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# ON THE EXIT TIME FROM A CONE FOR RANDOM WALKS WITH DRIFT

RODOLPHE GARBIT AND KILIAN RASCHEL

ABSTRACT. In this article we consider  $d$ -dimensional random walks such that the law of the increments has all exponential moments. For a large class of cones, we compute the exponential decay of the probability for such random walks to stay in the cone up to time  $n$ , as  $n$  goes to infinity. We show that the latter equals the global minimum, on the dual cone, of the Laplace transform of the random walk increments. Our results find applications in the counting of walks in orthants, a classical domain in enumerative combinatorics.

## 1. INTRODUCTION AND MAIN RESULTS

1.1. **General context.** For general random processes in  $\mathbb{R}^d$ ,  $d \geq 1$  (including in particular Brownian motion and random walks), it is at once important and natural to study the first exit times  $\tau_K$  from certain domains  $K$ . Precisely, for discrete-time random processes  $(S_n)_{n \geq 0}$ ,  $\tau_K$  is defined by

$$(1) \quad \tau_K = \inf\{n \geq 1 : S_n \notin K\}.$$

Indeed, these random times carry many valuable informations on the process. As an example, the fruitful theory of random walks fluctuations (see, e.g., Spitzer [20]) is based on the analysis of the  $\tau_K$  for compact domains  $K$ .

In a recent past (1990 to present), the case of cones  $K$  has arisen a great interest in the mathematical community, due to interactions with many areas: First, certain random walks in conical domains can be treated with representation theory [2, 3] (in that case, the cones are Weyl chambers related to Lie algebras). Further, the exit times  $\tau_K$  are crucial to construct conditioned random walks in cones, which appear in the theory of quantum random walks [2, 3], random matrices [13], non-colliding random walks [8, 14], etc. In another direction, the probability

$$(2) \quad \mathbb{P}^x[\tau_K > n]$$

admits a direct combinatorial interpretation in terms of the number of walks starting from  $x$  and staying in the cone  $K$  up to time  $n$ . These counting numbers are particularly important in enumerative combinatorics [6, 15, 18], and are the topic of many recent studies. Our results in particular solve a conjecture on these numbers stated in [18].

For processes with no drift, the exit times  $\tau_K$  from cones are now well studied in the literature. The case of Brownian motion was solved by DeBlassie [7] (see also Bañuelos

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and Smits [1]): He showed that the probability (2) satisfies a certain partial differential equation, and he solved it in terms of hypergeometric functions. Concerning discrete-time random processes, in the one-dimensional case, the asymptotic behavior of the probability (2) is well known, as well as that of

$$(3) \quad \mathbb{P}^x[S_n = y, \tau_K > n]$$

(called a local limit theorem), thanks to the theory of fluctuations of random walks [20]. In higher dimension, some sporadic cases have first been analyzed: We may cite [11], for which there exists a strong underlying algebraic structure (certain reflexion groups are finite), or the case of Weyl chambers, which has been considered in [8, 14]. For more general cones, but essentially for random walks with increments having a finite support, Varopoulos [21] gave lower and upper bounds for the probability (2). The first author of the present article showed in [16] that for general random walks, the probability (3) does not decay exponentially fast. More recently, Denisov and Wachtel [9] provided the exact asymptotics for both (2) and (3).

For processes with drift, much less is known. Concerning Brownian motion, one of the first significant results is due to Biane, Bougerol and O'Connell [4], who derived the asymptotics of the non-exit probability (2) in the case of Weyl chambers of type A, when the drift is inside of the cone. Later on, by using different techniques, Puchała and Rolski [19] obtained (also in the context of Weyl chambers) the asymptotics of (2) without any hypothesis on the drift. As for random walks  $(S_n)_{n \geq 0}$  with increments having a law noted  $\mu$ , the exponential decay of (3) is known: It equals the global minimum on  $\mathbb{R}^d$  of the Laplace transform of  $\mu$

$$(4) \quad L_\mu(x) = \mathbb{E}_\mu[e^{\langle x, S_{n+1} - S_n \rangle}] = \int_{\mathbb{R}^d} e^{\langle x, y \rangle} \mu(dy).$$

This was first proved by Iglehart [17] for one-dimensional random walks. For more general walks, this was shown and used by many authors (see, e.g., [9, 16]). Regarding now the asymptotic behavior of the probability (2), the case  $d = 1$  is known, see [10].

It is the aim of this paper to give, for a broad class of random walks and cones in any dimension, the exponential decay of the non-exit probability (2). Our results are valid whatever the position of the drift with respect to the cone is. We shall also relate the exponential decay to the Laplace transform (4), by proving that it equals the minimum of the Laplace transform on the dual cone; we give the exact statement (Theorem 1) in Subsection 1.3.

