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ABSTRACT. We compute the exponential decay of the probability that a given multi-dimensional random walk stays in a convex cone up to time n, as n goes to infinity. We show that the latter equals the minimum, on the dual cone, of the Laplace transform of the random walk increments. As an example, our results find applications in the counting of walks in orthants, a classical domain in enumerative combinatorics.

1. Introduction and main results

1.1. General context. For general random processes in $\mathbb{R}^d$, $d \geqslant 1$ (including in particular Brownian motion and random walks), it is at once important and natural to study the first exit times τ_K from certain domains K. Precisely, for discrete-time random processes $(S_n)_{n\geqslant 0}$, τ_K is defined by

$$\tau_K := \inf\{n \geqslant 1 : S_n \notin K\}. \tag{1}$$

Indeed, these random times carry many valuable informations on the process. As an example, the fruitful theory of random walks fluctuations (see, e.g., Spitzer [22]) is based on the analysis of the τ_K for compact domains K.

In a recent past (1990 to present), the case of cones K has arisen a great interest in the mathematical community, due to interactions with many areas: First, certain random walks in conical domains can be treated with representation theory [2, 3] (in that case, the cones are Weyl chambers related to Lie algebras). Further, the exit times τ_K are crucial to construct conditioned random walks in cones, which appear in the theory of quantum random walks [2, 3], random matrices [13], non-colliding random walks [8, 14], etc. In another direction, the probability

$$\mathbb{P}^x[\tau_K > n] \tag{2}$$

admits a direct combinatorial interpretation in terms of the number of walks starting from x and staying in the cone K up to time n. These counting numbers are particularly important in enumerative combinatorics [6, 15, 19], and are the topic of many recent studies.

For processes with no drift, the exit times τ_K from cones are now well studied in the literature. The case of Brownian motion was solved by DeBlassie [7] (see also Bañuelos and Smits [1]): He showed that the probability (2) satisfies a certain partial differential

Date: May 27, 2014.
2000 *Mathematics Subject Classification.* 60G40; 60G50; 05A16.
Key words and phrases. Random walk; Cones; Exit time; Laplace transform.

equation (the heat equation), and he solved it in terms of hypergeometric functions. Concerning discrete-time random processes, in the one-dimensional case, the asymptotic behavior of the non-exit probability (2) is well known, as well as that of

$$\mathbb{P}^x[S_n = y, \tau_K > n] \tag{3}$$

(called a local limit theorem), thanks to the theory of fluctuations of random walks [22]. In higher dimension, some sporadic cases have first been analyzed: We may cite [11], for which there exists a strong underlying algebraic structure (certain reflexion groups are finite), or the case of Weyl chambers, which has been considered in [8, 14]. For more general cones, but essentially for random walks with increments having a finite support, Varopoulos [23] gave lower and upper bounds for the probability (2). The first author of the present article showed in [16] that for general random walks, the probability (2) does not decay exponentially fast. More recently, Denisov and Wachtel [9] provided the exact asymptotics for both (2) and (3).

For processes with drift, much less is known. Concerning Brownian motion, one of the first significant results is due to Biane, Bougerol and O'Connell [4], who derived the asymptotics of the non-exit probability (2) in the case of Weyl chambers of type A, when the drift is inside of the cone. Later on, by using different techniques, Puchała and Rolski [20] obtained (also in the context of Weyl chambers) the asymptotics of (2) without any hypothesis on the drift. In [17] we gave, for Brownian motion with a given arbitrary drift, the asymptotics of (2) for a large class of cones.

As for random walks $(S_n)_{n\geqslant 0}$ with increments having a common distribution μ, the exponential decay of (3) is known: It equals the global minimum on $\mathbb{R}^d$ of the Laplace transform of μ:

$$L_\mu(x) := \mathbb{E}_\mu[e^{\langle x, S_{n+1}-S_n\rangle}] = \int_{\mathbb{R}^d} e^{\langle x,y\rangle}\mu(\mathrm{d}y). \tag{4}$$

This was first proved by Iglehart [18] for one-dimensional random walks. For more general walks, this was shown and used by many authors (see, e.g., [9, 16]). Regarding now the asymptotic behavior of the probability (2), the case $d = 1$ is known, see [10].

> *It is the aim of this paper to give, for a very broad class of random walks and cones, in any dimension, the exponential decay of the non-exit probability (2). We shall also relate its value to the Laplace transform, by proving that it equals the minimum of this function* on the dual cone*; we give the exact statement (Theorem 1) in Subsection 1.4.*

Our main motivation comes from the possible applications to lattice path enumeration. Indeed, our results provide the first unified treatment of the question of determining the growth constant for the number of lattice paths confined to the positive orthant. They also solve a conjecture on these numbers stated in [19]. However, we would like to emphasize that our results are much more general (see Section 1.3).

Simultaneously and independently of us, Duraj [12] obtained in some particular case the exact asymptotics of the non-exit probability (2) for lattice random walks. In the following section we introduce our main ideas and tools, and we discuss the difference between our results and his.

1.2. **Preliminary discussion.** Let $(S_n)_{n\geqslant 0} = (S_n^{(1)},\dots,S_n^{(d)})_{n\geqslant 0}$ be the canonical random walk on $\mathbb{R}^d$. Given any probability measure μ on $\mathbb{R}^d$ and $x\in\mathbb{R}^d$, we denote by $\mathbb{P}_\mu^x$ the probability measure under which $(S_n)_{n\geqslant 0}$ is a random walk started at x whose independent increments $(S_{n+1}-S_n)_{n\geqslant 0}$ have common distribution μ.

The standard idea to handle the case of random walks with non-zero drift is to carry out an exponential change of measure. More precisely, if z is a point in $\mathbb{R}^d$ such that $L_\mu(z)$ is finite, then we can consider the new probability measure

$$\mu_z(\mathrm{d}y)=\frac{e^{\langle z,y\rangle}}{L_\mu(z)}\mu(\mathrm{d}y).$$

It is *theoretically* possible to compare the behavior of the random walk under $\mathbb{P}_\mu$ with its behavior under $\mathbb{P}_{\mu_z}$ thanks to Cramér's formula (see Lemma 3). For example, for the local probabilities, this formula gives

$$\mathbb{P}_\mu^x[S_n=y,\tau_K>n]=L_\mu(z)^n e^{\langle z,x-y\rangle}\mathbb{P}_{\mu_z}^x[S_n=y,\tau_K>n].$$

Since the asymptotic behavior of those probabilities are now well known when the random walk has no drift (see [9]), the general problem can be solved if one can find a point z such that the distribution μ_z is *centered*. It is also well known that this condition is fulfilled if and only if z is a critical point for L_μ (under the assumption that L_μ be finite in a neighborhood of z). By convexity of L_μ, this means that one has to find a local, hence *global minimum point* $z=x_0$ in $\mathbb{R}^d$.

This approach is used by Duraj in [12] to analyze the non-exit probability (2). Indeed, for a lattice random walk, one can sum the contribution of each y to eventually obtain (below μ_0 is an abbreviation for μ_{x_0})

$$\mathbb{P}_\mu^x[\tau_K>n]=L_\mu(x_0)^n\sum_{y\in K\cap\mathbb{Z}^d}e^{\langle x_0,x-y\rangle}\mathbb{P}_{\mu_0}^x[S_n=y,\tau_K>n].$$

But then, one needs to impose an additional condition on the position of the global minimum point x_0 with respect to K so as to ensure that the infinite sum of asymptotics will be convergent as well. This technical assumption on x_0 done in [12] happens to have a very natural interpretation in the light of our analysis. Indeed, for the non-exit probability, Cramér's formula (applied with any z) gives

$$\mathbb{P}_\mu^x[\tau_K>n]=L_\mu(z)^n e^{\langle z,x\rangle}\mathbb{E}_{\mu_0}^x[e^{-\langle z,S_n\rangle},\tau_K>n],\tag{5}$$

and one sees that the main difficulty will arise because of the exponential term inside the expectation.

