFUSION IN HARMONIC SYSTEM WITH CONSERVATIVE NOISE

GIADA BASILE AND STEFANO OLLA

ABSTRACT. We prove diffusive behaviour of the energy fluctuations in a system of harmonic oscillators with a stochastic perturbation of the dynamics that conserves energy and momentum. The results concern pinned systems or lattice dimension  $d \geq 3$ , where the thermal diffusivity is finite.

## 1. INTRODUCTION

Lattice networks of oscillators have been considered for a long time as good models for studying macroscopic energy transport and its diffusion, i.e. for obtaining, on a macroscopic space-time scale, heat equation and Fourier law of conduction ([1]). It is well understood that the diffusive behavior of the energy is due to the non-linearity of the interactions, and that purely deterministic harmonic systems have a ballistic transport of energy (cf. [14]). On the other hand, non-linear dynamics are very difficult to study and even the convergence of the Green-Kubo formula defining the macroscopic thermal conductivity is an open problem. In fact in some cases, like in one dimensional unpinned systems, it is expected (and observed numerically) an infinite conductivity and a superdiffusion of the energy (cf. [12, 11]).

In order to model the phonon scattering due to the dynamics, various stochastic perturbation of the harmonic dynamics have been proposed, where the added random dynamics conserves the energy. In [7] Fourier Law is proven for an harmonic chain with stochastic dynamics that conserves only energy. In [2, 3] is studied the Green-Kubo formula for a stochastic perturbation that conserves energy *and* momentum. It is proven there that conductivity is finite for pinned systems, or unpinned if dimension is greater than 2. Qualitatively this agrees to what it is expected for deterministic anharmonic dynamics.

In this article we consider the same stochastic dynamics as in [2, 3] and we prove that in the cases when conductivity is finite, the energy

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*Date:* June 29, 2013.

*2000 Mathematics Subject Classification.* ...

*Key words and phrases.* ...

Dedicated to Herbert Spohn for his 65th birthday.

This paper has been partially supported by the European Advanced Grant *Macroscopic Laws and Dynamical Systems* (MALADY) (ERC AdG 246953).

behave diffusively in the sense that energy fluctuations of the system in equilibrium evolve according to a linear heat equation.

The key point in proving such diffusive behavior is to obtain a fluctuation-dissipation decomposition of the microscopic energy currents  $j_{\mathbf{x},\mathbf{y}}$  between two adjacent atoms  $\mathbf{x}, \mathbf{y}$ . This means to be able to approximate  $j_{\mathbf{x},\mathbf{y}}$  by a function of the form  $\kappa(e_{\mathbf{x}} - e_{\mathbf{y}}) + LF$ , where  $e_{\mathbf{x}}$  is the energy of atom  $\mathbf{x}$ ,  $L$  is the generator of the dynamics and  $F$  a function in its domain, possibly local. With such decomposition it is possible to close macroscopically the evolution of the energy and  $\kappa$  identify the thermal diffusivity.

In the harmonic chain with noise that conserves only energy, there is an exact fluctuation-dissipation decomposition with a local function  $F$  (cf. [7]). For anharmonic system this decomposition is non-local, and much harder to obtain. In fact there exist only results with an *elliptic* noise that acts also on the positions of the atoms (cf. [13]).

In the model considered here, the noise is of hypoelliptic type, i.e. acts only on the velocity. But because of the additional conservation of momentum, the fluctuation-dissipation decomposition of the currents is non-local. Thanks to the linearity of the interaction, we can use resolvent approximations instead of local approximations for this decomposition.

The corresponding results for the cases when thermal conductivity is infinite, i.e. the unpinned model in dimension 1 and 2, remains open problems. In dimension 2 the divergence of the conductivity is of *logarithmic* type, and we expect a diffusive behavior of the energy fluctuations under the proper space-time scale.

In dimension 1, also to establish a conjecture is hard, as the behavior is really superdiffusive. Under a weak noise limit the local spectral energy density (Wigner distribution) behaves following a linear Boltzmann type equation (cf. [5]). Under proper rescaling of this equation, is proven the convergence to a superdiffusive Levy process governed by a fractional laplacian (cf. [4, 8]. See also [6] for diffusive behaviour in dimension 2). A possible guess is that the fractional laplacian behavior will govern energy fluctuations without taking first a weak noise limit. For what concerns this problem in one dimensional unpinned anharmonic chains, see the recent article by Herbert Spohn [15].

## 2. THE DYNAMICS

The Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2} \sum_{\mathbf{x}} [\mathbf{p}_{\mathbf{x}}^2 + \mathbf{q}_{\mathbf{x}} \cdot (\nu I - \alpha \Delta) \mathbf{q}_{\mathbf{x}}].$$

The atoms are labeled by  $\mathbf{x} \in \mathbb{Z}^d$  and  $\{\mathbf{q}_{\mathbf{x}}\}$  are the displacements of the atoms from their equilibrium positions. We denote with  $\nabla, \nabla^*$

and  $\Delta = \nabla^* \cdot \nabla$  respectively the discrete gradient, its adjoint and the discrete Laplacian on  $\mathbb{Z}^d$ . These are defined as

$$(2.1) \quad \nabla_{\mathbf{e}_j} f(\mathbf{x}) = f(\mathbf{x} + \mathbf{e}_j) - f(\mathbf{x})$$

and

$$(2.2) \quad \nabla_{\mathbf{e}_j}^* f(\mathbf{x}) = f(\mathbf{x} - \mathbf{e}_j) - f(\mathbf{x}).$$

The parameter  $\alpha > 0$  is the strength of the interparticles springs, and  $\nu \geq 0$  is the strength of the pinning (on-site potential).

We consider the stochastic dynamics corresponding to the Fokker-Planck equation

$$(2.3) \quad \frac{\partial P}{\partial t} = (-A + \gamma S)P = LP .$$

where  $A$  is the usual Hamiltonian vector field

$$A = \sum_{\mathbf{x}} \{ \mathbf{p}_{\mathbf{x}} \cdot \partial_{\mathbf{q}_{\mathbf{x}}} + [(\alpha \Delta - \nu I) \mathbf{q}_{\mathbf{x}}] \cdot \partial_{\mathbf{p}_{\mathbf{x}}} \}$$

while  $S$  is the generator of the stochastic perturbation and  $\gamma > 0$  is a positive parameter that regulates its strength. The operator  $S$  acts only on the momentums  $\{ \mathbf{p}_{\mathbf{x}} \}$  and generates a diffusion on the surface of constant kinetic energy and constant momentum. This is defined as follows. For every nearest neighbor atoms  $\mathbf{x}$  and  $\mathbf{z}$ , consider the  $d - 1$  dimensional surface of constant kinetic energy and momentum

$$\mathbb{S}_{e,\mathbf{p}} = \left\{ (\mathbf{p}_{\mathbf{x}}, \mathbf{p}_{\mathbf{z}}) \in \mathbb{R}^{2d} : \frac{1}{2} (\mathbf{p}_{\mathbf{x}}^2 + \mathbf{p}_{\mathbf{z}}^2) = e ; \mathbf{p}_{\mathbf{x}} + \mathbf{p}_{\mathbf{z}} = \mathbf{p} \right\} .$$

The following vector fields are tangent to  $\mathbb{S}_{e,\mathbf{p}}$

$$X_{\mathbf{x},\mathbf{z}}^{i,j} = (p_{\mathbf{z}}^j - p_{\mathbf{x}}^j)(\partial_{p_{\mathbf{z}}^i} - \partial_{p_{\mathbf{x}}^i}) - (p_{\mathbf{z}}^i - p_{\mathbf{x}}^i)(\partial_{p_{\mathbf{z}}^j} - \partial_{p_{\mathbf{x}}^j}).$$

so  $\sum_{i,j=1}^d (X_{\mathbf{x},\mathbf{z}}^{i,j})^2$  generates a diffusion on  $\mathbb{S}_{e,\mathbf{p}}$ . In  $d \geq 2$  we define

$$\begin{aligned} S &= \frac{1}{2(d-1)} \sum_{\mathbf{x}} \sum_{i,j,k}^d (X_{\mathbf{x},\mathbf{x}+\mathbf{e}_k}^{i,j})^2 \\ &= \frac{1}{4(d-1)} \sum_{\substack{\mathbf{x},\mathbf{z} \in \mathbb{Z}_N^d \\ \|\mathbf{x}-\mathbf{z}\|=1}} \sum_{i,j} (X_{\mathbf{x},\mathbf{z}}^{i,j})^2 \end{aligned}$$

where  $\mathbf{e}_1, \dots, \mathbf{e}_d$  is canonical basis of  $\mathbb{Z}^d$ .