A few days before completing the present work, we learnt about the article [12], written simultaneously and independently of ours, in which Duraj obtained the exact asymptotics of (2) and (3) for lattice random walks in cones with exponential moments, under the assumption that the minimum of the Laplace transform on the dual cone is reached in its interior. For that particular case, we easily find the exponential decay of the non-exit probability, see Subsection 2.3. In contrast with [12], our results are still valid when the minimum point lies on the boundary of the cone, which is the most interesting situation from the point of view of our analysis.

**1.2. Cones and random walks considered.** We will restrict ourselves to a certain class of cones, namely those which are finite intersections of half-spaces. Hence, we will assume that there exists a family of  $r \in \llbracket 1, d \rrbracket$  linearly independent vectors  $\mathcal{F} = \{f_1, \dots, f_r\}$  such that denoting  $\langle \cdot, \cdot \rangle$  the canonical inner product,

$$K = \{x \in \mathbb{R}^d : \langle x, f_i \rangle \geq 0, \forall i \in \llbracket 1, r \rrbracket\}.$$

Further, to each cone  $K$  we associate its dual cone

$$K^* = \{x \in \mathbb{R}^d : \langle x, y \rangle \geq 0, \forall y \in K\},$$

which turns out to be particularly relevant for our problem. We also set

$$K_\delta = K + \delta v,$$

where  $\delta \in \mathbb{R}$  and  $v$  is some fixed vector in  $K^o$ , the interior of  $K$ .

When  $r = d$  and  $\mathcal{F}$  is the canonical basis  $\{e_1, \dots, e_d\}$  of  $\mathbb{R}^d$ , the cone  $K$  is the positive orthant. In this case, we shall use the letter  $Q$  (corresponding to the quarter-plane when  $d = 2$ ) instead. In this case we have  $Q^* = Q$ .

Let  $(S_n)_{n \geq 0} = (S_n^{(1)}, \dots, S_n^{(d)})_{n \geq 0}$  be the canonical random walk on  $\mathbb{R}^d$ . Given any probability measure  $\mu$  on  $\mathbb{R}^d$  and  $x \in \mathbb{R}^d$ , we note  $\mathbb{P}_\mu^x$  the probability measure under which  $(S_n)_{n \geq 0}$  is a random walk started at  $x$  whose independent increments  $(S_{n+1} - S_n)_{n \geq 0}$  have common distribution  $\mu$ . Throughout this paper, we shall make the assumption that  $\mu$  is truly  $d$ -dimensional, that is:

(H1) The support of the probability measure  $\mu$  is not included in any linear hyperplane.

For a square-integrable probability measure  $\mu$  with mean  $m$  and variance-covariance matrix  $\Gamma$ , it is well-known that the minimal (with respect to inclusion) affine subspace  $A$  such that  $\mu(A) = 1$  is  $m + (\ker \Gamma)^\perp$ . Hence, the condition in (H1) holds if and only if  $m + (\ker \Gamma)^\perp$  is not included in any hyperplane (or equivalently, if and only if  $\Gamma$  is non-degenerate or  $\dim(\ker \Gamma) = 1$  and  $m \notin (\ker \Gamma)^\perp$ ). Notice that in the case where  $m = 0$ , (H1) is equivalent to  $\ker \Gamma = \{0\}$ , i.e.,  $\Gamma$  is non-degenerate. We will also assume that:

(H2) The probability measure  $\mu$  has all exponential moments, and its support is not included in any half-space  $u^- = \{x \in \mathbb{R}^d : \langle u, x \rangle \leq 0\}$  with  $u \in K^* \setminus \{0\}$ .

The reason of assuming (H2) is the following: To compute the exponential decay of the non-exit probability (2), our strategy is to perform a Cramér transformation (i.e., an exponential change of measure) with the help of the Laplace transform. Indeed, this will enable us to compare the non-exit probability (2) under  $\mu$  to (quite) the same probability under another probability measure  $\mu_0$  whose mean belongs to the cone; see Section 2 for details.

Under (H2), the Laplace transform (4) of  $\mu$  is defined and finite for all  $x \in \mathbb{R}^d$ . As it will be seen in Section 2, the geometric condition about the support of  $\mu$  in (H2) is equivalent to the fact that  $L_\mu$  reaches a global minimum on  $K^*$ .

**1.3. Main results.** We are now in position to state our main result:

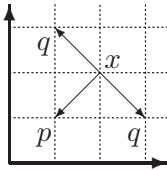


FIGURE 1. Random walks considered in Section 4 ( $p+2q = 1$ ), for different starting points  $x$

**Theorem 1.** *Suppose  $\mu$  satisfies (H1) and (H2). Then its Laplace transform (4) reaches a global minimum on  $K^*$  at a unique point  $x_0$ , and*

$$\lim_{n \rightarrow \infty} \mathbb{P}_\mu^x[\tau_K > n]^{1/n} = L_\mu(x_0),$$

for all  $x \in K_\delta$ , for some constant  $\delta \geq 0$ .