Let K^* denote the *dual* cone associated with K, that is, the closed convex cone defined by

$$K^*:=\{z\in\mathbb{R}^d:\langle x,z\rangle\geqslant 0,\forall x\in K\},\tag{6}$$

where $\langle x,z\rangle$ denotes the standard inner product. If z belongs to K^*, it immediately follows from (5) that

$$\mathbb{P}_\mu^x[\tau_K>n]\leqslant L_\mu(z)^n e^{\langle z,x\rangle}.$$

Hence, the infimum ρ of the Laplace transform on K^* is always an upper bound of the exponential rate, i.e.,

$$\limsup_{n\to\infty} \mathbb{P}_\mu^x[\tau_K > n]^{1/n} \leqslant \rho := \inf_{K^*} L_\mu.$$

Our main result shows that, in fact, when the infimum ρ is a minimum, it is also a lower bound of the above quantity. Thus ρ is the value of the exponential decreasing rate of the non-exit probability. In the light of equation (5), this gives the heuristic rule:

The right location to perform Cramér's transformation is the point $x^* \in K^*$ where the Laplace transform reaches its minimum on the dual cone K^*.

It is now easily seen that the condition in [12] on the global minimum x_0 is that it belongs to the interior $(K^*)^o$ of the dual cone (and in this case, clearly, $x_0 = x^*$). In [12] the author then obtains the precise asymptotics of the non-exit probability (2) in this specific case. Note also that the bigger the cone, the smaller the dual cone.

The general philosophy of our work is different (and in a sense complementary). We shall only focus on the exponential rate

$$\rho_x := \limsup_{n\to\infty} \mathbb{P}_\mu^x[\tau_K > n]^{1/n},$$

and we answer completely the question of determining its value under fairly broad assumptions, regardless the position of the global minimum point x_0.

1.3. **Cones and random walks considered.** In this work, we consider a *closed convex cone* K with *non-empty interior*. Recall that we denote by K^* its *dual* cone, which turns out to be particularly relevant for our problem. It is the closed convex cone defined in (6). We also set

$$K_\delta := K + \delta v,$$

where $\delta \in \mathbb{R}$ and v is some fixed vector in K^o, the interior of K.

Throughout this paper, we shall make the assumption that μ is truly d-dimensional in the following sense:

(H1) The support of the probability measure μ is not included in any linear hyperplane.

For a square-integrable probability measure μ with mean m and variance-covariance matrix Γ, it is well known that the minimal (with respect to inclusion) affine subspace A such that $\mu(A) = 1$ is $m + (\ker \Gamma)^\perp$. Hence, the condition in (H1) holds if and only if $m + (\ker \Gamma)^\perp$ is not included in any hyperplane (or equivalently, if and only if Γ is non-degenerate or $\dim(\ker \Gamma) = 1$ and $m \notin (\ker \Gamma)^\perp$). Notice that in the case where $m = 0$, the assumption (H1) is equivalent to $\ker \Gamma = \{0\}$, i.e., Γ is non-degenerate.

As pointed out in the preceding section, our analysis of the decreasing rate of the non-exit probability requires the existence of a minimum point for the Laplace transform L_μ on the dual cone. Thus we shall impose the following technical-looking condition:

(H2) There exists a point $x^* \in K^*$ and an open neighborhood V of x^* in $\mathbb{R}^d$ such that $L_\mu(x)$ is finite for all $x \in V$, and x^* is a minimum point of L_μ restricted to $K^* \cap V$.

In view of applications, we will prove in Subsection 2.3 that for random walks with all exponential moments (i.e., $L_\mu(x)$ is finite for all $x \in \mathbb{R}^d$), the condition (H2) is equivalent to the more geometric-flavoured condition:

(H2') The support of μ is not included in any half-space $u^- := \{x \in \mathbb{R}^d : \langle u, x\rangle \leqslant 0\}$ with $u \in K^* \setminus \{0\}$.

1.4. **Main results.** We are now in position to state our main result:

Theorem 1. *Suppose μ satisfies (H1) and (H2). Then,*

$$\lim_{n\to\infty} \mathbb{P}_\mu^x[\tau_K > n]^{1/n} = L_\mu(x^*),$$

for all $x \in K_\delta$, for some constant $\delta \geqslant 0$.

For a large class of random walks and cones, Theorem 1 gives the universal recipe to compute the exponential decay of the non-exit probability. Notice that the latter is independent of the starting point x. This is not the case when (H2) is not satisfied and we shall illustrate this phenomenon in Section 4, with the walks in the quarter-plane having transition probabilities as in Figure 1.

The theorem in itself does not provide any explicit value for δ, but such a value can be found a posteriori thanks to the following:

Proposition 2. *The statement in Theorem 1 holds for any $\delta \geqslant 0$ for which there exists $n_0 \geqslant 1$ such that*

$$\mathbb{P}_\mu^0[\tau_{K_{-\delta}} > n_0, S_{n_0} \in K^o] > 0.$$

In order to illustrate Theorem 1, it is interesting to compare its content with the corresponding result known for Brownian motion with drift, in the light of the recent paper [17]. It is proved there that, for Brownian motion $(B_t)_{t\geqslant 0}$ with drift $a \in \mathbb{R}^d$, the non-exit probability admits the asymptotics (in the continuous case, the exit time from K is defined by $\tau_K := \inf\{t > 0 : B_t \notin K\}$)

$$\mathbb{P}^x[\tau_K > t] = \kappa h(x) t^{-\alpha} e^{-\gamma t}(1 + o(1)), \quad t \to \infty, \tag{7}$$

where $\gamma := d(a, K)^2/2$. Therefore

$$\lim_{t\to\infty} \mathbb{P}^x[\tau_K > t]^{1/t} = e^{-d(a,K)^2/2}.$$

Let us compare with the value given by Theorem 1 for the random walk $(B_n)_{n\geqslant 0}$. Its distribution μ is Gaussian with mean a and identity variance-covariance matrix; therefore

$$L_\mu(x) = e^{|x|^2/2 + \langle x, a\rangle}.$$

The minimum on K^* of $|x|^2/2 + \langle x, a\rangle$ is obviously the minimum on the *polar* cone $K^\sharp = -K^*$ of

$$|x|^2/2 - \langle x, a\rangle = |x - a|^2/2 - |a|^2/2.$$

It is reached at $x = p_{K^\sharp}^\perp(a)$, the orthogonal projection of a on $K^\sharp$, and an easy computation shows that the minimum value is

$$|p_{K^\sharp}^\perp(a) - a|^2/2 - |a|^2/2 = -|a - p_K^\perp(a)|^2/2 = -d(a, K)^2/2,$$

where we have used Moreau's decomposition theorem which asserts that, for any convex cone K, a is the orthogonal sum of $p_K^\perp(a)$ and $p_{K^\sharp}^\perp(a)$. We thus have

$$\min_{K^*} L_\mu = e^{-d(a,K)^2/2},$$

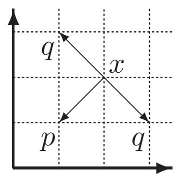

Figure 1. Random walks considered in Section 4 ($p+2q=1$), for different starting points x

which means that the exponential decreasing rate is the same for Brownian motion $(B_t)_{t\geqslant 0}$ and for the "sampled" Brownian motion $(B_n)_{n\geqslant 0}$, as one could expect.

