



**HAL**  
open science

## Lyapunov stability of a singularly perturbed system of two conservation laws

Ying Tang, Christophe Prieur, Antoine Girard

► **To cite this version:**

Ying Tang, Christophe Prieur, Antoine Girard. Lyapunov stability of a singularly perturbed system of two conservation laws. CPDE 2013 - 1st IFAC Workshop on Control of Systems Modeled by Partial Differential Equations, Sep 2013, Paris, France. pp.227-232, 10.3182/20130925-3-FR-4043.00050 . hal-00839621

**HAL Id: hal-00839621**

**<https://hal.science/hal-00839621>**

Submitted on 11 Feb 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Lyapunov stability of a singularly perturbed system of two conservation laws

Ying TANG\* Christophe PRIEUR\*<sup>1</sup> Antoine GIRARD\*\*

\* *UJF-Grenoble 1/CNRS, Grenoble Image Parole Signal Automatique (GIPSA-lab), UMR 5216, B.P. 46, F-38402, St Martin d'Hères, France, ying.tang@gipsa-lab.fr, christophe.prieur@gipsa-lab.fr.*

\*\* *Laboratoire Jean Kuntzmann, Université de Grenoble, B.P. 53, 38041 Grenoble, France, antoine.girard@imag.fr.*

---

**Abstract:** This paper is concerned with a class of singularly perturbed systems of two conservation laws. A small perturbation parameter is introduced in the dynamics and the boundary conditions. By setting the perturbation parameter to zero, the singularly perturbed system of conservation laws can be treated as two subsystems of one conservation law: the reduced system and the boundary-layer system. The asymptotic stability of the complete system is investigated via Lyapunov techniques. A Lyapunov function for the singularly perturbed system is obtained as a weighted sum of two Lyapunov functions of the subsystems.

Keywords: Partial differential equation, Lyapunov function, Singularly perturbed system, Reduced system, Boundary-layer.

---

## 1. INTRODUCTION

The singular perturbation techniques occurred at the beginning of the 20th century (see Kokotovic et al. [1986]). The development of this method led to the efficient use in various fields in mathematical physics and engineering, for instance, fluid mechanics, fluid dynamics, elasticity, quantum mechanics, chemical-reactor, aerodynamics etc. (see Kadalbajoo and Patidar [2003] for a survey). Due to Habets [1974], Chow [1978], Grujic [1981], and Chow and Kokotovic [1981], the decomposition of a singularly perturbed system into lower order subsystems, the reduced system and the boundary-layer system, provides a powerful tool for stability analysis.

The research works in the singularly perturbed partial differential equations (PDEs) have started from late 1980s. This kind of systems is interesting for analysis as far as it describes numerous important phenomenon in many domains, for example, a two-dimensional motion of an incompressible viscous fluid problem in Nefedov [1988]. The model of gas transport in a constant section tube from the work of Castillo et al. [2012] provides the principal motivation for this paper. This model contains two time scales for propagation speed, precisely the propagation speed of gas is much slower than the sound speed, which can be considered as a singular perturbation problem.

Lyapunov methods are quite usual for stability analysis of dynamical systems. This is also true for those dynamics with a small perturbation parameter. Lyapunov functions are employed to show the asymptotic stability of the equilibrium of a singularly perturbed system, for a small enough perturbation parameter, by investigating

the equilibria of the subsystems (as in Klimushchev and Krasovskii [1961]). Wilde and Kokotovic [1972] give similar results for linear systems. In the work of Saberi and Khalil [1984], a quadratic-type Lyapunov function has been investigated for singularly perturbed ODEs systems. They have established a quadratic-type Lyapunov function for each subsystem, and these Lyapunov functions are used to build a Lyapunov function for the overall system. It is worth to mention that a strict Lyapunov function (i.e. a Lyapunov function whose derivative, along the trajectories of the system, is negative definite) is usually considered to demonstrate asymptotic stability (as in Coron et al. [2008]).

In this paper, we consider systems modelled by singularly perturbed partial differential equations (PDEs). More precisely a class systems of two conservation laws with a small perturbation parameter  $\varepsilon$  is investigated. The main idea is to consider the two subsystems: reduced system and boundary-layer system by setting  $\varepsilon = 0$ . Each of the two systems has a Lyapunov function and is asymptotically stable under suitable boundary conditions. For a sufficiently small perturbation parameter, the stability of the singularly perturbed system of conservation laws can be obtained by a Lyapunov function which is given by a convex combination of the Lyapunov function of the reduced system and the boundary-layer system.

The paper is organized as follows. Section 2 introduces the singularly perturbed system of conservation laws. Section 3 presents the stability of the reduced and boundary-layer systems. Each system has a Lyapunov function. Section 4 shows the stability for the overall singularly perturbed system. In Section 5, an illustrative example is provided to show the main result. Finally, concluding remarks end the paper.