For a large class of random walks and cones, Theorem 1 gives the universal recipe to compute the exponential decay of the non-exit probability. Notice that the latter is independent of the starting point  $x$ . The theorem in itself does not provide any explicit value for  $\delta$ , but such a value can be found a posteriori thanks to the following:

**Proposition 2.** *The statement in Theorem 1 holds for any  $\delta \geq 0$  for which there exists  $n_0 \geq 1$  such that*

$$\mathbb{P}_\mu^0[\tau_{K-\delta} > n_0, S_{n_0} \in K^o] > 0.$$

If (H2) is not satisfied, then the exponential rate might depend on the starting point  $x$ . We shall illustrate this phenomenon in Section 4, with the walks in the quarter-plane having transition probabilities as in Figure 1.

**1.4. Plan of the paper.** The rest of our article has a simple organization: In Section 2 we prove Theorem 1. In Section 3 we present an important consequence of Theorem 1 in the counting of walks in orthants (a topical domain in enumerative combinatorics), see Corollaries 9 and 10. In Section 4 we consider the walks of Figure 1, for which we prove that contrary to the walks satisfying to hypothesis (H2), the exponential decay depends on the starting point  $x$ . Finally in Appendix A we prove the non-exponential decay of the non-exit probability for random walks with drift in the cone (a refinement of some theorems of [16]), which is needed for proving our main results.

## 2. LAPLACE TRANSFORM, CRAMÉR TRANSFORM AND PROOF OF THE MAIN RESULTS

In this section we prove Theorem 1 and Proposition 2, as follows: In Subsections 2.1 and 2.2, we review some elementary properties of the Laplace and Cramér transforms, which will be needed for proving our main results. In Subsection 2.3 we sketch the proof of Theorem 1 in a simple case, in order to give some intuition. Finally, in Subsection 2.4 we provide the full details of the general proof.

**2.1. Basic properties of the Laplace transform.** The Laplace transform of a probability distribution  $\mu$  is the function  $L_\mu$  defined for  $x \in \mathbb{R}^d$  by

$$L_\mu(x) = \int_{\mathbb{R}^d} e^{\langle x, y \rangle} \mu(dy).$$

The probability measure  $\mu$  is said to have all exponential moments if  $L_\mu(x)$  is finite for all  $x$ , or equivalently if

$$\int_{\mathbb{R}^d} e^{t|y|} \mu(dy)$$

is finite for all real  $t$  (above and throughout,  $|\cdot|$  stands for the norm associated with the inner product  $\langle \cdot, \cdot \rangle$ ). From now on, we assume that this condition is satisfied.

A direct application of Leibniz's rule for differentiation under the integral sign shows that  $L_\mu$  is then (infinitely) differentiable, and that its partial derivatives are given by

$$\frac{\partial L_\mu(x)}{\partial x_i} = \int_{\mathbb{R}^d} y_i e^{\langle x, y \rangle} \mu(dy), \quad \forall i \in \llbracket 1, d \rrbracket.$$

Therefore, the expectation of  $\mu$  is equal to the gradient of  $L_\mu$  at the origin

$$\mathbb{E}[\mu] = \nabla L_\mu(0),$$

and  $\mu$  is centered if and only if 0 is a critical point of  $L_\mu$ . The strict convexity of the exponential function ensures that

$$(5) \quad L_\mu(ax_1 + bx_2) \leq aL_\mu(x_1) + bL_\mu(x_2),$$

for all  $x_1 \neq x_2$  and  $a, b > 0$  with  $a + b = 1$ , and that equality occurs in (5) if and only if

$$\mu((x_1 - x_2)^\perp) = 1,$$

where  $(x_1 - x_2)^\perp$  denotes the hyperplane orthogonal to  $x_1 - x_2$ . Thus, if (H1) is satisfied, the equality in (5) never occurs and  $L_\mu$  is strictly convex.

Let  $C$  be a closed cone and  $\mathbb{S}^{d-1}$  denote the unit sphere of  $\mathbb{R}^d$ . Standard arguments involving the convexity of  $L_\mu$  and the compactness of  $C \cap \mathbb{S}^{d-1}$  show that  $L_\mu(x)$  goes to infinity as  $|x| \rightarrow \infty$  uniformly on  $C$  if and only if

$$(6) \quad \lim_{t \rightarrow \infty} L_\mu(tu) = \infty, \quad \forall u \in C \cap \mathbb{S}^{d-1}.$$

Hence, the condition (6) is sufficient for the existence of a global minimum on  $C$ . Indeed, if it is satisfied, then there exists  $R > 0$  such that  $L_\mu(x) \geq 1$  for all  $x \in C$  with  $|x| \geq R$ . By continuity,  $L_\mu$  reaches a minimum on  $\overline{B(0, R)} \cap C$  which is less than or equal to 1 (since  $L_\mu(0) = 1$ ), thus it is a global minimum on  $C$ .