1.5. **Plan of the paper.** The rest of our article has a simple organization: In Section 2 we prove Theorem 1. In Section 3 we present an important consequence of Theorem 1 in the counting of walks in orthants (a topical domain in enumerative combinatorics), see Corollaries 8 and 9. In Section 4 we consider the walks of Figure 1, for which we prove that contrary to the walks satisfying hypothesis (H2), the exponential decay depends on the starting point x. Finally, in Section 5 we prove the non-exponential decay of the non-exit probability for random walks with drift in the cone (a refinement of a theorem of [16]), which is needed for proving our main result.

2. Proof of the main results

This section is organized as follows: In Subsection 2.1, we review some elementary properties of the Laplace transform and present Cramér's formula. Then, we prove Theorem 1 and Proposition 2 in Subsection 2.2. In Subsection 2.3 we provide a geometric interpretation of our main assumption on the random walk distribution.

2.1. **Cramér's formula.** The Laplace transform of a probability distribution μ is the function L_μ defined for $x\in\mathbb{R}^d$ by

$$L_\mu(x) := \int_{\mathbb{R}^d} e^{\langle x,y\rangle}\mu(\mathrm{d}y).$$

It is clearly a convex function. If L_μ is finite in a neighborhood of the origin, say $\overline{B(0,r)}$, then it is well known that L_μ is (infinitely) differentiable in $B(0,r)$, and that its partial derivatives are given by

$$\frac{\partial L_\mu(x)}{\partial x_i} = \int_{\mathbb{R}^d} y_i e^{\langle x,y\rangle}\mu(\mathrm{d}y),\quad \forall i\in [\![1,d]\!].$$

Therefore, the expectation of μ is equal to the gradient of L_μ at the origin: $\mathbb{E}[\mu]=\nabla L_\mu(0)$. Notice that μ is centered if and only if 0 is a critical point of L_μ.

Suppose now that z is a point where L_μ is finite, and let μ_z denote the probability measure defined by

$$\mu_z(\mathrm{d}y) := \frac{e^{\langle z,y\rangle}}{L_\mu(z)}\mu(\mathrm{d}y). \tag{8}$$

The Laplace transform of μ_z is related to that of μ by the formula

$$L_{\mu_z}(x) = \frac{L_\mu(z+x)}{L_\mu(z)}.$$

If in addition L_μ is finite in some ball $\overline{B(z,r)}$, then L_{μ_z} is finite in $\overline{B(0,r)}$. By consequence, L_{μ_z} is differentiable in $B(0,r)$ and

$$\mathbb{E}[\mu_z] = \nabla L_{\mu_z}(0) = \frac{\nabla L_\mu(z)}{L_\mu(z)}.$$

The distribution of the random walk under $\mathbb{P}_{\mu_z}$ is linked to the initial distribution by the following:

Lemma 3 (Cramér's formula)**.** *For any measurable and positive function $F : \mathbb{R}^n \to \mathbb{R}$, we have*

$$\mathbb{E}^x_\mu[F(S_1,\dots,S_n)] = L_\mu(z)^n e^{\langle z,x\rangle}\mathbb{E}^x_{\mu_z}[e^{-\langle z,S_n\rangle}F(S_1,\dots,S_n)].$$

Proof. It follows directly from the definition (8) of μ_z that

$$\mu^{\otimes n}(\mathrm{d}y_1,\dots,\mathrm{d}y_n) = L_\mu(z)^n e^{-\langle z,\sum_{i=1}^n y_i\rangle}\mu_z^{\otimes n}(\mathrm{d}y_1,\dots,\mathrm{d}y_n).$$

The conclusion is then straightforward. □

Applied to the function $F(s_1,\dots,s_n) = \Pi_{i=1}^n \mathbb{1}_K(s_i)$, Cramér's formula reads

$$\mathbb{P}^x_\mu[\tau_K > n] = L_\mu(z)^n e^{\langle z,x\rangle}\mathbb{E}^x_{\mu_z}[e^{-\langle z,S_n\rangle}, \tau_K > n],$$

and this implies that for all $x \in \mathbb{R}^d$,

$$\limsup_{n\to\infty} \mathbb{P}^x_\mu[\tau_K > n]^{1/n} \leqslant \inf_{K^*} L_\mu, \tag{9}$$

as already observed at the end of Subsection 1.2.

2.2. **Proofs of Theorem 1 and Proposition 2.** In order to obtain the lower bound for the non-exit probability, we shall use Cramér's formula at $z = x^*$. The following lemma gives some useful information on the position of the drift of the random walk under the measure changed at x^*.

Lemma 4. *Suppose μ satisfies (H1) and (H2). Then, the gradient $\nabla L_\mu(x^*)$ belongs to K and is orthogonal to x^*.*

Proof. We first notice that under (H2), the Laplace transform is finite, hence differentiable, in some neighborhood of x^*. It is well known that the equality $(K^*)^* = K$ holds for any closed convex cone, see [21, Theorem 14.1]. Hence $\nabla L_\mu(x^*)$ belongs to K if and only if

$$\langle \nabla L_\mu(x^*), y\rangle \geqslant 0, \quad \forall y \in K^*.$$

So, let $y \in K^*$. Since K^* is a convex cone and $x^* \in K^*$, $x^* + ty$ also belongs to K^* for all $t \geqslant 0$. Hence, thanks to (H2), the function

$$t \in [0,\infty) \mapsto f_y(t) := L_\mu(x^* + ty)$$

is differentiable in some neighborhood of $t = 0$ and reaches its minimum at $t = 0$. This implies

$$\langle \nabla L_\mu(x^*), y\rangle = f'_y(0) \geqslant 0.$$

Now, if we take $y = x^*$, we have a stronger result since $x^* + tx^* = (1+t)x^*$ belongs to K^* for all $t \geqslant -1$: the function f_{x^*} is differentiable in some open neighborhood of $t = 0$ and has a local minimum point on $[-1, \infty)$ at $t = 0$, hence

$$\langle \nabla L_\mu(x^*), x^* \rangle = f'_{x^*}(0) = 0.$$

The proof is completed. □

We are now in position to conclude the proof of Theorem 1.

Proof of Theorem 1. As already observed, $L_\mu(x^*)$ is an upper bound for the exponential decreasing rate

$$\limsup_{n\to\infty} \mathbb{P}^x_\mu[\tau_K > n]^{1/n},$$

and it remains to prove that it is also the right lower bound. By performing the Cramér transformation at x^*, we get

$$\begin{aligned}\mathbb{P}^x_\mu[\tau_K > n] &= \rho^n e^{\langle x^*, x\rangle} \mathbb{E}^x_{\mu_*}[e^{-\langle x^*, S_n\rangle}, \tau_K > n] \\ &\geqslant \rho^n e^{\langle x^*, x\rangle} e^{-\alpha\sqrt{n}} \mathbb{P}^x_{\mu_*}[|\langle x^*, S_n\rangle| \leqslant \alpha\sqrt{n}, \tau_K > n],\end{aligned}$$

where $\rho = L_\mu(x^*)$, $\mu_*(\mathrm{d}y) = \rho^{-1} e^{\langle x^*, y\rangle}\mu(\mathrm{d}y)$, and α is any positive number. Notice that μ_* is truly d-dimensional, because μ has this property and both measures have the same support. Since the new drift

$$\mathbb{E}[\mu_*] = m_* = \rho^{-1}\nabla L_\mu(x^*)$$

belongs to K and is orthogonal to x^* (by Lemma 4), it follows from Theorem 12 in Section 5 that there exist $\alpha > 0$ and $\delta \geqslant 0$ such that

$$\lim_{n\to\infty} \mathbb{P}^x_{\mu_*}[|\langle x^*, S_n\rangle| \leqslant \alpha\sqrt{n}, \tau_K > n]^{1/n} = 1, \quad \forall x \in K_\delta.$$

Hence, we reach the conclusion that

$$\liminf_{n\to\infty} \mathbb{P}^x_\mu[\tau_K > n]^{1/n} \geqslant \rho$$

for all $x \in K_\delta$, and the theorem is proved. □

Remark 5. If the drift of the original random walk belongs to the cone K, the minimum of the Laplace transform on K^* is at the origin of the cone:

$$\lim_{n\to\infty} \mathbb{P}^x_\mu[\tau_K > n]^{1/n} = 1 = L_\mu(0) = \min_{K^*} L_\mu.$$

In other cases, when the drift is not inside the cone K, there is no direct link between the location of the minimum x^* and the location of the drift.