---

<sup>1</sup> Work supported by HYCON2 Network of Excellence "Highly-Complex and Networked Control Systems", grant agreement 257462.

**Notation.** For a matrix  $M$ ,  $M^T$  denotes the transpose. For a symmetric matrix  $P$ ,  $P \geq 0$  means that  $P$  is positive semidefinite. A continuous function  $\alpha: [0, \infty) \rightarrow [0, \infty)$  belongs to class  $\mathcal{K}$  provided it is increasing and  $\alpha(0) = 0$ . It belongs to class  $\mathcal{K}_\infty$  if in addition  $\alpha(k) \rightarrow \infty$  as  $k \rightarrow \infty$ .

## 2. SINGULARLY PERTURBED SYSTEM OF CONSERVATION LAWS

Consider the following singularly perturbed system of conservation laws for a small positive perturbation parameter  $\varepsilon$ :

$$\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} y(x, t) \\ z(x, t) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} y(x, t) \\ z(x, t) \end{pmatrix} = 0, \quad (1)$$

where  $x \in [0, 1]$ ,  $t \in [0, +\infty)$ ,  $y: [0, 1] \times [0, +\infty) \rightarrow \mathbb{R}$ ,  $z: [0, 1] \times [0, +\infty) \rightarrow \mathbb{R}$ .

Let us consider the following boundary conditions:

$$\begin{pmatrix} y(0, t) \\ z(0, t) \end{pmatrix} = G \begin{pmatrix} y(1, t) \\ z(1, t) \end{pmatrix}, \quad (2)$$

where  $G = \begin{pmatrix} G_{11} & G_{12} \\ \varepsilon G_{21} & G_{22} \end{pmatrix}$  is a  $2 \times 2$  matrix.

Given two continuous functions  $y^0: [0, 1] \rightarrow \mathbb{R}$  and  $z^0: [0, 1] \rightarrow \mathbb{R}$ , the initial conditions are:

$$\begin{pmatrix} y(x, 0) \\ z(x, 0) \end{pmatrix} = \begin{pmatrix} y^0 \\ z^0 \end{pmatrix}. \quad (3)$$

Let us define the reduced system and the boundary-layer system of (1) and (2). By setting  $\varepsilon = 0$  in system (1), we get:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} y(x, t) \\ z(x, t) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} y(x, t) \\ z(x, t) \end{pmatrix} = 0, \quad (4)$$

and setting  $\varepsilon = 0$  in boundary conditions (2) yields:

$$\begin{pmatrix} y(0, t) \\ z(0, t) \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{pmatrix} \begin{pmatrix} y(1, t) \\ z(1, t) \end{pmatrix}. \quad (5)$$

Assuming that  $G_{22} \neq 1$  in (5) and using  $\frac{\partial}{\partial x} z(x, t) = 0$  (due to (4)), we have:

$$z(., t) = 0. \quad (6)$$

We substitute (6) into (4) and (5), the reduced system is defined as:

$$\frac{\partial}{\partial t} \bar{y}(x, t) + \frac{\partial}{\partial x} \bar{y}(x, t) = 0, \quad (7)$$

with boundary condition:

$$\bar{y}(0, t) = G_{11} \bar{y}(1, t). \quad (8)$$

To define the boundary-layer system, we consider  $y(x, t)$  as constant with respect to time, which yields:

$$\frac{\partial}{\partial \tau} \bar{z}(x, \tau) + \frac{\partial}{\partial x} \bar{z}(x, \tau) = 0, \quad (9)$$

with boundary condition:

$$\bar{z}(0, \tau) = G_{22} \bar{z}(1, \tau), \quad (10)$$

where  $\tau = \frac{t}{\varepsilon}$  is a stretching time scale.

Now let us introduce the notion of strict Lyapunov function that is considered in this paper (see Luo et al. [1999] for the definition of a Lyapunov function).