Observe that this noise conserves the total momentum  $\sum_{\mathbf{x}} \mathbf{p}_{\mathbf{x}}$  and energy  $\mathcal{H}_N$ , i.e.

$$S \sum_{\mathbf{x}} \mathbf{p}_{\mathbf{x}} = 0 , \quad S \mathcal{H}_N = 0$$

In dimension 1, in order to conserve total momentum and total kinetic energy, we have to consider a random exchange of momentum

between three consecutive atoms, and we define

$$S = \frac{1}{6} \sum_{x \in \mathbb{T}_N^1} (Y_x)^2$$

where

$$Y_x = (p_x - p_{x+1})\partial_{p_{x-1}} + (p_{x+1} - p_{x-1})\partial_{p_x} + (p_{x-1} - p_x)\partial_{p_{x+1}}$$

which is vector field tangent to the surface of constant energy and momentum of the three particles involved. The Fokker-Planck equation (2.3) gives the time evolution of the probability distribution  $P(\mathbf{q}, \mathbf{p}, t)$ , given an initial distribution  $P(\mathbf{q}, \mathbf{p}, 0)$ . It correspond to the law at time  $t$  of the solution of the following stochastic differential equations:

$$(2.4) \quad \begin{aligned} d\mathbf{q}_x &= \mathbf{p}_x dt \\ d\mathbf{p}_x &= -(\nu I - \alpha \Delta)\mathbf{q}_x dt + 2\gamma \Delta \mathbf{p}_x dt \\ &\quad + \frac{\sqrt{\gamma}}{2\sqrt{d-1}} \sum_{\mathbf{z}: \|\mathbf{z}-\mathbf{x}\|=1} \sum_{i,j=1}^d (X_{\mathbf{x},\mathbf{z}}^{i,j} \mathbf{p}_x) dw_{\mathbf{x},\mathbf{z}}^{i,j}(t) \end{aligned}$$

where  $\{w_{\mathbf{x},\mathbf{y}}^{i,j} = w_{\mathbf{y},\mathbf{x}}^{i,j}; \mathbf{x}, \mathbf{y} \in \mathbb{Z}^d; i, j = 1, \dots, d; \|\mathbf{y} - \mathbf{x}\| = 1\}$  are independent standard Wiener processes. In  $d = 1$  the sde are:

$$(2.5) \quad \begin{aligned} dp_x &= -(\nu I - \alpha \Delta)q_x dt + \frac{\gamma}{6} \Delta(4p_x + p_{x-1} + p_{x+1})dt \\ &\quad + \sqrt{\frac{\gamma}{3}} \sum_{k=-1,0,1} (Y_{x+k} p_x) dw_{x+k}(t) \end{aligned}$$

where here  $\{w_x(t), x = 1, \dots, N\}$  are independent standard Wiener processes.

Defining the energy of the atom  $\mathbf{x}$  as

$$e_{\mathbf{x}} = \frac{1}{2} \mathbf{p}_{\mathbf{x}}^2 + \frac{\alpha}{4} \sum_{\mathbf{y}: \|\mathbf{y}-\mathbf{x}\|=1} (\mathbf{q}_{\mathbf{y}} - \mathbf{q}_{\mathbf{x}})^2 + \frac{\nu}{2} \mathbf{q}_{\mathbf{x}}^2,$$

the energy conservation law can be read locally as

$$e_{\mathbf{x}}(t) - e_{\mathbf{x}}(0) = \sum_{k=1}^d (J_{\mathbf{x}-\mathbf{e}_k, \mathbf{x}}(t) - J_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}(t))$$

where  $J_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}(t)$  is the total energy current between  $\mathbf{x}$  and  $\mathbf{x} + \mathbf{e}_k$  up to time  $t$ . This can be written as

$$(2.6) \quad J_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}(t) = \int_0^t j_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}(s) ds + M_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}(t).$$

In the above  $M_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}(t)$  are martingales that can be written explicitly as Ito stochastic integrals

$$(2.7) \quad M_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}(t) = \sqrt{\frac{\gamma}{(d-1)}} \sum_{i,j} \int_0^t (X_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}^{i,j} e_{\mathbf{x}})(s) dw_{\mathbf{x}, \mathbf{x}+\mathbf{e}_k}^{i,j}(s)$$

The stationary equilibrium probability measures for this dynamics are given by the corresponding Gibbs measure, defined through the DLR equations. Because of the linearity of the interaction and the conservation laws of the stochastic perturbation, these are gaussian measures on  $(\mathbb{R}^{2d})^{\mathbb{Z}^d}$  with covariance given by

$$(2.8) \quad \begin{aligned} \langle (p_{\mathbf{x}}^i - v^i)(p_{\mathbf{y}}^j - v^j) \rangle &= \beta^{-1} \delta_{i,j} \delta_{\mathbf{x},\mathbf{y}}, & \langle (p_{\mathbf{x}}^i - v^i)q_{\mathbf{y}}^j \rangle &= 0, \\ \langle q_{\mathbf{x}}^i q_{\mathbf{y}}^j \rangle &= \frac{1}{d} \Gamma(\mathbf{x} - \mathbf{y}) \delta_{i,j} \end{aligned}$$

where  $\Gamma(\mathbf{x}) = (\nu I - \alpha \Delta)^{-1}(\mathbf{x})$ . In the pinned case,  $\nu = 0$ , momentum is not conserved and we have to set  $\mathbf{v} = 0$ . Since in the unpinned case the parameter  $\mathbf{v}$  represent a trivial translation invariance, for simplicity we choose  $\mathbf{v} = 0$ .

We consider this dynamics starting with an equilibrium distribution at a given temperature  $\beta^{-1}$ . The existence of the infinite dynamics under this initial distribution can be proven by standard techniques (for example see [10]).

Given two continuous functions  $F, H$  on  $\mathbb{R}^d$  with compact support, and  $\varepsilon > 0$ , we look at the evolution of the following quantity

$$\begin{aligned} \sigma_{\varepsilon,t}(F, H) &= \varepsilon^d \sum_{\mathbf{x},\mathbf{y}} F(\varepsilon \mathbf{x}) H(\varepsilon \mathbf{y}) \langle (e_{\mathbf{x}}(\varepsilon^{-2}t) - \beta^{-1})(e_{\mathbf{y}}(0) - \beta^{-1}) \rangle \\ &= \varepsilon^d \sum_{\mathbf{y},\mathbf{z}} F(\varepsilon(\mathbf{y} + \mathbf{z})) H(\varepsilon \mathbf{y}) \langle e_{\mathbf{z}}(\varepsilon^{-2}t)(e_{\mathbf{0}}(0) - \beta^{-1}) \rangle, \end{aligned}$$

where  $\langle \cdot \rangle$  is the expectation value with respect to the equilibrium measure.

**Theorem 2.1.**

$$(2.9) \quad \lim_{\varepsilon \rightarrow 0} \sigma_{\varepsilon,t}(F, H) = \iint d\mathbf{u} d\mathbf{v} F(\mathbf{u}) G(\mathbf{v}) \frac{e^{-|\mathbf{u}-\mathbf{v}|^2/2t\kappa}}{(2\pi t\kappa)^{d/2}}$$

where the diffusion coefficient  $\kappa$  is given by:

$$(2.10) \quad \kappa = \frac{1}{8\pi^2\gamma} \int_{\mathbb{T}^d} d\xi \frac{(\partial_1 \omega(\xi))^2}{\Phi(\xi)} + \gamma$$

Here  $\omega$  is the dispersion relation

$$\omega(\xi) = \left( \nu + 4\alpha \sum_{j=1}^d \sin^2(\pi \xi_j) \right)^{1/2}$$

and  $\Phi$  is the scattering rate

$$(2.11) \quad \Phi(\mathbf{k}) = \begin{cases} 8 \sum_{j=1}^d \sin^2(\pi k_j) & d \geq 2 \\ \frac{4}{3} \sin^2(\pi k) (1 + 2 \cos^2(\pi k)) & d = 1 \end{cases}$$

With a little more work one can prove that the fluctuation field

$$(2.12) \quad Y_t^\varepsilon(F) = \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) [e_{\mathbf{z}}(\varepsilon^{-2}t) - \beta^{-1}]$$

converges in law to the infinite dimensional Ornstein Uhlenbeck  $Y_t$  solution of the linear stochastic PDE:

$$(2.13) \quad \partial_t Y = \frac{\kappa}{2} \Delta Y + \beta^{-1/2} \nabla W$$

where  $W(\mathbf{x}, t)$  is the standard space time white noise on  $\mathbb{R}^{d+1}$ . This extension is standard and we will expose here only the proof of Theorem 2.1.