To conclude this section, let us explain how to find a $\delta \geqslant 0$ for which the statement of Theorem 1 holds.

Proof of Proposition 2. Recall that $v \in K^o$ is fixed and $K_\delta = K + \delta v$. We assume that there exist $\delta \geqslant 0$ and $k \geqslant 1$ such that

$$\mathbb{P}^0_\mu[\tau_{K_{-\delta}} > k, S_k \in K^o] > 0.$$

Therefore, we can find $\epsilon > 0$ such that

$$\mathbb{P}_\mu^0[\tau_{K_{-\delta}} > k, S_k \in K_\epsilon] = \gamma > 0,$$

and since K is a convex cone, it satisfies the relation $K + K \subset K$, thus

$$\mathbb{P}_\mu^x[\tau_K > k, S_k - x \in K_\epsilon] \geqslant \gamma,$$

for all $x \in K_\delta$ (by inclusion of events). From this, we shall deduce by induction that

$$p_\ell := \mathbb{P}_\mu^x[\tau_K > \ell k, S_{\ell k} - x \in K_{\ell\epsilon}] \geqslant \gamma^\ell, \tag{10}$$

for all $\ell \geqslant 1$ and $x \in K_\delta$. Indeed, by the Markov property of the random walk,

$$\begin{aligned} p_{\ell+1} &\geqslant \mathbb{E}_\mu^x[\tau_K > \ell k, S_{\ell k} - x \in K_{\ell\epsilon}, \mathbb{P}_\mu^{S_{\ell k}}[\tau_K > k, S_k - x \in K_{(\ell+1)\epsilon}]] \\ &\geqslant p_\ell \cdot \inf_y \mathbb{P}_\mu^y[\tau_K > k, S_k - x \in K_{(\ell+1)\epsilon}], \end{aligned}$$

where the infimum is taken over all $y \in K$ such that $y - x \in K_{\ell\epsilon}$. Noting that $y - x \in K_{\ell\epsilon}$ and $S_k - y \in K_\epsilon$ imply $S_k - x \in K_{(\ell+1)\epsilon}$, we obtain

$$p_{\ell+1} \geqslant p_\ell \cdot \inf_y \mathbb{P}_\mu^y[\tau_K > k, S_k - y \in K_\epsilon].$$

But $x \in K_\delta$ and $y - x \in K_{\ell\epsilon} \subset K$ imply $y \in K_\delta$. Hence $p_{\ell+1} \geqslant p_\ell \cdot \gamma$ and (10) is proved.

Now Theorem 1 asserts the existence of some $\delta_0 \geqslant 0$ such that

$$\lim_{n\to\infty} \mathbb{P}_\mu^y[\tau_K > n]^{1/n} = L_\mu(x^*),$$

for all $y \in K_{\delta_0}$, and we want to prove that the result also holds for $x \in K_\delta$. To do this, we shall simply use (10) in order to push the walk from K_δ to K_{δ_0}. More precisely, choose $\ell \geqslant 1$ such that $\delta + \ell\epsilon \geqslant \delta_0$. Then for all $x \in K_\delta$ the inclusion $x + K_{\ell\epsilon} \subset K_{\delta_0}$ holds, and thanks to (10),

$$\mathbb{P}_\mu^x[\tau_K > m, S_m \in K_{\delta_0}] \geqslant \gamma^\ell,$$

for $m = k\ell$. By the Markov property, for all $n \geqslant m$, we have

$$\begin{aligned} \mathbb{P}_\mu^x[\tau_K > n] &\geqslant \mathbb{E}_\mu^x[\tau_K > m, S_m \in K_{\delta_0}, \mathbb{P}_\mu^{S_m}[\tau_K > n - m]] \\ &\geqslant \gamma^\ell \cdot \inf_{y \in K_{\delta_0}} \mathbb{P}_\mu^y[\tau_K > n - m] \\ &\geqslant \gamma^\ell \cdot \mathbb{P}_\mu^{\delta_0 v}[\tau_K > n - m], \end{aligned}$$

where the last inequality follows by inclusion of events. This implies immediately

$$\liminf_{n\to\infty} \mathbb{P}_\mu^x[\tau_K > n]^{1/n} \geqslant L_\mu(x^*).$$

But since the inequality

$$\limsup_{n\to\infty} \mathbb{P}_\mu^x[\tau_K > n]^{1/n} \leqslant L_\mu(x^*)$$

holds for all x (see (9)), Proposition 2 is proved. □

2.3. **Geometric interpretation of condition (H2).** The aim of this subsection is to give a geometric interpretation of condition (H2) under some additional condition on the exponential moments. Throughout this section, we fix a closed convex cone C and assume that μ has all C-exponential moments, that is,

$$L_\mu(x) < \infty, \quad \forall x \in C.$$

The strict convexity of the exponential function ensures that

$$L_\mu(ax_1 + bx_2) \leqslant aL_\mu(x_1) + bL_\mu(x_2), \tag{11}$$

for all $x_1 \neq x_2 \in C$ and $a, b > 0$ with $a + b = 1$, and that equality occurs if and only if

$$\mu((x_1 - x_2)^\perp) = 1,$$

where $(x_1 - x_2)^\perp$ denotes the hyperplane orthogonal to $x_1 - x_2$. Thus, if (H1) is satisfied, the equality in (11) never occurs and L_μ is strictly convex on C.

Let $\mathbb{S}^{d-1}$ denote the unit sphere of $\mathbb{R}^d$. Standard arguments involving the convexity of L_μ and the compactness of $C\cap\mathbb{S}^{d-1}$ show that $L_\mu(x)$ goes to infinity as $|x| \to \infty$ uniformly on C if and only if

$$\lim_{t\to\infty} L_\mu(tu) = \infty, \quad \forall u \in C \cap \mathbb{S}^{d-1}. \tag{12}$$

Hence, the condition (12) is sufficient for the existence of a global minimum on C. Indeed, if it is satisfied, then there exists $R > 0$ such that $L_\mu(x) \geqslant 1$ for all $x \in C$ with $|x| \geqslant R$. By continuity, L_μ reaches a minimum on $\overline{B(0, R)}\cap C$ which is less than or equal to 1 (since $L_\mu(0) = 1$), thus it is a global minimum on C.

The next lemma gives some interesting informations on the behavior at infinity of L_μ. Recall that u^- denotes the half-space $\{y \in \mathbb{R}^d : \langle u, y\rangle \leqslant 0\}$.

