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ANALYTIC PRECONDITIONERS FOR THE ITERATIVE SOLUTION OF ELASTIC SCATTERING PROBLEMS

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Abstract. We construct and analyze a family of well-conditioned boundary integral equations for the Krylov iterative solution of three-dimensional elastic scattering problems by a bounded rigid obstacle. We develop a new potential theory using a rewriting of the Somigliana integral representation formula. From these results, we generalize to linear elasticity the well-known Brakhage-Werner and Combined Field Integral Equation formulations. We use a suitable approximation of the Dirichlet-to-Neumann (DtN) map as a regularizing operator in the proposed boundary integral equations. The construction of the approximate DtN map is inspired by the On-Surface Radiation Conditions method. We prove that the associated integral equations are uniquely solvable and possess very interesting spectral properties. Promising numerical experiments with the elastic sphere are provided.

1. Introduction

The paper is concerned with the preconditioning of iterative methods for the scattering problem of time-harmonic elastic waves by a three-dimensional rigid obstacle. In recent years, interest has grown considerably in developing effective numerical methods for solving these problems due to the various biomedical or industrial applications such as non-destructive testing of materials and seismic exploration.

We assume the obstacle be represented by a bounded domain Ω^- in $\mathbb{R}^3$. Let Ω^+ denote the exterior domain $\mathbb{R}^3\backslash\overline{\Omega^-}$ and $\boldsymbol{n}$ the outer unit normal vector to the boundary $\Gamma := \partial\Omega^-$. The Lamé parameters μ and λ and the density ρ are positive constants. The propagation of time-harmonic waves in a three-dimensional isotropic and homogeneous elastic medium is governed by the Navier equation

$$\operatorname{div}\sigma(\boldsymbol{u}) + \rho\omega^2\boldsymbol{u} = 0, \tag{1.1a}$$

where $\omega > 0$ is the frequency. Here

$$\sigma(\boldsymbol{u}) = \lambda(\operatorname{div}\boldsymbol{u})\mathrm{I}_3 + 2\mu\varepsilon(\boldsymbol{u}) \quad \text{and} \quad \varepsilon(\boldsymbol{u}) = \frac{1}{2}\big([\boldsymbol{\nabla u}] + [\boldsymbol{\nabla u}]^{\mathsf{T}}\big)$$

denote the stress tensor and the strain tensor respectively. Notice that I_3 is the 3-by-3 identity matrix and $[\boldsymbol{\nabla u}]$ is the matrix whose the j-th column is the gradient of the j-th component of $\boldsymbol{u}$. The scattering problem is formulated as follows : Given an incident wave $\boldsymbol{u}^{inc}$ which is assumed to solve the Navier equation in the absence of any scatterer, find the solution $\boldsymbol{u}$ to the Navier equation (1.1a) in Ω^+ which satisfies the Dirichlet boundary condition

$$\boldsymbol{u} = -\boldsymbol{u}^{inc} \quad \text{on } \Gamma. \tag{1.1b}$$

In addition the field $\boldsymbol{u}$ has to satisfy the Kupradze radiation conditions

$$\lim_{r\to\infty} r\left(\frac{\partial \boldsymbol{u}_p}{\partial r} - i\kappa_p\boldsymbol{u}_p\right) = 0, \qquad \lim_{r\to\infty} r\left(\frac{\partial \boldsymbol{u}_s}{\partial r} - i\kappa_s\boldsymbol{u}_s\right) = 0, \quad r = |\boldsymbol{x}|, \tag{1.1c}$$

uniformly in all directions. The field $\boldsymbol{u}$ can be decomposed into a longitudinal field $\boldsymbol{u}_p$ with vanishing curl and a transverse field $\boldsymbol{u}_s$, with vanishing divergence, solutions respectively to

$$\Delta\boldsymbol{u}_p + \kappa_p^2\boldsymbol{u}_p = 0 \text{ and } \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_s - \kappa_s^2\boldsymbol{u}_s = 0, \tag{1.2}$$

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with $\kappa_p^2 = \rho\omega^2(\lambda + 2\mu)^{-1}$ and $\kappa_s^2 = \rho\omega^2\mu^{-1}$.

The main difficulty for simulating such exterior boundary-value problems is related to the unboundedness of the computational domain Ω^+. The method of boundary integral equations (BIEs) is one of the principal tools to overcome this concern (see e.g. [9, 26, 31, 30]). This approach applies the Green's function formalism to reduce equivalently the governing boundary value problem to an integral equation on the surface of the scatterer. Its efficiency has been widely demonstrated for low and middle frequencies using collocation or boundary element methods (see e.g. [9, 10]). However, solving the exterior Navier problem in the high-frequency regime remains a very challenging problem for a few reasons. The discretization of boundary integral formulations leads to fully-populated and in general, non-hermitian complex linear systems. To retain a certain level of accuracy in the high-oscillatory solution, one has to consider a sufficient number of points per wavelength (about 10 points). Moreover, the vector nature of the unknown field greatly increases the size of the scattering-matrices by comparison to the acoustic case. The solution of these large and dense linear systems are handled by Krylov-subspaces iterative solvers (e.g. GMRES [39]). To make possible the use of iterative solvers for scattering problems, two kinds of techniques have to be investigated: fast methods for the computation of matrix-vector products and preconditioners to speed up the convergence of the solver.

The Fast Multipole Method (FMM) is one of the most efficient methods used to perform matrix-vector products and accelerate the resolution of the linear system. The method has been introduced by Rokhlin *et al* (e.g. [16]) and was adapted to integral equations of wave propagation in the 90's (e.g. [15, 21, 22]). Recently, FMM have been extended to linear elasticity by Bonnet, Chaillat and Semblat [13, 14]. The theoretical complexity of the multi-level FMM is $\mathcal{O}(N\log N)$ per iteration both for CPU time and memory, where N is the number of degrees of freedom.

Preconditioners are prescribed to yield fast convergence independently of both mesh size and frequency. A recent class of analytic preconditioners has been derived to regularize the classical BIEs. The idea is to consider a judicious integral representation of the scattered field which naturally incorporates a regularizing operator. This operator is an approximation of the Dirichlet-to-Neumann (DtN) map, when considering Dirichlet boundary value problems. The BIEs arising from this representation are compact perturbations of the identity operator. Such integral formulations can be interpreted as generalizations of the well-known Brakhage-Werner (BW) [11] and Combined Field Integral Equations (CFIE) [25]. Several well-conditioned integral equations based on this formalism have already been proposed in acoustic and electromagnetic scattering (e.g. [3, 5, 6, 12, 19, 33, 37]). A pseudo inverse of the principal classical symbol of the single layer boundary integral operator - or equivalently the principal classical symbol of the Neuman trace of the double layer boundary integral operator - is used to approach the DtN map [7, 6, 19] in the framework of the On-Surface Radiation Condition (OSRC) methods (e.g. [29, 27, 4]). This is intuitively natural in view of the Calderón formulas and the compactness of the double layer boundary integral operator. The resulting preconditioner can be expressed analytically by a simple square root of the form $i\sqrt{\Delta_\Gamma + \kappa^2 \mathrm{I}}$. More precisely, a regularization of the square root, by considering a small complex perturbation of the exterior wavenumber κ, is considered instead. This allows to model the creeping waves and provides the existence and uniqueness of the solution to the BIEs for any positive real values of the frequency. Furthermore, a complex Padé-approximation of the square root operator [35] leads to a sparse matrix involving only the mass and rigidity finite element matrices. This makes the implementation of the analytic preconditioner rather easy with a low additional computational cost.

It is the purpose of this paper to extend to linear elasticity such analytic preconditioning techniques. In this work we encounter some new difficulties for many reasons. First of all, one knows that the double layer boundary integral operator and its adjoint are not compact. This implies that regularizing the BIEs via a pseudo inverse of the single layer boundary integral operator is not sufficient to obtain well-conditioned boundary integral equations. We have to take into account the principal part of the double layer boundary integral operator or its adjoint. Secondly, we have to deal with longitudinal and transversal waves and the expression of the preconditioner is, by consequence, more complicated than in the acoustic and electromagnetic cases. Finally, the

resulting preconditioner - expressed in a suitable vector function basis - is not diagonal and consists in sums and products of square roots and their inverse and of surface differential operators. Here, we suggest a more appropriate preconditioner for a numerical implementation by considering a rewriting of the Somigliana integral representation formula. According to this representation, we introduce a new Neumann-type trace by adding a surface differential operator to the traction trace. This allows to develop a new potential theory for the Navier equation. The expression of the preconditioner is greatly simplified and previous works on the implementation of such OSRC-like preconditioner can be applied. The associated preconditioned BIEs are well posed and show very interesting spectral properties. Contrary to the acoustic and electromagnetic cases, defining the Neumann-to-Dirichlet preconditioner as the inverse of the Dirichlet-to-Neumann preconditioner is not sufficient to construct well-conditioned BIEs for the Neumann (or cavity) scattering problem. A more extensive analysis of the DtN and NtD maps has to be realized in the transition region corresponding to the grazing modes.

The paper is organized as follows: In section 2, we recall several results on the trace and potential theory in linear elasticity and review the standard Brakhage-Werner-type and combined field integral equations. In section 3, we present two preconditioning techniques based on a DtN approach and a modified DtN approach for the Brakhage-Werner-type (BW) BIEs. The computation of the principal parts of the elementary boundary integral operators is described in detail. This step is the key one to obtain the approximations of the DtN and modified DtN maps. We also prove existence and uniqueness for the proposed preconditioned formulations. Section 4 is devoted to a numerical investigation of the eigenvalues of the classical and preconditioned BW integral operators in the case of the elastic sphere. Furthermore, we study the convergence behavior of GMRES in function of both a mesh refinement and a frequency increase. Preconditioned CFIEs are built in section 5. We give a sketch of the numerical implementation of these novel BIEs in section 6. Finally, we draw concluding remarks and we discuss possible research lines in section 7.

2. Classical boundary integral equations

In a first part, we review some well-known results about traces of vector fields, and integral representations of time-harmonic elastic fields. Details can be found in [2, 18, 31, 30]. We introduce a new Neumann-type trace and develop the corresponding potential theory. In a second part, we describe the classical direct and indirect approaches usually used to solve time-harmonic exterior elastic scattering problems.

2.1. Traces and elastic potentials.

We assume Ω^- be a bounded domain with a closed boundary Γ of class $\mathscr{C}^2$ at least. Throughout the paper we denote by $H^s(\Omega^-)$, $H^s_{loc}(\overline{\Omega^+})$ and $H^s(\Gamma)$ the standard (local in the case of the exterior domain) complex valued, Hilbertian Sobolev space of order $s \in \mathbb{R}$ defined on Ω^-, $\overline{\Omega^+}$ and Γ respectively (with the convention $H^0 = L^2$). Spaces of vector functions will be denoted by boldface letters, thus $\boldsymbol{H}^s = (H^s)^3$. We set

$$\Delta^* \boldsymbol{u} := \operatorname{div} \sigma(\boldsymbol{u}) = \mu \Delta \boldsymbol{u} + (\lambda + \mu) \boldsymbol{\nabla} \operatorname{div} \boldsymbol{u}$$

and introduce the following energy spaces

$$\begin{aligned} \boldsymbol{H}^1(\Omega^-, \Delta^*) &:= \left\{ \boldsymbol{u} \in \boldsymbol{H}^1(\Omega^-) : \ \Delta^* \boldsymbol{u} \in \boldsymbol{L}^2(\Omega^-) \right\}, \\ \boldsymbol{H}^1_{loc}(\Omega^+, \Delta^*) &:= \left\{ \boldsymbol{u} \in \boldsymbol{H}^1_{loc}(\overline{\Omega^+}) : \ \Delta^* \boldsymbol{u} \in \boldsymbol{L}^2_{loc}(\overline{\Omega^+}) \right\}. \end{aligned}$$

Definition 2.1. *For a vector function $\boldsymbol{u} \in (\mathscr{C}^\infty(\overline{\Omega^-}))^3$ we define the following interior traces :*

$$\begin{aligned} \gamma_0^- \boldsymbol{u} &= \boldsymbol{u}_{|\Gamma} \ \textit{(Dirichlet trace)} \\ \gamma_1^- \boldsymbol{u} &= \boldsymbol{T}\boldsymbol{u} \ \textit{(Neumann or traction trace)} \\ \gamma_{1,\alpha}^- \boldsymbol{u} &= \boldsymbol{T}\boldsymbol{u} - \alpha \boldsymbol{\mathcal{M}} \boldsymbol{u}_{|\Gamma} \ \textit{(Neumann-type trace)}, \end{aligned}$$

where $\boldsymbol{T}$ denotes the traction operator defined by

$$\boldsymbol{T} = 2\mu \frac{\partial}{\partial \boldsymbol{n}} + \lambda \boldsymbol{n} \operatorname{div} + \mu \, \boldsymbol{n} \times \mathbf{curl}\,.$$

The parameter α is a real-valued constant and the surface operator $\boldsymbol{\mathcal{M}}$ is the tangential Günter derivative defined by

$$\boldsymbol{\mathcal{M}} = \frac{\partial}{\partial \boldsymbol{n}} - \boldsymbol{n}\,\mathrm{div} + \boldsymbol{n} \times \mathbf{curl}\,.$$

The following trace maps

$$\begin{aligned} \gamma_0^- : \boldsymbol{H}^1(\Omega^-, \Delta^*) &\to \boldsymbol{H}^{\frac{1}{2}}(\Gamma) \\ \gamma_1^- : \boldsymbol{H}^1(\Omega^-, \Delta^*) &\to \boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \\ \gamma_{1,\alpha}^- : \boldsymbol{H}^1(\Omega^-, \Delta^*) &\to \boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \end{aligned}$$

are continuous. For $\boldsymbol{u} \in \boldsymbol{H}^1_{loc}(\Omega^+, \Delta^*)$ we define the exterior traces $\gamma_0^+\boldsymbol{u}$, $\gamma_1^+\boldsymbol{u}$ and $\gamma_{1,\alpha}^+\boldsymbol{u}$ in the same way and the same mapping properties hold true.

For two (3×3) matrices A and B whose columns are denoted by (a_1, a_2, a_3) and (b_1, b_2, b_3), respectively, we set $A : B = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$. The following lemma is a consequence of the Gauss divergence theorem and the identity

$$\mathrm{div}\,\big(\sigma(\boldsymbol{u})\boldsymbol{v}\big) = \Delta^*\boldsymbol{u} \cdot \boldsymbol{v} + \sigma(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}). \tag{2.1}$$

Lemma 2.2. *For vector functions $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{H}^1(\Omega, \Delta^*)$, we have the first Green formula*

$$\int_\Omega \Delta^*\boldsymbol{u} \cdot \boldsymbol{v}\, dx = \int_\Gamma T\boldsymbol{u} \cdot \boldsymbol{v}\, ds - \int_\Omega \sigma(\boldsymbol{u}) : \varepsilon(\boldsymbol{v})\, dx, \tag{2.2}$$

where $\sigma(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) = \lambda(\mathrm{div}\,\boldsymbol{u})(\mathrm{div}\,\boldsymbol{v}) + 2\mu\,\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) = \sigma(\boldsymbol{v}) : \varepsilon(\boldsymbol{u})$.

We use the surface differential operators: The tangential gradient $\boldsymbol{\nabla}_\Gamma$, the surface divergence div_Γ, the tangential vector curl $\mathbf{curl}_\Gamma$, the surface scalar curl curl_Γ, the surface scalar Laplace-Beltrami operator Δ_Γ and the vector Laplace-Beltrami operator $\boldsymbol{\Delta}_\Gamma$. For their definitions we refer to [36, pages 68-75]. We recall some well-known surface decompositions that are useful in the sequel for the determination of the principal part of the boundary integral operators.

Lemma 2.3. *For a vector function $\boldsymbol{u}$ and a scalar function v both defined in the neighborhood of Γ and sufficiently smooth, we have the following decompositions on Γ:*

$$\boldsymbol{\nabla} v = \boldsymbol{\nabla}_\Gamma v + \frac{\partial v}{\partial \boldsymbol{n}}, \tag{2.3}$$

$$\mathrm{div}\,\boldsymbol{u} = \mathrm{div}_\Gamma\big(\boldsymbol{n} \times (\boldsymbol{u} \times \boldsymbol{n})\big) + 2\mathcal{H}(\boldsymbol{u} \cdot \boldsymbol{n}) + \boldsymbol{n} \cdot \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}, \tag{2.4}$$

$$\boldsymbol{n} \times \mathbf{curl}\,\boldsymbol{u} = \boldsymbol{\nabla}_\Gamma(\boldsymbol{u} \cdot \boldsymbol{n}) + \boldsymbol{n} \times \big(\boldsymbol{\mathcal{R}} - 2\mathcal{H}\,\mathrm{I}_3\big)(\boldsymbol{u} \times \boldsymbol{n}) + \boldsymbol{n} \times \Big(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \times \boldsymbol{n}\Big), \tag{2.5}$$

$$\Delta v = \Delta_\Gamma v + 2\mathcal{H}\,\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + \frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{n}^2}, \tag{2.6}$$

where $\mathcal{H}$ is the mean curvature and $\boldsymbol{\mathcal{R}}$ is the curvature operator.

Remark 2.4. *The tangential Günter derivative can be rewritten as follows:*

$$\boldsymbol{\mathcal{M}}\boldsymbol{u} = \big([\boldsymbol{\nabla}_\Gamma \boldsymbol{u}] - (\mathrm{div}_\Gamma\,\boldsymbol{u}) \cdot \mathrm{I}_3\big)\boldsymbol{n}.$$

Notice that $[\boldsymbol{\nabla}_\Gamma \boldsymbol{u}]$ is the matrix whose the j-th column is the tangential gradient of the j-th component of $\boldsymbol{u}$.

For any complex number κ such that $\mathrm{Im}(\kappa) \geq 0$, let

$$G(\kappa, \boldsymbol{x} - \boldsymbol{y}) = \frac{e^{i\kappa|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}$$

be the fundamental solution of the Helmholtz equation $\Delta v + \kappa^2 v = 0$. Then, the fundamental solution of the Navier equation can be written as follows

$$\Phi(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{\rho\omega^2}\left(\mathbf{curl}\,\mathbf{curl}_{\boldsymbol{x}}\left\{G(\kappa_s, \boldsymbol{x} - \boldsymbol{y})\,\mathrm{I}_3\right\} - \boldsymbol{\nabla}_{\boldsymbol{x}}\,\mathrm{div}_{\boldsymbol{x}}\,\left\{G(\kappa_p, \boldsymbol{x} - \boldsymbol{y})\mathrm{I}_3\right\}\right).$$

It is a 3-by-3 matrix-valued function and we have $\Phi(\boldsymbol{x},\boldsymbol{y}) = \Phi(\boldsymbol{x},\boldsymbol{y})^{\mathsf{T}} = \Phi(\boldsymbol{y},\boldsymbol{x})$. The single and double layer potential operators are defined by

$$\mathcal{S}\boldsymbol{\varphi} = \int_\Gamma \Phi(\cdot\,,\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})ds(\boldsymbol{y}) \quad \text{and} \quad \mathcal{D}\boldsymbol{\psi} = \int_\Gamma [\boldsymbol{T}_{\boldsymbol{y}}\Phi(\cdot\,,\boldsymbol{y})]^{\mathsf{T}}\boldsymbol{\psi}(\boldsymbol{y})ds(\boldsymbol{y}), \tag{2.7}$$

where $\boldsymbol{T}_{\boldsymbol{y}} = \boldsymbol{T}(\boldsymbol{n}(\boldsymbol{y}),\partial_{\boldsymbol{y}})$ and $\boldsymbol{T}_{\boldsymbol{y}}\Phi(\boldsymbol{x},\boldsymbol{y})$ is the tensor obtained by applying the traction operator $\boldsymbol{T}_{\boldsymbol{y}}$ to each column of $\Phi(\boldsymbol{x},\boldsymbol{y})$.

Theorem 2.5. *The potential operators $\mathcal{S}$ and $\mathcal{D}$ are continuous from $\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ and $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ respectively to $\boldsymbol{H}^1(\Omega^-,\Delta^*) \cup \boldsymbol{H}^1_{loc}(\Omega^+,\Delta^*)$. For any $\boldsymbol{\varphi} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ and $\boldsymbol{\psi} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, the potentials $\mathcal{S}\boldsymbol{\varphi}$ and $\mathcal{D}\boldsymbol{\psi}$ solve the Navier equation* (1.1a) *and satisfy the Kupraze radiation conditions* (1.1c)*.*

The Dirichlet and traction traces can be applied to $\mathcal{S}$ and $\mathcal{D}$ and we obtain the following identities

$$\gamma_0^{\pm}\mathcal{S} = S, \quad \gamma_1^{\pm}\mathcal{S} = \mp\frac{1}{2}\mathrm{I} + D', \quad \gamma_0^{\pm}\mathcal{D} = \pm\frac{1}{2}\mathrm{I} + D, \text{ and } \gamma_1^{\pm}\mathcal{D} = N, \tag{2.8}$$

where I is the identity operator and the elementary boundary integral operators S, D, D' and N are defined, for $\boldsymbol{x} \in \Gamma$, by

$$\begin{aligned}
S\boldsymbol{\varphi}(\boldsymbol{x}) &= \int_\Gamma \Phi(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\, ds(\boldsymbol{y}),\\
D\boldsymbol{\psi}(\boldsymbol{x}) &= \int_\Gamma [\boldsymbol{T}_{\boldsymbol{y}}\Phi(\boldsymbol{x},\boldsymbol{y})]^{\mathsf{T}}\boldsymbol{\psi}(\boldsymbol{y})\, ds(\boldsymbol{y}),\\
D'\boldsymbol{\varphi}(\boldsymbol{x}) &= \int_\Gamma \boldsymbol{T}_{\boldsymbol{x}}\left\{\Phi(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\right\} ds(\boldsymbol{y}),\\
N\boldsymbol{\psi}(\boldsymbol{x}) &= \int_\Gamma \boldsymbol{T}_{\boldsymbol{x}}\left\{[\boldsymbol{T}_{\boldsymbol{y}}\Phi(\boldsymbol{x},\boldsymbol{y})]^{\mathsf{T}}\boldsymbol{\psi}(\boldsymbol{y})\right\} ds(\boldsymbol{y}).
\end{aligned}$$

The operator S is a pseudo-differential operator of order -1, i.e it is bounded from $\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ to $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ and compact from $\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ to itself. The operators D and D' are of order 0, i.e. they have a strongly singular kernel and are bounded from $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ and $\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ to themselves, respectively. The operator N is of order $+1$, i.e. it has a hypersingular kernel and is bounded from $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ to $\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$.