The next lemma gives some interesting informations on the behavior at infinity of  $L_\mu$ . Recall that  $u^-$  denotes the half-space  $\{y \in \mathbb{R}^d : \langle u, y \rangle \leq 0\}$ .

**Lemma 3.** *Suppose that  $\mu$  has all exponential moments. The following dichotomy holds:*

(1) *If  $\mu(u^-) < 1$ , then*

$$\lim_{t \rightarrow \infty} L_\mu(x + tu) = \infty, \quad \forall x \in \mathbb{R}^d.$$

(2) If  $\mu(u^-) = 1$ , then

$$\lim_{t \rightarrow \infty} L_\mu(x + tu) = \int_{u^\perp} e^{\langle x, y \rangle} \mu(dy), \quad \forall x \in \mathbb{R}^d.$$

*Proof.* If  $\mu(u^-) < 1$ , then we can find  $\epsilon > 0$  such that the set  $\{y \in \mathbb{R}^d : \langle u, y \rangle \geq \epsilon\}$  has positive measure, and the inequality

$$L_\mu(x + tu) \geq \int_{\{y \in \mathbb{R}^d : \langle u, y \rangle \geq \epsilon\}} e^{\langle x+tu, y \rangle} \mu(dy) \geq e^{t\epsilon} \int_{\{y \in \mathbb{R}^d : \langle u, y \rangle \geq \epsilon\}} e^{\langle x, y \rangle} \mu(dy) \geq C e^{t\epsilon},$$

proves the first assertion of Lemma 3. Suppose now on the contrary that  $\mu(u^-) = 1$ . We then may write

$$L_\mu(x + tu) = \int_{u^\perp} e^{\langle x, y \rangle} \mu(dy) + \int_{\{y \in \mathbb{R}^d : \langle u, y \rangle < 0\}} e^{\langle x+tu, y \rangle} \mu(dy).$$

The second integral on the right-hand side of the above equation goes to zero as  $t$  goes to infinity by the dominated convergence theorem, thus proving the second assertion of Lemma 3.  $\square$

**Lemma 4.** *Suppose that  $\mu$  satisfies (H1) and has all exponential moments. Then the Laplace transform  $L_\mu$  has a global minimum on the closed convex cone  $C$  if and only if there does not exist any  $u \neq 0$  in  $C$  such that  $\mu(u^-) = 1$ .*

*Proof.* If  $\mu(u^-) < 1$  for all  $u \neq 0$  in  $C$ , then  $\lim_{t \rightarrow \infty} L_\mu(tu) = \infty$  by Lemma 3, and the function  $L_\mu$  has a global minimum on  $C$  as explained earlier. Now, suppose on the contrary that  $\mu(u^-) = 1$  for some  $u \neq 0$  in  $C$ . Then, by Lemma 3 again, the limit

$$h(x) := \lim_{t \rightarrow \infty} L_\mu(x + tu)$$

exists and is finite for all  $x$ . Since any convex cone is a semi-group,  $x + tu \in C$  for all  $x \in C$  and  $t \geq 0$ , and consequently

$$h(x) \geq \inf_C L_\mu, \quad \forall x \in C.$$

But our assumption that  $\mu$  satisfies (H1) implies that  $L_\mu$  is strictly convex, so that  $L_\mu(x) > h(x)$  for all  $x \in C$  (for else the strictly convex function  $t \mapsto L_\mu(x + tu)$  on  $[0, \infty[$  would not have a finite limit). We thus reach the conclusion that  $L_\mu(x) > \inf_C L_\mu$  for all  $x \in C$ , thereby proving that  $L_\mu$  has no global minimum on  $C$ .  $\square$

**Remark 5.** In view of Lemma 4, assuming (H1), we can now restate (H2) in terms of the Laplace transform of  $\mu$ : The probability measure  $\mu$  satisfies (H2) if and only if  $L_\mu$  is finite everywhere and has a global minimum on the dual cone  $K^*$ .

**2.2. Cramér transformation.** Given  $x_0 \in \mathbb{R}^d$ , let  $\mu_0$  denote the probability measure defined by

$$(7) \quad \mu_0(dy) = \frac{e^{\langle x_0, y \rangle}}{L_\mu(x_0)} \mu(dy).$$

Note that  $\mu$  and  $\mu_0$  are absolutely continuous relatively to each other, hence they have the same support, and  $\mu_0$  is also truly  $d$ -dimensional.