Lemma 6. *Suppose that μ has all C-exponential moments. For every $u \in C \cap \mathbb{S}^{d-1}$, the following dichotomy holds:*

(1) *If $\mu(u^-) < 1$, then*

$$\lim_{t\to\infty} L_\mu(x + tu) = \infty, \quad \forall x \in C.$$

(2) *If $\mu(u^-) = 1$, then*

$$\lim_{t\to\infty} L_\mu(x + tu) = \int_{u^\perp} e^{\langle x,y\rangle}\mu(\mathrm{d}y), \quad \forall x \in C.$$

Proof. If $\mu(u^-) < 1$, then we can find $\epsilon > 0$ such that the set $\{y \in \mathbb{R}^d : \langle u, y\rangle \geqslant \epsilon\}$ has positive measure, and the inequality

$$L_\mu(x + tu) \geqslant \int_{\{y\in\mathbb{R}^d:\langle u,y\rangle\geqslant\epsilon\}} e^{\langle x,y\rangle}e^{t\epsilon}\mu(\mathrm{d}y) \geqslant ce^{t\epsilon},$$

proves the first assertion of Lemma 6. Suppose now on the contrary that $\mu(u^-) = 1$. We then may write

$$L_\mu(x + tu) = \int_{u^\perp} e^{\langle x,y\rangle}\mu(\mathrm{d}y) + \int_{\{y\in\mathbb{R}^d:\langle u,y\rangle<0\}} e^{\langle x+tu,y\rangle}\mu(\mathrm{d}y).$$

The second integral on the right-hand side of the above equation goes to zero as t goes to infinity by the dominated convergence theorem, thus proving the second assertion of Lemma 6. □

Lemma 7. *Suppose that μ satisfies (H1) and has all C-exponential moments. Then the Laplace transform L_μ has a global minimum on the closed convex cone C if and only if there does not exist any $u \neq 0$ in C such that $\mu(u^-) = 1$.*

Proof. If $\mu(u^-) < 1$ for all $u \neq 0$ in C, then $\lim_{t\to\infty} L_\mu(tu) = \infty$ by Lemma 6, and the function L_μ has a global minimum on C as explained earlier. Now, suppose on the contrary that $\mu(u^-) = 1$ for some $u \neq 0$ in C. Then, by Lemma 6 again, the limit

$$h(x) := \lim_{t\to\infty} L_\mu(x + tu)$$

exists and is finite for all x. Since any convex cone is a semi-group, $x + tu \in C$ for all $x \in C$ and $t \geqslant 0$, and consequently

$$h(x) \geqslant \inf_C L_\mu, \quad \forall x \in C.$$

But our assumption that μ satisfies (H1) implies that L_μ is strictly convex, so that $L_\mu(x) > h(x)$ for all $x \in C$ (for else the strictly convex function $t \mapsto L_\mu(x + tu)$ on $[0, \infty)$ would not have a finite limit). We thus reach the conclusion that $L_\mu(x) > \inf_C L_\mu$ for all $x \in C$, thereby proving that L_μ has no global minimum on C. □

For random walks with all exponential moments, the equivalence between conditions (H2) and (H2') under (H1) is now an easy consequence of Lemma 7.

3. Application to lattice path enumeration

In this section we present an application of our main result (Theorem 1) in enumerative combinatorics: Given a finite set $\mathfrak{S}$ of allowed steps, a now classical problem is to study $\mathfrak{S}$-walks in the orthant

$$Q := (\mathbb{R}^+)^d = \{x \in \mathbb{R}^d : x_i \geqslant 0, \forall i \in [\![1, d]\!]\},$$

that is walks confined to Q, starting at a fixed point x (often the origin) and using steps in $\mathfrak{S}$ only. Denote by $f_\mathfrak{S}(x, y; n)$ the number of such walks that end at y and use exactly n steps. Many properties of the counting numbers $f_\mathfrak{S}(x, y; n)$ have been recently analyzed (the seminal work in this area is [6]). First, exact properties of them were derived, via the study of their generating function (exact expression and algebraic nature). Such properties are now well established for the case of small steps walks in the quarter-plane, meaning that the step set $\mathfrak{S}$ is included in $\{0, \pm 1\}^2$. More qualitative properties of the $f_\mathfrak{S}(x, y; n)$ were also investigated, such as the asymptotic behavior, as $n \to \infty$, of the number of excursions $f_\mathfrak{S}(x, y; n)$ for fixed y, or that of the total number of walks,

$$f_\mathfrak{S}(x; n) := \sum_{y\in Q} f_\mathfrak{S}(x, y; n). \tag{13}$$

Concerning the excursions, several small steps cases have been treated by Bousquet-Mélou and Mishna [6] and by Fayolle and Raschel [15]. Later on, Denisov and Wachtel [9] obtained the very precise asymptotics of the excursions, for a quite large class of step sets and cones. As for the total number of walks (13), only very particular cases are solved, see again [6, 15]. In a most recent work [19], Johnson, Mishna and Yeats obtained an upper bound for the exponential growth constant, namely,

$$\limsup_{n\to\infty} f_{\mathfrak{S}}(x;n)^{1/n},$$

and proved by comparison with results of [15] that these bounds are tight for all small steps models in the quarter-plane. In the present article, we find the exponential growth constant of the total number of walks (13) in any dimension for any model such that:

(H1") The step set $\mathfrak{S}$ is not included in a linear hyperplane;
(H2") The step set $\mathfrak{S}$ is not included in a half-space u^-, with $u \in Q \setminus \{0\}$.

Our results provide the first unified treatment of this problem of determining the growth constant for the number of lattice paths confined to the positive orthant. In the sequel we shall say that a step set $\mathfrak{S}$ is proper if it satisfies to (H1") and (H2"). Note in particular that the well-known 79 models of walks in the quarter-plane studied in [6, 15] (including the so-called 5 singular walks) satisfy both hypotheses above.

Corollary 8. *Let $\mathfrak{S}$ be any proper step set. The Laplace transform of $\mathfrak{S}$,*

$$L_{\mathfrak{S}}(x) := \sum_{s\in\mathfrak{S}} e^{\langle x,s\rangle},$$

reaches a global minimum on Q at a unique point x_0, and there exists $\delta \geqslant 0$ such that for any starting point $x \in Q_\delta$,

$$\lim_{n\to\infty} f_{\mathfrak{S}}(x;n)^{1/n} = L_{\mathfrak{S}}(x_0).$$

Suppose that in addition:

(H3") The step set allows a path staying in Q from the origin to some point in the interior of Q.

Then it follows from Proposition 2 that the result in Corollary 8 holds with $\delta = 0$, i.e., it is valid for all $x \in Q$. Note that this assumption is not restrictive from a combinatorial point of view, since if (H3") is not satisfied, the counting problem is obvious.

Proof of Corollary 8. Consider a random walk $(S_n)_{n\geqslant 0}$ starting from x such that μ is the uniform law on $\mathfrak{S}$. Let then τ_Q denote the first exit time from Q. The enumeration problem is related to probabilities in a simple way:

$$\mathbb{P}^x_\mu[\tau_Q > n] = \frac{f_{\mathfrak{S}}(x;n)}{|\mathfrak{S}|^n}. \tag{14}$$

Further, it is immediate from our definitions that $L_{\mathfrak{S}}(x) = |\mathfrak{S}|L_\mu(x)$. Corollary 8 then follows from Theorem 1 and from the fact that $Q^* = Q$. □

As a consequence, we obtain the following result, which was conjectured in [19]:

Corollary 9. *Let $\mathfrak{S} \subset \mathbb{Z}^d$ be a proper step set (hypotheses (H1") and (H2")), which besides satisfies (H3"), and let $K_{\mathfrak{S}}$ be the growth constant for the total number of walks (13). Let $\mathcal{P}$ be the set of hyperplanes through the origin in $\mathbb{R}^d$ which do not meet the interior of the first orthant. Given $p \in \mathcal{P}$, let $K_{\mathfrak{S}}(p)$ be the growth constant of the walks on $\mathfrak{S}$ which are restricted to the side of p which includes the first orthant. Then $K_{\mathfrak{S}} = \min_{p\in\mathcal{P}} K_{\mathfrak{S}}(p)$.*