Theorem 2.6. *For a solution $\boldsymbol{u} \in \boldsymbol{H}^1_{loc}(\Omega^+,\Delta^*)$ to the Navier equation* (1.1a) *in Ω^+ that satisfies the Kupradze radiation conditions, one can derive the Somigliana integral representation formula for $\boldsymbol{x} \in \Omega^+$:*

$$\boldsymbol{u}(\boldsymbol{x}) = \mathcal{D}\gamma_0^+\boldsymbol{u}(\boldsymbol{x}) - \mathcal{S}\gamma_1^+\boldsymbol{u}(\boldsymbol{x}). \tag{2.9}$$

For a solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega^-,\Delta^*)$ to the Navier equation (1.1a) *in Ω^-, the Somigliana integral representation formula is given for $\boldsymbol{x} \in \Omega^-$ by:*

$$\boldsymbol{u}(\boldsymbol{x}) = \mathcal{S}\gamma_1^-\boldsymbol{u}(\boldsymbol{x}) - \mathcal{D}\gamma_0^-\boldsymbol{u}(\boldsymbol{x}). \tag{2.10}$$

The Calderón projectors for the time-harmonic Navier equation are

$$P^{\pm} = \begin{pmatrix} \pm\frac{1}{2}\mathrm{I} + D & -S \\ N & \pm\frac{1}{2}\mathrm{I} - D' \end{pmatrix}.$$

We have $P^+ \circ P^- = P^- \circ P^+ = 0$ and thus the relations

$$\begin{aligned}
SD' = DS\,, \quad D'N = ND\,,\\
SN = D^2 - \frac{1}{4}\mathrm{I}\,, \quad NS = D'^2 - \frac{1}{4}\mathrm{I}.
\end{aligned} \tag{2.11}$$

Now we show that similar results remain true if we consider the Neumann-type trace $\gamma_{1,\alpha}$ instead of the traction trace γ_1.

Theorem 2.7. *We introduce the potential operator $\mathcal{D}_\alpha$ defined as $\mathcal{D}$ by replacing the traction operator $\boldsymbol{T}$ by $\boldsymbol{T}-\alpha\boldsymbol{\mathcal{M}}$. In the same way we construct the boundary integral operators D_α, D'_α, and N_α. Then we have the following results:*

(i) *$\mathcal{D}_\alpha = \mathcal{D}-\alpha\mathcal{S}\boldsymbol{\mathcal{M}}$ and for any $\boldsymbol{\psi}\in\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ the potential $\mathcal{D}_\alpha\boldsymbol{\psi}\in\boldsymbol{H}^1(\Omega^-,\Delta^*)\cup\boldsymbol{H}^1_{loc}(\Omega^+,\Delta^*)$ satisfies the Navier equation* (1.1a) *and the Kupradze radiation conditions* (1.1c)*;*

(ii) *$\gamma^\pm_{1,\alpha}\mathcal{S} = \mp\frac{1}{2}\mathrm{I}+D'_\alpha$, $\gamma^\pm_0\mathcal{D}_\alpha = \pm\frac{1}{2}\mathrm{I}+D_\alpha$ and $\gamma^\pm_{1,\alpha}\mathcal{D}_\alpha = N_\alpha$;*

(iii) *$\boldsymbol{u} = \mathcal{D}_\alpha\gamma^+_0\boldsymbol{u} - \mathcal{S}\gamma^+_{1,\alpha}\boldsymbol{u}$ for any solution $\boldsymbol{u}\in\boldsymbol{H}^1_{loc}(\Omega^+,\Delta^*)$ to the Navier equation* (1.1a) *in Ω^+ that satisfies the Kupraze radiation conditions;*

(iv) *$\boldsymbol{u} = \mathcal{S}\gamma^-_{1,\alpha}\boldsymbol{u} - \mathcal{D}_\alpha\gamma^-_0\boldsymbol{u}$ for any solution $\boldsymbol{u}\in\boldsymbol{H}^1(\Omega^-,\Delta^*)$ to the Navier equation* (1.1a) *in Ω^-;*

(v) *$SN_\alpha = D^2_\alpha - \frac{1}{4}\mathrm{I}$ and $N_\alpha S = {D'_\alpha}^2 - \frac{1}{4}\mathrm{I}$.*

We have $\mathcal{D}_0=\mathcal{D}$, $D_0=D$, $D'_0=D$ and $N_0=N$.

Proof. (i) The operator $\mathcal{D}_\alpha$ applied to any vector density $\boldsymbol{\psi}$ writes for $\boldsymbol{x}\in\Omega^\pm$

$$\mathcal{D}_\alpha\boldsymbol{\psi}(\boldsymbol{x}) = \int_\Gamma \Big[\Big(\boldsymbol{T}_{\boldsymbol{y}}-\alpha\boldsymbol{\mathcal{M}}\Big)\Phi(\boldsymbol{x},\boldsymbol{y})\Big]^\mathsf{T}\boldsymbol{\psi}(\boldsymbol{y})ds(\boldsymbol{y}).$$

Using that for two vector densities $\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_2$ we have [26]

$$\int_\Gamma(\boldsymbol{\mathcal{M}}\boldsymbol{\psi}_1)\cdot\boldsymbol{\psi}_2\,ds = \int_\Gamma\boldsymbol{\psi}_1\cdot(\boldsymbol{\mathcal{M}}\boldsymbol{\psi}_2)\,ds,$$

we obtain that

$$\begin{aligned}\mathcal{D}_\alpha\boldsymbol{\psi}(\boldsymbol{x}) &= \int_\Gamma[\boldsymbol{T}_{\boldsymbol{y}}\Phi(\boldsymbol{x},\boldsymbol{y})]^\mathsf{T}\boldsymbol{\psi}(\boldsymbol{y})ds(\boldsymbol{y}) - \alpha\int_\Gamma[\boldsymbol{\mathcal{M}}\Phi(\boldsymbol{x},\boldsymbol{y})]^\mathsf{T}\boldsymbol{\psi}(\boldsymbol{y})ds(\boldsymbol{y})\\ &= \int_\Gamma[\boldsymbol{T}_{\boldsymbol{y}}\Phi(\boldsymbol{x},\boldsymbol{y})]^\mathsf{T}\boldsymbol{\psi}(\boldsymbol{y})ds(\boldsymbol{y}) - \alpha\int_\Gamma\Phi(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\mathcal{M}}\boldsymbol{\psi}(\boldsymbol{y})ds(\boldsymbol{y})\\ &= \mathcal{D}\boldsymbol{\psi}(\boldsymbol{x}) - \alpha\mathcal{S}\boldsymbol{\mathcal{M}}\boldsymbol{\psi}(\boldsymbol{x}).\end{aligned}$$

(ii) We decompose the traces as follows

$$\begin{aligned}\gamma^\pm_{1,\alpha}\mathcal{S} &= \gamma^\pm_1\mathcal{S}-\alpha\boldsymbol{\mathcal{M}}\gamma^\pm_0\mathcal{S}\\ \gamma^\pm_0\mathcal{D}_\alpha &= \gamma^\pm_0\mathcal{D}-\alpha\gamma^\pm_0\mathcal{S}\boldsymbol{\mathcal{M}}\\ \gamma^\pm_{1,\alpha}\mathcal{D}_\alpha &= \gamma^\pm_1\mathcal{D}_\alpha-\alpha\boldsymbol{\mathcal{M}}\gamma^\pm_0\mathcal{D}_\alpha\end{aligned}$$

and we apply the equalities (2.8). The first two above equalities in (ii) are obvious. We develop the third one. We have

$$\begin{aligned}\gamma^\pm_{1,\alpha}\mathcal{D}_\alpha &= \gamma^\pm_1(\mathcal{D}-\alpha\mathcal{S}\boldsymbol{\mathcal{M}}) - \alpha\boldsymbol{\mathcal{M}}\gamma^\pm_0(\mathcal{D}-\alpha\mathcal{S}\boldsymbol{\mathcal{M}})\\ &= N-\alpha\left(\mp\frac{1}{2}\mathrm{I}+D'\right)\boldsymbol{\mathcal{M}} - \alpha\boldsymbol{\mathcal{M}}\left(\pm\frac{1}{2}\mathrm{I}+D\right)+\alpha^2\boldsymbol{\mathcal{M}}\mathcal{S}\boldsymbol{\mathcal{M}}\\ &= N-\alpha D'\boldsymbol{\mathcal{M}}-\alpha\boldsymbol{\mathcal{M}}D+\alpha^2\boldsymbol{\mathcal{M}}\mathcal{S}\boldsymbol{\mathcal{M}}\\ &= N_\alpha.\end{aligned}$$

(iii) To obtain the new integral representation formula we use (i) and write, for any solution $\boldsymbol{u}\in\boldsymbol{H}^1_{loc}(\Omega^+,\Delta^*)$,

$$\begin{aligned}\mathcal{D}_\alpha\gamma^+_0\boldsymbol{u} - \mathcal{S}\gamma^+_{1,\alpha}\boldsymbol{u} &= (\mathcal{D}-\alpha\mathcal{S}\boldsymbol{\mathcal{M}})\gamma^+_0\boldsymbol{u} - \mathcal{S}\left(\gamma^+_1\boldsymbol{u}-\alpha\boldsymbol{\mathcal{M}}\gamma^+_0\boldsymbol{u}\right)\\ &= \mathcal{D}\gamma^+_0\boldsymbol{u} - \mathcal{S}\gamma^+_1\boldsymbol{u}\\ &= \boldsymbol{u}.\end{aligned}$$

The assertion (iv) is obtained in the same way.

(v) We use the above decomposition of N_α in terms of S, D, D', N, and $\boldsymbol{\mathcal{M}}$, the equalities

$D_\alpha = D - \alpha S\boldsymbol{\mathcal{M}}$ and $D'_\alpha = D' - \alpha\boldsymbol{\mathcal{M}}S$ and the Calderón formulas (2.11). We get

$$\begin{aligned} SN_\alpha &= SN - \alpha SD'\boldsymbol{\mathcal{M}} - \alpha S\boldsymbol{\mathcal{M}}D + \alpha^2 S\boldsymbol{\mathcal{M}}\mathcal{S}\boldsymbol{\mathcal{M}} \\ &= SN - \alpha DS\boldsymbol{\mathcal{M}} - \alpha S\boldsymbol{\mathcal{M}}D + \alpha^2 S\boldsymbol{\mathcal{M}}\mathcal{S}\boldsymbol{\mathcal{M}} \\ &= -\frac{1}{4}\mathrm{I} + D^2 - \alpha DS\boldsymbol{\mathcal{M}} - \alpha S\boldsymbol{\mathcal{M}}D + \alpha^2 S\boldsymbol{\mathcal{M}}\mathcal{S}\boldsymbol{\mathcal{M}} \\ &= -\frac{1}{4}\mathrm{I} + D_\alpha^2\,, \end{aligned}$$

and finally

$$\begin{aligned} N_\alpha S &= NS - \alpha D'\boldsymbol{\mathcal{M}}S - \alpha\boldsymbol{\mathcal{M}}DS + \alpha^2\boldsymbol{\mathcal{M}}\mathcal{S}\boldsymbol{\mathcal{M}}S \\ &= NS - \alpha D'\boldsymbol{\mathcal{M}}S - \alpha\boldsymbol{\mathcal{M}}SD' + \alpha^2\boldsymbol{\mathcal{M}}\mathcal{S}\boldsymbol{\mathcal{M}}S \\ &= -\frac{1}{4}\mathrm{I} + D'^2 - \alpha D'\boldsymbol{\mathcal{M}}S - \alpha\boldsymbol{\mathcal{M}}SD' + \alpha^2\boldsymbol{\mathcal{M}}\mathcal{S}\boldsymbol{\mathcal{M}}S \\ &= -\frac{1}{4}\mathrm{I} + D_\alpha'^{\,2}. \end{aligned}$$

□

2.2. **The standard BW-type and combined field integral equations.** Applying the potential theory, the elastic scattering problem (1.1a)-(1.1b)-(1.1c) can be reduced, via direct and indirect approaches, to a single uniquely solvable boundary integral equation [31].

The direct formulation consists in seeking the scattered field under the form

$$\boldsymbol{u}(\boldsymbol{x}) = \mathcal{S}\boldsymbol{\varphi}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega^+, \tag{2.12}$$

setting $\boldsymbol{\varphi} = -\gamma_1^+\big(\boldsymbol{u} + \boldsymbol{u}^{inc}\big)$. Taking the interior Dirichlet and Neumann-type traces of the right hand side of the above equality, we obtain

$$S\boldsymbol{\varphi} = -\gamma_0^-\boldsymbol{u}^{inc} \quad \text{and} \quad \big(\frac{\mathrm{I}}{2} + D'_\alpha\big)\boldsymbol{\varphi} = -\gamma_{1,\alpha}^-\boldsymbol{u}^{inc}, \quad \text{on } \Gamma.$$

Under certain compatibility conditions for the incident field $\boldsymbol{u}^{inc}$ [26, 31, 30], these integral equations are uniquely solvable in $\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$. To avoid any compatibility conditions, an idea is to consider the CFIE

$$(\frac{\mathrm{I}}{2} + D'_\alpha + i\eta S)\boldsymbol{\varphi} = -\big(\gamma_{1,\alpha}^+\boldsymbol{u}^{inc} + i\eta\gamma_0^+\boldsymbol{u}^{inc}\big), \quad \text{on } \Gamma, \tag{2.13}$$

where η is a non-zero real constant. Then, the scattered field $\boldsymbol{u}$ given by (2.12) solves the Dirichlet boundary value problem (1.1a)-(1.1b)-(1.1c) provided the physical unknown $\boldsymbol{\varphi}$ is solution of the integral equation (2.13). It can be shown that the homogenous equation (2.13) has only the trivial solution $\boldsymbol{\varphi} = \boldsymbol{0}$. The integral operator $(\frac{\mathrm{I}}{2} + D'_\alpha + i\eta S)$ is not, in general, a compact perturbation of the identity operator (close to a constant). However, an integral equation of the second kind equivalent to (2.13) can be obtained using a left equivalent regularizer [28, 30, 31]. Then we can use Riesz theory to prove existence of a unique solution to (2.13).

The indirect formulations are based on the following ansatz for the integral representation of the scattered elastic field

$$\boldsymbol{u}(\boldsymbol{x}) = \mathcal{D}_\alpha\boldsymbol{\psi}(\boldsymbol{x}) + i\eta\mathcal{S}\boldsymbol{\psi}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega^+, \tag{2.14}$$

where η is a non-zero real constant to choose and $\boldsymbol{\psi}$ is a fictitious density in $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$. Applying the exterior Dirichlet trace to (2.14) and expressing the boundary condition (1.1b), we get the BW-type integral equation

$$(\frac{\mathrm{I}}{2} + D_\alpha + i\eta S)\boldsymbol{\psi} = -\gamma_0^+\boldsymbol{u}^{inc}, \quad \text{on } \Gamma. \tag{2.15}$$

Existence and uniqueness can be shown by the use of a right equivalent regularizer [28, 30, 31] for the integral equations (2.15). We mention that in the special case $\alpha = \dfrac{2\mu^2}{\lambda + 3\mu}$, it can be shown that the integral equations (2.13) and (2.15) are of the second kind [24] for any real parameter η.

3. Preconditioned BW-type integral equations

This section is devoted to the construction of well-conditioned indirect boundary integral formulations for the solution of the Dirichlet exterior problem in elasticity. We analyze two approaches: the first one by applying the analytic preconditioning technique to the classical BW integral equation, namely equation (2.15) with $\alpha = 0$, and the second one by applying the same preconditioning technique to the equation (2.15) with $\alpha = 2\mu$.

3.1. DtN approach ($\boldsymbol{\alpha = 0}$).

Here the field $\boldsymbol{u}$ represents a solution to the rigid body problem (1.1a)-(1.1b)-(1.1c). Assume that the exact Dirichlet-to-Neumann map

$$\boldsymbol{\Lambda}^{\text{ex}} : \gamma_0^+ \boldsymbol{u} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma) \mapsto \boldsymbol{\Lambda}^{\text{ex}} \gamma_0^+ \boldsymbol{u} := \gamma_1^+ \boldsymbol{u} = \boldsymbol{T}\boldsymbol{u} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$$

is known. The Somigliana integral representation (2.9) can be rewritten under this form

$$\boldsymbol{u}(\boldsymbol{x}) = \mathcal{D}\gamma_0^+ \boldsymbol{u}(\boldsymbol{x}) - \mathcal{S}\boldsymbol{\Lambda}^{\text{ex}} \gamma_0^+ \boldsymbol{u}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega^+.$$

Taking the Dirichlet trace, we get the integral identity on Γ:

$$\frac{1}{2}\mathrm{I} + D - S\boldsymbol{\Lambda}^{\text{ex}} = \mathrm{I}. \tag{3.1}$$

In this ideal configuration, the solution is computed directly. However, an expression of the exact DtN operator is not available for a general surface Γ. Instead, an approximation $\boldsymbol{\Lambda}$ of $\boldsymbol{\Lambda}^{\text{ex}}$ is introduced. We propose to consider the following ansatz which is inspired by the Brakhage-Werner approach

$$\boldsymbol{u}(\boldsymbol{x}) = \mathcal{D}\boldsymbol{\psi}(\boldsymbol{x}) - \mathcal{S}\boldsymbol{\Lambda}\boldsymbol{\psi}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega^+,$$

where $\boldsymbol{\psi}$ is a fictive density. The scattered field $\boldsymbol{u}$ solves the Dirichlet boundary value problem if $\boldsymbol{\psi} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ is a solution of the integral equation

$$\Big(\frac{1}{2}\mathrm{I} + D - S\boldsymbol{\Lambda}\Big)\boldsymbol{\psi} = -\gamma_0^+ \boldsymbol{u}^{inc}, \quad \text{on } \Gamma. \tag{3.2}$$

The key task is now to propose an efficient approximation $\boldsymbol{\Lambda}$ of the exact DtN $\boldsymbol{\Lambda}^{\text{ex}}$. The following lemma [2] gives a hint for constructing $\boldsymbol{\Lambda}$.

Lemma 3.1. *Assume that ω is not an eigenfrequency of the Navier equation* (1.1a) *then we have*

$$\boldsymbol{\Lambda}^{\text{ex}} = -S^{-1}(\frac{1}{2}\mathrm{I} - D) = N(\frac{1}{2}\mathrm{I} + D)^{-1}.$$

Proof. By the use of (3.1) and the Calderón formula (2.11) . □

We choose to construct the operator $\boldsymbol{\Lambda}$ as the pseudo-differential operator associated with the principal classical symbol of the operator $\boldsymbol{\Lambda}^{\text{ex}}$. From Lemma 3.1, we have to compute the principal parts of the operators S or N. Since D and D' are operators of order 0, it is necessary to take into account their principal part too, contrary to the acoustic and electromagnetic cases. This is realized in the following three lemmas.

Lemma 3.2. *The principal part $P_{-1}(S)$ of the boundary integral operator S is expressed by*

$$P_{-1}(S) = \frac{i}{2\rho\omega^2}\Big[\big(\boldsymbol{\Delta}_\Gamma + \kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\Big(\kappa_s^2 \mathrm{I}_{\boldsymbol{t}} + \boldsymbol{\nabla}_\Gamma \operatorname{div}_\Gamma \mathrm{I}_{\boldsymbol{t}}\Big) - \boldsymbol{\nabla}_\Gamma\big(\Delta_\Gamma + \kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}} \operatorname{div}_\Gamma \mathrm{I}_{\boldsymbol{t}}$$
$$+ \boldsymbol{n}\big(\Delta_\Gamma + \kappa_p^2\mathrm{I}\big)^{\frac{1}{2}}\big(\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}}\big) - \boldsymbol{n}\big(\Delta_\Gamma + \kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\Delta_\Gamma\big(\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}}\big)\Big],$$

where $\mathrm{I}_{\boldsymbol{n}} = \boldsymbol{n}\otimes\boldsymbol{n}$ and $\mathrm{I}_{\boldsymbol{t}} = \mathrm{I} - \mathrm{I}_{\boldsymbol{n}}$.

Proof. Since $\mathbf{curl}\,\mathbf{curl} = -\boldsymbol{\Delta} + \boldsymbol{\nabla}\operatorname{div}$, the fundamental solution of the Navier equation can be rewritten as follows

$$\Phi(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\rho\omega^2}\Big((-\boldsymbol{\Delta}_{\boldsymbol{x}} + \boldsymbol{\nabla}_{\boldsymbol{x}}\operatorname{div}_{\boldsymbol{x}})\big\{G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})\mathrm{I}_3\big\} - \boldsymbol{\nabla}_{\boldsymbol{x}}\operatorname{div}_{\boldsymbol{x}}\big\{G(\kappa_p,\boldsymbol{x}-\boldsymbol{y})\mathrm{I}_3\big\}\Big).$$

Using Lemma 2.3, we split the operator $\boldsymbol{\nabla}\,\mathrm{div}$ into a sum of surface differential operators acting either on tangential densities, or normal densities to Γ. We have

$$\begin{aligned}
\boldsymbol{\nabla}\,\mathrm{div} &= \left(\boldsymbol{\nabla}_\Gamma + \boldsymbol{n}\frac{\partial}{\partial \boldsymbol{n}}\right)\left(\mathrm{div}_\Gamma\,\mathrm{I}_{\boldsymbol{t}} + 2\mathcal{H}(\boldsymbol{n}\cdot\mathrm{I}) + \boldsymbol{n}\cdot\frac{\partial}{\partial \boldsymbol{n}}\right)\\
&= \boldsymbol{\nabla}_\Gamma\,\mathrm{div}_\Gamma\,\mathrm{I}_{\boldsymbol{t}} + \frac{\partial^2}{\partial \boldsymbol{n}^2}\,\mathrm{I}_{\boldsymbol{n}} + \boldsymbol{n}\frac{\partial}{\partial \boldsymbol{n}}\,\mathrm{div}_\Gamma + \boldsymbol{\nabla}_\Gamma\left(2\mathcal{H}(\boldsymbol{n}\cdot\mathrm{I}) + \boldsymbol{n}\cdot\frac{\partial}{\partial \boldsymbol{n}}\right)\\
&= \boldsymbol{\nabla}_\Gamma\,\mathrm{div}_\Gamma\,\mathrm{I}_{\boldsymbol{t}} + \left(\boldsymbol{\Delta} - \boldsymbol{\Delta}_\Gamma - 2\mathcal{H}\frac{\partial}{\partial \boldsymbol{n}}\right)\mathrm{I}_{\boldsymbol{n}} + \boldsymbol{n}\frac{\partial}{\partial \boldsymbol{n}}\,\mathrm{div}_\Gamma + \boldsymbol{\nabla}_\Gamma\left(2\mathcal{H}(\boldsymbol{n}\cdot\mathrm{I}) + \boldsymbol{n}\cdot\frac{\partial}{\partial \boldsymbol{n}}\right)\\
&= \boldsymbol{\nabla}_\Gamma\,\mathrm{div}_\Gamma\,\mathrm{I}_{\boldsymbol{t}} + (\boldsymbol{\Delta} - \boldsymbol{\Delta}_\Gamma)\,\mathrm{I}_{\boldsymbol{n}} - 2\mathcal{H}\frac{\partial}{\partial \boldsymbol{n}}\mathrm{I}_{\boldsymbol{n}} + \boldsymbol{n}\frac{\partial}{\partial \boldsymbol{n}}\,\mathrm{div}_\Gamma + \boldsymbol{\nabla}_\Gamma\left(2\mathcal{H}(\boldsymbol{n}\cdot\mathrm{I}) + \boldsymbol{n}\cdot\frac{\partial}{\partial \boldsymbol{n}}\right),
\end{aligned}$$

and

$$-\boldsymbol{\Delta} + \boldsymbol{\nabla}\,\mathrm{div} = (-\boldsymbol{\Delta} + \boldsymbol{\nabla}_\Gamma\,\mathrm{div}_\Gamma)\mathrm{I}_{\boldsymbol{t}} - \boldsymbol{\Delta}_\Gamma\,\mathrm{I}_{\boldsymbol{n}} - 2\mathcal{H}\frac{\partial}{\partial \boldsymbol{n}}\mathrm{I}_{\boldsymbol{n}} + \boldsymbol{n}\frac{\partial}{\partial \boldsymbol{n}}\,\mathrm{div}_\Gamma + \boldsymbol{\nabla}_\Gamma\left(2\mathcal{H}(\boldsymbol{n}\cdot\mathrm{I}) + \boldsymbol{n}\cdot\frac{\partial}{\partial \boldsymbol{n}}\right).$$

We use the following result [6].