**Definition 1.** Let  $V: L^2((0, 1), \mathbb{R}^2) \rightarrow \mathbb{R}$  be a continuously differentiable function. The function  $V$  is said to be a strict Lyapunov function for singularly perturbed system (1) and (2), if there are two functions  $\kappa_S$  and  $\kappa_M$  of class  $\mathcal{K}_\infty$  and one value  $\lambda > 0$  such that, for all functions  $\phi \in L^2((0, 1), \mathbb{R}^2)$ ,

$$\kappa_S(|\phi|_{L^2(0,1)}) \leq V(\phi) \leq \int_0^1 \kappa_M(|\phi(x)|) dx \quad (11)$$

and for all solutions of (1) and (2), for all  $t \geq 0$ ,

$$\frac{d}{dt} V \begin{pmatrix} y(., t) \\ z(., t) \end{pmatrix} \leq -\lambda V \begin{pmatrix} y(., t) \\ z(., t) \end{pmatrix}. \quad (12)$$

**Remark.** When a strict Lyapunov function exists, then the value of a strict Lyapunov function for (1) and (2) along the solutions of (1) and (2) exponentially decays to zero and therefore each solution  $\begin{pmatrix} y(x, t) \\ z(x, t) \end{pmatrix}$  satisfies

$\lim_{t \rightarrow +\infty} \begin{pmatrix} y(., t) \\ z(., t) \end{pmatrix}_{L^2(0,1)} = 0$ , that is (1) and (2) is asymptotically stable (in  $L^2(0, 1)$  norm).

## 3. ANALYSIS OF REDUCED SYSTEM AND BOUNDARY-LAYER SYSTEM

In this section, the asymptotic stability of reduced and boundary-layer systems is established.

**Proposition 1.** The reduced system (7) and (8) is asymptotically stable if and only if the boundary condition satisfies

$$|G_{11}| < 1. \quad (13)$$

Under this condition, a strict Lyapunov function is defined as:

$$V(\bar{y}) = \int_0^1 e^{-\mu x} \bar{y}^2 dx, \quad (14)$$

for all  $\bar{y} \in L^2((0, 1), \mathbb{R})$ , where  $\mu > 0$  satisfies

$$e^{-\mu} \geq G_{11}^2. \quad (15)$$

**Proof.** Let us consider the following solution for the reduced system (7) and (8):

$$\bar{y}(x, t) = u(t - x).$$

The solution at two boundaries is:

$$\begin{aligned} \bar{y}(0, t) &= u(t), \\ \bar{y}(1, t) &= u(t - 1). \end{aligned} \quad (16)$$

Applying the boundary condition (8) to (16) yields:

$$\bar{y}(x, t) = G_{11}\bar{y}(x, t - 1).$$

If the reduced system (7) and (8) is asymptotically stable, then  $|G_{11}| < 1$ .

Let us now check that the boundary condition (13) is sufficient for the asymptotic stability of the reduced system (7) with (8). There exists a positive value  $\beta_1$ , such that, for every  $\bar{y} \in L^2((0, 1), \mathbb{R})$ ,

$$\frac{1}{\beta_1} \int_0^1 \bar{y}^2 dx \leq V(\bar{y}) \leq \beta_1 \int_0^1 \bar{y}^2 dx. \quad (17)$$

The time derivative of  $V$  along the solutions to (7) and (8) is

$$\begin{aligned} \dot{V}(\bar{y}) &= - \int_0^1 2e^{-\mu x} \bar{y} \frac{\partial}{\partial x} \bar{y} dx \\ &= -[e^{-\mu x} \bar{y}^2] \Big|_{x=0}^{x=1} - \mu \int_0^1 e^{-\mu x} \bar{y}^2 dx \\ &= -[e^{-\mu x} \bar{y}^2] \Big|_{x=0}^{x=1} - \mu V(\bar{y}). \end{aligned} \quad (18)$$

Under the boundary condition (8), (18) becomes

$$\begin{aligned} \dot{V}(\bar{y}) &= -[e^{-\mu} \bar{y}^2(1) - \bar{y}^2(0)] - \mu V(\bar{y}) \\ &= -[e^{-\mu} - G_{11}^2] \bar{y}^2(1) - \mu V(\bar{y}). \end{aligned} \quad (19)$$

For a given  $\mu$  satisfying (15), and using the boundary condition (13), it is obtained:

$$\dot{V}(\bar{y}) \leq -\mu V(\bar{y}). \quad (20)$$

Therefore the Lyapunov function (14) is a strict Lyapunov function for reduced system (7) and (8), and the system is asymptotically stable. This concludes the proof of Proposition 1.  $\bullet$

**Proposition 2.** *The boundary-layer system (9) and (10) is asymptotically stable if and only if the boundary condition satisfies*

$$|G_{22}| < 1. \quad (21)$$

*Under this condition, a strict Lyapunov function is defined as:*

$$W(\bar{z}) = \int_0^1 e^{-\nu x} \bar{z}^2 dx, \quad (22)$$

for all  $\bar{z} \in L^2((0, 1), \mathbb{R})$ , where  $\nu > 0$  satisfies

$$e^{-\nu} \geq G_{22}^2. \quad (23)$$

The proof of this proposition is similar to the one of Proposition 1.