The Laplace transform of  $\mu_0$  is related to that of  $\mu$  by the formula

$$L_{\mu_0}(x) = \frac{L_\mu(x_0 + x)}{L_\mu(x_0)}, \quad \forall x \in \mathbb{R}^d.$$

Therefore, the expectation of  $\mu_0$  is given by

$$\mathbb{E}[\mu_0] = \nabla L_{\mu_0}(0) = \frac{\nabla L_\mu(x_0)}{L_\mu(x_0)}.$$

Hence, if  $L_\mu$  has a local minimum at  $x_0$ , then  $\nabla L_\mu(x_0) = 0$  and  $\mu_0$  is centered.

The link between probabilities involving random walks with increments distributions  $\mu$  and  $\mu_0$  is given by the following:

**Lemma 6** (Cramér's formula). *For any measurable and positive function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$ , we have*

$$\mathbb{E}_\mu^x[F(S_1, \dots, S_n)] = L_\mu(x_0)^n e^{\langle x_0, x \rangle} \mathbb{E}_{\mu_0}^x[e^{-\langle x_0, S_n \rangle} F(S_1, \dots, S_n)].$$

*Proof.* It follows directly from the definition (7) of  $\mu_0$  that

$$\mu^{\otimes n}(dy_1, \dots, dy_n) = L_\mu(x_0)^n e^{-\langle x_0, \sum_{i=1}^n y_i \rangle} \mu_0^{\otimes n}(dy_1, \dots, dy_n).$$

The conclusion is then straightforward.  $\square$

Applied to the function  $F(s_1, \dots, s_n) = \prod_{i=1}^n \mathbb{1}_K(s_i)$ , Cramér's formula reads

$$(8) \quad \mathbb{P}_\mu^x[\tau_K > n] = L_\mu(x_0)^n e^{\langle x_0, x \rangle} \mathbb{E}_{\mu_0}^x[e^{-\langle x_0, S_n \rangle}, \tau_K > n],$$

and it is immediate that

$$\mathbb{P}_\mu^x[\tau_K > n] \leq L_\mu(x_0)^n e^{\langle x_0, x \rangle},$$

for all  $x_0 \in K^*$ , since  $\langle x_0, S_n \rangle \geq 0$  for such  $x_0$  and  $S_n \in K$ . Hence, the infimum of the Laplace transform on  $K^*$  is always an upper bound of the exponential rate, i.e.,

$$(9) \quad \limsup_{n \rightarrow \infty} \mathbb{P}_\mu^x[\tau_K > n]^{1/n} \leq \inf_{y \in K^*} L_\mu(y),$$

for all  $x \in \mathbb{R}^d$ .

**2.3. Main idea of the proof.** In order to give the basic idea of the proof of Theorem 1, we shall now prove it in the simple case where  $L_\mu$  reaches a global minimum on  $K^*$  at some interior point  $x_0 \in (K^*)^\circ$ . In that case,  $x_0$  is a local minimum point, therefore  $\mu_0$  is centered, non-degenerate (since it satisfies (H1)), and it follows from [16, Theorem] that

$$\lim_{n \rightarrow \infty} \mathbb{P}_{\mu_0}^x[|S_n| \leq \sqrt{n}, \tau_K > n]^{1/n} = 1$$

for all  $x \in K_\delta$ , for some  $\delta \geq 0$ . Now, using (8), we get the lower bound

$$\mathbb{P}_\mu^x[\tau_K > n] \geq L_\mu(x_0)^n e^{\langle x_0, x \rangle} e^{-|x_0| \sqrt{n}} \mathbb{P}_{\mu_0}^x[|S_n| \leq \sqrt{n}, \tau_K > n],$$

and the conclusion follows from these two last equations.

To handle the case where  $L_\mu$  reaches a global minimum on  $K^*$  at some boundary point  $x_0$ , we will use a combination of Cramér's transformation at  $x_0$  and some refinement of [16, Theorem] for which a proof is given in Appendix A.



**2.4. Proofs of Theorem 1 and Proposition 2.** We will first prove the theorem in the case  $K = Q$ , and then derive the general result via an appropriate linear transformation. Let us begin with the following result, which gives the position of the drift of the Cramér transform performed at the minimum on  $Q$  of the Laplace transform.

**Lemma 7.** *Suppose  $\mu$  satisfies (H1) and (H2) with  $K = Q$ . Then its Laplace transform reaches a global minimum in  $Q$  at a unique point  $x_0$ . In addition, the gradient  $\nabla L_\mu(x_0)$  belongs to  $Q$  and*

$$\frac{\partial L_\mu}{\partial x_i}(x_0) = 0$$

for all  $i$  such that  $\langle x_0, e_i \rangle > 0$ .