Proposition 3.3. *The single layer boundary integral operator associated with the scalar Helmholtz equation with the wavenumber κ is a compact perturbation of the operator $\frac{i}{2}(\Delta_\Gamma + \kappa^2\mathrm{I})^{-\frac{1}{2}}$.*

To determine $P_{-1}(S)$, we first apply the above surface decomposition of $(-\boldsymbol{\Delta} + \boldsymbol{\nabla}\,\mathrm{div})$ to the single layer boundary integral operator associated with the vector Helmholtz equation with the wavenumber κ_s. Using Proposition 3.3 together with some well-known mapping properties of the single layer potential and its derivatives, we obtain the following top-order part

$$\frac{i}{2}\big(\boldsymbol{\Delta}_\Gamma + \kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\Big(\kappa_s^2\mathrm{I}_{\boldsymbol{t}} + \boldsymbol{\nabla}_\Gamma\,\mathrm{div}_\Gamma\,\mathrm{I}_{\boldsymbol{t}}\Big) - \frac{i}{2}\boldsymbol{n}\big(\Delta_\Gamma + \kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\Delta_\Gamma\,\big(\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}}\big).$$

Secondly, we apply the above surface decomposition of $\boldsymbol{\nabla}\,\mathrm{div}$ to the single layer boundary integral operator associated with the vector Helmholtz equation with the wavenumber κ_p. We obtain the top-order part

$$\frac{i}{2}\boldsymbol{\nabla}_\Gamma\big(\Delta_\Gamma + \kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}}\,\mathrm{div}_\Gamma\,\mathrm{I}_{\boldsymbol{t}} - \frac{i}{2}\boldsymbol{n}(\kappa_p^2\mathrm{I} + \Delta_\Gamma)^{\frac{1}{2}}\big(\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}}\big).$$

To conclude we substract the two above operators and multiply the result by $(\rho\omega^2)^{-1}$. □

Lemma 3.4. *The principal parts $P_0(D)$ and $P_0(D')$ of the boundary integral operators D and D' are given respectively by the following formulas*

$$P_0(D) = 2\mu P_{-1}(S)\boldsymbol{\mathcal{M}} + \frac{i}{2}\Big(\boldsymbol{n}\big(\Delta_\Gamma + \kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\,\mathrm{div}_\Gamma\,\mathrm{I}_{\boldsymbol{t}} - \boldsymbol{\nabla}_\Gamma\big(\Delta_\Gamma + \kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}}\,\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}}\Big),$$

and

$$P_0(D') = 2\mu\boldsymbol{\mathcal{M}}P_{-1}(S) + \frac{i}{2}\Big(\boldsymbol{n}\big(\Delta_\Gamma + \kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}}\,\mathrm{div}_\Gamma\,\mathrm{I}_{\boldsymbol{t}} - \boldsymbol{\nabla}_\Gamma\big(\Delta_\Gamma + \kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\,\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}}\Big).$$

Proof. We consider integral representations of the operators D and D' obtained in [32]

$$\begin{aligned}
(D\boldsymbol{\psi})(\boldsymbol{x}) = 2\mu S\boldsymbol{\mathcal{M}}\boldsymbol{\psi}(\boldsymbol{x}) &- \int_\Gamma \left[\boldsymbol{n}(\boldsymbol{y})\times\mathbf{curl}_{\boldsymbol{y}}\{G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})\mathrm{I}_3\}\right]^{\mathsf{T}}\boldsymbol{\psi}(\boldsymbol{y})ds(\boldsymbol{y})\\
&- \int_\Gamma \boldsymbol{\nabla}_{\boldsymbol{x}}G(\kappa_p,\boldsymbol{x}-\boldsymbol{y})\big(\boldsymbol{n}(\boldsymbol{y})\cdot\boldsymbol{\psi}(\boldsymbol{y})\big)\,ds(\boldsymbol{y}),\ \boldsymbol{x}\in\Gamma,
\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}
(D'\boldsymbol{\varphi})(\boldsymbol{x}) = 2\mu\boldsymbol{\mathcal{M}}S\boldsymbol{\varphi}(\boldsymbol{x}) &- \int_\Gamma \boldsymbol{n}(\boldsymbol{x})\times\mathbf{curl}_{\boldsymbol{x}}\{G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\}ds(\boldsymbol{y})\\
&- \boldsymbol{n}(\boldsymbol{x})\int_\Gamma \boldsymbol{\varphi}(\boldsymbol{y})\cdot\boldsymbol{\nabla}_{\boldsymbol{y}}G(\kappa_p,\boldsymbol{x}-\boldsymbol{y})ds(\boldsymbol{y}),\ \boldsymbol{x}\in\Gamma.
\end{aligned}\tag{3.4}$$

Let us work first on the operator D. The principal part of $2\mu S\boldsymbol{\mathcal{M}}$ is $2\mu P_{-1}(S)\boldsymbol{\mathcal{M}}$. In view of (2.5), the top-order term of the second term in the right hand side (with the minus sign) of (3.3) is

$$-\int_\Gamma [\boldsymbol{\nabla}^{\boldsymbol{y}}_\Gamma\{G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})\boldsymbol{n}(\boldsymbol{y})\}]^{\mathsf{T}}(\mathrm{I}_{\boldsymbol{t}}\boldsymbol{\psi})(\boldsymbol{y})ds(\boldsymbol{y}) = \int_\Gamma \boldsymbol{n}(\boldsymbol{y})\, G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})\,\mathrm{div}_\Gamma\,\mathrm{I}_{\boldsymbol{t}}\boldsymbol{\psi}(\boldsymbol{y})ds(\boldsymbol{y}).$$

By composition, the principal part of the above expression is $\dfrac{i}{2}\boldsymbol{n}\big(\Delta_\Gamma+\kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\,\mathrm{div}_\Gamma\,\mathrm{I}_{\boldsymbol{t}}$. Furthermore, we deduce from (2.3) that the top-order term of the third term in the right hand side (with the minus sign) of (3.3) is

$$-\int_\Gamma \boldsymbol{\nabla}^{\boldsymbol{x}}_\Gamma G(\kappa_p,\boldsymbol{x}-\boldsymbol{y})\big(\boldsymbol{n}(\boldsymbol{y})\cdot\boldsymbol{\psi}(\boldsymbol{y})\big)\,ds(\boldsymbol{y}) = -\boldsymbol{\nabla}_\Gamma\int_\Gamma G(\kappa_p,\boldsymbol{x}-\boldsymbol{y})\big(\boldsymbol{n}(\boldsymbol{y})\cdot\boldsymbol{\psi}(\boldsymbol{y})\big)\,ds(\boldsymbol{y}).$$

By composition, the principal part of the above expression is $-\dfrac{i}{2}\boldsymbol{\nabla}_\Gamma\big(\Delta_\Gamma+\kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}}\,\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}}$. Adding the three terms we obtain the principal part of $P_0(D)$.

Let us consider now the operator D'. The principal part of $2\mu\boldsymbol{\mathcal{M}}S$ is obviously $2\mu\boldsymbol{\mathcal{M}}P_{-1}(S)$. Using (2.5), the top-order term of the second term in the right hand side (with the minus sign) of (3.4) is expressed by

$$-\int_\Gamma \boldsymbol{\nabla}^{\boldsymbol{x}}_\Gamma\{G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})\big(\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{\varphi}(\boldsymbol{y})\big)\}ds(\boldsymbol{y}) = -\boldsymbol{\nabla}_\Gamma\int_\Gamma G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})\big(\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{\varphi}(\boldsymbol{y})\big)ds(\boldsymbol{y}).$$

By composition, the principal part of the above expression is $-\dfrac{i}{2}\boldsymbol{\nabla}_\Gamma\big(\Delta_\Gamma+\kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\,\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}}$. The top-order term of the third term in the right hand side (with the minus sign) of (3.4) is (see (2.3))

$$-\boldsymbol{n}(\boldsymbol{x})\int_\Gamma(\mathrm{I}_{\boldsymbol{t}}\boldsymbol{\varphi})(\boldsymbol{y})\cdot\boldsymbol{\nabla}^{\boldsymbol{y}}_\Gamma G(\kappa_p,\boldsymbol{x}-\boldsymbol{y})ds(\boldsymbol{y}) = \boldsymbol{n}(\boldsymbol{x})\int_\Gamma G(\kappa_p,\boldsymbol{x}-\boldsymbol{y})\,\mathrm{div}_\Gamma\,\mathrm{I}_{\boldsymbol{t}}\boldsymbol{\varphi}(\boldsymbol{y})ds(\boldsymbol{y}).$$

By composition, its principal part is $\dfrac{i}{2}\boldsymbol{n}\big(\Delta_\Gamma+\kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}}\,\mathrm{div}_\Gamma\,\mathrm{I}_{\boldsymbol{t}}$. The principal part $P_0(D')$ of D' is given by adding these three terms. □

Lemma 3.5. *The principal part $P_1(N)$ of the boundary integral operator N is expressed by*

$$\begin{aligned}P_1(N) = \;& 2\mu P_0(D')\boldsymbol{\mathcal{M}} + i\mu\boldsymbol{\mathcal{M}}\Big(\boldsymbol{n}\big(\Delta_\Gamma+\kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\,\mathrm{div}_\Gamma\,\mathrm{I}_{\boldsymbol{t}} - \boldsymbol{\nabla}_\Gamma\big(\Delta_\Gamma+\kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}}\,\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}}\Big)\\ &+\frac{i}{2}\Big[(\lambda+2\mu)\kappa_p^2\,\boldsymbol{n}\big(\Delta_\Gamma+\kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}}\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}} + \mu\big(\boldsymbol{\Delta}_\Gamma+\kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\Big(\kappa_s^2\mathrm{I}_{\boldsymbol{t}} - \mathbf{curl}_\Gamma\,\mathrm{curl}_\Gamma\Big)\Big].\end{aligned}$$

Proof. We use the following integral representation of the operator N

$$\begin{aligned}(N\boldsymbol{\psi})(\boldsymbol{x}) = 2\mu\mathcal{D}'\boldsymbol{\mathcal{M}}\boldsymbol{\psi}(\boldsymbol{x}) &- \int_\Gamma \boldsymbol{T}_{\boldsymbol{x}}[\boldsymbol{n}(\boldsymbol{y})\times\mathbf{curl}_{\boldsymbol{y}}\{G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})\mathrm{I}_3\}]^{\mathsf{T}}\boldsymbol{\psi}(\boldsymbol{y})ds(\boldsymbol{y})\\ &- \int_\Gamma \boldsymbol{T}_{\boldsymbol{x}}\boldsymbol{\nabla}_{\boldsymbol{x}}G(\kappa_p,\boldsymbol{x}-\boldsymbol{y})\big(\boldsymbol{n}(\boldsymbol{y})\cdot\boldsymbol{\psi}(\boldsymbol{y})\big)\,ds(\boldsymbol{y}),\end{aligned} \tag{3.5}$$

which is obtained by applying $\boldsymbol{T}_{\boldsymbol{x}}$ to the kernels of the right hand side in (3.3).

The principal part of $2\mu D'\boldsymbol{\mathcal{M}}$ is $2\mu P_0(D')\boldsymbol{\mathcal{M}}$. To recover the principal parts of the other terms in the operator N, we consider the following form for the operator $\boldsymbol{T}_{\boldsymbol{x}}$ [32].

$$\boldsymbol{T}_{\boldsymbol{x}} = 2\mu\boldsymbol{\mathcal{M}}_{\boldsymbol{x}} + (\lambda+2\mu)\boldsymbol{n}(\boldsymbol{x})\,\mathrm{div}_{\boldsymbol{x}} - \mu\boldsymbol{n}(\boldsymbol{x})\times\mathbf{curl}_{\boldsymbol{x}}\,.$$

We have to apply $\boldsymbol{T}_{\boldsymbol{x}}$ to these two terms

$$t_1 = -\int_\Gamma [\boldsymbol{n}(\boldsymbol{y})\times\mathbf{curl}_{\boldsymbol{y}}\{G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})\mathrm{I}_3\}]^{\mathsf{T}}\boldsymbol{\psi}(\boldsymbol{y})ds(\boldsymbol{y}), \tag{3.6}$$

$$t_2 = -\int_\Gamma \boldsymbol{\nabla}_{\boldsymbol{x}}G(\kappa_p,\boldsymbol{x}-\boldsymbol{y})\big(\boldsymbol{n}(\boldsymbol{y})\cdot\boldsymbol{\psi}(\boldsymbol{y})\big)\,ds(\boldsymbol{y}). \tag{3.7}$$

- First, from Lemma 3.4, applying $2\mu\boldsymbol{\mathcal{M}}_{\boldsymbol{x}}$ to (3.6) and (3.7), we obtain the principal part

$$2\mu\frac{i}{2}\boldsymbol{\mathcal{M}}\Big(\boldsymbol{n}\big(\Delta_\Gamma+\kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\operatorname{div}_\Gamma \mathrm{I}_{\boldsymbol{t}}-\boldsymbol{\nabla}_\Gamma\big(\Delta_\Gamma+\kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}}\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}}\Big).$$

- Secondly, we observe that

$$\begin{aligned}
&[\boldsymbol{n}(\boldsymbol{y})\times\mathbf{curl}_{\boldsymbol{y}}\{G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})\mathrm{I}_3\}]^{\mathsf{T}}\boldsymbol{\psi}(\boldsymbol{y})\\
=\;&[\mathbf{curl}_{\boldsymbol{y}}\{G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})\mathrm{I}_3\}]^{\mathsf{T}}(\boldsymbol{\psi}(\boldsymbol{y})\times\boldsymbol{n}(\boldsymbol{y}))\\
=\;&-[\mathbf{curl}_{\boldsymbol{y}}\{G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})\mathrm{I}_3\}]\,(\boldsymbol{\psi}(\boldsymbol{y})\times\boldsymbol{n}(\boldsymbol{y}))\\
=\;&[\mathbf{curl}_{\boldsymbol{x}}\{G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})\mathrm{I}_3\}](\boldsymbol{\psi}(\boldsymbol{y})\times\boldsymbol{n}(\boldsymbol{y}))\\
=\;&\mathbf{curl}_{\boldsymbol{x}}\left\{G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})(\boldsymbol{\psi}(\boldsymbol{y})\times\boldsymbol{n}(\boldsymbol{y}))\right\}.
\end{aligned}\tag{3.8}$$

  The composition of the operator $(\lambda+2\mu)\boldsymbol{n}(\boldsymbol{x})\operatorname{div}_{\boldsymbol{x}}$ with this term vanishes since one has $\operatorname{div}\mathbf{curl}=0$. Now we apply the operator $(\lambda+2\mu)\boldsymbol{n}(\boldsymbol{x})\operatorname{div}_{\boldsymbol{x}}$ to (3.7). We have

$$\begin{aligned}
&-(\lambda+2\mu)\boldsymbol{n}(\boldsymbol{x})\operatorname{div}_{\boldsymbol{x}}\boldsymbol{\nabla}_{\boldsymbol{x}}G(\kappa_p,\boldsymbol{x}-\boldsymbol{y})\big(\boldsymbol{n}(\boldsymbol{y})\cdot\boldsymbol{\psi}(\boldsymbol{y})\big)\\
=\;&-(\lambda+2\mu)\boldsymbol{n}(\boldsymbol{x})\Delta_{\boldsymbol{x}}G(\kappa_p,\boldsymbol{x}-\boldsymbol{y})\big(\boldsymbol{n}(\boldsymbol{y})\cdot\boldsymbol{\psi}(\boldsymbol{y})\big)\\
=\;&(\lambda+2\mu)\kappa_p^2\boldsymbol{n}(\boldsymbol{x})G(\kappa_p,\boldsymbol{x}-\boldsymbol{y})\big(\boldsymbol{n}(\boldsymbol{y})\cdot\boldsymbol{\psi}(\boldsymbol{y})\big).
\end{aligned}$$

  The principal part of the corresponding boundary integral operator is then given by

$$\frac{i}{2}(\lambda+2\mu)\kappa_p^2\boldsymbol{n}\big(\Delta_\Gamma+\kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}}\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}}.$$

- Finally, we apply the operator $\mu\boldsymbol{n}(\boldsymbol{x})\times\mathbf{curl}_{\boldsymbol{x}}$ to (3.8) and we obtain

$$\begin{aligned}
&\mu\boldsymbol{n}(\boldsymbol{x})\times\Big(\left(-\boldsymbol{\Delta}_{\boldsymbol{x}}+\boldsymbol{\nabla}_{\boldsymbol{x}}\operatorname{div}_{\boldsymbol{x}}\right)\left\{G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})(\boldsymbol{\psi}(\boldsymbol{y})\times\boldsymbol{n}(\boldsymbol{y}))\right\}\Big)\\
=\;&\mu\kappa_s^2\boldsymbol{n}(\boldsymbol{x})\times\left\{G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})(\boldsymbol{\psi}(\boldsymbol{y})\times\boldsymbol{n}(\boldsymbol{y}))\right\}-\mu\,\mathbf{curl}_\Gamma^{\boldsymbol{x}}\operatorname{div}_{\boldsymbol{x}}\left\{G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})(\boldsymbol{\psi}(\boldsymbol{y})\times\boldsymbol{n}(\boldsymbol{y}))\right\}\\
=\;&\mu\kappa_s^2\boldsymbol{n}(\boldsymbol{x})\times\left\{G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})(\boldsymbol{\psi}(\boldsymbol{y})\times\boldsymbol{n}(\boldsymbol{y}))\right\}-\mu\,\mathbf{curl}_\Gamma^{\boldsymbol{x}}\left\{-\boldsymbol{\nabla}_\Gamma^{y}G(\kappa_s,\boldsymbol{x}-\boldsymbol{y})\cdot(\boldsymbol{\psi}(\boldsymbol{y})\times\boldsymbol{n}(\boldsymbol{y}))\right\},
\end{aligned}\tag{3.9}$$

  since $\mathbf{curl}_{\boldsymbol{x}}\mathbf{curl}_{\boldsymbol{x}}=-\boldsymbol{\Delta}_{\boldsymbol{x}}+\boldsymbol{\nabla}_{\boldsymbol{x}}\operatorname{div}_{\boldsymbol{x}}$. The following integral part formula for a scalar density $\boldsymbol{\varphi}_1$ and a tangential vector density $\boldsymbol{\varphi}_2$

$$-\int_\Gamma(\boldsymbol{\nabla}_\Gamma\phi_1)\cdot\boldsymbol{\varphi}_2\,ds=+\int_\Gamma\boldsymbol{\varphi}_1\operatorname{div}_\Gamma\boldsymbol{\varphi}_2\,ds,$$

  and the identity $\operatorname{div}_\Gamma(\boldsymbol{\varphi}_2\times\boldsymbol{n})=\operatorname{curl}_\Gamma\boldsymbol{\varphi}_2$ give the principal part for the boundary integral operator associated with the kernel (3.9)

$$\mu\frac{i}{2}\big(\boldsymbol{\Delta}_\Gamma+\kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\Big(\kappa_s^2\mathrm{I}_{\boldsymbol{t}}-\mathbf{curl}_\Gamma\operatorname{curl}_\Gamma\Big).$$

  The composition of the operator $\mu\boldsymbol{n}(\boldsymbol{x})\times\mathbf{curl}_{\boldsymbol{x}}$ with the term (3.7) also vanishes since $\mathbf{curl}\,\boldsymbol{\nabla}=0$.