### 3. ENERGY CURRENTS

Let  $e_{\mathbf{x}}$  be the energy of the atom  $\mathbf{x}$ , which is equal to

$$e_{\mathbf{x}} = \frac{1}{2} \mathbf{p}_{\mathbf{x}}^2 - \frac{1}{2} \mathbf{q}_{\mathbf{x}} \cdot (\alpha \Delta - \nu) \mathbf{q}_{\mathbf{x}}.$$

When  $\nu = 0$  there is no pinning. We consider cases  $\nu > 0$ ,  $d \geq 1$  and  $\nu = 0$ ,  $d = 3$ . The instantaneous energy currents  $j_{\mathbf{x}, \mathbf{x}+\mathbf{e}_i}$ ,  $i = 1, \dots, d$ , satisfy the equation

$$L e_{\mathbf{x}} = \sum_{i=1}^d (j_{\mathbf{x}-\mathbf{e}_i, \mathbf{x}} - j_{\mathbf{x}, \mathbf{x}+\mathbf{e}_i}),$$

and can be written as

$$(3.1) \quad j_{\mathbf{x}, \mathbf{x}+\mathbf{e}_i} = j_{\mathbf{x}, \mathbf{x}+\mathbf{e}_i}^a + \gamma j_{\mathbf{x}, \mathbf{x}+\mathbf{e}_i}^s.$$

The first term is the Hamiltonian contribution to the energy current, namely

$$(3.2) \quad j_{\mathbf{x}, \mathbf{x}+\mathbf{e}_i}^a = -\frac{\alpha}{2} (\mathbf{q}_{\mathbf{x}+\mathbf{e}_i} - \mathbf{q}_{\mathbf{x}}) \cdot (\mathbf{p}_{\mathbf{x}+\mathbf{e}_i} + \mathbf{p}_{\mathbf{x}}),$$

while the noise contribution in  $d \geq 2$  is

$$(3.3) \quad \gamma j_{\mathbf{x}, \mathbf{x}+\mathbf{e}_i}^s = -\gamma \nabla_{\mathbf{e}_i} \mathbf{p}_{\mathbf{x}}^2.$$

In one dimension

$$(3.4) \quad \begin{aligned} \gamma j_{x, x+1}^s &= -\gamma \nabla \phi(p_{x-1}, p_x p_{x+1}), \\ \phi(p_{x-1}, p_x p_{x+1}) &= \frac{1}{6} [p_{x+1}^2 + 4p_x^2 + p_{x-1}^2 + p_{x+1} p_{x-1} \\ &\quad - 2p_{x+1} p_x - 2p_{x-1} p_x]. \end{aligned}$$

We denote with  $\phi_x := \phi(p_{x-1}, p_x, p_{x+1})$ .

Given  $F, H \in C_c^2(\mathbb{R}^d)$  (twice differentiable functions with compact support), and  $\varepsilon > 0$ , we look at the evolution of the following quantity

$$\begin{aligned}\sigma_{\varepsilon,t}(F, H) &= \varepsilon^d \sum_{\mathbf{x}, \mathbf{y}} F(\varepsilon \mathbf{x}) H(\varepsilon \mathbf{y}) \langle (e_{\mathbf{x}}(\varepsilon^{-2}t) - \beta^{-1}) (e_{\mathbf{y}}(0) - \beta^{-1}) \rangle \\ &= \varepsilon^d \sum_{\mathbf{y}, \mathbf{z}} F(\varepsilon(\mathbf{y} + \mathbf{z})) H(\varepsilon \mathbf{y}) \langle e_{\mathbf{z}}(\varepsilon^{-2}t) (e_{\mathbf{0}}(0) - \beta^{-1}) \rangle,\end{aligned}$$

where  $\langle \cdot \rangle$  is the expectation value with respect to the equilibrium measure. For  $d \geq 1$  we have

(3.5)

$$\begin{aligned}\sigma_{\varepsilon,t}(F, H) &= \sigma_{\varepsilon,0}(F, H) \\ &+ \varepsilon^d \sum_{\mathbf{y}, \mathbf{z}} \sum_{i=1}^d \nabla_{\mathbf{e}_i}^\varepsilon F(\varepsilon(\mathbf{y} + \mathbf{z})) H(\varepsilon \mathbf{y}) \frac{1}{\varepsilon} \int_0^t ds \langle j_{\mathbf{z}, \mathbf{z} + \mathbf{e}_i}(s/\varepsilon^2) (e_{\mathbf{0}}(0) - \beta^{-1}) \rangle.\end{aligned}$$

Accordingly to (3.1), we decompose the energy current  $j_{\mathbf{z}, \mathbf{z} + \mathbf{e}_i}$  in two parts and we treat separately the two integrals.

**Noise current.** For  $d \geq 2$ , using (3.3) we have

$$\begin{aligned}\gamma \varepsilon^d \sum_{\mathbf{y}, \mathbf{z}} \sum_{i=1}^d \nabla_{\mathbf{e}_i}^\varepsilon F(\varepsilon(\mathbf{y} + \mathbf{z})) H(\varepsilon \mathbf{y}) \frac{1}{\varepsilon} \int_0^t ds \langle j_{\mathbf{z}, \mathbf{z} + \mathbf{e}_i}^s(s/\varepsilon^2) (e_{\mathbf{0}}(0) - \beta^{-1}) \rangle \\ = \gamma \varepsilon^d \sum_{\mathbf{z}, \mathbf{y}} \Delta F(\varepsilon(\mathbf{y} + \mathbf{z})) H(\varepsilon \mathbf{y}) \int_0^t ds \langle \mathbf{p}_{\mathbf{z}}^2(s/\varepsilon^2) (e_{\mathbf{0}}(0) - \beta^{-1}) \rangle + \mathcal{O}(\varepsilon).\end{aligned}$$

In order to replace in the last expression  $p_{\mathbf{z}}^2$  with  $e_{\mathbf{z}}$ , we use the following Lemma:

**Lemma 3.1.** For every  $G \in L^2(\mathbb{R}^d)$ ,  $\forall d \geq 1$

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} \left\langle \left( \int_0^t ds \varepsilon^{d/2} \sum_{\mathbf{y}} G(\varepsilon \mathbf{y}) [\mathbf{p}_{\mathbf{y}}^2 - e_{\mathbf{y}}](s/\varepsilon^2) \right)^2 \right\rangle = 0.$$

*Proof.* Observe that

$$\mathbf{p}_{\mathbf{x}}^2 - e_{\mathbf{x}} = \frac{1}{2} \mathbf{p}_{\mathbf{x}}^2 + \frac{1}{2} \mathbf{q}_{\mathbf{x}} \cdot [\alpha \Delta - \nu I] \mathbf{q}_{\mathbf{x}} = \frac{1}{2} L[\mathbf{q}_{\mathbf{x}} \cdot \mathbf{p}_{\mathbf{x}}] - \frac{1}{2} \gamma S[\mathbf{q}_{\mathbf{x}} \cdot \mathbf{p}_{\mathbf{x}}]$$

Therefore

$$\begin{aligned}(3.6) \quad & \left\langle \left( \int_0^t ds \varepsilon^{d/2} \sum_{\mathbf{y}} G(\varepsilon \mathbf{y}) [\mathbf{p}_{\mathbf{y}}^2 - e_{\mathbf{y}}](s/\varepsilon^2) \right)^2 \right\rangle \\ & \leq \varepsilon^4 \left\langle \left( \int_0^t ds \frac{\varepsilon^{d/2}}{2} \sum_{\mathbf{y}} G(\varepsilon \mathbf{y}) \varepsilon^{-2} L[\mathbf{q}_{\mathbf{y}} \cdot \mathbf{p}_{\mathbf{y}}](s/\varepsilon^2) \right)^2 \right\rangle \\ & + \gamma^2 \left\langle \left( \int_0^t ds \frac{\varepsilon^{d/2}}{2} \sum_{\mathbf{y}} G(\varepsilon \mathbf{y}) S[\mathbf{q}_{\mathbf{y}} \cdot \mathbf{p}_{\mathbf{y}}](s/\varepsilon^2) \right)^2 \right\rangle.\end{aligned}$$



In the first term of the rhs of the above inequality, we can perform explicitly the time integration and we have

$$\begin{aligned} & \frac{\varepsilon^{d/2}}{2} \sum_{\mathbf{y}} G(\varepsilon \mathbf{y}) \int_0^t ds \varepsilon^{-2} L[\mathbf{q}_{\mathbf{y}} \cdot \mathbf{p}_{\mathbf{y}}](s/\varepsilon^2) \\ &= \frac{\varepsilon^{d/2}}{2} \sum_{\mathbf{y}} G(\varepsilon \mathbf{y}) ([\mathbf{q}_{\mathbf{y}} \cdot \mathbf{p}_{\mathbf{y}}](t/\varepsilon^2) - [\mathbf{q}_{\mathbf{y}} \cdot \mathbf{p}_{\mathbf{y}}](0)) + \mathcal{M}_\varepsilon(G, t) \end{aligned}$$

where  $\mathcal{M}_\varepsilon(G, t)$  is a martingale (given by a stochastic integral) whose quadratic variation is bounded by

$$\begin{aligned} [\mathcal{M}_\varepsilon(G, t)^2] &\leq \varepsilon \frac{\varepsilon^d}{4} \sum_{\mathbf{y}} |\nabla^\varepsilon G(\varepsilon \mathbf{y})|^2 \int_0^t \langle \mathbf{p}_{\mathbf{y}}^2(s/\varepsilon^2) \mathbf{q}_{\mathbf{y}}^2(s/\varepsilon^2) \rangle ds \\ &\leq \varepsilon C \|G\|^2 \beta^{-1} \langle \mathbf{q}_0^2 \rangle. \end{aligned}$$

Since  $\langle \mathbf{q}_0^2 \rangle$  is bounded in the pinned case and for  $d \geq 3$ , we conclude that the first term on the RHS of (3.6) converges to 0 as  $\varepsilon \rightarrow 0$ .