*Proof.* As seen in Lemma 4, it follows from (H2) that  $L_\mu$  has a global minimum on  $Q$ . The uniqueness of the global minimum point  $x_0$  is an easy consequence of the strict convexity of  $L_\mu$  which is due to (H1).

Let  $i \in \llbracket 1, d \rrbracket$  and set  $x_i = \langle x_0, e_i \rangle$ . Since  $x_0 \in Q$ , the whole half-line  $\{x_0 + te_i : t \geq -x_i\}$  is included in  $Q$ , and therefore the function

$$t \in [-x_i, \infty) \mapsto f_i(t) = L_\mu(x_0 + te_i)$$

reaches its minimum at  $t = 0$ . If  $x_i > 0$ , this implies that  $f'_i(0) = 0$ . If  $x_i = 0$ , the point  $t = 0$  may not be critical for  $f_i$  but at least it must satisfy  $f'_i(0) \geq 0$ . The conclusion follows since  $f'_i(0) = \frac{\partial L_\mu(x_0)}{\partial x_i}$ .  $\square$

*Proof of Theorem 1.* Let  $x_0$  be the global minimum point of  $L_\mu$  in  $Q$  as given in Lemma 7. As already observed,  $L_\mu(x_0)$  is an upper bound for  $\limsup_{n \rightarrow \infty} \mathbb{P}_\mu^x[\tau_Q > n]^{1/n}$ , and it remains to prove that it is also the right lower bound. By performing the Cramér transformation at  $x_0$ , we get

$$(10) \quad \mathbb{P}_\mu^x[\tau_Q > n] = L_\mu(x_0)^n e^{\langle x_0, x \rangle} \mathbb{E}_{\mu_0}^x[e^{-\langle x_0, S_n \rangle}, \tau_Q > n],$$

where  $\mu_0(dy) = L_\mu(x_0)^{-1} e^{\langle x_0, y \rangle} \mu(dy)$  is truly  $d$ -dimensional with mean

$$m_0 = \frac{\nabla L_\mu(x_0)}{L_\mu(x_0)} \in Q.$$

Define  $I = \{i \in \llbracket 1, d \rrbracket : x_i = \langle x_0, e_i \rangle > 0\}$ . Since

$$|\langle x_0, S_n \rangle| = \left| \sum_{i \in I} x_i S_n^{(i)} \right| \leq C \sqrt{n},$$

where  $C = \sum_{i \in I} x_i$ , as soon as  $|S_n^{(i)}| \leq \sqrt{n}$  for all  $i \in I$ , we obtain the lower bound

$$\mathbb{E}_{\mu_0}^x[e^{-\langle x_0, S_n \rangle}, \tau_Q > n] \geq e^{-C\sqrt{n}} \mathbb{P}_{\mu_0}^x[|S_n^{(i)}| \leq \sqrt{n}, \forall i \in I, \tau_Q > n].$$

Now, since  $m_0 \in Q$  and  $\mathbb{E}_{\mu_0}^0[S_n^{(i)}] = m_0^{(i)} = 0$  for all  $i \in I$ , it follows from Theorem 13 in Appendix A that for any  $e \in Q^o$  there exists a constant  $\delta \geq 0$  such that

$$\lim_{n \rightarrow \infty} \mathbb{P}_{\mu_0}^x[|S_n^{(i)}| \leq \sqrt{n}, \forall i \in I, \tau_Q > n]^{1/n} = 1,$$

for all  $x \in Q_\delta(e) = Q + \delta e$ . Hence, going backward to (10), we reach the conclusion that

$$\liminf_{n \rightarrow \infty} \mathbb{P}_\mu^x[\tau_Q > n]^{1/n} \geq L_\mu(x_0),$$

for all  $x \in Q_\delta(e)$ , and the theorem is proved for  $K = Q$ .

We now turn to the general case where

$$K = \{x \in \mathbb{R}^d : \langle x, f_i \rangle \geq 0, \forall i \in \llbracket 1, r \rrbracket\}$$

for some family  $\mathcal{F} = \{f_1, \dots, f_r\}$  of  $r \in \llbracket 1, d \rrbracket$  linearly independent vectors. To this family, we add vectors  $f_{r+1}, \dots, f_d$  so that  $\{f_1, \dots, f_d\}$  forms a basis of  $\mathbb{R}^d$ ; we denote by  $N$  the linear transformation that maps  $e_i$  to  $f_i$  for all  $i \in \llbracket 1, d \rrbracket$  and by  $M = N^\top$  its transpose. Some basic linear algebra computations show that