We conclude by collecting all the results. □

Remark 3.6. *The principal part $P_1(N)$ can be rewritten under the form*

$$\begin{aligned}
P_1(N)\;=\;&2\mu P_0(D')\boldsymbol{\mathcal{M}}+2\mu\boldsymbol{\mathcal{M}}P_0(D)-(2\mu)^2\boldsymbol{\mathcal{M}}P_{-1}(S)\boldsymbol{\mathcal{M}}\\
&+\frac{i}{2}\Big[(\lambda+2\mu)\kappa_p^2\,\boldsymbol{n}\big(\Delta_\Gamma+\kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}}\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}}+\mu\big(\boldsymbol{\Delta}_\Gamma+\kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\Big(\kappa_s^2\mathrm{I}_{\boldsymbol{t}}-\mathbf{curl}_\Gamma\operatorname{curl}_\Gamma\Big)\Big].
\end{aligned}$$

The approximate DtN is then given by

$$\boldsymbol{\Lambda}=-(P_{-1}(S))^{-1}\Big(\frac{I}{2}-P_0(D)\Big)\qquad\text{or}\qquad\boldsymbol{\Lambda}=P_1(N)\Big(\frac{I}{2}+P_0(D)\Big)^{-1}.\tag{3.10}$$

Square root operators of the form $(\Delta_\Gamma+\kappa^2\mathrm{I})^{1/2}$ or $(\boldsymbol{\Delta}_\Gamma+\kappa^2\mathrm{I})^{1/2}$ and their inverse appear in $\boldsymbol{\Lambda}$. Since Γ is a compact manifold, the scalar Laplace-Beltrami operator Δ_Γ admits a countable

increasing sequence of non-negative real eigenvalues $(\beta_j)_{j\in\mathbb{N}}$ associated to the normalized eigenfunctions $(Y_j)_{j\in\mathbb{N}}$ satisfying the eigenvalue problem $-\Delta_\Gamma Y_j = \beta_j Y_j$ in $L^2(\Gamma)$. For any $\kappa > 0$, the scalar square root $(\Delta_\Gamma + \kappa^2 \mathrm{I})^{\frac{1}{2}}$ and its inverse are defined by

$$(\Delta_\Gamma + \kappa^2 \mathrm{I})^{\frac{1}{2}} = \sum_{j=1}^{\infty} (\kappa^2 - \beta_j)^{\frac{1}{2}} \langle \cdot, Y_j \rangle_{L^2} Y_j \text{ and } (\Delta_\Gamma + \kappa^2 \mathrm{I})^{-\frac{1}{2}} = \sum_{j=1}^{\infty} (\kappa^2 - \beta_j)^{-\frac{1}{2}} \langle \cdot, Y_j \rangle_{L^2} Y_j. \tag{3.11}$$

For $j \geq 1$ we have $\beta_j \neq 0$. Since $\Delta_\Gamma = \mathrm{div}_\Gamma \boldsymbol{\nabla}_\Gamma = -\,\mathrm{curl}_\Gamma \mathbf{curl}_\Gamma$, the vectors $\boldsymbol{\nabla}_\Gamma Y_j$ and $\mathbf{curl}_\Gamma Y_j$ are eigenvectors of the vector Laplace-Beltrami operator $\boldsymbol{\Delta}_\Gamma$ with the eigenvalue β_j. For $j \geq 0$ we set $\boldsymbol{\mathcal{Y}}_j^{(3)} = \boldsymbol{n} Y_j$ and for $j \geq 1$ we set $\boldsymbol{\mathcal{Y}}_j^{(1)} = \beta_j^{-\frac{1}{2}} \boldsymbol{\nabla}_\Gamma Y_j$ and $\boldsymbol{\mathcal{Y}}_j^{(2)} = \beta_j^{-\frac{1}{2}} \mathbf{curl}_\Gamma Y_j$. The concatenation of the sequences $(\boldsymbol{\mathcal{Y}}_j^{(1)})_{j\in\mathbb{N}^*}$, $(\boldsymbol{\mathcal{Y}}_j^{(2)})_{j\in\mathbb{N}^*}$ and $(\boldsymbol{\mathcal{Y}}_j^{(3)})_{j\in\mathbb{N}}$ forms an orthonormal basis of $\boldsymbol{L}^2(\Gamma)$. The vector square root $(\boldsymbol{\Delta}_\Gamma + \kappa^2 \mathrm{I})^{\frac{1}{2}}$ and its inverse are defined for tangential fields by

$$(\boldsymbol{\Delta}_\Gamma + \kappa^2 \mathrm{I})^{\frac{1}{2}} = \sum_{k=1}^{2} \sum_{j=1}^{\infty} (\kappa^2 - \beta_j)^{\frac{1}{2}} \langle \cdot, \boldsymbol{\mathcal{Y}}_j^{(k)} \rangle_{\boldsymbol{L}^2} \boldsymbol{\mathcal{Y}}_j^{(k)} \qquad \text{and,}$$

$$(\boldsymbol{\Delta}_\Gamma + \kappa^2 \mathrm{I})^{-\frac{1}{2}} = \sum_{k=1}^{2} \sum_{j=1}^{\infty} (\kappa^2 - \beta_j)^{-\frac{1}{2}} \langle \cdot, \boldsymbol{\mathcal{Y}}_j^{(k)} \rangle_{\boldsymbol{L}^2} \boldsymbol{\mathcal{Y}}_j^{(k)}. \tag{3.12}$$

The modes j such that $\beta_j < \kappa_p^2$ correspond to the propagative modes while the ones given for $\beta_j > \kappa_s^2$ are linked to evanescent waves. For elastic scattering problems, a transition region corresponding to modes such that $\beta_j \approx \kappa_p^2$ or $\beta_j \approx \kappa_s^2$ includes the grazing modes. The artificial singularity of the square root operator does not yield a satisfactory representation of these modes. To approximately model this behavior, we use a regularization of $\boldsymbol{\Lambda}$ by introducing a small local damping parameter $\varepsilon > 0$ in the transition region. More precisely, we consider the following approximations of the DtN operator

$$\boldsymbol{\Lambda}_\varepsilon = -(P_{-1,\varepsilon}(S))^{-1}\Big(\frac{I}{2} - P_{0,\varepsilon}(D)\Big) \qquad \text{or} \qquad \boldsymbol{\Lambda}_\varepsilon = P_{1,\varepsilon}(N)\Big(\frac{I}{2} + P_{0,\varepsilon}(D)\Big)^{-1} \tag{3.13}$$

where $P_{-1,\varepsilon}(S)$, $P_{0,\varepsilon}(D)$, $P_{1,\varepsilon}(N)$ are defined in the same way as $P_{-1}(S)$, $P_0(D)$ and $P_1(N)$ by replacing κ_s and κ_p respectively with $\kappa_{s,\varepsilon}$ and $\kappa_{p,\varepsilon}$

$$\begin{cases} \kappa_{s,\varepsilon} = \kappa_s + i\varepsilon_s, & \text{with} \quad \varepsilon_s = 0.4\kappa_s^{\frac{1}{3}} R^{\frac{2}{3}}, \\ \kappa_{p,\varepsilon} = \kappa_p + i\varepsilon_p, & \text{with} \quad \varepsilon_p = 0.4\kappa_p^{\frac{1}{3}} R^{\frac{2}{3}}, \end{cases} \tag{3.14}$$

where R is the radius of the smallest sphere containing the scatterer [7]. The generalized BW integral equation for the Dirichlet boundary condition, called DtN-preconditioned BW integral equation, is given by

$$(\frac{1}{2}\mathrm{I} + D - S\boldsymbol{\Lambda}_\varepsilon)\boldsymbol{\psi} = -\gamma_0^+ \boldsymbol{u}^{inc}, \quad \text{on } \Gamma. \tag{3.15}$$

Remark 3.7. *We have constructed an On-Surface Radiation Condition (OSRC) defined by*

$$\gamma_1^+ \boldsymbol{u} = \boldsymbol{\Lambda}_\varepsilon \gamma_0^+ \boldsymbol{u}, \quad \text{on } \Gamma,$$

for the exterior Dirichlet elasticity problem. This is why this preconditioning approach can be also called OSRC preconditioning.

We have the following existence and uniqueness result.

Proposition 3.8. *The DtN-preconditioned BW integral equation* (3.15) *is uniquely solvable in* $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ *for any frequency* $\omega > 0$ *and damping parameters* $\varepsilon_s > 0$ *and* $\varepsilon_p > 0$.

Proof. We first prove that the boundary integral operator $(\frac{1}{2}\mathrm{I}+D-S\mathbf{\Lambda}_\varepsilon)$ can be written as $A+K$, where A is an invertible operator and K is a compact operator from $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ to itself. To this end, we set $\boldsymbol{\psi}=\psi_0^{(3)}\,\boldsymbol{\mathcal{Y}}_0^{(3)}+\sum\limits_{j=1}^{+\infty}\Big(\psi_j^{(1)}\,\boldsymbol{\mathcal{Y}}_j^{(1)}+\psi_j^{(2)}\,\boldsymbol{\mathcal{Y}}_j^{(2)}+\psi_j^{(3)}\,\boldsymbol{\mathcal{Y}}_j^{(3)}\Big)$ and compute

$$\Big(\frac{1}{2}\mathrm{I}+P_0(D)-P_{-1}(S)\mathbf{\Lambda}_\varepsilon\Big)\boldsymbol{\psi}=\frac{1}{2}\Big(\mathrm{I}+P_{-1}(S)\big(P_{-1,\varepsilon}(S)\big)^{-1}\Big)\boldsymbol{\psi}+\Big(P_0(D)-P_{-1}(S)\big(P_{-1,\varepsilon}(S)\big)^{-1}P_{0,\varepsilon}(D)\Big)\boldsymbol{\psi}.$$

For $j\in\mathbb{N}^*$, we set $\boldsymbol{\psi}_j=(\boldsymbol{\psi}_j^{(1)},\boldsymbol{\psi}_j^{(2)},\boldsymbol{\psi}_j^{(3)})^{\mathsf{T}}$ and we introduce a family of operators $\boldsymbol{\mathcal{L}}_j$ defined by $\boldsymbol{\mathcal{L}}_j\boldsymbol{\psi}_j=\psi_j^{(1)}\,\boldsymbol{\mathcal{Y}}_j^{(1)}+\psi_j^{(2)}\,\boldsymbol{\mathcal{Y}}_j^{(2)}+\psi_j^{(3)}\,\boldsymbol{\mathcal{Y}}_j^{(3)}$. We can thus write $\boldsymbol{\psi}=\psi_0^{(3)}\,\boldsymbol{\mathcal{Y}}_0^{(3)}+\sum\limits_{j=1}^{+\infty}\boldsymbol{\mathcal{L}}_j\boldsymbol{\psi}_j$.

From Lemma 3.2 and the spectral expansions (3.11) and (3.12) of the square root operators, we obtain the following spectral decomposition for the operator $P_{-1}(S)$:

$$P_{-1}(S)\boldsymbol{\psi}=s_0^{(3,3)}\psi_0^{(3)}\,\boldsymbol{\mathcal{Y}}_0^{(3)}+\sum_{j=1}^{+\infty}\frac{i}{2\rho\omega^2}\boldsymbol{\mathcal{L}}_j\begin{pmatrix}s_j^{(1,1)}&0&0\\0&s_j^{(2,2)}&0\\0&0&s_j^{(3,3)}\end{pmatrix}\begin{pmatrix}\boldsymbol{\psi}_j^{(1)}\\\boldsymbol{\psi}_j^{(2)}\\\boldsymbol{\psi}_j^{(3)}\end{pmatrix},$$

with $s_0^{(3,3)}=i\kappa_p\big[2\rho\omega^2\big]^{-1}$ and for $j\geq 1$, $s_j^{(1,1)}=(\kappa_s^2-\beta_j)^{\frac{1}{2}}+\beta_j(\kappa_p^2-\beta_j)^{-\frac{1}{2}}$, $s_j^{(2,2)}=\kappa_s^2(\kappa_s^2-\beta_j)^{-\frac{1}{2}}$ and $s_j^{(3,3)}=(\kappa_p^2-\beta_j)^{\frac{1}{2}}+\beta_j(\kappa_s^2-\beta_j)^{-\frac{1}{2}}$. From Lemma 3.4, we deduce

$$P_0(D)\boldsymbol{\psi}=2\mu P_{-1}(S)\boldsymbol{\mathcal{M}}\boldsymbol{\psi}-\sum_{j=1}^{+\infty}\frac{1}{2}\boldsymbol{\mathcal{L}}_j\begin{pmatrix}0&0&\tilde{d}_j^{(1,3)}\\0&0&0\\\tilde{d}_j^{(3,1)}&0&0\end{pmatrix}\begin{pmatrix}\boldsymbol{\psi}_j^{(1)}\\\boldsymbol{\psi}_j^{(2)}\\\boldsymbol{\psi}_j^{(3)}\end{pmatrix},$$

with $\tilde{d}_j^{(1,3)}=(-\beta_j)^{\frac{1}{2}}(\kappa_p^2-\beta_j)^{-\frac{1}{2}}$ and $\tilde{d}_j^{(3,1)}=(-\beta_j)^{\frac{1}{2}}(\kappa_s^2-\beta_j)^{-\frac{1}{2}}$. The spectral decomposition of $P_{-1,\varepsilon}(S)$ and $P_{0,\varepsilon}(D)$ are given by the same formulas as here above with κ_s and κ_p replaced by $\kappa_{s,\varepsilon}$ and $\kappa_{p,\varepsilon}$. We denote by $s_{j,\varepsilon}$ and $\tilde{d}_{j,\varepsilon}$ their spectral coefficients. When $j\to\infty$ (i.e $\beta_j\to+\infty$), we have

$$\begin{aligned}
s_j^{(1,1)}\big(s_{j,\varepsilon}^{(1,1)}\big)^{-1}&=\frac{\kappa_s^2+\kappa_p^2}{\kappa_{s,\varepsilon}^2+\kappa_{p,\varepsilon}^2}+\mathcal{O}\left(\frac{1}{\beta_j}\right),\\
s_j^{(2,2)}\big(s_{j,\varepsilon}^{(2,2)}\big)^{-1}&=\frac{\kappa_s}{\kappa_{s,\varepsilon}}+\mathcal{O}\left(\frac{1}{\beta_j}\right),\\
s_j^{(3,3)}\big(s_{j,\varepsilon}^{(3,3)}\big)^{-1}&=\frac{\kappa_s^2+\kappa_p^2}{\kappa_{s,\varepsilon}^2+\kappa_{p,\varepsilon}^2}+\mathcal{O}\left(\frac{1}{\beta_j}\right).
\end{aligned}$$

This implies

$$P_{-1}(S)(P_{-1,\varepsilon}(S))^{-1}\boldsymbol{\psi}=\frac{\kappa_p}{\kappa_{p,\varepsilon}}\psi_0^{(3)}\,\boldsymbol{\mathcal{Y}}_0^{(3)}+\sum_{j=1}^{+\infty}\boldsymbol{\mathcal{L}}_j\begin{pmatrix}\frac{\kappa_s^2+\kappa_p^2}{\kappa_{s,\varepsilon}^2+\kappa_{p,\varepsilon}^2}&0&0\\0&\frac{\kappa_s}{\kappa_{s,\varepsilon}}&0\\0&0&\frac{\kappa_s^2+\kappa_p^2}{\kappa_{s,\varepsilon}^2+\kappa_{p,\varepsilon}^2}\end{pmatrix}\begin{pmatrix}\psi_j^{(1)}\\\psi_j^{(2)}\\\psi_j^{(3)}\end{pmatrix}+\widetilde{K}_1\boldsymbol{\psi},$$

with $\widetilde{K}_1$ a compact operator of negative order.

The tangential Günter derivative can be expressed in the spectral function basis as follows

$$\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{Y}}_j^{(1)}=-\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{Y}}_j^{(1)}+\beta_j^{\frac{1}{2}}\boldsymbol{\mathcal{Y}}_j^{(3)},\quad \boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{Y}}_j^{(2)}=-\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{Y}}_j^{(2)}\quad\text{and}\quad \boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{Y}}_j^{(3)}=\beta_j^{\frac{1}{2}}\boldsymbol{\mathcal{Y}}_j^{(1)}-2\mathcal{H}\boldsymbol{\mathcal{Y}}_j^{(3)}.$$

From this, we deduce the following asymptotic expansions when $j\to\infty$ (i.e $\beta_j\to+\infty$)

$$\begin{aligned}
\frac{2i\mu}{\rho\omega^2}s_j^{(1,1)}\beta_j^{\frac{1}{2}}-\frac{1}{2}\tilde{d}_j^{(1,3)}&=\frac{\kappa_p^2}{4\kappa_s^2}+\mathcal{O}\left(\frac{1}{\beta_j}\right),\\
\frac{2i\mu}{\rho\omega^2}s_j^{(3,3)}\beta_j^{\frac{1}{2}}-\frac{1}{2}\tilde{d}_j^{(3,1)}&=\frac{\kappa_p^2}{4\kappa_s^2}+\mathcal{O}\left(\frac{1}{\beta_j}\right).
\end{aligned}$$

And we have

$$P_0(D)\boldsymbol{\psi} = \sum_{j=1}^{+\infty} \boldsymbol{\mathcal{L}}_j \begin{pmatrix} 0 & 0 & \frac{\kappa_p^2}{4\kappa_s^2} \\ 0 & 0 & 0 \\ \frac{\kappa_p^2}{4\kappa_s^2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_j^{(1)} \\ \psi_j^{(2)} \\ \psi_j^{(3)} \end{pmatrix} + \widetilde{K}_2\boldsymbol{\psi},$$

where $\widetilde{K}_2$ is a compact operator of negative order.

Finally, the integral operator $(\frac{1}{2}\mathrm{I} + D - S\boldsymbol{\Lambda}_\varepsilon)$ is a compact perturbation of the operator A defined by

$$A\boldsymbol{\psi} = (\frac{1}{2}+\frac{1}{2}\frac{\kappa_p}{\kappa_{p,\varepsilon}})\psi_0^{(3)}\,\boldsymbol{\mathcal{Y}}_0^{(3)} + \frac{1}{2}\sum_{j=1}^{+\infty} \boldsymbol{\mathcal{L}}_j \begin{pmatrix} 1+\frac{\kappa_s^2+\kappa_p^2}{\kappa_{s,\varepsilon}^2+\kappa_{p,\varepsilon}^2} & 0 & \frac{\kappa_p^2}{2\kappa_s^2} - \frac{\kappa_s^2+\kappa_p^2}{\kappa_{s,\varepsilon}^2+\kappa_{p,\varepsilon}^2}\frac{\kappa_{p,\varepsilon}^2}{2\kappa_{s,\varepsilon}^2} \\ 0 & 1+\frac{\kappa_s}{\kappa_{s,\varepsilon}} & 0 \\ \frac{\kappa_p^2}{2\kappa_s^2} - \frac{\kappa_s^2+\kappa_p^2}{\kappa_{s,\varepsilon}^2+\kappa_{p,\varepsilon}^2}\frac{\kappa_{p,\varepsilon}^2}{2\kappa_{s,\varepsilon}^2} & 0 & 1+\frac{\kappa_s^2+\kappa_p^2}{\kappa_{s,\varepsilon}^2+\kappa_{p,\varepsilon}^2} \end{pmatrix} \begin{pmatrix} \psi_j^{(1)} \\ \psi_j^{(2)} \\ \psi_j^{(3)} \end{pmatrix}$$

which is invertible since the coefficients of the 3-by-3 matrices are diagonally dominant. It follows that th e operator is a Fredholm operator of index zero and the Riez theory yields its invertibility if the operator is injective. Therefore, it is sufficient to prove that the homogeneous equation associated to (3.15) admits $\boldsymbol{\psi} = 0$ as unique solution to get the existence and uniqueness of the solution to the generalized BW integral equation. Let $\boldsymbol{\psi} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ be a solution to $(\frac{1}{2}\mathrm{I} + D - S\boldsymbol{\Lambda}_\varepsilon)\boldsymbol{\psi} = 0$. Then, the function $\boldsymbol{u}$ defined by

$$\boldsymbol{u}^+(\boldsymbol{x}) = \mathcal{D}\boldsymbol{\psi}(\boldsymbol{x}) - \mathcal{S}\boldsymbol{\Lambda}_\varepsilon\boldsymbol{\psi}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega^+,$$

solves the homogeneous exterior Dirichlet problem and $\boldsymbol{u}^+ = 0$ in Ω^+. Now we set

$$\boldsymbol{u}^-(\boldsymbol{x}) = \mathcal{D}\boldsymbol{\psi}(\boldsymbol{x}) - \mathcal{S}\boldsymbol{\Lambda}_\varepsilon\boldsymbol{\psi}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega^-.$$

Then we have $-\gamma_0^-\boldsymbol{u}^- = \boldsymbol{\psi}$ and $-\gamma_1^-\boldsymbol{u}^- = \boldsymbol{\Lambda}_\varepsilon\boldsymbol{\psi}$. The first Green formula (2.2) with $\boldsymbol{u} = \boldsymbol{u}^-$ and $\boldsymbol{v} = \overline{\boldsymbol{u}^-}$ in the interior domain Ω^- gives

$$\int_\Omega \Big(\lambda|\operatorname{div}\boldsymbol{u}^-|^2 + 2\mu\,|\varepsilon(\boldsymbol{u}^-)|^2\Big)\,dx - \rho\omega^2\int_\Omega |\boldsymbol{u}^-|^2\,dx = \int_\Gamma \overline{\boldsymbol{\psi}}\cdot\boldsymbol{\Lambda}_\varepsilon\boldsymbol{\psi}\,ds. \tag{3.16}$$

If we take the imaginary part of this equation, we obtain the equality

$$\operatorname{Im}\left(\int_\Gamma \overline{\boldsymbol{\psi}}\cdot\boldsymbol{\Lambda}_\varepsilon\boldsymbol{\psi}\,ds\right) = 0.$$

We have the spectral decomposition for the operator $\boldsymbol{\Lambda}_\varepsilon$

$$\begin{aligned}\boldsymbol{\Lambda}_\varepsilon\boldsymbol{\psi} = {} & 2\mu\boldsymbol{\mathcal{M}}\boldsymbol{\psi} + i\frac{\rho\omega^2}{\kappa_{p,\varepsilon}}\psi_0^{(3)}\,\boldsymbol{\mathcal{Y}}_0^{(3)} \\ & + i\rho\omega^2\sum_{j=1}^{+\infty}\boldsymbol{\mathcal{L}}_j \begin{pmatrix} \big(s_{j,\varepsilon}^{(1,1)}\big)^{-1} & 0 & -\big(s_{j,\varepsilon}^{(1,1)}\big)^{-1}\tilde{d}_{j,\varepsilon}^{(1,3)} \\ 0 & \big(s_{j,\varepsilon}^{(2,2)}\big)^{-1} & 0 \\ -\big(s_{j,\varepsilon}^{(3,3)}\big)^{-1}\tilde{d}_{j,\varepsilon}^{(3,1)} & 0 & \big(s_{j,\varepsilon}^{(3,3)}\big)^{-1} \end{pmatrix} \begin{pmatrix} \psi_j^{(1)} \\ \psi_j^{(2)} \\ \psi_j^{(3)} \end{pmatrix}.\end{aligned}$$

- Let $\boldsymbol{\psi} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$. We can write

  $$\int_\Gamma \overline{\boldsymbol{\psi}}\cdot\boldsymbol{\mathcal{M}}\boldsymbol{\psi}\,ds = \int_\Gamma \boldsymbol{\mathcal{M}}\overline{\boldsymbol{\psi}}\cdot\boldsymbol{\psi} ds = \overline{\int_\Gamma \boldsymbol{\mathcal{M}}\boldsymbol{\psi}\cdot\overline{\boldsymbol{\psi}}\,ds}.$$