#### 4. ASYMPTOTIC STABILITY OF SINGULARLY PERTURBED SYSTEM OF CONSERVATION LAWS

The aim of this section is to state the asymptotic stability of the singularly perturbed system of conservation laws (1) and (2), for small  $\varepsilon > 0$ , from that of the reduced system (7) and (8) and the boundary-layer system (9) and (10).

Our result shows that as soon as the reduced system (7) and (8) and the boundary-layer system (9) and (10) are

asymptotically stable, for sufficiently small  $\varepsilon$ , any weighted sum of the Lyapunov functions of the reduced and the boundary-layer systems is a Lyapunov function for the system (1) and (2) (in Saberi and Khalil [1984], a composite Lyapunov function has been investigated for a singularly perturbed finite dimensional nonlinear system).

Let us start by stating our assumptions.

**Assumption 1.** *The reduced system (7) with boundary condition (8) is asymptotically stable.*

**Assumption 2.** *The boundary-layer system (9) with boundary condition (10) is asymptotically stable.*

Due to Proposition 1 (resp. Proposition 2), Assumption 1 (resp. Assumption 2) is equivalent to  $|G_{11}| < 1$  (resp.  $|G_{22}| < 1$ ).

We can state the following theorem:

**Theorem 1.** *Under Assumptions 1 and 2, let  $d$  be a positive value such that  $0 < d < 1$ ,  $\mu > 0$  (resp.  $\nu > 0$ ) such that  $e^{-\mu} > G_{11}^2$  (resp.  $e^{-\nu} > G_{22}^2$ ), and the positive value  $\varepsilon^*(d) \in (0, +\infty]$  be given by:*

*Case 1: If  $G_{12} \neq 0$  or  $G_{21} \neq 0$*

$$\varepsilon^*(d) = \frac{d(1-d)K_1K_2}{(1-d)^2G_{12}^2K_1 + d^2G_{21}^2K_2 + ((1-d)K_3 + dK_4)^2} \quad (24)$$

*Case 2: If  $G_{12} = 0$  and  $G_{21} = 0$*

$$\varepsilon^*(d) = +\infty \quad (25)$$

where  $K_1 = e^{-\mu} - G_{11}^2$ ,  $K_2 = e^{-\nu} - G_{22}^2$ ,  $K_3 = G_{11}G_{12}$ ,  $K_4 = G_{21}G_{22}$ .

*Then, for all  $0 < \varepsilon \leq \varepsilon^*(d)$  and  $\varepsilon < +\infty$ , the singularly perturbed system of conservation laws (1) and (2) is asymptotically stable and it has a strict Lyapunov function:*

$$L(y, z) = (1-d)V(y) + dW(z) \quad (26)$$

where  $V$  and  $W$  are given by (14) and (22).

**Proof.** Let us compute the time derivative of  $L(y, z)$  along the solutions to (1) and (2):

$$\begin{aligned} \dot{L}(y, z) &= (1-d)\dot{V} + d\dot{W} \\ &= -(1-d) \int_0^1 2e^{-\mu x} y \frac{\partial}{\partial x} y dx \\ &\quad - \frac{d}{\varepsilon} \int_0^1 2e^{-\nu x} z \frac{\partial}{\partial x} z dx \\ &= -(1-d)[e^{-\mu x} y^2] \Big|_{x=0}^{x=1} - (1-d)\mu \int_0^1 e^{-\mu x} y^2 dx \\ &\quad - \frac{d}{\varepsilon}[e^{-\nu x} z^2] \Big|_{x=0}^{x=1} - \frac{d}{\varepsilon}\nu \int_0^1 e^{-\nu x} z^2 dx \\ &= -(1-d)[e^{-\mu x} y^2] \Big|_{x=0}^{x=1} - \frac{d}{\varepsilon}[e^{-\nu x} z^2] \Big|_{x=0}^{x=1} \\ &\quad - \left( \mu(1-d)V(y) + \frac{\nu}{\varepsilon}dW(z) \right). \end{aligned} \quad (27)$$

Employing the boundary condition (2) to the above equation (27) it follows:

$$\begin{aligned} \dot{L}(y, z) &= -(1-d)(e^{-\mu}y^2(1) - (G_{11}y(1) + G_{12}z(1))^2) \\ &\quad - \frac{d}{\varepsilon}(e^{-\nu}z^2(1) - (\varepsilon G_{21}y(1) + G_{22}z(1))^2) \\ &\quad - \left( \mu(1-d)V(y) + \frac{\nu}{\varepsilon}dW(z) \right) \\ &= - \begin{pmatrix} y(1) \\ z(1) \end{pmatrix}^T M \begin{pmatrix} y(1) \\ z(1) \end{pmatrix} \\ &\quad - \left( \mu(1-d)V(y) + \frac{\nu}{\varepsilon}dW(z) \right), \end{aligned} \quad (28)$$

where

$$M = \begin{pmatrix} (1-d)K_1 - d\varepsilon G_{21}^2 & -((1-d)K_3 + dK_4) \\ -((1-d)K_3 + dK_4) & \frac{d}{\varepsilon}K_2 - (1-d)G_{12}^2 \end{pmatrix}.$$

To prove that the singularly perturbed system of conservation laws (1) and (2) is asymptotically stable, it is sufficient to require that the matrix  $M \geq 0$ .