$$(11) \quad MK = Q^r \times \mathbb{R}^{d-r},$$

where  $Q^r$  denotes the positive orthant of  $\mathbb{R}^r$ . Therefore,

$$\mathbb{P}_\mu^x[\tau_K > n] = \mathbb{P}_\mu^{Mx}[MS_k \in MK, \forall k \in \llbracket 1, n \rrbracket] = \mathbb{P}_\mu^{Mx}[\tilde{S}_k \in Q^r, \forall k \in \llbracket 1, n \rrbracket],$$

where  $\tilde{S}_k = \Pi_r(MS_k)$  and where  $\Pi_r$  is the projection onto  $\mathbb{R}^r$ , defined by  $\Pi_r(x_1, \dots, x_d) = (x_1, \dots, x_r)$ . Since the process  $(\tilde{S}_n)_{n \geq 0}$  is an  $r$ -dimensional random walk whose increments have common distribution

$$\nu(dy) = \mu(\Pi_r M \in dy),$$

we may finally write

$$(12) \quad \mathbb{P}_\mu^x[\tau_K > n] = \mathbb{P}_\nu^y[\tau_{Q^r} > n],$$

where  $y = \Pi_r Mx$ . We shall now apply the result obtained in the first part of the proof to our  $r$ -dimensional random walk. It is clear that  $\nu$  is truly  $r$ -dimensional and has all exponential moments. Noting that  $(MK)^* = (M^\top)^{-1}K^*$  and  $(Q^r \times \mathbb{R}^{d-r})^* = Q^r \times \{0\}^{d-r}$ , we obtain from (11) that

$$(13) \quad K^* = M^\top(Q^r \times \{0\}^{d-r}).$$

In addition, we observe that for any  $w \in Q^r \times \{0\}^{d-r}$  and  $v = \Pi_r(w)$ , we have the equality  $(\Pi_r M)^{-1}(v^-) = (M^\top w)^-$ . Thus, if there were some  $v \in Q^r$  such that  $\nu(v^-) = 1$ , we would also have  $1 = \mu((\Pi_r M)^{-1}(v^-)) = \mu((M^\top w)^-) = \mu(u^-)$  for  $u = M^\top w \in K^*$ , and this would contradict (H2). Therefore, the probability measure  $\nu$  also satisfies (H2) (with respect to the cone  $Q^r$ ) and we conclude that for any  $e \in (Q^r)^\circ$  there exists  $\delta \geq 0$  such that

$$(14) \quad \lim_{n \rightarrow \infty} \mathbb{P}_\nu^y[\tau_{Q^r} > n]^{1/n} = \min_{Q^r} L_\nu,$$

for all  $y \in Q_\delta^r(e)$ . Another simple computation shows that for all  $x \in Q^r \times \{0\}^{d-r}$ ,

$$L_\nu(\Pi_r(x)) = L_\mu(M^\top x),$$

and hence the equality

$$(15) \quad \min_{Q^r} L_\nu = \min_{K^*} L_\mu$$

follows from (13). The combination of Equations (12), (14) and (15) implies

$$\lim_{n \rightarrow \infty} \mathbb{P}_\mu^x[\tau_K > n]^{1/n} = \min_{K^*} L_\mu,$$

for all  $x$  such that  $y = \Pi_r Mx \in Q_\delta^r(e)$ ; but thanks to (11), for any fixed  $v \in K^o$ ,  $e = \Pi_r Mv$  belongs to  $(Q^r)^o$  and the condition  $\Pi_r Mx \in Q_\delta^r(e)$  is equivalent to  $x \in K + \delta v$ . This completes the proof of Theorem 1.  $\square$

**Remark 8.** If the drift of the random walk belongs to the cone  $K$ , the minimum of the Laplace transform on  $K^*$  is at the origin of the cone:

$$\lim_{n \rightarrow \infty} \mathbb{P}_\mu^x[\tau_K > n]^{1/n} = 1 = L_\mu(0) = \min_{K^*} L_\mu.$$

On the other hand, when the drift is not inside the cone  $K$ , there is no direct link between the location of the minimum  $x_0$  and the location of the drift.

To conclude this section, let us explain how to find a  $\delta \geq 0$  for which the statement of Theorem 1 holds.