  We deduce that the integral $\displaystyle\int_\Gamma \overline{\boldsymbol{\psi}}\cdot(2\mu\boldsymbol{\mathcal{M}}\boldsymbol{\psi})\,ds$ is a real number.
- For any $a \in \mathbb{C}$, we have $\operatorname{Im}(ia) = \operatorname{Re}(a)$. It follows that

  $$\operatorname{Im}\left(\int_\Gamma \overline{\psi_0^{(3)}}\boldsymbol{\mathcal{Y}}_0^{(3)}\cdot(i\frac{\rho\omega^2}{\kappa_{p,\varepsilon}}\psi_0^{(3)}\,\boldsymbol{\mathcal{Y}}_0^{(3)})ds\right) = \rho\omega^2\operatorname{Re}\left(\frac{1}{\kappa_{p,\varepsilon}}\right)|\psi_0^{(3)}|^2,$$

  and $\operatorname{Re}\left(\dfrac{1}{\kappa_{p,\varepsilon}}\right) = \operatorname{Re}\left(\dfrac{\kappa_p - i\varepsilon_p}{|\kappa_{p,\varepsilon}|^2}\right) > 0.$

- It remains to analyze the imaginary part of

$$\int_\Gamma \left(\sum_{j=1}^{\infty} \overline{\boldsymbol{\mathcal{L}}_j \boldsymbol{\psi}_j}\right) \cdot \left[i\rho\omega^2 \sum_{j=1}^{+\infty} \boldsymbol{\mathcal{L}}_j \begin{pmatrix} \big(s_{j,\varepsilon}^{(1,1)}\big)^{-1} & 0 & -\big(s_{j,\varepsilon}^{(1,1)}\big)^{-1} \tilde{d}_{j,\varepsilon}^{(1,3)} \\ 0 & \big(s_{j,\varepsilon}^{(2,2)}\big)^{-1} & 0 \\ -\big(s_{j,\varepsilon}^{(3,3)}\big)^{-1} \tilde{d}_{j,\varepsilon}^{(3,1)} & 0 & \big(s_{j,\varepsilon}^{(3,3)}\big)^{-1} \end{pmatrix} \begin{pmatrix} \psi_j^{(1)} \\ \psi_j^{(2)} \\ \psi_j^{(3)} \end{pmatrix} \right] ds,$$

which is the real part of

$$\sum_{j=1}^{+\infty} \begin{pmatrix} \overline{\psi_j^{(1)}} & \overline{\psi_j^{(2)}} & \overline{\psi_j^{(3)}} \end{pmatrix} \begin{pmatrix} \big(s_{j,\varepsilon}^{(1,1)}\big)^{-1} & 0 & -\big(s_{j,\varepsilon}^{(1,1)}\big)^{-1} \tilde{d}_{j,\varepsilon}^{(1,3)} \\ 0 & \big(s_{j,\varepsilon}^{(2,2)}\big)^{-1} & 0 \\ -\big(s_{j,\varepsilon}^{(3,3)}\big)^{-1} \tilde{d}_{j,\varepsilon}^{(3,1)} & 0 & \big(s_{j,\varepsilon}^{(3,3)}\big)^{-1} \end{pmatrix} \begin{pmatrix} \psi_j^{(1)} \\ \psi_j^{(2)} \\ \psi_j^{(3)} \end{pmatrix},$$

that we rewrite

$$\sum_{j=1}^{+\infty} \big(s_{j,\varepsilon}^{(2,2)}\big)^{-1} |\psi_j^{(2)}|^2 + \sum_{j=1}^{+\infty} \begin{pmatrix} \overline{\psi_j^{(1)}} & \overline{\psi_j^{(3)}} \end{pmatrix} \begin{pmatrix} \big(s_{j,\varepsilon}^{(1,1)}\big)^{-1} & -\big(s_{j,\varepsilon}^{(1,1)}\big)^{-1} \tilde{d}_{j,\varepsilon}^{(1,3)} \\ -\big(s_{j,\varepsilon}^{(3,3)}\big)^{-1} \tilde{d}_{j,\varepsilon}^{(3,1)} & \big(s_{j,\varepsilon}^{(3,3)}\big)^{-1} \end{pmatrix} \begin{pmatrix} \psi_j^{(1)} \\ \psi_j^{(3)} \end{pmatrix}.$$

We have to prove that $\mathrm{Re}\left(\big(s_{j,\varepsilon}^{(2,2)}\big)^{-1}\right) > 0$ and that the real parts of the above 2-by-2 matrices are positive definite matrices, for all $j \geq 1$.

We have $\big(s_{j,\varepsilon}^{(2,2)}\big)^{-1} = \kappa_{s,\epsilon}^{-1}\big(1 - \frac{\beta_j}{\kappa_{s,\varepsilon}^2}\big)^{\frac{1}{2}}$. Let us set $\big(1 - \frac{\beta_j}{\kappa_{s,\varepsilon}^2}\big)^{\frac{1}{2}} = a_s + ib_s$ with $a_s, b_s \in \mathbb{R}$ and $a_s \geq 0$. Then we have $2a_s b_s = \mathrm{Im}\big(1 - \frac{\beta_j}{\kappa_{s,\varepsilon}^2}\big) = -\,\mathrm{Im}\big(\frac{\beta_j}{\kappa_{s,\varepsilon}^2}\big) = \beta_j \frac{2\kappa_s \varepsilon_s}{|\kappa_{s,\varepsilon}|^4} > 0$. It follows that $a_s > 0$ and $b_s > 0$ and

$$\mathrm{Re}\left(\big(s_{j,\varepsilon}^{(2,2)}\big)^{-1}\right) = \frac{\kappa_s a_s}{|\kappa_{s,\varepsilon}|^2} + \frac{\varepsilon_s b_s}{|\kappa_{s,\varepsilon}|^2} > 0.$$

We have $\big(s_{j,\varepsilon}^{(1,1)}\big)^{-1} \tilde{d}_{j,\varepsilon}^{(1,3)} = \big(s_{j,\varepsilon}^{(3,3)}\big)^{-1} \tilde{d}_{j,\varepsilon}^{(3,1)} = \frac{(-\beta_j)^{\frac{1}{2}}}{\beta_j + (\kappa_{s,\varepsilon}^2 - \beta_j)^{\frac{1}{2}} (\kappa_{p,\varepsilon}^2 - \beta_j)^{\frac{1}{2}}}$. Therefore the real parts of the 2-by-2 matrices are symmetric and diagonalizable. These are positive definite matrices if and only if their eigenvalues are positive real constants. This is realized if the trace and the determinant of 2-by-2 matrices are positive real constants. We have

$$\mathrm{Re}\left(\big(s_{j,\varepsilon}^{(1,1)}\big)^{-1}\right) = \frac{\mathrm{Re}\,\big(s_{j,\varepsilon}^{(1,1)}\big)}{\big|s_{j,\varepsilon}^{(1,1)}\big|^2} = \frac{1}{\big|s_{j,\varepsilon}^{(1,1)}\big|^2}\left(\mathrm{Re}\left(\big(\kappa_{s,\varepsilon}^2 - \beta_j\big)^{\frac{1}{2}}\right) + \beta_j \frac{\mathrm{Re}\left(\big(\kappa_{p,\varepsilon}^2 - \beta_j\big)^{\frac{1}{2}}\right)}{\big|\big(\kappa_{p,\varepsilon}^2 - \beta_j\big)^{\frac{1}{2}}\big|^2}\right) \geq 0.$$

Assume $\mathrm{Re}\left(\big(\kappa_{s,\varepsilon}^2 - \beta_j\big)^{\frac{1}{2}}\right) = 0$. It follows that $\mathrm{Im}(\kappa_{s,\varepsilon}^2 - \beta_j) = 2\kappa_s\varepsilon_s = 0$. This leads to a contradiction with the hypothesis that $\varepsilon_s > 0$. We deduce that $\mathrm{Re}\left(\big(\kappa_{s,\varepsilon}^2 - \beta_j\big)^{\frac{1}{2}})\right) > 0$ and $\mathrm{Re}\left(\big(\kappa_{p,\varepsilon}^2 - \beta_j\big)^{\frac{1}{2}}\right) > 0$. We have proved that $\mathrm{Re}\left(\big(s_{j,\varepsilon}^{(1,1)}\big)^{-1}\right) > 0$ and with the same arguments we also hav e $\mathrm{Re}\left(\big(s_{j,\varepsilon}^{(3,3)}\big)^{-1}\right) > 0$. We deduce that the trace of the 2-by-2 matrices are positive real constants. From long and similar calculus, we obtain the positivity of the determinant. Collecting all the results we conclude that

$$\mathrm{Im}\left(\int_\Gamma \overline{\boldsymbol{\psi}} \cdot \boldsymbol{\Lambda}_\varepsilon \boldsymbol{\psi}\, ds\right) = 0$$

implies $\boldsymbol{\psi} = 0$. The injectivity of the BIE operator is proved. □

The proof of the previous Proposition 3.8 highlights the good spectral behavior of the DtN-preconditioned BW-type integral operator $\frac{1}{2}\mathrm{I} + D - S\boldsymbol{\Lambda}_\varepsilon$. From a numerical point of view, the application of the regularizing operator $\boldsymbol{\Lambda}_\varepsilon$ needs the discretization of the operator $P_{0,\varepsilon}(D)$ and of either $P_{-1,\varepsilon}(S)$ or $P_{1,\varepsilon}(N)$. The expressions of these operators (see Lemmas 3.2, 3.4 and 3.5) show that the associated matrices after discretization do not enjoy interesting properties. The aim is to construct a preconditioner which is easier to implement. When we look at the definition of the operator $P_0(D)$, we would like to delete the term $2\mu P_{-1}(S)\boldsymbol{\mathcal{M}}$. In the same way, we would like to delete the term $R = 2\mu(P_0(D')\boldsymbol{\mathcal{M}} + \boldsymbol{\mathcal{M}}P_0(D)) - 4\mu^2\boldsymbol{\mathcal{M}}P_{-1}(S)\boldsymbol{\mathcal{M}}$ in the definition of $P_1(N)$

(cf. Remark 3.6) so that this operator is finally diagonal. The operator $P_0(D) - 2\mu P_{-1}(S)\boldsymbol{\mathcal{M}}$ is nothing else than the principal part of the integral operator $D_{2\mu}$, and $P_1(N) - R$ the principal part of $N_{2\mu}$. These observations have motivated the following section.

3.2. **Modified DtN approach ($\boldsymbol{\alpha = 2\mu}$).** We propose to rewrite the Somigliana representation formula as follows:

$$\begin{aligned}\boldsymbol{u}(\boldsymbol{x}) &= \mathcal{D}\gamma_0^+\boldsymbol{u}(\boldsymbol{x}) - \mathcal{S}\gamma_1^+\boldsymbol{u}(\boldsymbol{x}) \\ &= \mathcal{D}\gamma_0^+\boldsymbol{u}(\boldsymbol{x}) - 2\mu\mathcal{S}\boldsymbol{\mathcal{M}}\gamma_0^+\boldsymbol{u}(\boldsymbol{x}) - \mathcal{S}\gamma_1^+\boldsymbol{u}(\boldsymbol{x}) + 2\mu\mathcal{S}\boldsymbol{\mathcal{M}}\gamma_0^+\boldsymbol{u}(\boldsymbol{x}) \\ &= (\mathcal{D} - 2\mu\mathcal{S}\boldsymbol{\mathcal{M}})\gamma_0^+\boldsymbol{u}(\boldsymbol{x}) - \mathcal{S}(\boldsymbol{\Lambda}^{\rm ex} - 2\mu\boldsymbol{\mathcal{M}})\gamma_0^+\boldsymbol{u}(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega^+.\end{aligned}$$

Here we consider the Modified Dirichlet-to-Neumann (MDtN) map

$$\boldsymbol{\Lambda}_{2\mu}^{\rm ex} : \gamma_0^+\boldsymbol{u} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma) \mapsto \boldsymbol{\Lambda}_{2\mu}^{\rm ex}\gamma_0^+\boldsymbol{u} := \gamma_{1,2\mu}^+\boldsymbol{u} = (\boldsymbol{\Lambda}^{\rm ex} - 2\mu\boldsymbol{\mathcal{M}})\boldsymbol{u} = (\boldsymbol{T} - 2\mu\boldsymbol{\mathcal{M}})\boldsymbol{u} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma),$$

and we have

$$\boldsymbol{u}(\boldsymbol{x}) = \mathcal{D}_{2\mu}\gamma_0^+\boldsymbol{u}(\boldsymbol{x}) - \mathcal{S}\boldsymbol{\Lambda}_{2\mu}^{\rm ex}\gamma_0^+\boldsymbol{u}(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega^+.$$

As previously, an approximation $\boldsymbol{\Lambda}_{2\mu}$ of $\boldsymbol{\Lambda}_{2\mu}^{\rm ex}$ is introduced to construct the regularized integral equation

$$(\frac{1}{2}\mathrm{I} + D_{2\mu} - S\boldsymbol{\Lambda}_{2\mu})\boldsymbol{\psi} = -\gamma_0^+\boldsymbol{u}^{inc}, \quad \text{on } \Gamma, \tag{3.17}$$

where $\boldsymbol{\psi}$ is a fictive density. Then the scattered field $\boldsymbol{u}$ expressed by

$$\boldsymbol{u}(\boldsymbol{x}) = \mathcal{D}_{2\mu}\boldsymbol{\psi}(\boldsymbol{x}) - \mathcal{S}\boldsymbol{\Lambda}_{2\mu}\boldsymbol{\psi}(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega^+, \tag{3.18}$$

solves the Dirichlet elastic boundary value problem provided the fictive density $\boldsymbol{\psi} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ is solution of the integral equation (3.17).

Lemma 3.9. *Assume that ω is not an eigenfrequency of the Navier equation* (1.1a) *then we have*

$$\boldsymbol{\Lambda}_{2\mu}^{\rm ex} = -S^{-1}(\frac{1}{2}\mathrm{I} - D_{2\mu}) = N_{2\mu}(\frac{1}{2}\mathrm{I} + D_{2\mu})^{-1}.$$

Proof. We have $\boldsymbol{\Lambda}_{2\mu}^{\rm ex} = \boldsymbol{\Lambda}^{\rm ex} - 2\mu\boldsymbol{\mathcal{M}}$. We use Lemma 3.1 to obtain

$$\begin{aligned}\boldsymbol{\Lambda}_{2\mu}^{\rm ex} &= -S^{-1}\Big(\frac{1}{2}\mathrm{I} - D\Big) - 2\mu S^{-1}S\boldsymbol{\mathcal{M}} \\ &= -S^{-1}\Big(\frac{1}{2}\mathrm{I} - (D - 2\mu S\boldsymbol{\mathcal{M}})\Big) \\ &= -S^{-1}\Big(\frac{1}{2}\mathrm{I} - D_{2\mu}\Big).\end{aligned} \tag{3.19}$$

For the second equality we use Theorem 2.7 (v) and it remains to prove that $N_{2\mu}$ is invertible which ensures that $(D_{2\mu}^2 - \frac{1}{4}\mathrm{I})$ is invertible too. From the proof of Lemma 3.5, we obtain that

$$\begin{aligned}N_{2\mu}\boldsymbol{\psi}(\boldsymbol{x}) = (\lambda+2\mu)\kappa_p^2 \int_\Gamma \boldsymbol{n}(\boldsymbol{x})G(\kappa_p, \boldsymbol{x}-\boldsymbol{y})\big(\boldsymbol{n}(\boldsymbol{y})\cdot\boldsymbol{\psi}(\boldsymbol{y})\big)ds(\boldsymbol{y}) \\ + \mu\kappa_s^2 \int_\Gamma \boldsymbol{n}(\boldsymbol{x}) \times \mathbf{curl}_{\boldsymbol{x}}\,\mathbf{curl}_{\boldsymbol{x}}\left\{G(\kappa_s, \boldsymbol{x}-\boldsymbol{y})(\boldsymbol{\psi}(\boldsymbol{y})\times\boldsymbol{n}(\boldsymbol{y}))\right\} ds(\boldsymbol{y}),\ \boldsymbol{x}\in\Gamma.\end{aligned} \tag{3.20}$$

The frequency ω is not an eigenfrequency for the Navier equation means that the wavenumber κ_p is not an eigenfrequency for the Helmholtz equation and that the wavenumber κ_s is not an eigenfrequency for the Maxwell's equation. In this case, one knows that the corresponding single layer boundary integral operators are invertible. The operator $N_{2\mu}$ being an orthogonal linear combination of the acoustic and electromagnetic single layer boundary integral operators, we deduce that it is invertible too. □

According to these results, we propose the following representation of the operator $\boldsymbol{\Lambda}_{2\mu}$

$$\boldsymbol{\Lambda}_{2\mu} = P_1(N_{2\mu})\Big(\frac{1}{2}\mathrm{I} + P_0(D_{2\mu})\Big)^{-1}, \tag{3.21}$$

where $P_0(D_{2\mu})$ and $P_1(N_{2\mu})$ are the principal parts of the integral operators $D_{2\mu}$ and $N_{2\mu}$ respectively. These operators are expressed by

$$P_0(D_{2\mu}) = \frac{i}{2}\Big(-\boldsymbol{\nabla}_\Gamma\big(\Delta_\Gamma + \kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}}\,\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}} + \boldsymbol{n}\big(\Delta_\Gamma + \kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\,\mathrm{div}_\Gamma\,\mathrm{I}_{\boldsymbol{t}}\Big),$$

$$\begin{aligned} P_1(N_{2\mu}) &= \frac{i}{2}\Big[(\lambda+2\mu)\kappa_p^2\,\boldsymbol{n}\big(\Delta_\Gamma+\kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}}\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}} + \mu\big(\boldsymbol{\Delta}_\Gamma+\kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\Big(\kappa_s^2\mathrm{I}_{\boldsymbol{t}} - \mathbf{curl}_\Gamma\,\mathrm{curl}_\Gamma\Big)\Big] \\ &= \frac{i\rho\omega^2}{2}\Big[\boldsymbol{n}\big(\Delta_\Gamma+\kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}}\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}} + \big(\boldsymbol{\Delta}_\Gamma+\kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\Big(\mathrm{I}_{\boldsymbol{t}} - \frac{1}{\kappa_s^2}\,\mathbf{curl}_\Gamma\,\mathrm{curl}_\Gamma\Big)\Big]. \end{aligned}$$

For the same reasons as mentioned previously we will consider instead the following MDtN-preconditioner

$$\boldsymbol{\Lambda}_{2\mu,\varepsilon} = P_{1,\varepsilon}(N_{2\mu})\Big(\frac{I}{2} + P_{0,\varepsilon}(D_{2\mu})\Big)^{-1} \tag{3.22}$$

where $P_{1,\varepsilon}(N_{2\mu})$ and $P_{0,\varepsilon}(D_{2\mu})$ are deduced from $P_1(N_{2\mu})$ and $P_0(D_{2\mu})$ respectively by substituting κ_s and κ_p for $\kappa_{s,\varepsilon}$ and $\kappa_{p,\varepsilon}$ (see (3.14)). We will see in Section 4.1 that this operator is more suited to a numerical implementation than $\boldsymbol{\Lambda}_\varepsilon$. We can use previous investigations in the implementation of acoustic and electromagnetic OSRC-like preconditioners [7]. The new boundary integral equation for the Dirichlet boundary condition is given by

$$(\frac{1}{2}\mathrm{I} + D_{2\mu} - S\boldsymbol{\Lambda}_{2\mu,\varepsilon})\boldsymbol{\psi} = -\gamma_0^+\boldsymbol{u}^{inc}, \quad \text{on } \Gamma, \tag{3.23}$$

and is called MDtN-preconditioned BW integral equation. Following the steps of the proof of Proposition 3.8, we can prove the existence and uniqueness of the solution of (3.23).

Proposition 3.10. *The MDtN-preconditioned BW integral equation (3.23) is uniquely solvable in $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ for any frequency $\omega > 0$ and damping parameters $\varepsilon_s > 0$ and $\varepsilon_p > 0$.*

4. Spectral study for the spherical case

In this section, we present a numerical investigation of the eigenvalues of the standard and the DtN- and MDtN-preconditioned BW integral operators for the spherical case. The use of complex Padé rational approximants to localize the DtN and MDtN-preconditioners is discussed. Furthermore, we analyze the convergence properties of the GMRES algorithm for solving the different BW-type integral equations.

4.1. Asymptotic analysis.

Let Γ the sphere of radius R. Spectral decomposition in terms of the vector spherical harmonics of the elementary boundary integral operators S, D and N, and of their principal parts can be obtained (cf. Appendix A). Let us denote by $B_m^{(j,l)}$, $(j,l) \in \{1,2,3\}\times\{1,2,3\}$, the spectral coefficients of the decomposition of an integral operator B ($B = S, D$ or N). Each coefficient is of multiplicity $2m+1$. We study their asymptotic behavior. To this end, we distinguish three zones of modes: the hyperbolic zone for $|\kappa_p R| \to +\infty$ and $m \ll \kappa_p R$ (propagative modes), the elliptic zone when $m \to +\infty$ and $m \gg \kappa_s R$ (evanescent modes), and the transition zone of physical surface modes between $m \approx \kappa_p R$ and $m \approx \kappa_s R$. First, let us consider the hyperbolic zone. We have the following results.