Moreover, by the estimate in appendix A

$$\left\langle \left( \int_0^t ds \varepsilon^{d/2} \sum_{\mathbf{y}} G(\varepsilon \mathbf{y}) S[\mathbf{q}_{\mathbf{y}} \cdot \mathbf{p}_{\mathbf{y}}](s/\varepsilon^2) \right)^2 \right\rangle \leq Ct\varepsilon^2$$

which vanishes as  $\varepsilon \rightarrow 0$ . □

Using Cauchy-Schwartz, the quantity

$$\begin{aligned} & \gamma \varepsilon^d \sum_{\mathbf{z}, \mathbf{y}} \Delta F(\varepsilon(\mathbf{y} + \mathbf{z})) H(\varepsilon \mathbf{y}) \int_0^t ds \langle [\mathbf{p}_{\mathbf{z}}^2 - e_{\mathbf{z}}](s/\varepsilon^2) (e_{\mathbf{0}}(0) - \beta^{-1}) \rangle \\ &= \gamma \varepsilon^d \sum_{\mathbf{y}', \mathbf{y}} \Delta F(\varepsilon \mathbf{y}') H(\varepsilon \mathbf{y}) \int_0^t ds \langle [\mathbf{p}_{\mathbf{y}'}^2 - e_{\mathbf{y}'}](s/\varepsilon^2) (e_{\mathbf{y}}(0) - \beta^{-1}) \rangle \end{aligned}$$

is bounded in absolute value by

$$\gamma \|H\| \langle (e_{\mathbf{0}} - \beta^{-1})^2 \rangle^{1/2} \left\langle \left( \int_0^t ds \varepsilon^{d/2} \sum_{\mathbf{y}} \Delta F(\varepsilon \mathbf{y}) [\mathbf{p}_{\mathbf{y}}^2 - e_{\mathbf{y}}](s/\varepsilon^2) \right)^2 \right\rangle^{1/2},$$

which vanishes as  $\varepsilon \rightarrow 0$  in view of the previous lemma. Then the contribution of the noise in the evolution of the energy fluctuations is

given by

$$\begin{aligned}
(3.7) \quad & \gamma \varepsilon^d \sum_{\mathbf{y}, \mathbf{z}} \sum_{i=1}^d \nabla_{\mathbf{e}_i}^\varepsilon F(\varepsilon(\mathbf{y} + \mathbf{z})) H(\varepsilon \mathbf{y}) \frac{1}{\varepsilon} \int_0^t ds \langle j_{\mathbf{z}, \mathbf{z} + \mathbf{e}_i}^s(s/\varepsilon^2) (e_0(0) - \beta^{-1}) \rangle \\
& = \gamma \varepsilon^d \sum_{\mathbf{z}, \mathbf{y}} \Delta F(\varepsilon(\mathbf{y} + \mathbf{z})) H(\varepsilon \mathbf{y}) \int_0^t ds \langle e_{\mathbf{z}}(s/\varepsilon^2) (e_0(0) - \beta^{-1}) \rangle + \mathcal{O}(\varepsilon) \\
& = \sigma_{\varepsilon, t}(\gamma \Delta F, G) + \mathcal{O}(\varepsilon).
\end{aligned}$$

In  $d = 1$  we have

$$\begin{aligned}
& \gamma \varepsilon \sum_{\mathbf{y}, \mathbf{z}} \nabla^\varepsilon F(\varepsilon(y + z)) H(\varepsilon y) \frac{1}{\varepsilon} \int_0^t ds \langle j_{z, z+1}^s(s/\varepsilon^2) (e_0(0) - \beta^{-1}) \rangle \\
& = \gamma \varepsilon \sum_y \sum_z F''(\varepsilon(y + z)) H(\varepsilon y) \int_0^t ds \langle \phi_z(s/\varepsilon^2) (e_0(0) - \beta^{-1}) \rangle + \mathcal{O}(\varepsilon),
\end{aligned}$$

with  $\phi$  defined in (3.4). The last quantity can be written as

$$\begin{aligned}
& = \gamma \varepsilon \sum_y \sum_z F'''(\varepsilon(y + z)) H(\varepsilon y) \int_0^t ds \left\{ \langle p_z^2(s/\varepsilon^2) (e_0(0) - \beta^{-1}) \rangle \right. \\
& \quad \left. + \frac{1}{6} \langle (p_{z+1} p_{z-1} - 2p_z p_{z+1} - 2p_z p_{z-1})(s/\varepsilon^2) (e_0(0) - \beta^{-1}) \rangle \right\} + \mathcal{O}(\varepsilon),
\end{aligned}$$

where we can replace  $p_z^2$  with  $e_z$  using same arguments as in  $d \geq 2$ . We will prove in Section 4 that the second integral vanishes as  $\varepsilon \rightarrow 0$ , then formula (3.7) holds in all dimensions.

**Hamiltonian currents.** Notice that

$$(3.8) \quad j_{0, \mathbf{e}_i}^a = \frac{\alpha}{2} (\mathbf{p}_{\mathbf{e}_i} - \mathbf{p}_{-\mathbf{e}_i}) \cdot \mathbf{q}_{\mathbf{e}_i} - \frac{\alpha}{2} \nabla_i (\mathbf{p}_0 \cdot \mathbf{q}_0 + \mathbf{p}_{-\mathbf{e}_i} \cdot \mathbf{q}_0)$$

it is easy to see that the contribution of the gradient terms in (3.5) vanish as  $\varepsilon \rightarrow 0$ , so we have only to consider the first term.

Let  $g_\lambda^j$ ,  $j = 1, \dots, d$  be the solution of the equation

$$(3.9) \quad (\lambda - 2\Delta) g_\lambda^j(\mathbf{z}) = \delta(\mathbf{z} + \mathbf{e}_j) - \delta(\mathbf{z} - \mathbf{e}_j), \quad d \geq 2,$$

with  $\lambda > 0$ . In  $d = 1$  the equation reads

$$(3.10) \quad -\frac{1}{3} \Delta [4g_\lambda(z) + g_\lambda(z+1) + g_\lambda(z-1)] + \lambda g_\lambda(z) = \delta(z+1) - \delta(z-1).$$

We observe that  $g_\lambda^j$  decays exponentially fast, and in particular the  $\ell^2$ -norm is finite. We by  $\hat{g}_\lambda^j$  its Fourier transform, namely

$$(3.11) \quad \hat{g}_\lambda^j(\mathbf{k}) = \frac{2i \sin(2\pi k_j)}{\Phi(\mathbf{k}) + \lambda},$$

where  $\Phi$  is defined by (2.11). We define

$$(3.12) \quad u_\lambda^i = \sum_{\mathbf{x}} g_\lambda^i(\mathbf{x}) \mathbf{q}_0 \cdot \mathbf{p}_\mathbf{x}, \quad i = 1, \dots, d.$$

so that

$$\alpha (\mathbf{p}_{\mathbf{e}_i} - \mathbf{p}_{-\mathbf{e}_i}) \cdot \mathbf{q}_{\mathbf{e}_i} = \alpha (\lambda - S) u_\lambda^i = \alpha \lambda u_\lambda^i - \frac{\alpha}{\gamma} L u_\lambda^i + \frac{\alpha}{\gamma} A u_\lambda^i.$$

Now we show that the contribution of the first two terms of the above to (3.5) will vanish as  $\varepsilon \rightarrow 0$ ,  $\lambda \rightarrow 0$ . We will use the following Lemma.