Proof. Let us first notice that $\mathcal{P}$ can be described as the set of hyperplanes $u^\perp$ such that $u \in Q \cap \mathbb{S}^{d-1}$, and that the side of $p = u^\perp$ which includes the first orthant is then the half-space $u^+ = \{x \in \mathbb{R}^d : \langle x, u\rangle \geqslant 0\}$. By Theorem 1, the exponential rate for the random walk associated to the step set $\mathfrak{S}$ and confined to u^+ is the minimum of $L_\mathfrak{S}/|\mathfrak{S}|$ on the dual cone $(u^+)^* = \{tu : t \geqslant 0\}$. Therefore, the growth constant $K_\mathfrak{S}(p)$ equals $\min_{t\geqslant 0} L_\mathfrak{S}(tu)$, and the equality

$$\min_{x\in Q} L_\mathfrak{S}(x) = \min_{u\in Q\cap\mathbb{S}^{d-1}} \min_{t\geqslant 0} L_\mathfrak{S}(tu)$$

immediately translates into

$$K_\mathfrak{S} = \min_{p\in\mathcal{P}} K_\mathfrak{S}(p).$$

The proof of Corollary 9 is completed. □

4. An example of half-space walks

In this section we illustrate the following phenomenon: If the support of the random walk is included in a certain half-space (chosen so as contradicting (H2)), such a universal result as Theorem 1 does not hold; in particular, the exponential decay of the non-exit probability may depend on the starting point.

Let $(S_n)_{n\geqslant 0}$ be the random walk on Q starting at x and with transition probabilities to $(1,-1)$, $(-1,1)$ and $(-1,-1)$, with respective probabilities q, q and p, where $p+2q = 1$, see Figures 1 and 2. Let τ_Q be the exit time (1) of this random walk from the quarter-plane Q. Finally, define for fixed N the segment $D_{2N} = \{(i,j) \in Q : i+j = 2N\}$.

Proposition 10. *For any $N \geqslant 1$ and any $x \in D_{2N}$, we have*

$$\lim_{n\to\infty} \mathbb{P}^x[\tau_Q > n]^{1/n} = 2q\cos\left(\frac{\pi}{2N+2}\right). \tag{15}$$

We shall need the following result on the simple symmetric random walk on $\mathbb{Z}$ (we refer to [22, page 243] for a proof):

Lemma 11 ([22])**.** *For the simple symmetric random walk $(\widetilde{S}_n)_{n\geqslant 0}$ on $\mathbb{Z}$ (with jumps to the left and to the right with equal probabilities $1/2$), we have, for any $x \in [\![0, 2N]\!]$,*

$$\lim_{n\to\infty} \mathbb{P}^x[\widetilde{S}_1, \ldots, \widetilde{S}_n \in [\![0, 2N]\!]]^{1/n} = \cos\left(\frac{\pi}{2N+2}\right).$$

Proof of Proposition 10. We shall prove Proposition 10 by induction over $N \geqslant 1$. For $N = 1$, we have three choices for x (see Figure 2). We write the proof when $x = (1,1)$, since the arguments for other values of x are quite similar. For this choice of x, the origin $(0,0)$ can be reached only at odd times n, and in that event, the random walk gets out of Q at time $n+1$. From this simple remark we deduce that (below, we note $(X_k)_{k\geqslant 1}$ the increments of the random walk $(S_n)_{n\geqslant 0}$)

$$\begin{aligned}\mathbb{P}^x[\tau_Q > 2n] &= \mathbb{P}^x[\tau_Q > 2n, X_k \neq (-1,-1), \forall k \in [\![1, n]\!]] \\ &= \mathbb{P}^x[\tau_Q > 2n |\, X_k \neq (-1,-1), \forall k \in [\![1, n]\!]](2q)^n.\end{aligned}$$

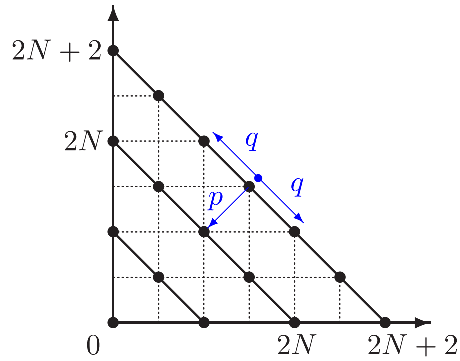

FIGURE 2. Random walks considered in the proof of Proposition 10 on the lines $\{(i,j) \in Q : i+j = 2N\}$ for $N \geqslant 0$

Further, the random walk conditioned on never making the jump $(-1,-1)$ is a simple symmetric random walk on the segment D_2. Therefore,

$$\mathbb{P}^x[\tau_Q > 2n] = \mathbb{P}^1[\widetilde{S}_1, \ldots, \widetilde{S}_n \in [\![0,2]\!]](2q)^{2n}.$$

Using Lemma 11, we conclude that for $x = (1,1)$,

$$\lim_{n\to\infty} \mathbb{P}^x[\tau_Q > 2n]^{1/(2n)} = 2q \cos\left(\frac{\pi}{4}\right).$$

The fact that $\mathbb{P}^x[\tau > n]$ is decreasing in n implies that the above equation holds with $2n+1$ instead of $2n$. This achieves the proof of Proposition 10 for $N = 1$.

Let us now assume that equation (15) holds for a fixed value of $N \geqslant 1$. For $x \in D_{2N+2}$, introduce

$$H := \inf\{n > 0 : S_n \in D_{2N}\}$$

the hitting time of the set D_{2N}, see Figure 2. We can write

$$\mathbb{P}^x[\tau_Q > n] = \mathbb{P}^x[\tau_Q > n, H > n] + \sum_{k=1}^{n} \mathbb{P}^x[\tau_Q > n, H = k]. \tag{16}$$

The first term in the right-hand side of (16) can be written as

$$\mathbb{P}^x[\tau_Q > n, H > n] = \mathbb{P}^x[\tau_Q > n | H > n](2q)^n,$$

where (for the same reasons as for the case $N = 1$)

$$\lim_{n\to\infty} \mathbb{P}^x[\tau_Q > n | H > n]^{1/n} = \cos\left(\frac{\pi}{2N+4}\right).$$

As for the second term in the right-hand side of (16),

$$\begin{aligned}
\mathbb{P}^x[\tau_Q > n, H = k] &= \mathbb{E}^x[\tau_Q > k, K = k, \mathbb{P}^{S_k}[\tau_Q > n-k]] \\
&\leqslant C \cdot \mathbb{P}^x[\tau_Q > k, H = k]\mathbb{P}^{x_0}[\tau_Q > n-k] \\
&\leqslant C \cdot \mathbb{P}^x[\tau_Q > k-1, H > k-1]\mathbb{P}^{x_0}[\tau_Q > n-k] \\
&:= C a_k b_{n-k}.
\end{aligned}$$

The first equality above comes from the strong Markov property. The first inequality follows from the fact that for any fixed $x_0 \in D_{2N}$, there exists a constant $C > 0$ such that, for any $n \geqslant 0$ and any y, $\mathbb{P}^y[\tau_Q > n] \leqslant C\mathbb{P}^{x_0}[\tau_Q > n]$. The second inequality is obvious, and the last line has to be read as a definition.