The first diagonal term of  $M$  is non negative if

$$(1-d)K_1 - d\varepsilon G_{21}^2 \geq 0. \quad (29)$$

Due to Assumption 1 and Proposition 1, there always exists  $\mu > 0$  such that  $e^{-\mu} > G_{11}^2$ , which means  $K_1 > 0$ . To ensure that (29) is satisfied, pick a positive value  $\varepsilon$  such that

$$0 < \varepsilon \leq \frac{(1-d)K_1}{dG_{21}^2}, \quad (30)$$

if  $G_{21} \neq 0$ , (and  $\varepsilon > 0$  if  $G_{21} = 0$ ).

The determinant of  $M$  is non negative if

$$\begin{aligned} \det(M) &= \frac{d(1-d)K_1K_2}{\varepsilon} + \varepsilon d(1-d)G_{12}^2G_{21}^2 \\ &\quad - [(1-d)^2K_1G_{12}^2 + d^2K_2G_{21}^2 \\ &\quad + ((1-d)K_3 + dK_4)^2] \geq 0. \end{aligned} \quad (31)$$

As  $\varepsilon d(1-d)G_{12}^2G_{21}^2 \geq 0$ , (31) implies that the determinant of  $M$  is non negative as soon as

$$\begin{aligned} \frac{d(1-d)K_1K_2}{\varepsilon} - [(1-d)^2K_1G_{12}^2 + d^2K_2G_{21}^2 \\ + ((1-d)K_3 + dK_4)^2] \geq 0. \end{aligned} \quad (32)$$

To ensure that (32) is satisfied, according to Assumption 1 and Proposition 1 (resp. Assumption 2 and Proposition 2), there always exists  $\mu > 0$  (resp.  $\nu > 0$ ) such that  $e^{-\mu} > G_{11}^2$  (resp.  $e^{-\nu} > G_{22}^2$ ), which means  $K_1 > 0$  (resp.  $K_2 > 0$ ), pick  $\varepsilon$  such that

$$0 < \varepsilon \leq \frac{d(1-d)K_1K_2}{(1-d)^2G_{12}^2K_1 + d^2G_{21}^2K_2 + ((1-d)K_3 + dK_4)^2}, \quad (33)$$

if  $G_{12} \neq 0$  or  $G_{21} \neq 0$ , (and  $\varepsilon > 0$  if  $G_{12} = 0$  and  $G_{21} = 0$ ).

$\varepsilon^*(d)$  is chosen as the minimum value of (30) and (33). The following calculations show that the value defined by the right-hand side of (33) is the minimum value.

$$\begin{aligned} \varepsilon &\leq \frac{d(1-d)K_1K_2}{(1-d)^2G_{12}^2K_1 + d^2G_{21}^2K_2 + ((1-d)K_3 + dK_4)^2} \\ &\leq \frac{d(1-d)K_1K_2}{d^2G_{21}^2K_2} = \frac{(1-d)K_1}{dG_{21}^2}, \end{aligned}$$

this means (33) is always smaller than (30).

Under Assumptions 1 and 2, for any choice of  $0 < d < 1$ , there exists a positive value  $\varepsilon^*(d)$ , for  $0 < \varepsilon \leq \varepsilon^*(d)$ , the conditions (29) and (31) are satisfied,  $M$  is positive semidefinite.

This concludes the proof of Theorem 1.  $\bullet$

**Remark.** Note that the second diagonal term of  $M$  is non negative if

$$\frac{d}{\varepsilon}K_2 - (1-d)G_{12}^2 \geq 0. \quad (34)$$

Due to Assumption 2 and Proposition 2, there always exists  $\nu > 0$  such that  $e^{-\nu} > G_{22}^2$ , which means  $K_2 > 0$ . To ensure that (34) is satisfied, pick a positive value  $\varepsilon$  such that

$$0 < \varepsilon \leq \frac{dK_2}{(1-d)G_{12}^2}, \quad (35)$$

if  $G_{12} \neq 0$ , (and  $\varepsilon > 0$  if  $G_{12} = 0$ ). This condition is implied by (31).