**Lemma 3.2.** *For every  $F \in L^2(\mathbb{R}^d)$ ,  $\forall i = 1, \dots, d$*

$$(3.13) \quad \left\langle \left( \varepsilon^{d/2} \sum_{\mathbf{x}} F(\varepsilon \mathbf{x}) \tau_{\mathbf{x}} u_\lambda^i \right)^2 \right\rangle \leq \beta^{-1} \|F\|^2 \int d\mathbf{k} \hat{\Gamma}(\mathbf{k}) |\hat{g}_\lambda^i(\mathbf{k})|^2.$$

*Proof.* By Schwarz inequality

$$\left\langle \left( \varepsilon^{d/2} \sum_{\mathbf{x}} F(\varepsilon \mathbf{x}) \tau_{\mathbf{x}} u_\lambda^i \right)^2 \right\rangle \leq \|F\|^2 \sum_{\mathbf{x}} \langle u_\lambda^i \tau_{\mathbf{x}} u_\lambda^i \rangle,$$

where

$$\begin{aligned} \sum_{\mathbf{x}} \langle u_\lambda^i \tau_{\mathbf{x}} u_\lambda^i \rangle &= \beta^{-1} \sum_{\mathbf{x}} \Gamma(\mathbf{x}) \sum_i \sum_{\mathbf{z}} g_\lambda^i(\mathbf{z}) g_\lambda^i(\mathbf{z} + \mathbf{x}) \\ &= \beta^{-1} \int d\mathbf{k} \hat{\Gamma}(\mathbf{k}) |\hat{g}_\lambda^i(\mathbf{k})|^2 \end{aligned}$$

□

Notice that if  $d \geq 3$  or  $\nu > 0$  the quantity  $\int d\mathbf{k} \hat{\Gamma}(\mathbf{k}) |\hat{g}_\lambda^i(\mathbf{k})|^2$  is bounded.

In order to prove that  $L u_\lambda^i$  does not contribute in the limit we have just to show that

$$\left\langle \left( \varepsilon^{-1} \int_0^t \varepsilon^{d/2} \sum_{\mathbf{x}} F(\varepsilon \mathbf{x}) \tau_{\mathbf{x}} L u_\lambda^i(s/\varepsilon^2) \right)^2 \right\rangle \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

We can perform the time integration and we deduce

$$\begin{aligned} &\varepsilon^{-1} \int_0^t \varepsilon^{d/2} \sum_{\mathbf{x}} F(\varepsilon \mathbf{x}) \tau_{\mathbf{x}} L u_\lambda^i(s/\varepsilon^2) \\ &= \varepsilon \varepsilon^{d/2} \sum_{\mathbf{x}} F(\varepsilon \mathbf{x}) \tau_{\mathbf{x}} [u_\lambda^i(t/\varepsilon^2) - u_\lambda^i(0)] + \varepsilon \mathcal{M}_\varepsilon(F, t), \end{aligned}$$

where  $\mathcal{M}_\varepsilon(F, t)$  is a martingale given by a stochastic integral. By Lemma 3.2 the first term on the rhs of the previous equality is of order  $\varepsilon$ . Moreover,  $\mathcal{M}_\varepsilon(F, t)$  has bounded quadratic variation, therefore the contribution of  $\varepsilon \mathcal{M}_\varepsilon(F, t)$  vanishes as  $\varepsilon \rightarrow 0$ .

**Lemma 3.3.** For every  $F \in L^2(\mathbb{R}^d)$ ,  $\forall i = 1, \dots, d$

$$\lim_{\lambda \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \left\langle \left( \frac{\lambda}{\varepsilon} \int_0^t ds \varepsilon^{d/2} \sum_{\mathbf{z}} F(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} u_{\lambda}^i(s/\varepsilon^2) \right)^2 \right\rangle = 0$$

*Proof.* The following inequality holds:

$$\begin{aligned} & \sup_{t \in [0, T]} \left\langle \left( \frac{\lambda}{\varepsilon} \int_0^t ds \varepsilon^{d/2} \sum_{\mathbf{z}} F(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} u_{\lambda}^i(s/\varepsilon^2) \right)^2 \right\rangle \\ & \leq CT \sup_f \left\{ 2 \frac{\lambda}{\varepsilon} \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \langle f \tau_{\mathbf{y}} u_{\lambda}^i \rangle - \frac{1}{\varepsilon^2} \langle f(-Sf) \rangle \right\}. \end{aligned}$$

Observe that

$$u_{\lambda}^i = \frac{1}{d-1} \sum_{\mathbf{x}} \sum_{k,j} G_{\lambda}(\mathbf{x}) q_0^j [X_{\mathbf{x}-\mathbf{e}_i}^{k,j} p_{\mathbf{x}-\mathbf{e}_i}^k + X_{\mathbf{x}+\mathbf{e}_i}^{k,j} p_{\mathbf{x}+\mathbf{e}_i}^k]$$

for  $d \geq 2$ , where  $G_{\lambda}(\mathbf{x})$  solves  $(\lambda - 2\Delta)G_{\lambda}(\mathbf{x}) = \delta(\mathbf{x})$ . A similar formula holds for  $d = 1$ . Then

$$\begin{aligned} & \frac{\lambda}{\varepsilon} \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \langle f \tau_{\mathbf{y}} u_{\lambda}^i \rangle \\ & = \frac{1}{d-1} \frac{\lambda}{\varepsilon} \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \sum_{\mathbf{x}} \sum_{k,j} G_{\lambda}(\mathbf{x} - \mathbf{y}) \{ \langle (X_{\mathbf{x}-\mathbf{e}_i}^{k,j} f) q_{\mathbf{y}}^j p_{\mathbf{x}-\mathbf{e}_i}^k \rangle \\ & \quad + \langle (X_{\mathbf{x}-\mathbf{e}_i}^{k,j} f) q_{\mathbf{y}}^j p_{\mathbf{x}-\mathbf{e}_i}^k \rangle \} \end{aligned}$$

which is bounded in absolute value by

$$\begin{aligned} & C \left( \lambda^2 \beta^{-1} \sum_{\mathbf{x}} \sum_j \left\langle \left( \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) G_{\lambda}(\mathbf{x} - \mathbf{y}) q_{\mathbf{y}}^j \right)^2 \right\rangle \right)^{1/2} \\ & \quad \times \left( \varepsilon^{-2} \langle f(-Sf) \rangle \right)^{1/2}. \end{aligned}$$

Observe that

$$\begin{aligned} & \lambda^2 \sum_{\mathbf{x}} \sum_j \left\langle \left( \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) G_{\lambda}(\mathbf{x} - \mathbf{y}) q_{\mathbf{y}}^j \right)^2 \right\rangle \\ & = \lambda^2 \varepsilon^d \sum_{\mathbf{y}, \mathbf{z}} F(\varepsilon \mathbf{y}) F(\varepsilon(\mathbf{y} + \mathbf{z})) \sum_{\mathbf{x}} G_{\lambda}(\mathbf{x}) G_{\lambda}(\mathbf{x} - \mathbf{z}) \langle \mathbf{q}_{\mathbf{z}} \cdot \mathbf{q}_0 \rangle \\ & \leq \lambda^2 \|F\|^2 \sum_{\mathbf{x}, \mathbf{z}} G_{\lambda}(\mathbf{x}) G_{\lambda}(\mathbf{x} - \mathbf{z}) \Gamma(\mathbf{z}), \end{aligned}$$

where in the last step we used Schwarz inequality. In the Fourier space

$$\lambda^2 \sum_{\mathbf{x}, \mathbf{z}} G_{\lambda}(\mathbf{x}) G_{\lambda}(\mathbf{x} - \mathbf{z}) \Gamma(\mathbf{z}) = \lambda^2 \int_{\mathbb{T}^d} d\mathbf{k} \frac{\hat{\Gamma}(\mathbf{k})}{(\lambda + \Phi(\mathbf{k}))^2},$$

which vanishes as  $\lambda \rightarrow 0$  for  $\nu > 0$  or  $d \geq 3$ . Therefore

$$\begin{aligned} & \sup_f \left\{ 2 \frac{\lambda}{\varepsilon} \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \langle f \tau_{\mathbf{y}} u_{\lambda}^i \rangle - \frac{1}{\varepsilon^2} \langle f(-Sf) \rangle \right\} \\ & \leq C_0 \|F\|^2 \lambda^2 \sum_{\mathbf{x}, \mathbf{z}} G_{\lambda}(\mathbf{x}) G_{\lambda}(\mathbf{x} - \mathbf{z}) \Gamma(\mathbf{z}) \end{aligned}$$

which vanishes for  $\lambda \rightarrow 0$ . □

Observe that

$$\frac{\alpha}{\gamma} A u_{\lambda}^i = \frac{\alpha}{\gamma} \sum_{\mathbf{x}} g_{\lambda}^i(\mathbf{x}) \left\{ \mathbf{p}_0 \cdot \mathbf{p}_{\mathbf{x}} + \mathbf{q}_0 \cdot (\alpha \Delta - \nu I) \mathbf{q}_{\mathbf{x}} \right\}.$$