Lemma 4.1. *In the hyperbolic zone, that is $|\omega R| \to +\infty$, the following asymptotic behaviors hold:*

- *for the operator S :*

$$\begin{aligned} S_m^{(1,1)} &= \frac{i}{\mu\kappa_s}\left(\sin^2(\kappa_s R - (m+1)\tfrac{\pi}{2}) - \tfrac{i}{2}\sin\left(2(\kappa_s R - (m+1)\tfrac{\pi}{2})\right)\right) + \mathcal{O}\left(\frac{1}{|\omega R|}\right), \quad m \geq 1, \\ S_m^{(2,2)} &= \frac{i}{\mu\kappa_s}\left(\cos^2(\kappa_s R - (m+1)\tfrac{\pi}{2}) + \tfrac{i}{2}\sin\left(2(\kappa_s R - (m+1)\tfrac{\pi}{2})\right)\right) + \mathcal{O}\left(\frac{1}{|\omega R|}\right), \quad m \geq 1, \\ S_m^{(3,3)} &= \frac{i}{(\lambda+2\mu)\kappa_p}\left(\sin^2(\kappa_p R - (m+1)\tfrac{\pi}{2}) - \tfrac{i}{2}\sin\left(2(\kappa_p R - (m+1)\tfrac{\pi}{2})\right)\right) + \mathcal{O}\left(\frac{1}{|\omega R|}\right), \ m \geq 0, \\ S_m^{(1,3)} &= \mathcal{O}\left(\frac{1}{|\omega R|}\right), \quad m \geq 1, \end{aligned}$$

- *for the operator D :*

$$\begin{aligned}
D_m^{(1,1)} &= \frac{1}{2} - \sin^2(\kappa_s R - (m+1)\tfrac{\pi}{2}) + \tfrac{i}{2}\sin\big(2(\kappa_s R - (m+1)\tfrac{\pi}{2})\big) + \mathcal{O}\left(\frac{1}{|\omega R|}\right), \quad m \geq 1,\\
D_m^{(2,2)} &= \frac{1}{2} - \cos^2(\kappa_s R - (m+1)\tfrac{\pi}{2}) - \tfrac{i}{2}\sin\big(2(\kappa_s R - (m+1)\tfrac{\pi}{2})\big) + \mathcal{O}\left(\frac{1}{|\omega R|}\right), \quad m \geq 1,\\
D_m^{(3,3)} &= \frac{1}{2} - \sin^2(\kappa_p R - (m+1)\tfrac{\pi}{2}) + \tfrac{i}{2}\sin\big(2(\kappa_p R - (m+1)\tfrac{\pi}{2})\big) + \mathcal{O}\left(\frac{1}{|\omega R|}\right), \quad m \geq 1,\\
D_m^{(1,3)} &= \mathcal{O}\left(\frac{1}{|\omega R|}\right), \quad D_m^{(3,1)}(\omega R) = \mathcal{O}\left(\frac{1}{|\omega R|}\right), \quad m \geq 1,
\end{aligned}$$

- *and for the operator N :*

$$\begin{aligned}
N_m^{(1,1)} &= \frac{\mu\kappa_s}{i}\left(-\cos^2(\kappa_s R - (m+1)\tfrac{\pi}{2}) - \tfrac{i}{2}\sin\big(2(\kappa_s R - (m+1)\tfrac{\pi}{2})\big)\right) + \mathcal{O}\left(\frac{1}{|\omega R|}\right), \quad m \geq 1,\\
N_m^{(2,2)} &= \frac{\mu\kappa_s}{i}\left(-\sin^2(\kappa_s R - (m+1)\tfrac{\pi}{2}) + \tfrac{i}{2}\sin\big(2(\kappa_s R - (m+1)\tfrac{\pi}{2})\big)\right) + \mathcal{O}\left(\frac{1}{|\omega R|}\right), \quad m \geq 1,\\
N_m^{(3,3)} &= i(\lambda+2\mu)\kappa_p\left(\cos^2(\kappa_p R - (m+1)\tfrac{\pi}{2}) + \tfrac{i}{2}\sin\big(2(\kappa_p R - (m+1)\tfrac{\pi}{2})\big)\right) + \mathcal{O}\left(\frac{1}{|\omega R|}\right), \ m \geq 0,\\
N_m^{(1,3)} &= \mathcal{O}\left(\frac{1}{|\omega R|}\right), \quad N_m^{(3,1)}(\omega R) = \mathcal{O}\left(\frac{1}{|\omega R|}\right), \quad m \geq 1.
\end{aligned}$$

Proof. We use asymptotic expansions for the spherical Bessel and Hankel functions expressed in [1]. □

And in the elliptic zone, we have

Lemma 4.2. *In the elliptic zone, that is $m \to +\infty$, the following asymptotic behaviors hold :*

- *for the operator S :*

$$\begin{aligned}
S_m^{(1,1)} &= \frac{1}{\mu}\frac{R}{2(2m+1)} + \frac{1}{(\lambda+2\mu)}\frac{R}{2(2m+1)} + \mathcal{O}\left(\frac{1}{m^3}\right),\\
S_m^{(2,2)} &= \frac{1}{\mu}\frac{R}{2m+1} + O\left(\frac{1}{m^3}\right),\\
S_m^{(3,3)} &= \frac{1}{\mu}\frac{R}{2(2m+1)} + \frac{1}{(\lambda+2\mu)}\frac{R}{2(2m+1)} + \mathcal{O}\left(\frac{1}{m^3}\right),\\
S_m^{(1,3)} &= \mathcal{O}\left(\frac{1}{m^2}\right),
\end{aligned}$$

- *for the operator D :*

$$\begin{aligned}
D_m^{(1,1)} &= -\frac{\mu}{\lambda+2\mu}\frac{1}{2(2m+1)} + \mathcal{O}\left(\frac{1}{m^3}\right),\\
D_m^{(2,2)} &= -\frac{3}{2(2m+1)} + O\left(\frac{1}{m^3}\right),\\
D_m^{(3,3)} &= -\frac{3\mu}{\lambda+2\mu}\frac{1}{2(2m+1)} + \mathcal{O}\left(\frac{1}{m^3}\right),\\
D_m^{(1,3)} &= \frac{\mu}{2(\lambda+2\mu)} + \mathcal{O}\left(\frac{1}{m^2}\right),
\end{aligned}$$

- *and for the operator N :*

$$\begin{aligned}
N_m^{(1,1)} &= \frac{-2\mu(\lambda+\mu)}{\lambda+2\mu}\frac{m+\frac{1}{2}}{2R} + \mathcal{O}\left(\frac{1}{m}\right),\\
N_m^{(2,2)} &= -\mu\frac{m+\frac{1}{2}}{2R} + \mathcal{O}\left(\frac{1}{m}\right),\\
N_m^{(3,3)} &= \frac{-2\mu(\lambda+\mu)}{\lambda+2\mu}\frac{m+\frac{1}{2}}{2R} + \mathcal{O}\left(\frac{1}{m}\right),\\
N_m^{(1,3)} &= N_m^{(3,1)} = \frac{\mu(3\lambda+\mu)}{2R(\lambda+2\mu)} + \mathcal{O}\left(\frac{1}{m^2}\right).
\end{aligned}$$

Proof. We use asymptotic expansions given in [1]. □

Let us validate both these asymptotic estimates and the principal parts of the elementary boundary integral operators proposed in Section 3.1. We keep only the modes m such that $|m| \leq m_{\max}$ with $m_{\max} = [4\kappa_s]$ (cf. Section 4.2). The value $[x]$ denotes the integer part of a real number x. We set the physical parameters $\kappa_s = 16\pi$, $\rho = \mu = 1$, and $\lambda = 2$. This corresponds to the Poisson's ratio $\nu = 1/3$, and to the relation $\kappa_s = 2\kappa_p$. We can see in Fig. 1 that the estimates obtained in the elliptic zone (cf. Lemma 4.2) well predict the behavior of the compact operator S. The principal part of the operator S is given by the diagonal coefficients. Note that the operator S and the proposed operator $P_{-1,\varepsilon}(S)$ exhibit the same asymptotic behavior in the elliptic zone. Furthermore, their spectral coefficients have the same magnitude order in the hyperbolic and transition parts. We conclude that the operator $P_{-1,\varepsilon}(S)$ represents a good approximation of the boundary integral operator S.

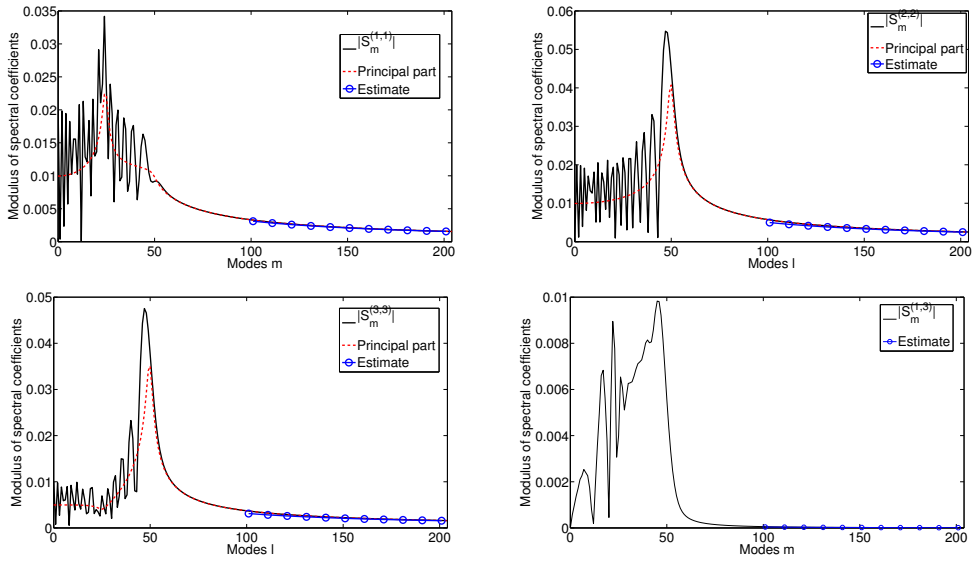


FIGURE 1. Asymptotic behavior: comparison between the integral operator S and the principal part $P_{-1,\varepsilon}(S)$, for $\kappa_s = 16\pi$ and $m_{\max} = [4\kappa_s]$.

We report in Fig. 2 numerical comparisons for the integral operator D. Same conclusions hold. The obtained estimates in the elliptic zone are validated and the operator $P_{0,\varepsilon}(D)$ contains the necessary spectral informations of the operator D. Here, the principal part of the operator D is contained in the extra-diagonal coefficients. Finally, we consider the integral operator N in Fig. 3. The operator N is a first-order pseudodifferential operator. The linear dependance of the diagonal coefficients on m in the elliptic zone shows this property. The operator $P_{1,\varepsilon}(N)$ have an identical behavior. However, in view of a numerical implementation, we would like to delete the diagonal components of $P_{0,\varepsilon}(D)$, and the extra-diagonal ones of $P_{1,\varepsilon}(N)$ in the elliptic part. This is realized by considering $P_{0,\varepsilon}(D_{2\mu})$ and $P_{1,\varepsilon}(N_{2\mu})$ (cf. Fig. 4 and 5). Furthermore, note that diagonal coefficients of $P_{0,\varepsilon}(D)$ associated with propagative and creeping modes are regularized. Spectral extra-diagonal coefficients of $P_{1,\varepsilon}(N_{2\mu})$ are equal to zero for any mode.

4.2. **Spectral comparison of standard and preconditioned BW integral formulations.** We consider the unit sphere as the scatterer. For the following numerical tests, the constitutive properties of the elastic obstacle are given by $\rho = 1$, $\nu = 1/3$, $\mu = 1$ and $\lambda = 2$. The wavenumbers satisfy $\kappa_s = 2\kappa_p$. This subsection is devoted to analyze the effect of the preconditioning on the eigenvalue clustering of the BW-type integral operators. In a finite dimensional approximation, we keep only the modes m such that $m \leq m_{\max}$. The number of modes $m_{\max}$ to retain must be large enough to capture the hyperbolic and transition parts of the spectrum ($m_{\max} \geq \kappa_p$) but also

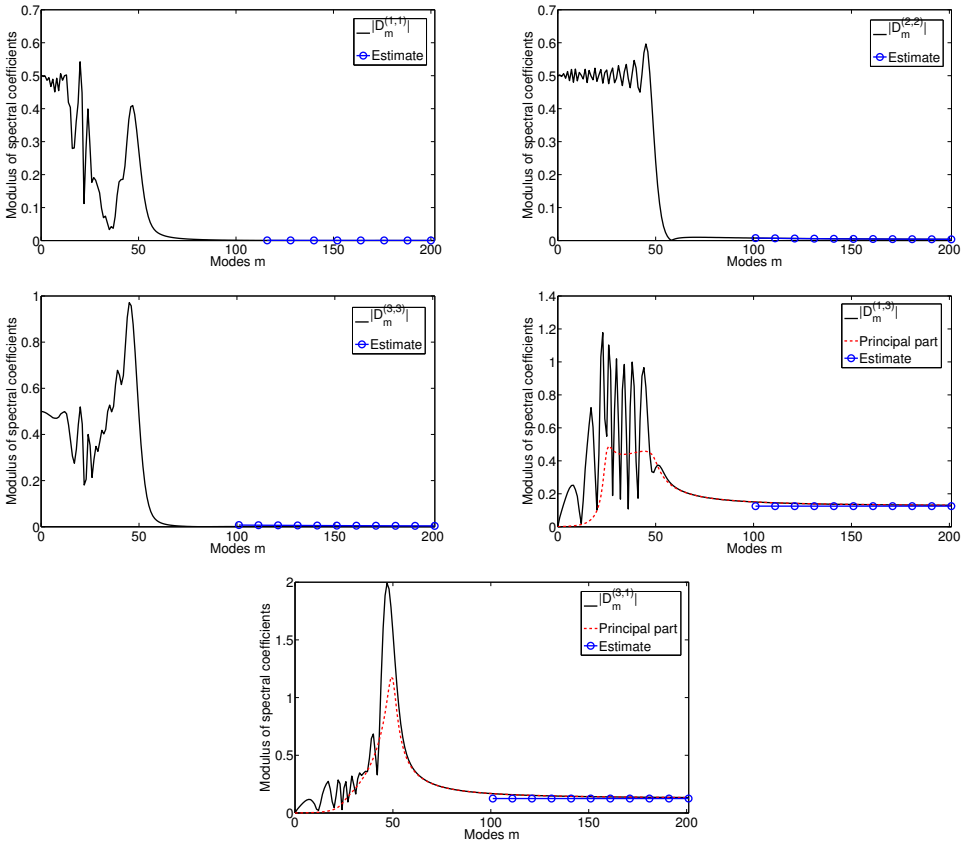


FIGURE 2. Asymptotic behavior: comparison between the integral operator D and the principal part $P_{0,\varepsilon}(D)$, for $\kappa_s = 16\pi$ and $m_{\max} = [4\kappa_s]$.

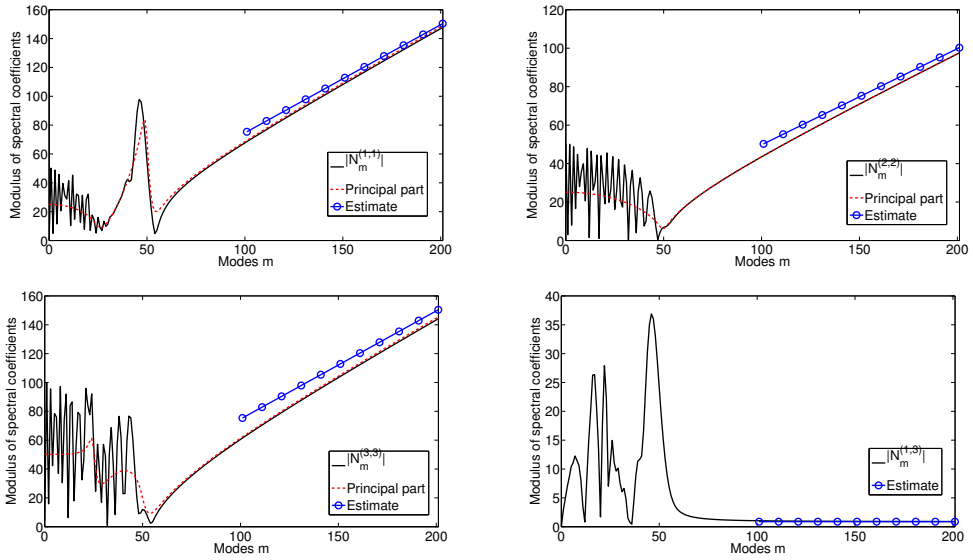


FIGURE 3. Asymptotic behavior: comparison between the integral operator N and the principal part $P_{1,\varepsilon}(N)$, for $\kappa_s = 16\pi$ and $m_{\max} = [4\kappa_s]$.

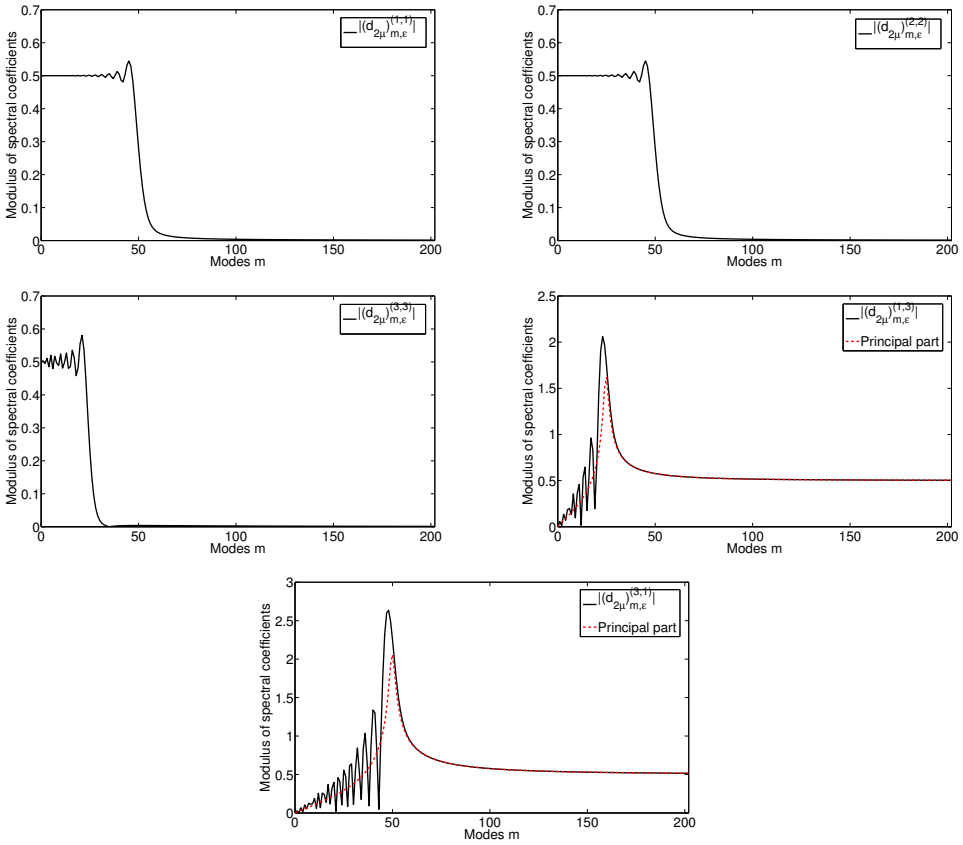


FIGURE 4. Asymptotic behavior: comparison between the integral operator $D_{2\mu}$ and the principal part $P_{0,\varepsilon}(D_{2\mu})$, for $\kappa_s = 16\pi$ and $m_{\max} = [4\kappa_s]$.

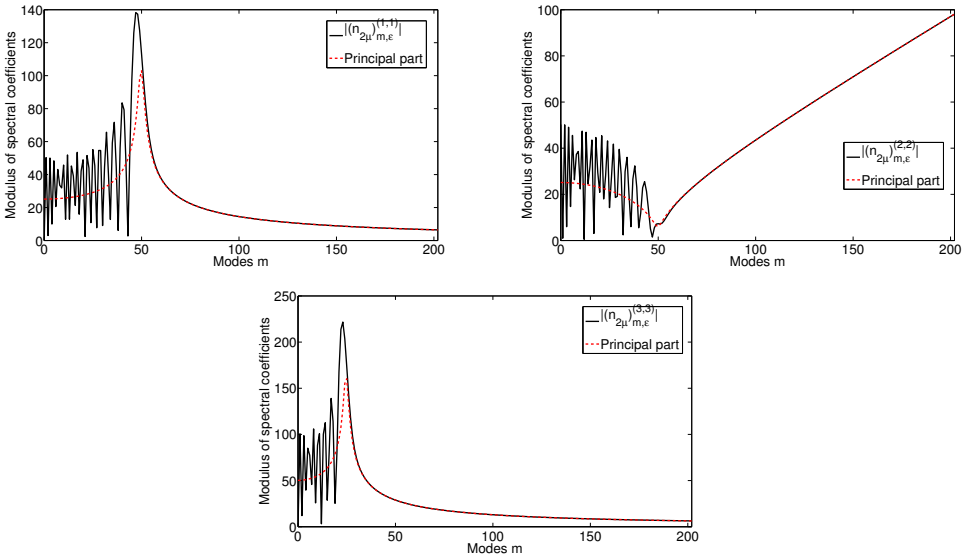


FIGURE 5. Asymptotic behavior: comparison between the integral operator $N_{2\mu}$ and the principal part $P_{1,\varepsilon}(N_{2\mu})$, for $\kappa_s = 16\pi$ and $m_{\max} = [4\kappa_s]$.

the elliptic part ($m_{\max} \geq \kappa_s$), while avoiding the divergence of the spherical Bessel and Hankel functions. For our simulations, we will always set $\kappa_s + 7.5\log_{10}(\kappa_s + \pi) \leq m_{\max} \leq [4\kappa_s]$. The lower bound is proposed in the context of multipole methods [14]. This truncation parameter $m_{\max}$ represents the mesh refinement.

First, we numerically exhibit an optimal parameter η_{opt} which minimizes the condition number of the usual BW integral operator (2.15). A good choice is around the value $\eta_{\text{opt}} = (\kappa_s + \kappa_p)/2$. We illustrate this observation in Fig. 6. We will consider this value in the sequel, and (2.15) with this choice is the reference integral equation.

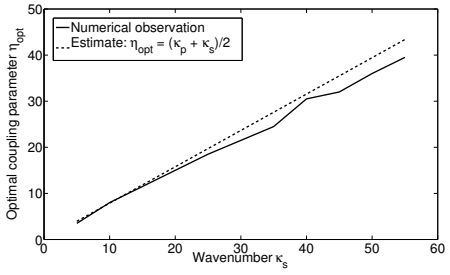


FIGURE 6. Optimal coupling parameter η_{opt} for the classical BW integral operator versus κ_s, taking $\kappa_s = 2\kappa_p$ and $m_{\max} = [4\kappa_s]$.

Now, we study the eigenvalue behavior of the standard (2.15) and preconditioned BW integral equations (3.15) and (3.23). We also consider the integral equation (3.2) with a DtN approximation proposed by Gächter and Grote in [23], called GGDtN approach. This approximation is constructed in the framework of non-reflecting boundary conditions for the spherical case. In Fig. 7, we compare the analytic eigenvalues of the different BW integral operators for $\kappa_s = 32\pi$ and $m_{\max} = [3\kappa_s]$. We zoom in on the eigenvalues of the DtN- and MDtN-preconditioned BW integral operators in Fig. 8. As we can see, according to the theoretical results, these two operators are the same.

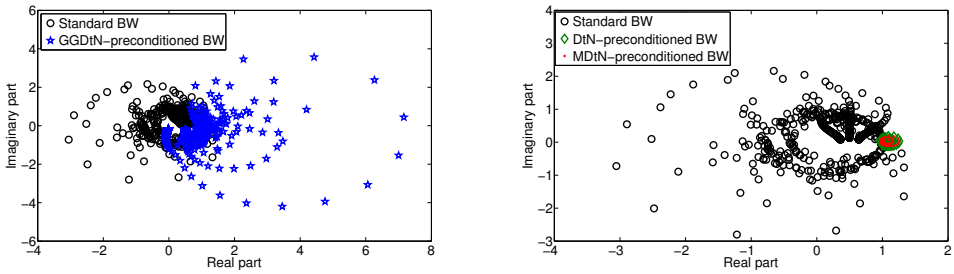


FIGURE 7. Distribution of the eigenvalues of the standard and preconditioned BW integral operators for $\kappa_s = 32\pi$ and $m_{\max} = [3\kappa_s]$.