Theorem 1 shows that taking  $d$  as any value in the interval  $(0, 1)$ , the composite of Lyapunov function  $L(y, z)$  covers all the possible linear combinations of  $V(y)$  and  $W(z)$ . Theorem 1 gives us an upper bounded value  $\varepsilon^*(d)$  for any  $d$ . To find the maximal value  $\bar{\varepsilon}$  at  $\bar{d}$ , let us compute the derivative of  $\varepsilon^*(d)$  with respect to  $d$ , denoted by  $\varepsilon'(d)$ :

$$\begin{aligned} \varepsilon'(d) &= \frac{(1-2d)K_1K_2}{(1-d)^2G_{12}^2K_1 + d^2G_{21}^2K_2 + ((1-d)K_3 + dK_4)^2} \\ &\quad - \frac{-2d(1-d)^2K_1^2K_2G_{12}^2 + 2d^2(1-d)K_2^2K_1G_{21}^2}{[(1-d)^2G_{12}^2K_1 + d^2G_{21}^2K_2 + ((1-d)K_3 + dK_4)^2]^2} \\ &\quad - \frac{2(K_4 - K_3)K_1K_2(d(1-d)^2K_3 + d^2(1-d)K_4)}{[(1-d)^2G_{12}^2K_1 + d^2G_{21}^2K_2 + ((1-d)K_3 + dK_4)^2]^2}. \end{aligned}$$

For  $G_{12} \neq 0$  or  $G_{21} \neq 0$ , then  $\varepsilon'(d) = 0$  if

$$\begin{aligned} 0 &= (1-2d)(1-d)^2K_1^2K_2G_{12}^2 + (1-2d)d^2K_2^2K_1G_{21}^2 \\ &\quad + (1-2d)K_1K_2((1-d)K_3 + dK_4)^2 \\ &\quad + 2d(1-d)^2K_1^2K_2G_{12}^2 - 2d^2(1-d)K_2^2K_1G_{21}^2 \\ &\quad - 2(K_4 - K_3)K_1K_2(d(1-d)^2K_3 + d^2(1-d)K_4) \\ &= (1-d)^2K_1^2K_2G_{12}^2 - d^2K_2^2K_1G_{21}^2 \\ &\quad + K_1K_2((1-d)K_3 + dK_4)((1-d)K_3 - dK_4) \\ &= (1-d)^2K_1K_2(K_1G_{12}^2 + K_3) - d^2K_1K_2(K_2G_{21}^2 + K_4), \end{aligned}$$

this is the case for  $d = \bar{d}$  with:

$$\bar{d} = \frac{\sqrt{K_1G_{12}^2 + K_3}}{\sqrt{K_1G_{12}^2 + K_3} + \sqrt{K_2G_{21}^2 + K_4}}. \quad (36)$$

For such a value of  $d$ , it holds:

$$\begin{aligned}\bar{\varepsilon} &= \varepsilon^*(\bar{d}) \\ &= \frac{n}{m},\end{aligned}\quad (37)$$

with:

$$\begin{aligned}n &= K_1 K_2 \sqrt{K_1 G_{12}^2 + K_3^2} \sqrt{K_2 G_{21}^2 + K_4^2} \\ m &= ((K_2 G_{21}^2 + K_4^2) K_1 G_{12}^2 + (K_1 G_{12}^2 + K_3^2) K_2 G_{21}^2 \\ &\quad + (K_3 \sqrt{K_2 G_{21}^2 + K_4^2} + K_4 \sqrt{K_1 G_{12}^2 + K_3^2})^2).\end{aligned}$$

**Corollary 1.** *Under Assumptions 1 and 2, and  $\varepsilon \leq \bar{\varepsilon}$ , the singularly perturbed system (1) and (2) is asymptotically stable.*

## 5. ILLUSTRATING EXAMPLE

In this section, we consider the following boundary condition for the singularly perturbed system (1):

$$G = \begin{pmatrix} -0.9 & 2 \\ 2\varepsilon & 0.9 \end{pmatrix}, \quad (38)$$

where Assumptions 1 and 2 hold. To compute the admissible perturbation parameter  $\varepsilon$ , let take  $\mu = \nu = 0.01$ , and following the statement of Theorem 1 it is computed:  $K_1 = K_2 = 0.18$ ,  $K_3 = -1.8$ ,  $K_4 = 1.8$ . According to (36) and (37), the maximal value of perturbation parameter  $\bar{\varepsilon}$  is computed:

$$\begin{aligned}\bar{d} &= 0.5, \\ \bar{\varepsilon} &= 0.0225.\end{aligned}\quad (39)$$

Take the admissible  $\varepsilon = 0.02$ , the boundary condition (38) becomes:

$$G_2 = \begin{pmatrix} -0.9 & 2 \\ 0.04 & 0.9 \end{pmatrix}. \quad (40)$$

Applying Corollary 1, the system (1) and (40) is asymptotically stable. Considering a diagonal positive definite matrix  $\Delta = \begin{pmatrix} 1 & 0 \\ 0 & 7.8 \end{pmatrix}$ , the inequality  $\rho_1(G_2) < 1$  holds.