So we have to look at

$$\begin{aligned} & \frac{\alpha}{\gamma} \varepsilon^{d-1} \sum_i \sum_{\mathbf{y}, \mathbf{z}} \nabla_{\mathbf{e}_i}^{\varepsilon} F(\varepsilon(\mathbf{y} + \mathbf{z})) H(\varepsilon \mathbf{y}) \tau_{\mathbf{z}} A u_{\lambda}^i \\ & = \frac{\alpha}{\gamma} \varepsilon^{d-1} \sum_i \sum_{\mathbf{y}, \mathbf{z}} \nabla_{\mathbf{e}_i}^{\varepsilon} F(\varepsilon(\mathbf{y} + \mathbf{z})) H(\varepsilon \mathbf{y}) \sum_{\mathbf{x}} g_{\lambda}^i(\mathbf{x}) \left[ \mathbf{p}_{\mathbf{z}} \cdot \mathbf{p}_{\mathbf{x}+\mathbf{z}} + \mathbf{q}_{\mathbf{z}} \cdot (\alpha \Delta - \nu I) \mathbf{q}_{\mathbf{x}+\mathbf{z}} \right] \\ & = \frac{\alpha}{\gamma} \varepsilon^{d-1} \sum_i \sum_{\mathbf{y}, \mathbf{z}} \nabla_{\mathbf{e}_i}^{\varepsilon} F(\varepsilon(\mathbf{y} + \mathbf{z})) H(\varepsilon \mathbf{y}) \sum_{\mathbf{x}} g_{\lambda}^i(\mathbf{x} - \mathbf{z}) \left[ \mathbf{p}_{\mathbf{z}} \cdot \mathbf{p}_{\mathbf{x}} + \mathbf{q}_{\mathbf{z}} \cdot (\alpha \Delta - \nu I) \mathbf{q}_{\mathbf{x}} \right] \end{aligned}$$

We are left to study:

$$\begin{aligned} K_2 & = \sum_i \frac{\alpha}{\gamma} \varepsilon^d \sum_{\mathbf{y}, \mathbf{z}} H(\varepsilon \mathbf{y}) \nabla_{\mathbf{e}_i}^{\varepsilon} F(\varepsilon(\mathbf{y} + \mathbf{z})) \sum_{\mathbf{x}} g_{\lambda}^i(\mathbf{x}) \\ & \quad \times \frac{1}{\varepsilon} \int_0^t ds \langle \tau_{\mathbf{z}} [\mathbf{p}_0 \cdot \mathbf{p}_{\mathbf{x}} + \mathbf{q}_0 \cdot (\alpha \Delta - \nu I) \mathbf{q}_{\mathbf{x}}] (s/\varepsilon^2) (e_0(0) - \beta^{-1}) \rangle. \end{aligned}$$

Remark that

$$A \frac{1}{2} [\mathbf{p}_0 \cdot \mathbf{q}_{\mathbf{x}} + \mathbf{q}_0 \cdot \mathbf{p}_{\mathbf{x}}] = \frac{1}{2} [\mathbf{q}_0 \cdot (\alpha \Delta - \nu I) \mathbf{q}_{\mathbf{x}} + (\alpha \Delta - \nu I) \mathbf{q}_0 \cdot \mathbf{q}_{\mathbf{x}}] + \mathbf{p}_0 \cdot \mathbf{p}_{\mathbf{x}},$$

that implies

$$\begin{aligned} & \mathbf{p}_0 \cdot \mathbf{p}_{\mathbf{x}} + \mathbf{q}_0 \cdot (\alpha \Delta - \nu I) \mathbf{q}_{\mathbf{x}} \\ & = \frac{1}{2} [\mathbf{q}_0 \cdot (\alpha \Delta - \nu I) \mathbf{q}_{\mathbf{x}} - (\alpha \Delta - \nu I) \mathbf{q}_0 \cdot \mathbf{q}_{\mathbf{x}}] + \frac{1}{2} A [\mathbf{p}_0 \cdot \mathbf{q}_{\mathbf{x}} + \mathbf{q}_0 \cdot \mathbf{p}_{\mathbf{x}}] \\ & = \frac{1}{2} \sum_j \nabla_{\mathbf{e}_j} [\mathbf{q}_{\mathbf{x}} \cdot \mathbf{q}_{-\mathbf{e}_j} - \mathbf{q}_0 \cdot \mathbf{q}_{\mathbf{x}-\mathbf{e}_j}] + \frac{1}{2} (L - \gamma S) [\mathbf{p}_0 \cdot \mathbf{q}_{\mathbf{x}} + \mathbf{q}_0 \cdot \mathbf{p}_{\mathbf{x}}] \end{aligned}$$

The last term on the right hand side gives a negligible contribution as  $\varepsilon \rightarrow 0$ . The first term is a gradient and by summation by part gives

$$\begin{aligned}
 (3.14) \quad & \frac{\alpha}{\gamma} \varepsilon^d \sum_{i,j} \sum_{\mathbf{y}, \mathbf{z}} H(\varepsilon \mathbf{y}) \nabla_j^{\varepsilon, *} \nabla_i^\varepsilon F(\varepsilon(\mathbf{y} + \mathbf{z})) \sum_{\mathbf{x}} g_\lambda^i(\mathbf{x}) \\
 & \int_0^t ds \langle \tau_{\mathbf{z}}[\mathbf{q}_{\mathbf{x}} \cdot \mathbf{q}_{-\mathbf{e}_j} - \mathbf{q}_0 \cdot \mathbf{q}_{\mathbf{x}-\mathbf{e}_j}](s/\varepsilon^2)(e_0(0) - \beta^{-1}) \rangle \\
 & = \frac{\alpha}{\gamma} \varepsilon^d \sum_{i,j} \sum_{\mathbf{y}, \mathbf{z}} H(\varepsilon \mathbf{y}) \nabla_j^{\varepsilon, *} \nabla_i^\varepsilon F(\varepsilon \mathbf{z}) \sum_{\mathbf{x}} g_\lambda^i(\mathbf{x}) \\
 & \int_0^t ds \langle \tau_{\mathbf{z}}[\mathbf{q}_{\mathbf{x}} \cdot \mathbf{q}_{-\mathbf{e}_j} - \mathbf{q}_0 \cdot \mathbf{q}_{\mathbf{x}-\mathbf{e}_j}](s/\varepsilon^2)(e_0(\mathbf{y}) - \beta^{-1}) \rangle
 \end{aligned}$$

Observe that

$$\begin{aligned}
 (3.15) \quad & \sum_{\mathbf{z}} G(\varepsilon \mathbf{z}) \sum_{\mathbf{x}} g_\lambda^j(\mathbf{x}) \tau_{\mathbf{z}}[\mathbf{q}_{\mathbf{x}} \cdot \mathbf{q}_{-\mathbf{e}_i} - \mathbf{q}_0 \cdot \mathbf{q}_{\mathbf{x}-\mathbf{e}_i}] \\
 & = \sum_{\mathbf{z}} G(\varepsilon \mathbf{z}) \sum_{\mathbf{x}} [g_\lambda^j(\mathbf{x} - \mathbf{e}_i) - g_\lambda^j(\mathbf{x} + \mathbf{e}_i)] \tau_{\mathbf{z}}[\mathbf{q}_{\mathbf{x}} \cdot \mathbf{q}_0] (1 + \mathcal{O}(\varepsilon)).
 \end{aligned}$$

We denote by  $c_\lambda^{i,j}(\mathbf{x}) := g_\lambda^j(\mathbf{x} - \mathbf{e}_i) - g_\lambda^j(\mathbf{x} + \mathbf{e}_i)$ . Let  $f(\mathbf{z})$  be the function satisfying

$$(\alpha \Delta - \nu I) f(\mathbf{z}) = \delta(\mathbf{z}).$$

By direct computation

$$\begin{aligned}
 (3.16) \quad & \sum_{\mathbf{x}} c_\lambda^{i,j}(\mathbf{x}) \mathbf{q}_{\mathbf{x}} \cdot \mathbf{q}_0 = (L - \gamma S) [\mathbf{q}_0 \cdot \sum_{\mathbf{z}} \sum_{\mathbf{x}} c_\lambda^{i,j}(\mathbf{x}) f(\mathbf{z} - \mathbf{x}) \mathbf{p}_{\mathbf{z}}] \\
 & - \mathbf{p}_0 \cdot \sum_{\mathbf{z}} \sum_{\mathbf{x}} c_\lambda^{i,j}(\mathbf{x}) f(\mathbf{z} - \mathbf{x}) \mathbf{p}_{\mathbf{z}}.
 \end{aligned}$$

The function  $\tilde{\kappa}_\lambda^{i,j}(\mathbf{z}) := -\sum_{\mathbf{x}} c_\lambda^{i,j}(\mathbf{x}) f(\mathbf{z} - \mathbf{x})$  decay exponentially fast, since the Fourier transform is given by

$$\frac{4 \sin(2\pi k_i) \sin(2\pi k_j)}{\omega^2(\mathbf{k})} \frac{1}{\Phi(\mathbf{k}) + \lambda}.$$