Using on the one hand the same reasoning as for the case $N = 1$, and on the other hand the induction hypothesis, we obtain

$$\lim_{n\to\infty} a_n^{1/n} = 2q\cos\left(\frac{\pi}{2N+4}\right) > 2q\cos\left(\frac{\pi}{2N+2}\right) = \lim_{n\to\infty} b_n^{1/n}.$$

Standard properties of the Cauchy product then lead to

$$\limsup_{n\to\infty}\left(\sum_{k=1}^{n} a_k b_{n-k}\right)^{1/n} \leqslant 2q\cos\left(\frac{\pi}{2N+4}\right).$$

To summarize, with the help of (16) we have written $\mathbb{P}^x[\tau_Q > n] = A_n + B_n$, where

$$\lim_{n\to\infty} A_n^{1/n} = 2q\cos\left(\frac{\pi}{2N+4}\right), \quad \limsup_{n\to\infty} B_n^{1/n} \leqslant 2q\cos\left(\frac{\pi}{2N+4}\right).$$

The formula (15) therefore holds for $N + 1$, and Proposition 10 is proved. □

5. The case of random walks with drift in the cone

In this section we refine a result announced in [16] concerning the non-exponential decay of the non-exit probability (2) from a cone for a centered, square-integrable and non-degenerate multidimensional random walk. We prove that the result in [16] still holds if the random walk has a drift in the cone and if the hypothesis that its variance-covariance matrix is non-degenerate is weakened to (H1). This result is one of the main ingredients of the proof of Theorem 1. We would like to notice that the proof of (a weakened form of) Theorem 12 below was only sketched in [16], so that the extended proof we shall give here is new.

As before, we only assume that K is a closed convex cone with non-empty interior K^o. We fix some $v \in K^o$ and define $K_\delta = K + \delta v$. In this setting, we shall prove the following:

Theorem 12. *Assume that the distribution μ of the random walk increments is square-integrable and truly d-dimensional (H1). Suppose in addition that the drift $m = \mathbb{E}^0[S_1]$ belongs to the cone K and that v is a vector orthogonal to m. Then there exists $\alpha > 0$ and $\delta \geqslant 0$ such that, for all $x \in K_\delta$,*

$$\lim_{n\to\infty} \mathbb{P}^x[\tau_K > n, |\langle v, S_n\rangle| \leqslant \alpha\sqrt{n}]^{1/n} = 1.$$

If $m = 0$, as we have already explained, (H1) is equivalent to the fact that the variance-covariance matrix of the increments distribution is non-degenerate. Hence, we exactly recover [16, Theorem]. However, if $m \neq 0$, Theorem 12 can not be derived from [16, Theorem]. Indeed, under the hypothesis of Theorem 12, it is clear that

$$\mathbb{P}^x[\tau_K > n, |\langle v, S_n\rangle| \leqslant \alpha\sqrt{n}] \geqslant \mathbb{P}^x[\tau_K(\widetilde{S}) > n, |\langle v, \widetilde{S}_n\rangle| \leqslant \alpha\sqrt{n}],$$

where $(\widetilde{S}_n = S_n - nm)_{n\geqslant 0}$ is the centered random walk associated with $(S_n)_{n\geqslant 0}$. But the variance-covariance matrix of $\widetilde{S}_1$ is equal to that of S_1 and might be degenerate, so that [16, Theorem] would not apply to the walk $(\widetilde{S}_n)_{n\geqslant 0}$. This is for example the case when $(S_n)_{n\geqslant 0}$ is the uniform two-dimensional random walk with step set $\mathfrak{S} = \{(0,1),(1,0)\}$.

In order to prove Theorem 12, we will need a series of lemmas. We begin with some geometric considerations.

Lemma 13. *Let m be a point in K and V be a linear subspace of $\mathbb{R}^d$. If $(m+V)\cap K^o=\emptyset$, then $m+V$ is included in a hyperplane.*

Proof. Clearly, $\dim V < d$, for else $m+V=\mathbb{R}^d$ would intersect K^o. Since $m+V$ is included in the linear subspace generated by m and V, whose dimension is less or equal to $\dim V+1$, the affine space $m+V$ is always included in a hyperplane if $\dim V < d-1$. Thus, it remains to consider the case where V is a hyperplane, that is,

$$V = u^\perp = \{x\in\mathbb{R}^d : \langle u,x\rangle = 0\}$$

for some $u\neq 0$. Assume $m\notin V$. Possibly changing u to $-u$, we can assume in addition that $\langle m,u\rangle > 0$. Then,

$$\bigcup_{\lambda>0}(\lambda m+V) = \bigcup_{\lambda>0}(\lambda u+V) = \{x\in\mathbb{R}^d : \langle u,x\rangle > 0\} =: u_*^+.$$

Further, by homogeneity of V and K^o, we have $(\lambda m+V)\cap K^o=\emptyset$ for every $\lambda>0$. Hence, K^o does not intersect u_*^+, and is therefore included in $u^- = \{x\in\mathbb{R}^d : \langle u,x\rangle \leqslant 0\}$. This is a contradiction since $m\in K=\overline{(K^o)}$ (this equality holds for any convex set with non-empty interior) and $\langle m,u\rangle>0$. Thus m belongs to V and $m+V=V$ is a hyperplane. □

Lemma 14. *Let $Y\in\mathbb{R}^d$ be a random vector with Gaussian distribution $\mathcal{N}(m,\Gamma)$. If m belongs to K and if $m+(\ker\Gamma)^\perp$ is not included in a hyperplane, then*

$$\mathbb{P}[Y\in K^o]>0.$$

Proof. It is well known that the Gaussian distribution $\mathcal{N}(m,\Gamma)$ admits a positive density with respect to Lebesgue measure on the affine space $m+(\ker\Gamma)^\perp$. Thus, it suffices to show that $(m+(\ker\Gamma)^\perp)\cap K^o$ is a non-empty open set in $m+(\ker\Gamma)^\perp$. But this follows from Lemma 13 since $m+(\ker\Gamma)^\perp$ is not contained in a hyperplane. □

Lemma 15. *Under the hypotheses of Theorem 12, there exist $k\geqslant 1$ and $\delta\geqslant 0$ such that*

$$\mathbb{P}[\tau_{K_{-\delta}}>k, S_k\in K^o]>0.$$

Proof. For $n\geqslant 1$ define $Z_n := (S_n-nm)/\sqrt{n}$. The homogeneity and convexity properties of K ensure that

$$\{S_n\in K^o\} = \{Z_n\in K^o-\sqrt{n}m\}\supset\{Z_n\in K^o-m\}$$

Let Γ denote the variance-covariance matrix of μ. We notice that $m+(\ker\Gamma)^\perp$ is not included in a hyperplane since μ satisfies (H1). The central limit theorem asserts that $(Z_n)_{n\geqslant 1}$ converges in distribution to a random vector Y with Gaussian distribution $\mathcal{N}(0,\Gamma)$. Hence, applying the Portmanteau theorem [5, Theorem 2.1], we obtain the inequality

$$\liminf_{n\to\infty}\mathbb{P}[S_n\in K^o]\geqslant\mathbb{P}[Y\in K^o-m],$$

where the right-hand side is positive according to Lemma 14. Now, to conclude it suffices to fix k so that $\mathbb{P}[S_k\in K^o]=2\epsilon>0$, and then choose δ so large that $\mathbb{P}[\tau_{K_{-\delta}}>k]\geqslant 1-\epsilon$ (this is possible since $K_{-\delta}\uparrow\mathbb{R}^d$ as $\delta\uparrow\infty$). □

Proof of Theorem 12. The proof follows the same kind of arguments as in [16]. We shall first use Lemma 15 in order to push the random walk inside the cone at a distance $\sqrt{n}$ from the boundary, and then apply the functional central limit theorem.