Therefore with (Coron et al. [2008]), we recover the asymptotic stability of system (1) and (40). In other words, the stability condition of (Coron et al. [2008]) applies for system (1) and (40). However  $\rho(|G_2|) > 1$ , the stability condition of (Li [1994] Lemma 2.4, page 146) does not apply.

Let us check the asymptotic stability in numerical simulation of (1) and (40). Discretize the equation (1) using a two-step variant of the Lax-Wendroff method which is presented in Shampine [2005a] and the solver on Matlab in Shampine [2005b]. More precisely, we divide the space domain  $[0, 1]$  into 100 intervals of identical length, and 30 as final time. We choose a time-step that satisfies the CFL condition for the stability and select the following initial functions:

$$\begin{aligned}y(x, 0) &= \sin(4\pi x) \\ z(x, 0) &= \sin(5\pi x)\end{aligned}$$

for all  $x \in [0, 1]$ .

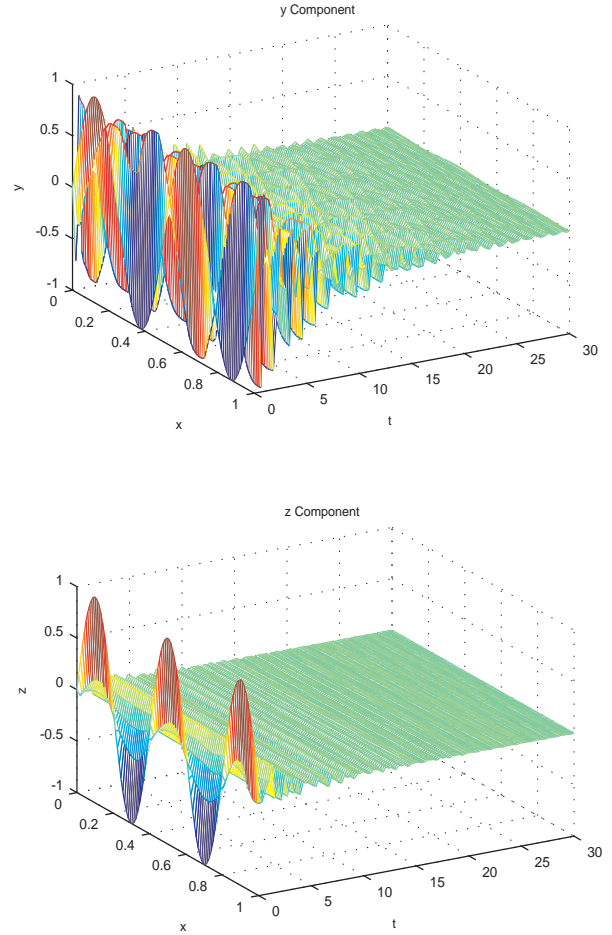


Fig. 1. Time evolutions of the first component  $y$  (top) and of the second component  $z$  (bottom) of the solution of the system (1) and (40)

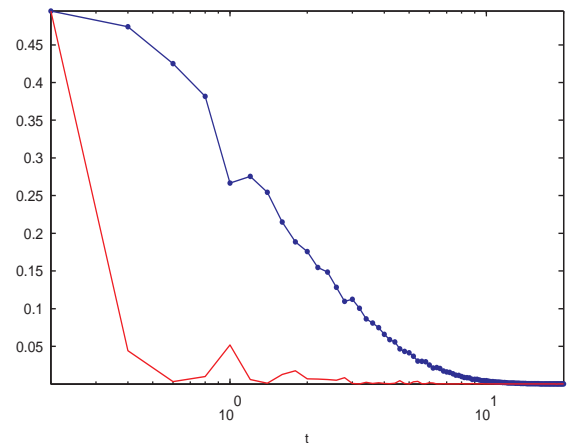


Fig. 2. Time evolutions of square of  $L^2$ -norm  $y$  (blue dotted line) and square of  $L^2$ -norm  $z$  (red plain line) in  $\log$  scale

Figure 1 shows that the components  $y$  and  $z$  of system (1) with the boundary condition (40) converge to 0 as  $t$  increases. It is observed from Figure 2 that the  $z$

component converges faster to the origin than the  $y$  component.