Therefore one can show that the the term  $(L - \gamma S)(\cdot)$  gives a contribution of order  $\varepsilon$  and (3.14) is equal to

$$\begin{aligned}
 (3.17) \quad & \frac{1}{8\pi^2 \gamma} \varepsilon^d \sum_{\mathbf{y}, \mathbf{z}} \sum_{j,\ell} H(\varepsilon \mathbf{y}) \nabla_j^{\varepsilon, *} \nabla_\ell^\varepsilon F(\varepsilon(\mathbf{y} + \mathbf{z})) \int_0^t ds \sum_{\mathbf{x}} \kappa_\lambda^{j,\ell}(\mathbf{x}) \\
 & \times \langle [\mathbf{p}_{\mathbf{z}} \cdot \mathbf{p}_{\mathbf{x}+\mathbf{z}}](s/\varepsilon^2)(e_0(0) - \beta^{-1}) \rangle + \mathcal{O}(\varepsilon),
 \end{aligned}$$

where

$$\kappa_\lambda^{j,\ell}(\mathbf{x}) = \int_{\mathbb{T}^d} d\xi \frac{\partial_j \omega^2(\xi) \partial_\ell \omega^2(\xi)}{\omega^2(\xi)} \frac{1}{\Phi(\xi) + \lambda} e^{2i\pi \mathbf{x} \cdot \xi}, \quad \ell, i = 1, \dots, d.$$

Consequently the quantity (3.17) is asymptotic to

$$\begin{aligned}
(3.18) \quad & \frac{1}{8\pi^2 \gamma} \varepsilon^d \sum_{\mathbf{y}, \mathbf{z}} \sum_{j, \ell} H(\varepsilon \mathbf{y}) \nabla_j^{\varepsilon, *} \nabla_\ell^\varepsilon F(\varepsilon(\mathbf{y} + \mathbf{z})) \\
& \times \int_0^t ds \kappa_\lambda^{j, \ell}(\mathbf{0}) \langle e_{\mathbf{z}}(s/\varepsilon^2) (e_{\mathbf{0}}(0) - \beta^{-1}) \rangle \\
& + \frac{1}{8\pi^2 \gamma} \varepsilon^d \sum_{\mathbf{y}, \mathbf{z}} \sum_{j, \ell} H(\varepsilon \mathbf{y}) \nabla_j^{\varepsilon, *} \nabla_\ell^\varepsilon F(\varepsilon(\mathbf{y} + \mathbf{z})) \\
& \times \int_0^t ds \sum_{\mathbf{x} \neq \mathbf{0}} \kappa_\lambda^{j, \ell}(\mathbf{x}) \langle [\mathbf{p}_{\mathbf{z}} \cdot \mathbf{p}_{\mathbf{x}+\mathbf{z}}](s/\varepsilon^2) (e_{\mathbf{0}}(0) - \beta^{-1}) \rangle.
\end{aligned}$$

We observe that  $\kappa_\lambda^{j, \ell}(\mathbf{0}) = 0$  if  $j \neq \ell$  and  $\kappa_\lambda^{j, j}(\mathbf{0}) = \kappa_\lambda^{1, 1}(\mathbf{0}) \forall j = 1, \dots, d$ . We set

$$(3.19) \quad \kappa_\lambda := \kappa_\lambda^{1, 1}(\mathbf{0}) = \int_{\mathbb{T}^d} d\xi \left( \frac{\partial_1 \omega^2(\xi)}{\omega(\xi)} \right)^2 \frac{1}{\Phi(\xi) + \lambda}.$$

Then (3.18) is equal to

$$\begin{aligned}
(3.20) \quad & \frac{1}{8\pi^2 \gamma} \varepsilon^d \sum_{\mathbf{y}, \mathbf{y}'} H(\varepsilon \mathbf{y}) \kappa_\lambda \Delta F(\varepsilon \mathbf{y}') \int_0^t ds \langle e_{\mathbf{y}'}(s/\varepsilon^2) (e_{\mathbf{y}}(0) - \beta^{-1}) \rangle \\
& + \frac{1}{8\pi^2 \gamma} \varepsilon^d \sum_{\mathbf{y}, \mathbf{z}} \sum_{j, \ell} H(\varepsilon \mathbf{y}) \nabla_j^{\varepsilon, *} \nabla_\ell^\varepsilon F(\varepsilon(\mathbf{y} + \mathbf{z})) \\
& \times \int_0^t ds \sum_{\mathbf{x} \neq \mathbf{0}} \kappa_\lambda^{j, \ell}(\mathbf{x}) \langle [\mathbf{p}_{\mathbf{z}} \cdot \mathbf{p}_{\mathbf{x}+\mathbf{z}}](s/\varepsilon^2) (e_{\mathbf{0}}(0) - \beta^{-1}) \rangle.
\end{aligned}$$

We will prove in the next section that the second term vanishes as  $\varepsilon \rightarrow 0$ ,  $\lambda \rightarrow 0$ .

#### 4. BOLTZMANN-GIBBS PRINCIPLE

By using as above the Schwarz inequality, all we need to prove is that

$$(4.1) \quad \sum_{\mathbf{x} \neq \mathbf{0}} |\kappa_\lambda^{i, \ell}(\mathbf{x})| \left\langle \left( \int_0^t ds \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \mathbf{p}_{\mathbf{y}} \cdot \mathbf{p}_{\mathbf{y}+\mathbf{x}}(s/\varepsilon^2) \right)^2 \right\rangle^{1/2}$$

is negligible as  $\varepsilon \rightarrow 0$ .

We denote the cube in  $\mathbb{Z}^d$  of size  $2\ell + 1$  by  $\Lambda_\ell := \{\mathbf{z} \in \mathbb{Z}^d : |z^j| \leq \ell, j = 1, \dots, d\}$ , and for every  $x \in \mathbb{Z}$  we define  $\Psi_{\ell, x} := \frac{1}{|\Lambda_\ell|} \sum_{\mathbf{z} \in \Lambda_\ell} \mathbf{p}_{\mathbf{z}} \cdot \mathbf{p}_{\mathbf{z}+\mathbf{x}}$ . We observe that in (4.1) we can replace  $\mathbf{p}_{\mathbf{y}} \cdot \mathbf{p}_{\mathbf{y}+\mathbf{x}}$  with  $\tau_{\mathbf{y}} \Psi_{\ell, \mathbf{x}}$ , with a difference  $\sim \varepsilon |\Lambda_\ell| \|\kappa_\lambda\|_{\ell^1}$ . We denote by  $\langle \cdot \rangle_{\Lambda_K} = \langle \cdot \rangle_{\Lambda_K, \mathcal{T}_K, \mathcal{P}_K}$  the micro-canonical expectation in the box  $\Lambda_K$ , with  $\mathcal{T}_K = \sum_{\mathbf{x} \in \Lambda_K} \mathbf{p}_{\mathbf{x}}^2$  and

$\mathcal{P}_K = \sum_{\mathbf{x} \in \Lambda_K} \mathbf{p}_x$ . We define  $\tilde{\Psi}_{\ell, \mathbf{x}} := \Psi_{\ell, \mathbf{x}} - \langle \Psi_{\ell, \mathbf{x}} \rangle_{\Lambda_{\ell+|\mathbf{x}|}}$ . Then

$$(4.2) \quad \left\langle \left( \int_0^t ds \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} \tilde{\Psi}_{\ell, \mathbf{x}}(s/\varepsilon^2) \right)^2 \right\rangle \\ \leq Ct \sup_f \left\{ \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \langle f \tau_{\mathbf{y}} \tilde{\Psi}_{\ell, \mathbf{x}} \rangle - \varepsilon^{-2} \langle f(-Sf) \rangle \right\}.$$

Introduce  $S_{\Lambda_K} = \frac{1}{4(d-1)} \sum_{\substack{\mathbf{x}, \mathbf{z} \in \Lambda_K \\ \|\mathbf{x}-\mathbf{z}\|=1}} \sum_{i,j} (X_{\mathbf{x}, \mathbf{z}}^{i,j})^2$ . By the spectral gap of

$S_K$ , there exists  $\tilde{\Psi}_{\ell, \mathbf{x}} = S_{\Lambda_{\ell+|\mathbf{x}|}} U_{\ell, \mathbf{x}}$ ,  $\forall \mathbf{x}, \forall \ell$ . Moreover, since the spectral gap of  $S_{\Lambda_K}$  is bounded below by  $CK^{-2}$ , we have that  $\langle U_{\ell, \mathbf{x}} \tilde{\Psi}_{\ell, \mathbf{x}} \rangle^2 \leq C(\ell + |\mathbf{x}|)^2 \langle \tilde{\Psi}_{\ell, \mathbf{x}}^2 \rangle \leq C(\ell + |\mathbf{x}|)^2 \beta^{-1} \ell^{-d}$ .