Recall that $v \in K^o$ is fixed and that $K_\delta = K + \delta v$. We know by Lemma 15 that there exist $\delta \geqslant 0$ and $k \geqslant 1$ such that

$$\mathbb{P}[\tau_{K_{-\delta}} > k, S_k \in K^o] > 0.$$

Therefore, we can find a closed ball $B := \overline{B(z,\epsilon)} \subset K^o$, with center at $z \in K^o$ and radius $\epsilon > 0$, such that

$$\mathbb{P}[\tau_{K_{-\delta}} > k, S_k \in B] = \gamma > 0.$$

Since K is a convex cone, it satisfies the relation $K + K \subset K$, thus

$$\mathbb{P}^x[\tau_K > k, S_k - x \in B] \geqslant \gamma,$$

for all $x \in K_\delta$ (by inclusion of events). From this, we shall deduce by induction that

$$p_\ell := \mathbb{P}^x[\tau_K > \ell k, S_{\ell k} - x \in \ell B] \geqslant \gamma^\ell, \tag{17}$$

for all $\ell \geqslant 1$ and $x \in K_\delta$. Indeed, by the Markov property of the random walk,

$$\begin{aligned} p_{\ell+1} &\geqslant \mathbb{E}^x[\tau_K > \ell k, S_{\ell k} - x \in \ell B, \mathbb{P}^{S_{\ell k}}[\tau_K > k, S_k - x \in (\ell+1)B]] \\ &\geqslant p_\ell \cdot \inf_{\{y \in K : y - x \in \ell B\}} \mathbb{P}^y[\tau_K > k, S_k - x \in (\ell+1)B]. \end{aligned}$$

Noticing that $y - x \in \ell B$ and $S_k - y \in B$ yields $S_k - x \in (\ell+1)B$, we obtain

$$p_{\ell+1} \geqslant p_\ell \cdot \inf_{\{y \in K : y - x \in \ell B\}} \mathbb{P}^y[\tau_K > k, S_k - y \in B].$$

But $x \in K_\delta$ and $y - x \in \ell B \subset K$ imply $y \in K_\delta$. Hence $p_{\ell+1} \geqslant p_\ell \cdot \gamma$ and (17) is proved.

Now, for $x \in K_\delta$, define

$$\widetilde{p}_n := \mathbb{P}^x[\tau_K > n, |\langle v, S_n \rangle| \leqslant \alpha\sqrt{n}],$$

where $\alpha > 0$ will be fixed latter. Write $\ell = \lfloor \sqrt{n} \rfloor$ for the lower integer part of $\sqrt{n}$. Using the Markov property at time ℓk and the estimate in (17) leads to

$$\begin{aligned} \widetilde{p}_n &\geqslant \mathbb{P}^x[\tau_K > n, S_{\ell k} - x \in \ell B, |\langle v, S_n \rangle| \leqslant \alpha\sqrt{n}] \\ &\geqslant \mathbb{E}^x[\tau_K > \ell k, S_{\ell k} - x \in \ell B, \mathbb{P}^{S_{\ell k}}[\tau_K > n - \ell k, |\langle v, S_{n-\ell k} \rangle| \leqslant \alpha\sqrt{n}]] \\ &\geqslant \gamma^\ell \cdot \inf_{\{y \in K : y - x \in \ell B\}} \mathbb{P}^y[\tau_K > n - \ell k, |\langle v, S_{n-\ell k} \rangle| \leqslant \alpha\sqrt{n}]. \end{aligned}$$

Therefore, Theorem 12 will follow from the fact that

$$\liminf_{n\to\infty} \inf_{\{y \in K : y - x \in \ell B\}} \mathbb{P}^y[\tau_K > n - \ell k, |\langle v, S_{n-\ell k} \rangle| \leqslant \alpha\sqrt{n}] > 0,$$

which we shall prove now. Since $\ell k \ll n$ does not play any significant role in the last probability, we will neglect it in order to simplify notations. Also, for any $\epsilon' > \epsilon$, we have

$$x + \ell B = \ell\left(\frac{x}{\ell} + \overline{B(z,\epsilon)}\right) \subset \ell\overline{B(z,\epsilon')}$$

for all large enough n. Since an $\epsilon' > \epsilon$ can be found so that $\overline{B(z,\epsilon')} \subset K^o$, we may replace $x + \ell B$ by ℓB without loss of generality. Finally, since $\ell \leqslant \sqrt{n}$, we may replace ℓB by $\sqrt{n}B$. With these simplifications, it remains to consider

$$q_n := \inf_{\{y\in K : y\in\sqrt{n}B\}} \mathbb{P}^y[\tau_K > n, |\langle v, S_n\rangle| \leqslant \alpha\sqrt{n}].$$

By mapping y to $y/\sqrt{n}$, we may write

$$q_n = \inf_{y\in B} q_n(y),$$

where

$$q_n(y) := \mathbb{P}^0[\tau_K(y\sqrt{n} + S) > n, |\langle v, y\sqrt{n} + S_n\rangle| \leqslant \alpha\sqrt{n}].$$

Let $\widetilde{S} = (\widetilde{S}_n = S_n - nm)_{n\geqslant 0}$ denote the centered random walk associated with $S = (S_n)_{n\geqslant 0}$. By inclusion of events we get the lower bound

$$q_n(y) \geqslant \mathbb{P}^0[\tau_K(y\sqrt{n} + \widetilde{S}) > n, |\langle v, y\sqrt{n} + \widetilde{S}_n\rangle| \leqslant \alpha\sqrt{n}],$$

where we used the fact that $m \in K$ and $\langle v, m\rangle = 0$. Finally, let us denote by $Z_n = (Z_n(t))_t$ the random process with continuous paths that coincides with $\widetilde{S}_k/\sqrt{n}$ for $t = k/n$ and which is linearly interpolated elsewhere. By definition of Z_n and convexity of K, the last inequality immediately rewrites

$$q_n(y) \geqslant \mathbb{P}^0[\tau_K(y + Z_n(t)) > 1, |\langle v, y + Z_n(1)\rangle| \leqslant \alpha].$$

The functional central limit theorem ensures that Z_n converges in distribution to a Brownian motion $(b(t))_t$ with variance-covariance matrix Γ. Suppose that the sequence $(y_n)_{n\geqslant 0}$ converges to some $y \in B$. Then $(y_n + Z_n)_{n\geqslant 0}$ converges in distribution to $y + b$, and it follows from the Portmanteau theorem that

$$\liminf_{n\to\infty} q_n(y_n) \geqslant \mathbb{P}^0[\tau_{K^o}(y + b(t)) > 1, |\langle v, y + b(1)\rangle| < \alpha]. \tag{18}$$

Now, it is time to choose α. To do this, first recall that $B = \overline{B(z,\epsilon)} \subset K^o$. Choose $\eta > \epsilon$ so that $\overline{B(z,\eta)} \subset K^o$ and set (notice that we could have done this at the very beginning of the proof)

$$\alpha = |v|(|z| + \eta).$$

If $|b(t)| < \eta - \epsilon$ for all $t \in [0,1]$, then $y + b(t) \in \overline{B(z,\eta)} \subset K^o$ for all $t \in [0,1]$. Furthermore,

$$|\langle v, y + b(1)\rangle| \leqslant |v|(|y| + |b(1)|) < |v|(|z| + \epsilon + \eta - \epsilon) = \alpha.$$

Therefore, the probability in (18) is bounded from below by the probability

$$\mathbb{P}\left[\max_{t\in[0,1]} |b(t)| < \eta - \epsilon\right]$$

that the Brownian motion $(b(t))_t$ stays near the origin for all $t \in [0,1]$, and this event happens with positive probability, regardless Γ be positive definite or not.

To summarize, we have proved that

$$\liminf_{n\to\infty} q_n(y_n) > 0,$$

for any sequence $(y_n)_{n\geqslant 0} \in B$ that converges to some y. Thus, by standard compactness arguments, we reach the conclusion that

$$\liminf_{n\to\infty} q_n = \liminf_{n\to\infty} \inf_{y\in B} q_n(y) > 0,$$

and Theorem 12 is proved. □

Acknowledgments

We thank Marni Mishna for motivating discussions. Many thanks also to Marc Peigné for his valuable comments and encouragement.

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