The GGDtN approach does not allow a better eigenvalue distribution than the reference BW one. The approximation proposed in [23] considers efficiently the hyperbolic part of the exact DtN operator but this information is not sufficient to regularize the usual BW integral equation. All the modes have to be taken account in the approximation. This is done by considering (3.13) and (3.22). The eigenvalues of the associated DtN- and MDtN-preconditioned BW operators (3.15) and (3.23) show a very interesting clustering around a point near $(1, 0)$. We can conclude to the efficient regularizing effect of such an analytic preconditioning technique. However, the DtN-preconditioner is more complicated to implement than the MDtN one. From now on, we only consider the MDtN approach.

The MDtN-preconditioner $\mathbf{\Lambda}_{2\mu,\varepsilon}$ (cf. (3.22)) is defined with the help of non-local pseudodifferential square root operators, of the form $\sqrt{1+z}$, and their inverses. We use complex Padé

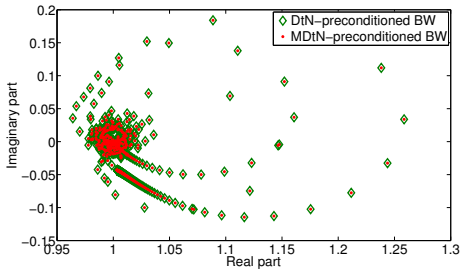


FIGURE 8. Distribution of the eigenvalues of the DtN- and MDtN-preconditioned BW integral operators for $\kappa_s = 32\pi$ and $m_{\max} = [3\kappa_s]$.

approximants with a rotating branch-cut technique [8, 34, 35, 7] to a suitable localization of these operators. We will discuss this in Section 6. We compare the analytical eigenvalues of the MDtN-preconditioned BW integral operator obtained with the exact square root operators, and the ones obtained considering Padé approximants. Efficiency of Padé approximants depends on different parameters to choose: their order $(2L+1)$ and the θ-rotation of the branch-cut. We fix $\theta = \pi/3$ ($\theta = \pi/2$ resp.) for the localization of the square root operator (of its inverse resp.). We study how the order impacts on the clustering of the eigenvalues. To this aim, we consider different Padé orders in Fig. 9.

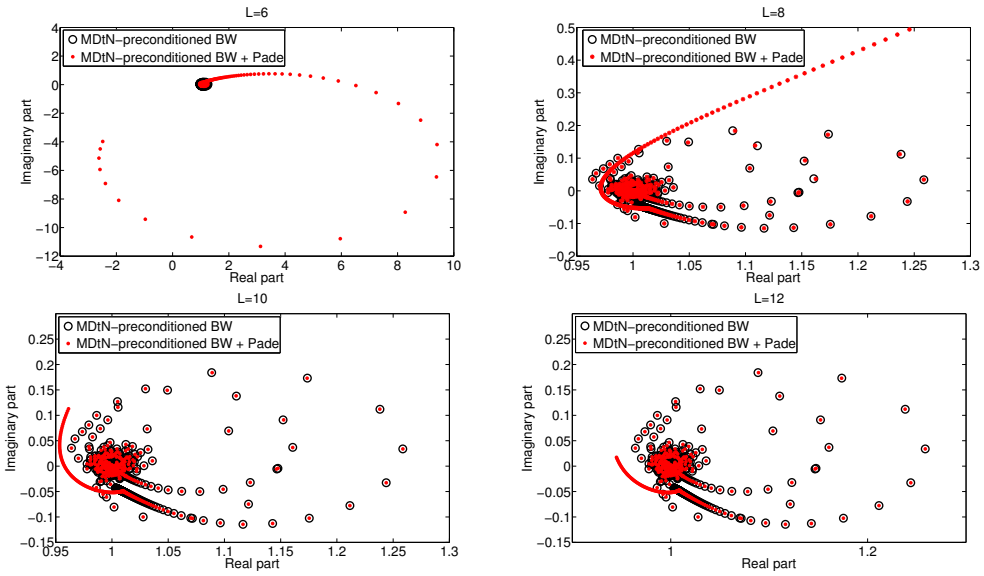
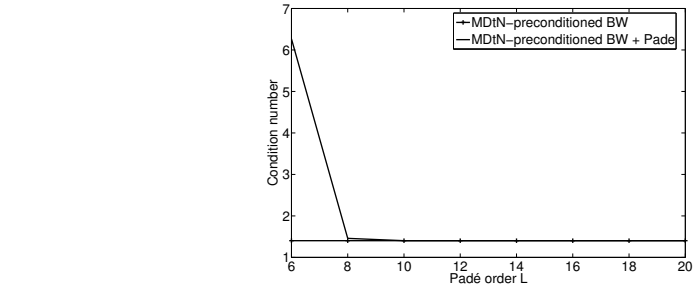
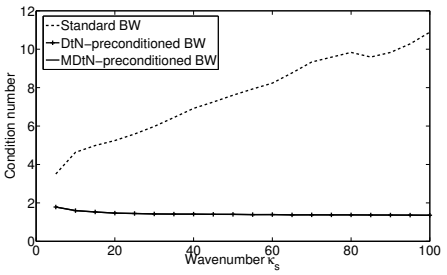


FIGURE 9. Distribution of the eigenvalues of the MDtN-preconditioned BW integral operator versus Padé order, $\kappa_s = 32\pi$, $m_{\max} = [3\kappa_s]$.

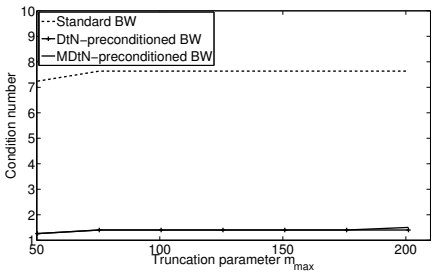
The clustering of the eigenvalues around the point $(1, 0)$ is better with higher Padé orders. We can see a spiral curve which is absorbed by the accumulation point when the Padé order increases. Complex Padé rational approximants for $L \geq 10$ provide an adequate tool to localize the MDtN-preconditioner $\mathbf{\Lambda}_{2\mu,\varepsilon}$. This is confirmed when we observe the condition number (cf. Fig. 10). The condition number of the MDtN-preconditioned integral equation - with Padé approximants or not- remains between 1.4 and 1.8 independently of the wavenumber κ_s (and hence κ_p too). This is not the case for the usual BW one. The dependance on the wavenumbers comes from the largest magnitude eigenvalues linked to a grazing mode. Note that the condition number of the standard BW IE is independent on the truncation parameter $m_{\max}$, as predicted by the asymptotic behavior of the eigenvalues in the elliptic part. The MDtN-preconditioned EI preserves this good property.



a) Vs. Padé order, $\kappa_s = 16\pi$, $m_{\max} = [3\kappa_s]$



b) Vs. κ_s, $L = 12$, $m_{\max} = [3\kappa_s]$

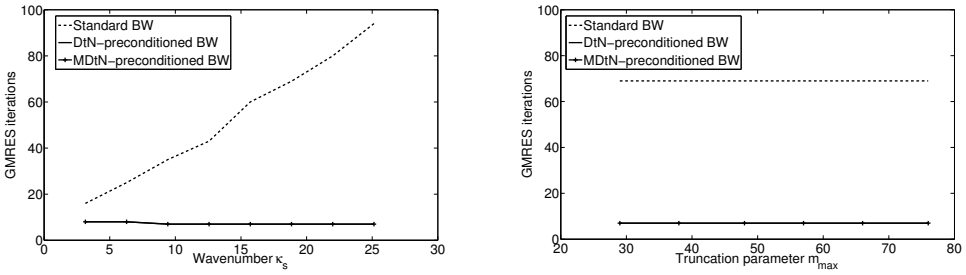


c) Vs. $m_{\max}$, $L = 12$, $\kappa_s = 16\pi$

FIGURE 10. Unit sphere: Condition number

4.3. **Convergence of the iterative solver GMRES.** We study the scattering by the unit sphere of an incident plane elastic wave of polarization $\mathbf{p}$ and propagating in the direction $\mathbf{d}$. Here, we consider shear plane waves. In this case, the polarization of the plane waves is orthogonal to the direction of propagation. We fix $\mathbf{p} = (1, 0, 0)^T$ and $\mathbf{d} = (0, 0, 1)^T$. For all the tests, we set $\rho = 1$, $\nu = 1/3$, $\mu = 1$ and $\lambda = 2$.

We use the iterative method GMRES with no restart for solving the linear systems of the standard and MDtN-preconditioned BW integral equations with or without the use of Padé approximants. The solution of the linear systems is expanded on the basis of normal and tangential vector spherical harmonics (see [32] for the coefficients of the incident wave). The size of the considered matrices is $N = (3(2m_{\max} + 1)^2 - 2)$. The GMRES iterations are stopped when the initial residual has decreased by a factor of 10^{-5}. Concerning the complex Padé approximants, we fix $L = 12$ and $\theta = \pi/2$. We examine the influence of an increase of the frequency and of the truncation parameter $m_{\max}$ on the GMRES convergence. In Fig. 11-left, the value of the wavenumber κ_s varies from π to 8π. The number of GMRES iterations increases with κ_s for the classical BW integral equation. The MDtN preconditioning leads to an independence of the GMRES convergence on κ_s, even for high values of κ_s. Only 7 iterations are required independently of κ_s. The size of the linear systems varies from 2698 to 17785 in the tests reported in Fig. 11-right. The number of GMRES iterations to solve the standard and MDtN-preconditioned BW integral equation is independent of the truncation parameter $m_{\max}$, and is efficiently decreasing using the MDtN preconditioning. The iterative solution needs only 7 iterations against 69 for the classical one. These results well fit with the previous spectral observations and validate the robustness of our approach.



a) Vs. κ_s, $L = 12$, $m_{\max} = [3\kappa_s]$ b) Vs. $m_{\max}$, $L = 12$, $\kappa_s = 6\pi$

FIGURE 11. Unit sphere: Convergence of GMRES solver

5. Preconditioned combined field integral equations

In this section, we derive in the same way preconditioned CFIEs. Recall that the standard CFIE is expressed by

$$(\frac{\mathrm{I}}{2} + D'_\alpha + i\eta S)\boldsymbol{\varphi} = -\big(\gamma^+_{1,\alpha}\boldsymbol{u}^{inc} + i\eta\gamma^+_0\boldsymbol{u}^{inc}\big), \quad \text{on } \Gamma,$$

with $\boldsymbol{\varphi} = -\gamma^+_1\big(\boldsymbol{u}+\boldsymbol{u}^{inc}\big)$ and η a non-zero real constant. The idea is to precondition the operator S with an approximation of the DtN or of the MDtN operator before combining it with the integral operator $\dfrac{\mathrm{I}}{2} + D'_\alpha$. We propose two preconditioned CFIEs:

- *A DtN-preconditioned CFIE ($\alpha = 0$)* Find $\boldsymbol{\varphi} = -\gamma^+_1\big(\boldsymbol{u}+\boldsymbol{u}^{inc}\big)$ solution to

$$(\frac{\mathrm{I}}{2} + D' - \boldsymbol{\Lambda}'_\varepsilon S)\boldsymbol{\varphi} = -\big(\gamma^+_1\boldsymbol{u}^{inc} - \boldsymbol{\Lambda}'_\varepsilon\gamma^+_0\boldsymbol{u}^{inc}\big), \quad \text{on } \Gamma.$$

- *A MDtN-preconditioned CFIE ($\alpha = 2\mu$):* Find $\boldsymbol{\varphi} = -\gamma^+_1\big(\boldsymbol{u}+\boldsymbol{u}^{inc}\big)$ solution to

$$(\frac{\mathrm{I}}{2} + D'_{2\mu} - \boldsymbol{\Lambda}'_{2\mu,\varepsilon} S)\boldsymbol{\varphi} = -\big(\gamma^+_{1,2\mu}\boldsymbol{u}^{inc} - \boldsymbol{\Lambda}'_{2\mu,\varepsilon}\gamma^+_0\boldsymbol{u}^{inc}\big), \quad \text{on } \Gamma.$$

We can remark that these integral operators are nothing else than the adjoint operators of the DtN- and MDtN-preconditioned BW integral operators (3.15) and (3.23) respectively. Consequently, the two proposed CFIEs are uniquely solvable for any frequency $\omega > 0$ and damping parameters $\varepsilon_s > 0$ and $\varepsilon_p > 0$. Furthermore, they show the same spectral properties than the proposed indirect preconditioned integral equations, namely an excellent eigenvalue clustering around the point $(1, 0)$ even for high frequencies, and a condition number independent of both frequency and mesh parameter. These novel CFIEs are therefore well-suited to an iterative Krylov solution, and here the unknown is the physical field.

6. Sketch of the numerical implementation

In this last section, we suggest ideas to implement the MDtN-preconditioned BW integral equation (3.23). We give elements of discretization to validate the ease of implementation of the proposed OSRC preconditioning approach. Let us denote by $[A]$ the matrix associated with the discretization by finite elements of a given integral operator A. To solve the MDtN-preconditioned BW integral equation with a GMRES iterative method, one needs to compute at each iteration

$$\mathbf{v} = \Big(\frac{1}{2}[\mathrm{I}] + [D_{2\mu}] - [S][\boldsymbol{\Lambda}_{2\mu,\varepsilon}]\Big)\mathbf{u},$$

with $\mathbf{v}, \mathbf{u} \in \mathbb{C}^N$, N the size of the approximation space. The matrix $[\mathrm{I}]$ represents the mass matrix on the surface mesh.

- First, we apply the MDtN-preconditioner $\mathbf{w} = [\boldsymbol{\Lambda}_{2\mu,\varepsilon}]\mathbf{u}$.

- Secondly, we compute the dense matrix-vector products $\mathbf{v} = \Big(\frac{1}{2}[\mathrm{I}] + [D_{2\mu}]\Big)\mathbf{u} - [S]\mathbf{w}$ coming from the discretization of the integral operators $D_{2\mu}$ and S. The integrals involve singular kernels. These singularities have to be carefully treated. Moreover, the use of FMM [13, 14] is prescribed to efficiently evaluate these products.

We focus on the application of the MDtN-preconditioner: computation of $\mathbf{w} = [\mathbf{\Lambda}_{2\mu,\varepsilon}]\mathbf{u}$. Recall that $\mathbf{\Lambda}_{2\mu,\varepsilon} = P_{1,\varepsilon}(N_{2\mu})\Big(\frac{I}{2} + P_{0,\varepsilon}(D_{2\mu})\Big)^{-1}$. First, we have to solve

$$\Big(\frac{[\mathrm{I}]}{2} + [P_{0,\varepsilon}(D_{2\mu})]\Big)\mathbf{q} = \mathbf{u}, \tag{6.1}$$

and secondly to realize

$$\mathbf{w} = [P_{1,\varepsilon}(N_{2\mu})]\mathbf{q}. \tag{6.2}$$

We explain the different steps to implement (6.1)-(6.2). Suppose that we know $\mathbf{u} \in RT$ (RT is well-known lower-order Raviart-Thomas space [38]).

- Step 1: We determine the Helmholtz decomposition of $\mathbf{u}$ such that
$$\mathbf{u} \approx \mathbf{curl}_\Gamma\, \varphi_\mathbf{u} + \boldsymbol{\nabla}_\Gamma \psi_\mathbf{u} + \mathbf{n}\phi_\mathbf{u}.$$
- Step 2: We propose to solve (6.1) with an iterative solver. At each iteration, we have to compute $\mathbf{y} = \Big(\frac{[\mathrm{I}]}{2} + [P_{0,\varepsilon}(D_{2\mu})]\Big)\mathbf{x}$. After determining the three scalar potentials $(\varphi_\mathbf{x}, \psi_\mathbf{x}, \phi_\mathbf{x})$ of $\mathbf{x}$, we obtain easily the potentials of $\mathbf{y}$ given by
$$\left\{\begin{array}{rcl} \varphi_\mathbf{y} &=& \frac{1}{2}\varphi_\mathbf{x}, \\ \psi_\mathbf{y} &=& \frac{1}{2}\Big(\psi_\mathbf{x} - \frac{1}{\kappa_{p,\varepsilon}}\Big([\mathrm{I}] + \frac{[\Delta_\Gamma]}{\kappa_{p,\varepsilon}^2}\Big)^{-1/2}\phi_\mathbf{x}\Big), \\ \phi_\mathbf{y} &=& \frac{1}{2}\Big(\phi_\mathbf{x} + \frac{i}{\kappa_{s,\varepsilon}}\Big([\mathrm{I}] + \frac{[\Delta_\Gamma]}{\kappa_{s,\varepsilon}^2}\Big)^{-1/2}[\Delta_\Gamma]\psi_\mathbf{x}\Big), \end{array}\right.$$
and we have
$$\mathbf{y} \approx \mathbf{curl}_\Gamma\, \varphi_\mathbf{y} + \boldsymbol{\nabla}_\Gamma \psi_\mathbf{y} + \mathbf{n}\phi_\mathbf{y}.$$
The matrix $-[\Delta_\Gamma]$ is the rigidity matrix on the surface mesh.
- Step 3: We have obtained $\mathbf{q}$ in the previous step. We determine three scalar potentials $(\varphi_\mathbf{q}, \psi_\mathbf{q}, \phi_\mathbf{q})$ of $\mathbf{q}$ such that
$$\mathbf{q} \approx \mathbf{curl}_\Gamma\, \varphi_\mathbf{q} + \boldsymbol{\nabla}_\Gamma \psi_\mathbf{q} + \mathbf{n}\phi_\mathbf{q}.$$
- Step 4: We solve (6.2) through
$$\left\{\begin{array}{rcl} \varphi_\mathbf{w} &=& \frac{i}{2\mu\kappa_{s,\varepsilon}}\Big([\mathrm{I}] + \frac{[\Delta_\Gamma]}{\kappa_{s,\varepsilon}^2}\Big)^{-1/2}\varphi_\mathbf{q}, \\ \psi_\mathbf{w} &=& \frac{i\rho\omega^2}{2\kappa_{s,\varepsilon}}\Big([\mathrm{I}] + \frac{[\Delta_\Gamma]}{\kappa_{s,\varepsilon}^2}\Big)^{-1/2}\psi_\mathbf{q}, \\ \phi_\mathbf{w} &=& \frac{i\rho\omega^2}{2\kappa_{p,\varepsilon}}\Big([\mathrm{I}] + \frac{[\Delta_\Gamma]}{\kappa_{p,\varepsilon}^2}\Big)^{-1/2}\phi_\mathbf{q}. \end{array}\right.$$
Finally, we derive
$$\mathbf{w} \approx \mathbf{curl}_\Gamma\, \varphi_\mathbf{w} + \boldsymbol{\nabla}_\Gamma \psi_\mathbf{w} + \mathbf{n}\phi_\mathbf{w}.$$

To efficiently localize the different inverse square root operators $(1+z)^{-1/2}$, we use complex Padé approximants of order $(2L+1)$ applying a rotating of the usual branch-cut $\{z \in \mathbb{R}, z < -1\}$ with an angle θ. We have

$$\Big([\mathrm{I}] + \frac{[\Delta_\Gamma]}{\kappa^2}\Big)^{-1/2} \;\approx\; r_0[\mathrm{I}] + \sum_{j=1}^{L} r_j\Big(\frac{[\Delta_\Gamma]}{\kappa^2} - q_j[\mathrm{I}]\Big)^{-1},$$

where r_j, $j = 0, \ldots, L$, and q_j, $j = 1, \ldots, L$, are complex coefficients depending on the angle θ [8, 34, 35, 7]. To conclude, the implementation of the MDtN-preconditioner requires only the solution

of sparse linear systems involving mass and rigidity matrices, and the numerical determination of the Helmholtz decomposition of a vector defined in RT.

7. Conclusion and perspectives

In this paper, we have built well-conditioned integral equations for the iterative solution of the Dirichlet elastic scattering problems. The preconditioning strategy is based on a rewriting of the Somigliana integral representation of the scattered field. A regularizing operator is incorporated and forces the integral equations arising from this representation to have very interesting spectral properties. Two regularizing operators are proposed: an approximation of the exact DtN operator and an approximation of a Modified DtN operator. The second preconditioner is more suited to a numerical implementation. It is expressed using the principal parts of the elementary boundary integral operators, namely surface differential operators and square root operators.

A numerical investigation of the eigenvalues of the standard and preconditioned integral operators for the spherical case illustrates the efficiency of the approach. The eigenvalues of the preconditioned formulations are well clustered and their condition number is close to 1 independently of both a frequency increase and a mesh refinement. This leads to a fast convergence of the GMRES iterative solver. These first results are very encouraging. Furthermore, notice that the proposed analytic preconditioner shows highly-desirable advantages: sparse structure, ease of implementation and a low additional computational cost using Padé approximants to localize the square root operators.

The numerical implementation and validation of this preconditioning approach for more general obstacles will be challenging future works. A combination with the FMM seems promising. Previous work in this area in acoustic scattering [20] has shown significant improvements for the computational time and cost of the solution. It would be very interesting to test first the numerical accuracy of the DtN and the MDtN approximations in the OSRC context, before applying them as preconditioners.