Then since

$$\langle f \tau_{\mathbf{y}} \tilde{\Psi}_{\ell, \mathbf{x}} \rangle \leq \langle U_{\ell, \mathbf{x}} \tilde{\Psi}_{\ell, \mathbf{x}} \rangle^{1/2} \langle \tau_{\mathbf{y}} f (-S_{\Lambda_{\ell+|\mathbf{x}|}} \tau_{\mathbf{y}} f) \rangle^{1/2}.$$

we can bound the right hand side of (4.2) by

$$Ct \sum_{\mathbf{y}} \sup_f \left\{ \varepsilon^{d/2} F(\varepsilon \mathbf{y}) \langle f \tau_{\mathbf{y}} \tilde{\Psi}_{\ell, \mathbf{x}} \rangle - \varepsilon^{-2} |\Lambda_{\ell+|\mathbf{x}|}|^{-1} \langle \tau_{\mathbf{y}} f (-S_{\Lambda_{\ell+|\mathbf{x}|}} \tau_{\mathbf{y}} f) \rangle \right\} \\ \leq Ct \sum_{\mathbf{y}} \sup_f \left\{ \varepsilon^{d/2} F(\varepsilon \mathbf{y}) C(\ell + |\mathbf{x}|) \langle \tilde{\Psi}_{\ell, \mathbf{x}}^2 \rangle^{1/2} \langle \tau_{\mathbf{y}} f (-S_{\Lambda_{\ell+|\mathbf{x}|}} \tau_{\mathbf{y}} f) \rangle^{1/2} \right. \\ \left. - \varepsilon^{-2} |\Lambda_{\ell+|\mathbf{x}|}|^{-1} \langle \tau_{\mathbf{y}} f (-S_{\Lambda_{\ell+|\mathbf{x}|}} \tau_{\mathbf{y}} f) \rangle \right\} \\ \leq C't \varepsilon^{d+2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y})^2 (\ell + |\mathbf{x}|)^{d+2} \langle \tilde{\Psi}_{\ell, \mathbf{x}}^2 \rangle.$$

Since  $\sum_{\mathbf{x}} \kappa_{\lambda}^{i, \ell}(\mathbf{x}) (\ell + |\mathbf{x}|)^{(d+2)/2} < \infty$  we conclude that

$$\lim_{\varepsilon \rightarrow 0} \sum_{\mathbf{x} \neq 0} |\kappa_{\lambda}^{i, \ell}(\mathbf{x})| \left\langle \left( \int_0^t ds \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} \tilde{\Psi}_{\ell, \mathbf{x}}(s/\varepsilon^2) \right)^2 \right\rangle^{1/2} = 0.$$

Setting  $\bar{\Psi}_{\ell, \mathbf{x}} := \langle \Psi_{\ell, \mathbf{x}} \rangle_{\Lambda_{\ell+|\mathbf{x}|}}$ , now we have just to show that

$$\lim_{\varepsilon \rightarrow 0} \sum_{\mathbf{x} \neq 0} |\kappa_{\lambda}(\mathbf{x})| \left\langle \left( \int_0^t ds \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} \bar{\Psi}_{\ell, \mathbf{x}}(s/\varepsilon^2) \right)^2 \right\rangle^{1/2} = 0.$$

The previous expression is bounded by

$$t \sum_{\mathbf{x} \neq 0} |\kappa_{\lambda}(\mathbf{x})| \left\langle \left( \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} \bar{\Psi}_{\ell, \mathbf{x}} \right)^2 \right\rangle^{1/2}.$$



We observe that

$$\begin{aligned}
& \left\langle \left( \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} \bar{\Psi}_{\ell, \mathbf{x}} \right)^2 \right\rangle \\
& \leq \varepsilon^d \sum_{\mathbf{y}, \mathbf{y}'} F(\varepsilon \mathbf{y}) F(\varepsilon \mathbf{y}') \langle \tau_{\mathbf{y}-\mathbf{y}'} \bar{\Psi}_{\ell, \mathbf{x}} \bar{\Psi}_{\ell, \mathbf{x}} \rangle \\
& \leq \frac{\varepsilon^d}{2} \sum_{\mathbf{y}, \mathbf{y}'} (F(\varepsilon \mathbf{y})^2 + F(\varepsilon \mathbf{y}')^2) \langle \tau_{\mathbf{y}-\mathbf{y}'} \bar{\Psi}_{\ell, \mathbf{x}} \bar{\Psi}_{\ell, \mathbf{x}} \rangle \\
& \leq \|F\|_{L^2(\mathbb{R}^d)}^2 \sum_{|\mathbf{z}| \leq 2(\ell + |\mathbf{x}|)} \langle \tau_{\mathbf{z}} \bar{\Psi}_{\ell, \mathbf{x}} \bar{\Psi}_{\ell, \mathbf{x}} \rangle \\
& \leq C \|F\|_{L^2(\mathbb{R}^d)}^2 (\ell + |\mathbf{x}|)^d \langle \bar{\Psi}_{\ell, \mathbf{x}}^2 \rangle.
\end{aligned}$$

By the properties of the microcanonical measure it holds  $\langle \bar{\Psi}_{\ell, \mathbf{x}}^2 \rangle \leq C_0 \beta^{-2} (\ell + |\mathbf{x}|)^{-2d}$ , and therefore

$$\begin{aligned}
& t \sum_{x, x \neq 0} |\kappa_\lambda(x)| \left\langle \left( \varepsilon^{d/2} \sum_{\mathbf{y}} F'(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} \bar{\Psi}_{\ell, \mathbf{x}} \right)^2 \right\rangle^{1/2} \\
& \leq C_1 T \beta^{-1} \|F'\|_{L^2(\mathbb{R}^d)} \|\kappa_\lambda\|_{\ell^1} \frac{1}{\ell^{d/2}},
\end{aligned}$$

which vanishes as  $\ell \rightarrow \infty$ .

## 5. APPENDIX A

Let  $\phi(\mathbf{x}, \mathbf{y})$  a local function on  $\mathbb{Z}^d \times \mathbb{Z}^d$  and define

$$\Phi = \sum_{\mathbf{x}, \mathbf{y}} \phi(\mathbf{x}, \mathbf{y}) \mathbf{p}_{\mathbf{x}} \cdot \mathbf{q}_{\mathbf{y}}$$

Consider  $F \in L^2(R^d)$ . We want to find a good upper bound for the variance:

$$(5.1) \quad \left\langle \left( \int_0^t ds \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} S \Phi(s/\varepsilon^2) \right)^2 \right\rangle$$

By lemma (2.4) in [9], pag. 48, we have

$$\begin{aligned}
& \left\langle \left( \sup_{0 \leq s \leq T} \int_0^t ds \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} S \Phi(s/\varepsilon^2) \right)^2 \right\rangle \\
& \leq 24T \varepsilon^2 \left\langle \left( \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} \Phi \right) (-S) \left( \varepsilon^{d/2} \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} \Phi \right) \right\rangle \\
& = \frac{48T}{\beta} \varepsilon^{d+2} \sum_{\mathbf{y}, \mathbf{y}'} F(\varepsilon \mathbf{y}) F(\varepsilon \mathbf{y}') \Xi(\mathbf{y} - \mathbf{y}')
\end{aligned}$$

where

$$(5.2) \quad \Xi(\mathbf{y}) = \sum_{\mathbf{x}, \mathbf{x}', \mathbf{z}} \phi(\mathbf{x}, \mathbf{x}') (\Delta_1 \phi)(\mathbf{x} + \mathbf{y}, \mathbf{z}) \Gamma(\mathbf{y} + \mathbf{x}' - \mathbf{z})$$

where  $\Delta_1 \phi$  indicates the discrete laplacian on the first variable of  $\phi$ .

Then by Schwarz inequality we can bound (5.1) by

$$(5.3) \quad \|F\|^2 \varepsilon^2 \sum_{\mathbf{y}} |\Xi(\mathbf{y})|.$$

## 6. FORMULAIRE

Some formulas we use:

$$(6.1) \quad S\mathbf{p}_x = 2\Delta\mathbf{p}_x \quad d \geq 2$$

$$(6.2) \quad Sp_x = \frac{1}{6}\Delta(4p_x + p_{x+1} + p_{x-1}) \quad d = 1$$

$$(6.3) \quad Se_x = S\mathbf{p}_x^2/2 = \Delta\mathbf{p}_x \quad d \geq 2$$

$$(6.4) \quad Se_x = Sp_x^2/2 = \frac{1}{6}\Delta(4p_x^2 + p_{x+1}^2 + p_{x-1}^2) \quad d = 1$$

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