Table 1. Evolutions of time integral of square of  $L^2$ -norms  $y$  and  $z$  for different  $\varepsilon$

$\varepsilon$	0.005	0.010	0.015	0.02
$L^2$ -norm $y$	0.7930	0.8380	1.0219	1.3153
$L^2$ -norm $z$	0.0073	0.0193	0.0377	0.0622

Table 1 indicates that the time integral of square of  $L^2$ -norms  $y$  and  $z$  decrease as  $\varepsilon$  decreases, moreover the time integral of square of  $L^2$ -norm  $z$  is close to 0 when  $\varepsilon$  goes to 0, which is implied by (28).

## 6. CONCLUSION

In this paper, a necessary and sufficient condition has been derived for the stability of the reduced and boundary-layer systems. The asymptotic stability analysis of the whole singularly perturbed system of conservation laws has been established via the weighted sum of two Lyapunov functions of the reduced and boundary-layer systems, for a sufficient small perturbation parameter  $\varepsilon$ . An upper bound of  $\varepsilon$  is established.

This work leaves many questions open. The case where the perturbation parameter  $\varepsilon$  is introduced only in the dynamics, will be studied in future works. The problem of Lyapunov stability for multidimensional singularly perturbed system of conservation laws and balance laws will also be considered. In addition, it would be of interest to consider some physical applications as in Dick et al. [2010], Dos Santos and Prieur [2008], and Colombo et al. [2009].

## REFERENCES

- F. Castillo, E. Witrant, and L. Dugard. Contrôle de Température dans un Flux de Poiseuille. *IEEE Conférence Internationale Francophone d'Automatique, Grenoble, France*, 2012.
- J.H. Chow. Asymptotic stability of a class of nonlinear singularly perturbed systems. *J.Franklin Inst.*, 305(5): 275–281, 1978.
- J.H. Chow and P.V. Kokotovic. A two-stage Lyapunov-Bellman feedback design of a class of nonlinear systems. *IEEE Trans. Automat. Contr.*, 26(3):656–663, 1981.
- R.M. Colombo, G. Guerra, M. Herty, and V. Schleper. Optimal control in networks of pipes and canals. *SIAM Journal on Control and Optimization*, 48:2032–2050, 2009.
- J-M. Coron, G. Bastin, and B. d'Andréa-Novel. Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. *SIAM Journal on Control and Optimization*, 47(3):1460–1498, 2008.
- M. Dick, M. Gugat, and G. Leugering. Classical solutions and feedback stabilization for the gas flow in a sequence of pipes. *Networks and Heterogeneous Media*, 5(4):691–709, 2010.
- V. Dos Santos and C. Prieur. Boundary control of open channels with numerical and experimental validations. *IEEE Trans. Control Syst. Tech.*, 16(6):1252–1264, 2008.
- L.T. Grujic. Uniform asymptotic stability of nonlinear singularly perturbed and large scale systems. *Int.J.Contr.*, 33(3):481–504, 1981.
- P. Habets. Stabilité asymptotique pour des problèmes de perturbations singulières. *Centro Internazionale Matematico Estimo*, 1974.
- M.K. Kadalbajoo and K.C. Patidar. Singularly perturbed problems in partial differential equations: a survey. *Applied Mathematics and Computation*, 134:371–429, 2003.
- A.I. Klimushchev and N.N. Krasovskii. Uniform asymptotic stability of systems of differential equations with a small parameter in the derivative terms. *PMM (J. Applied Math. and Mechanics)*, 25:680–690, 1961.
- P. Kokotovic, H.K. Khalil, and O'R. John. Singular perturbation methods in control: analysis and design. *Academic Press*, 1986.
- T.-T. Li. Global classical solutions for quasilinear hyperbolic systems. *RAM:Research in Applied Mathematics. Masson, Paris*, 32, 1994.
- Z.H. Luo, B.Z. Guo, and O. Morgul. Stability and stabilization of infinite dimensional systems and applications. *Communications and Control Engineering. Springer-Verlag, New York*, 1999.
- N.N. Nefedov. On some singularly perturbed problems for viscous stratified fluid. *Journal of Mathematical Analysis and Applications*, 131:118–126, 1988.
- A. Saberi and H.K. Khalil. Quadratic-type Lyapunov Functions for Singularly Perturbed Systems. *IEEE Transactions on Automatic Control*, AC-29(6):542–550, 1984.
- L.F. Shampine. Two-step Lax-Friedrichs method. *Applied Mathematics Letters*, 18:1134–1136, 2005a.
- L.F. Shampine. Solving hyperbolic PDEs in Matlab. *Appl. Numer. Anal. Comput. Math.*, 2:346–358, 2005b.
- R.R. Wilde and P.V. Kokotovic. Stability of singularly perturbed systems and networks with parasitics. *IEEE Trans. on Auto. Control*, AC-17:245–246, 1972.