Appendix A

In this appendix we give the spectral decomposition of the different elastic boundary integral operators in terms of the vector spherical harmonics. Let us introduce the spherical harmonics $Y_{m,\ell}$ as the functions of order m for $\ell = -m, \ldots, m$, with $m, \ell \in \mathbb{N}$, given by [17]

$$Y_{m,\ell}(\hat{\boldsymbol{x}}) = (-1)^{\frac{|\ell|+\ell}{2}} \sqrt{\frac{2m+1}{4\pi}\frac{(m-|\ell|)!}{(m+|\ell|)!}} P_m^{|\ell|}(\cos\theta) e^{i\ell\phi}, \quad \hat{\boldsymbol{x}}(\theta,\phi) \in \mathbb{S}^2,$$

where P_m^ℓ denote the ℓ-th associated Legendre polynomials of order m. The tangential vector spherical harmonics defined by

$$\boldsymbol{\mathcal{Y}}_{m,\ell}^{(1)} = \frac{1}{\sqrt{m(m+1)}} \boldsymbol{\nabla}_{\mathbb{S}^2} Y_{m,\ell}, \quad \boldsymbol{\mathcal{Y}}_{m,\ell}^{(2)} = \frac{1}{\sqrt{m(m+1)}} \mathbf{curl}_{\mathbb{S}^2} Y_{m,\ell} = \boldsymbol{\mathcal{Y}}_{m,\ell}^{(1)} \times \hat{\boldsymbol{x}},$$

for $\ell = -m, \ldots, m$ and $m = 1, 2, \ldots$ together with the normal vector spherical harmonics

$$\boldsymbol{\mathcal{Y}}_{m,\ell}^{(3)} = \hat{\boldsymbol{x}}\, Y_{m,\ell},$$

for $\ell = -m, \ldots, m$ and $m = 0, 1, 2, \ldots$ form a complete orthonormal system in $\boldsymbol{L}^2(\mathbb{S}^2)$ undowed with the hermitian product

$$(\boldsymbol{\varphi}_1 | \boldsymbol{\varphi}_2)_{\mathbb{S}^2} = \int_{\mathbb{S}^2} \boldsymbol{\varphi}_1(\hat{\boldsymbol{x}}) \cdot \overline{\boldsymbol{\varphi}_2(\hat{\boldsymbol{x}})} ds(\hat{\boldsymbol{x}}).$$

Let us denote by j_m the spherical Bessel function of order $m \in \mathbb{N}$ and by $h_m^{(1)}$ the spherical Hankel function of the first kind and order $m \in \mathbb{N}$. We set $z_m^{(1)} = j_m$ and $z_m^{(3)} = h_m^{(1)}$ and

$$u_{m,\ell}^{(k)}(\kappa, \boldsymbol{x}) = z_m^{(k)}(\kappa|\boldsymbol{x}|) Y_{m,\ell}(\hat{\boldsymbol{x}}),\ M_{m,\ell}^{(k)}(\kappa, \boldsymbol{x}) = \mathbf{curl}\left(\boldsymbol{x}\, u_{m,\ell}^{(k)}(\kappa, \boldsymbol{x})\right),\ N_{m,\ell}^{(k)}(\kappa, \boldsymbol{x}) = \frac{1}{i\kappa} \mathbf{curl}\, M_{m,\ell}^{(k)}(\kappa, \boldsymbol{x}).$$

Then, the fundamental solution of the Navier equation admits the following series expansion for $|\boldsymbol{x}| > |\boldsymbol{y}|$:

$$\begin{aligned}\Phi(\boldsymbol{x},\boldsymbol{y}) &= \frac{i\kappa_s}{\mu}\sum_{m=1}^{\infty}\frac{1}{m(m+1)}\sum_{\ell=-m}^{m} M_{m,\ell}^{(3)}(\kappa_s,\boldsymbol{x})\otimes\overline{M_{m,\ell}^{(1)}(\kappa_s,\boldsymbol{y})}\\ &+\frac{i\kappa_s}{\mu}\sum_{m=1}^{\infty}\frac{1}{m(m+1)}\sum_{\ell=-m}^{m} N_{m,\ell}^{(3)}(\kappa_s,\boldsymbol{x})\otimes\overline{N_{m,\ell}^{(1)}(\kappa_s,\boldsymbol{y})}\\ &+\frac{i}{\kappa_p(\lambda+2\mu)}\sum_{m=0}^{\infty}\sum_{\ell=-m}^{m}\boldsymbol{\nabla}u_{m,\ell}^{(3)}(\kappa_p,\boldsymbol{x})\otimes\boldsymbol{\nabla}\overline{u_{m,\ell}^{(1)}(\kappa_p,\boldsymbol{y})}\end{aligned}$$

where $\otimes$ denotes the tensor product. We denote by $\xi_m^{(1)}(t) = th_m^{(1)}(t)$ et $\psi_m(t) = tj_m(t)$ the Ricatti-Hankel and Ricatti-Bessel functions of order m respectively. In the case of the scattering by an elastic sphere of radius R, we have $S\boldsymbol{\mathcal{Y}}_{0,0}^{(3)} = S_0^{(3,3)}\boldsymbol{\mathcal{Y}}_{0,0}^{(3)}$ and for $m \geq 1$ and $\ell = -m,\ldots,m$,

$$S\begin{pmatrix}\boldsymbol{\mathcal{Y}}_{m,\ell}^{(1)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(2)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(3)}\end{pmatrix} = \begin{pmatrix}S_m^{(1,1)} & 0 & S_m^{(1,3)}\\ 0 & S_m^{(2,2)} & 0\\ S_m^{(3,1)} & 0 & S_m^{(3,3)}\end{pmatrix}\begin{pmatrix}\boldsymbol{\mathcal{Y}}_{m,\ell}^{(1)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(2)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(3)}\end{pmatrix}$$

with, for $m \geq 1$

$$\begin{aligned}S_m^{(1,1)} &= \frac{i}{\mu\kappa_s}\psi_m'(\kappa_sR)\xi_m^{(1)'}(\kappa_sR) + \frac{i}{\kappa_p(\lambda+2\mu)}m(m+1)j_m(\kappa_pR)h_m^{(1)}(\kappa_pR),\\ S_m^{(3,1)} &= \frac{i}{\mu\kappa_s}\frac{\sqrt{m(m+1)}}{2}\Big[j_m(\kappa_sR)\xi_m^{(1)'}(\kappa_sR) + h_m^{(1)}(\kappa_sR)\psi_m'(\kappa_sR)\Big]\\ &\quad+\frac{iR}{(\lambda+2\mu)}\frac{\sqrt{m(m+1)}}{2}\Big[h_m^{(1)'}(\kappa_pR)j_m(\kappa_pR) + j_m'(\kappa_pR)h_m^{(1)}(\kappa_pR)\Big],\\ S_m^{(2,2)} &= \frac{i}{\mu\kappa_s}\psi_m(\kappa_sR)\xi_m^{(1)}(\kappa_sR),\\ S_m^{(1,3)} &= S_m^{(3,1)},\end{aligned}$$

and for $m \geq 0$

$$S_m^{(3,3)} = \frac{i}{\mu\kappa_s}m(m+1)j_m(\kappa_sR)h_m^{(1)}(\kappa_sR) + \frac{i\kappa_pR^2}{(\lambda+2\mu)}j_m'(\kappa_pR)h_m^{(1)'}(\kappa_pR).$$

We also have $D' = D^T$, $D\boldsymbol{\mathcal{Y}}_{0,0}^{(3)} = D_0^{(3,3)}\boldsymbol{\mathcal{Y}}_{0,0}^{(3)}$ and for $m \geq 1$ and $\ell = -m,\ldots,m$,

$$D\begin{pmatrix}\boldsymbol{\mathcal{Y}}_{m,\ell}^{(1)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(2)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(3)}\end{pmatrix} = \begin{pmatrix}D_m^{(1,1)} & 0 & D_m^{(1,3)}\\ 0 & D_m^{(2,2)} & 0\\ D_m^{(3,1)} & 0 & D_m^{(3,3)}\end{pmatrix}\begin{pmatrix}\boldsymbol{\mathcal{Y}}_{m,\ell}^{(1)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(2)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(3)}\end{pmatrix}$$

with, for $m \geq 1$

$$
\begin{aligned}
D_m^{(1,1)} &= -\frac{i}{\kappa_s R}\Big[\Big(\big(\kappa_s^2R^2 - m(m+1)+1\big)h_m^{(1)}(\kappa_s R) + \kappa_s R h_m^{(1)'}(\kappa_s R)\Big)\psi_m'(\kappa_s R) \\
&\quad + \Big(\big(\kappa_s^2R^2 - m(m+1)+1\big)j_m(\kappa_s R) + \kappa_s R j_m'(\kappa_s R)\Big)\xi_m^{(1)'}(\kappa_s R)\Big] \\
&\quad + \frac{i\kappa_s R}{2}\Big(h_m^{(1)}(\kappa_s R)\psi_m'(\kappa_s R) + j_m(\kappa_s R)\xi_m^{(1)'}(\kappa_s R)\Big) \\
&\quad + \frac{i\mu\, m(m+1)}{\kappa_p R(\lambda+2\mu)}\Big[h_m^{(1)}(\kappa_p R)\psi_m'(\kappa_p R) + j_m(\kappa_p R)\xi_m^{(1)'}(\kappa_p R) - 4j_m(\kappa_p R)h_m^{(1)}(\kappa_p R)\Big], \\
D_m^{(1,3)} &= \frac{2i\sqrt{m(m+1)}}{\kappa_s R}\Big[\psi_m'(\kappa_s R)\xi_m^{(1)'}(\kappa_s R) - j_m(\kappa_s R)\xi_m^{(1)'}(\kappa_s R) - h_m^{(1)}(\kappa_s R)\psi_m'(\kappa_s R)\Big] \\
&\quad + \frac{\mu\sqrt{m(m+1)}}{i\kappa_p R(\lambda+2\mu)}\Big[j_m(\kappa_p R)\Big(\big(\kappa_p^2R^2 - m(m+1)\big)h_m^{(1)}(\kappa_p R) + 2\kappa_p R h_m^{(1)'}(\kappa_p R)\Big) \\
&\quad + h_m^{(1)}(\kappa_p R)\Big(\big(\kappa_p^2R^2 - m(m+1)\big)j_m(\kappa_p R) + 2\kappa_p R j_m'(\kappa_p R)\Big)\Big] \\
&\quad - \frac{i\lambda\kappa_p R\sqrt{m(m+1)}}{(\lambda+2\mu)}j_m(\kappa_p R)h_m^{(1)}(\kappa_p R), \\
D_m^{(2,2)} &= i(\kappa_s R)^2\Big(j_m'(\kappa_s R)h_m^{(1)}(\kappa_s R) + h_m^{(1)'}(\kappa_s R)j_m(\kappa_s R)\Big) \\
&\quad - \frac{i\kappa_s R}{2}\Big(j_m(\kappa_s R)\xi_m^{(1)'}(\kappa_s R) + h_m^{(1)}(\kappa_s R)\psi_m'(\kappa_s R)\Big), \\
D_m^{(3,1)} &= \frac{\sqrt{m(m+1)}}{i\kappa_s R}\Big[j_m(\kappa_s R)\Big(\big(\kappa_s^2R^2 - m(m+1)+1\big)h_m^{(1)}(\kappa_s R) + \kappa_s R h_m^{(1)'}(\kappa_s R)\Big) \\
&\quad + h_m^{(1)}(\kappa_s R)\Big(\big(\kappa_s^2R^2 - m(m+1)+1\big)j_m(\kappa_s R) + \kappa_s R j_m'(\kappa_s R)\Big)\Big] \\
&\quad + i\kappa_s R\sqrt{m(m+1)}j_m(\kappa_s R)h_m^{(1)}(\kappa_s R) \\
&\quad + \frac{i\mu\sqrt{m(m+1)}}{(\lambda+2\mu)}\Big[2\kappa_p R j_m'(\kappa_p R)h_m^{(1)'}(\kappa_p R) - j_m(\kappa_p R)h_m^{(1)'}(\kappa_p R) - j_m'(\kappa_p R)h_m^{(1)}(\kappa_p R)\Big],
\end{aligned}
$$

and for $m \geq 0$

$$
\begin{aligned}
D_m^{(3,3)} &= \frac{im(m+1)}{\kappa_s}\Big[h_m^{(1)}(\kappa_s R)\Big(\kappa_s j_m'(\kappa_s R) - \frac{j_m(\kappa_s R)}{R}\Big) + j_m(\kappa_s R)\Big(\kappa_s h_m^{(1)'}(\kappa_s R) - \frac{h_m^{(1)}(\kappa_s R)}{R}\Big)\Big] \\
&\quad - \frac{i\mu}{(\lambda+2\mu)}\Big[j_m'(\kappa_p R)\Big(\big(\kappa_p^2R^2 - m(m+1)\big)h_m^{(1)}(\kappa_p R) + 2\kappa_p R h_m^{(1)'}(\kappa_p R)\Big) \\
&\quad + h_m^{(1)'}(\kappa_p R)\Big(\big(\kappa_p^2R^2 - m(m+1)\big)j_m(\kappa_p R) + 2\kappa_p R j_m'(\kappa_p R)\Big)\Big] \\
&\quad - \frac{i(\kappa_p R)^2\lambda}{2(\lambda+2\mu)}\Big(j_m(\kappa_p R)h_m^{(1)'}(\kappa_p R) + j_m'(\kappa_p R)h_m^{(1)}(\kappa_p R)\Big).
\end{aligned}
$$

We have $N\boldsymbol{\mathcal{Y}}_{0,0}^{(3)} = N_0^{(3,3)}\boldsymbol{\mathcal{Y}}_{0,0}^{(3)}$ and for $m \geq 1$ and $\ell = -m, \ldots, m$

$$
N\begin{pmatrix}\boldsymbol{\mathcal{Y}}_{m,\ell}^{(1)} \\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(2)} \\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(3)}\end{pmatrix} = \begin{pmatrix} N_m^{(1,1)} & 0 & N_m^{(1,3)} \\ 0 & N_m^{(2,2)} & 0 \\ N_m^{(3,1)} & 0 & N_m^{(3,3)}\end{pmatrix}\begin{pmatrix}\boldsymbol{\mathcal{Y}}_{m,\ell}^{(1)} \\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(2)} \\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(3)}\end{pmatrix}
$$

where for $m \geq 1$

$$
\begin{aligned}
N_m^{(1,1)} &= \frac{i\mu}{\kappa_s R^2}\Big(\big(\kappa_s^2R^2-2m(m+1)+2\big)h_m^{(1)}(\kappa_sR)+2\kappa_sRh_m^{(1)'}(\kappa_sR)\Big)\\
&\qquad\times\Big(\big(\kappa_s^2R^2-2m(m+1)+2\big)j_m(\kappa_sR)+2\kappa_sRj_m'(\kappa_sR)\Big)\\
&\quad+\frac{4i\mu^2m(m+1)}{\kappa_p(\lambda+2\mu)}\Big(\kappa_ph_m^{(1)'}(\kappa_pR)-\frac{h_m^{(1)}(\kappa_pR)}{R}\Big)\Big(\kappa_pj_m'(\kappa_pR)-\frac{j_m(\kappa_pR)}{R}\Big),\\
N_m^{(3,1)} &= \frac{\mu\sqrt{m(m+1)}}{i\kappa_sR}\Big[\Big(\big(\kappa_s^2R^2-2m(m+1)\big)h_m^{(1)}(\kappa_sR)+2\xi_m^{(1)}(\kappa_sR)\Big)\Big(\kappa_sj_m'(\kappa_sR)-\frac{j_m(\kappa_sR)}{R}\Big)\\
&\quad+\Big(\big(\kappa_s^2R^2-2m(m+1)\big)j_m(\kappa_sR)+2\psi_m(\kappa_sR)\Big)\Big(\kappa_sh_m^{(1)'}(\kappa_sR)-\frac{h_m^{(1)}(\kappa_sR)}{R}\Big)\Big]\\
&\quad-\frac{i\mu R\sqrt{m(m+1)}}{\kappa_p(\lambda+2\mu)}\Big\{\Big[\frac{2\mu}{R^2}\Big(\big(\kappa_p^2R^2-m(m+1)\big)h_m^{(1)}(\kappa_pR)+2\kappa_pRh_m^{(1)'}(\kappa_pR)\Big)+\lambda\kappa_p^2h_m^{(1)}(\kappa_pR)\Big]\\
&\qquad\times\Big(\kappa_pj_m'(\kappa_pR)-\frac{j_m(\kappa_pR)}{R}\Big)\\
&\qquad+\Big[\frac{2\mu}{R^2}\Big(\big(\kappa_p^2R^2-m(m+1)\big)j_m(\kappa_pR)+2\kappa_pRj_m'(\kappa_pR)\Big)+\lambda\kappa_p^2j_m(\kappa_pR)\Big]\\
&\qquad\times\Big(\kappa_ph_m^{(1)'}(\kappa_pR)-\frac{h_m^{(1)}(\kappa_pR)}{R}\Big)\Big\},\\
N_m^{(2,2)} &= i\mu\kappa_sR^2\Big(\kappa_sj_m'(\kappa_sR)-\frac{j_m(\kappa_sR)}{R}\Big)\Big(\kappa_sh_m^{(1)'}(\kappa_sR)-\frac{h_m^{(1)}(\kappa_sR)}{R}\Big),\\
N_m^{(1,3)} &= N_m^{(3,1)},
\end{aligned}
$$

and for $m \geq 0$

$$
\begin{aligned}
N_m^{(3,3)} &= 4i\mu\frac{m(m+1)}{\kappa_s}\Big(\kappa_sh_m^{(1)'}(\kappa_sR)-\frac{h_m^{(1)}(\kappa_sR)}{R}\Big)\Big(\kappa_sj_m'(\kappa_sR)-\frac{j_m(\kappa_sR)}{R}\Big)\\
&\quad+\frac{iR^2}{\kappa_p(\lambda+2\mu)}\Big[\frac{2\mu}{R^2}\Big(\big(\kappa_p^2R^2-m(m+1)\big)h_m^{(1)}(\kappa_pR)+2\kappa_pRh_m^{(1)'}(\kappa_pR)\Big)+\lambda\kappa_p^2h_m^{(1)}(\kappa_pR)\Big]\\
&\qquad\times\Big[\frac{2\mu}{R^2}\Big(\big(\kappa_p^2R^2-m(m+1)\big)j_m(\kappa_pR)+2\kappa_pRj_m'(\kappa_pR)\Big)+\lambda\kappa_p^2j_m(\kappa_pR)\Big].
\end{aligned}
$$

Now, let us consider the principal parts $P_{-1}(S)$, $P_0(D)$ and $P_1(N)$ of the integral operators S, D and N respectively (see Section 2.1). By recalling that the eigenvalues of the surfacic operator Δ_Γ are given by $-\dfrac{m(m+1)}{R^2}$, we have the following results

$$
P_{-1}(S)\begin{pmatrix}\boldsymbol{\mathcal{Y}}_{m,\ell}^{(1)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(2)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(3)}\end{pmatrix}=\begin{pmatrix}s_m^{(1,1)} & 0 & 0\\ 0 & s_m^{(2,2)} & 0\\ 0 & 0 & s_m^{(3,3)}\end{pmatrix}\begin{pmatrix}\boldsymbol{\mathcal{Y}}_{m,\ell}^{(1)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(2)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(3)}\end{pmatrix},
$$

with

$$
\begin{aligned}
s_m^{(1,1)} &= \frac{i}{2\rho\omega^2}\left[\Big(\kappa_s^2-\frac{m(m+1)}{R^2}\Big)^{\frac{1}{2}}+\frac{m(m+1)}{R^2}\Big(\kappa_p^2-\frac{m(m+1)}{R^2}\Big)^{-\frac{1}{2}}\right],\\
s_m^{(2,2)} &= \frac{i}{2\rho\omega^2}\kappa_s^2\Big(\kappa_s^2-\frac{m(m+1)}{R^2}\Big)^{-\frac{1}{2}},\\
s_m^{(3,3)} &= \frac{i}{2\rho\omega^2}\left[\Big(\kappa_p^2-\frac{m(m+1)}{R^2}\Big)^{\frac{1}{2}}+\frac{m(m+1)}{R^2}\Big(\kappa_s^2-\frac{m(m+1)}{R^2}\Big)^{-\frac{1}{2}}\right],
\end{aligned}
$$

$$
P_0(D)\begin{pmatrix}\boldsymbol{\mathcal{Y}}_{m,\ell}^{(1)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(2)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(3)}\end{pmatrix}=\begin{pmatrix}0 & 0 & d_m^{(1,3)}\\ 0 & 0 & 0\\ d_m^{(3,1)} & 0 & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{\mathcal{Y}}_{m,\ell}^{(1)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(2)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(3)}\end{pmatrix},
$$

with

$$\begin{aligned} d_m^{(1,3)} &= \frac{i}{\kappa_s^2}\Big(\frac{m(m+1)}{R^2}\Big)^{1/2}\Bigg[\Big(\kappa_s^2-\frac{m(m+1)}{R^2}\Big)^{\frac12}+\frac{m(m+1)}{R^2}\Big(\kappa_p^2-\frac{m(m+1)}{R^2}\Big)^{-\frac12}\Bigg] \\ &\quad -\frac{i}{2}\Big(\frac{m(m+1)}{R^2}\Big)^{1/2}\Big(\kappa_p^2-\frac{m(m+1)}{R^2}\Big)^{-\frac12}, \\ d_m^{(3,1)} &= \frac{i}{\kappa_s^2}\Big(\frac{m(m+1)}{R^2}\Big)^{1/2}\Bigg[\Big(\kappa_p^2-\frac{m(m+1)}{R^2}\Big)^{\frac12}+\frac{m(m+1)}{R^2}\Big(\kappa_s^2-\frac{m(m+1)}{R^2}\Big)^{-\frac12}\Bigg] \\ &\quad -\frac{i}{2}\Big(\frac{m(m+1)}{R^2}\Big)^{1/2}\Big(\kappa_s^2-\frac{m(m+1)}{R^2}\Big)^{-\frac12}, \end{aligned}$$

and, finally,

$$P_1(N)\begin{pmatrix}\boldsymbol{\mathcal{Y}}_{m,\ell}^{(1)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(2)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(3)}\end{pmatrix}=\begin{pmatrix}n_m^{(1,1)} & 0 & 0\\ 0 & n_m^{(2,2)} & 0\\ 0 & 0 & n_m^{(3,3)}\end{pmatrix}\begin{pmatrix}\boldsymbol{\mathcal{Y}}_{m,\ell}^{(1)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(2)}\\ \boldsymbol{\mathcal{Y}}_{m,\ell}^{(3)}\end{pmatrix},$$

with

$$\begin{aligned} n_m^{(1,1)} &= \frac{i}{2\rho\omega^2}\Big(\kappa_s^2-\frac{m(m+1)}{R^2}\Big)^{\frac12}+\frac{2i\mu}{\kappa_s^2}\frac{m(m+1)}{R^2}\Bigg[\Big(\kappa_p^2-\frac{m(m+1)}{R^2}\Big)^{\frac12}+\frac{m(m+1)}{R^2}\Big(\kappa_s^2-\frac{m(m+1)}{R^2}\Big)^{-\frac12}\Bigg] \\ &\quad -2i\mu\Big(\frac{m(m+1)}{R^2}\Big)^{1/2}\Big(\kappa_s^2-\frac{m(m+1)}{R^2}\Big)^{-\frac12}, \\ n_m^{(2,2)} &= \frac{i}{2}\mu\Big(\kappa_s^2-\frac{m(m+1)}{R^2}\Big)^{\frac12}, \\ n_m^{(3,3)} &= \frac{i}{2\rho\omega^2}\Big(\kappa_p^2-\frac{m(m+1)}{R^2}\Big)^{\frac12}+\frac{2i\mu}{\kappa_s^2}\frac{m(m+1)}{R^2}\Bigg[\Big(\kappa_s^2-\frac{m(m+1)}{R^2}\Big)^{\frac12}+\frac{m(m+1)}{R^2}\Big(\kappa_p^2-\frac{m(m+1)}{R^2}\Big)^{-\frac12}\Bigg] \\ &\quad -2i\mu\Big(\frac{m(m+1)}{R^2}\Big)^{1/2}\Big(\kappa_p^2-\frac{m(m+1)}{R^2}\Big)^{-\frac12}. \end{aligned}$$

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