*Proof of Proposition 2.* Recall that  $v \in K^o$  is fixed and  $K_\delta = K + \delta v$ . We assume that there exist  $\delta \geq 0$  and  $k \geq 1$  such that

$$\mathbb{P}_\mu^0[\tau_{K-\delta} > k, S_k \in K^o] > 0.$$

Therefore, we can find  $\epsilon > 0$  such that

$$\mathbb{P}_\mu^0[\tau_{K-\delta} > k, S_k \in K_\epsilon] = \gamma > 0,$$

and since  $K$  is a convex cone, it satisfies the relation  $K + K \subset K$ , thus

$$\mathbb{P}_\mu^x[\tau_K > k, S_k - x \in K_\epsilon] \geq \gamma,$$

for all  $x \in K_\delta$  (by inclusion of events). From this, we shall deduce by induction that

$$(16) \quad p_\ell := \mathbb{P}_\mu^x[\tau_K > \ell k, S_{\ell k} - x \in K_{\ell\epsilon}] \geq \gamma^\ell,$$

for all  $\ell \geq 1$  and  $x \in K_\delta$ . Indeed, by the Markov property of the random walk,

$$\begin{aligned} p_{\ell+1} &\geq \mathbb{E}_\mu^x[\tau_K > \ell k, S_{\ell k} - x \in K_{\ell\epsilon}, \mathbb{P}_\mu^{S_{\ell k}}[\tau_K > k, S_k - x \in K_{(\ell+1)\epsilon}]] \\ &\geq p_\ell \cdot \inf_y \mathbb{P}_\mu^y[\tau_K > k, S_k - x \in K_{(\ell+1)\epsilon}], \end{aligned}$$

where the infimum is taken over all  $y \in K$  such that  $y - x \in K_{\ell\epsilon}$ . Noting that  $y - x \in K_{\ell\epsilon}$  and  $S_k - y \in K_\epsilon$  imply  $S_k - x \in K_{(\ell+1)\epsilon}$ , we obtain

$$p_{\ell+1} \geq p_\ell \cdot \inf_y \mathbb{P}_\mu^y[\tau_K > k, S_k - y \in K_\epsilon].$$

But  $x \in K_\delta$  and  $y - x \in K_{\ell\epsilon} \subset K$  imply  $y \in K_\delta$ . Hence  $p_{\ell+1} \geq p_\ell \cdot \gamma$  and (16) is proved.

Now Theorem 1 asserts the existence of some  $\delta_0 \geq 0$  such that

$$\lim_{n \rightarrow \infty} \mathbb{P}_\mu^y[\tau_K > n]^{1/n} = L_\mu(x_0),$$

for all  $y \in K_{\delta_0}$ , and we want to prove that the result also holds for  $x \in K_\delta$ . To do this, we shall simply use (16) in order to push the walk from  $K_\delta$  to  $K_{\delta_0}$ . More precisely, choose

$\ell \geq 1$  such that  $\delta + \ell\epsilon \geq \delta_0$ . Then for all  $x \in K_\delta$  the inclusion  $x + K_{\ell\epsilon} \subset K_{\delta_0}$  holds, and thanks to (16),

$$\mathbb{P}_\mu^x[\tau_K > m, S_m \in K_{\delta_0}] \geq \gamma^\ell,$$

for  $m = k\ell$ . By the Markov property, for all  $n \geq m$ , we have

$$\begin{aligned} \mathbb{P}_\mu^x[\tau_K > n] &\geq \mathbb{E}_\mu^x[\tau_K > m, S_m \in K_{\delta_0}, \mathbb{P}_\mu^{S_m}[\tau_K > n - m]] \\ &\geq \gamma^\ell \cdot \inf_{y \in K_{\delta_0}} \mathbb{P}_\mu^y[\tau_K > n - m] \\ &\geq \gamma^\ell \cdot \mathbb{P}_\mu^{\delta_0 v}[\tau_K > n - m], \end{aligned}$$

where the last inequality follows by inclusion of events. This implies immediately

$$\liminf_{n \rightarrow \infty} \mathbb{P}_\mu^x[\tau_K > n]^{1/n} \geq L_\mu(x_0).$$

But since the inequality

$$\limsup_{n \rightarrow \infty} \mathbb{P}_\mu^x[\tau_K > n]^{1/n} \leq L_\mu(x_0)$$

holds for all  $x$  (see (9)), Proposition 2 is proved.  $\square$

### 3. APPLICATION TO LATTICE PATH ENUMERATION

In this section we present an application of our main result (Theorem 1) in enumerative combinatorics: Given a finite set  $\mathfrak{S}$  of allowed steps, a now classical problem is to study  $\mathfrak{S}$ -walks in the orthant  $Q$ , that is walks confined to  $Q$ , starting at a fixed point  $x$  (often the origin) and using steps in  $\mathfrak{S}$  only. Denote by  $f_\mathfrak{S}(x, y; n)$  the number of such walks that end at  $y$  and use exactly  $n$  steps. Many properties of the counting numbers  $f_\mathfrak{S}(x, y; n)$  have been recently analyzed (the seminal work in this area is [6]). First, exact properties of them were derived, via the study of their generating function (exact expression and algebraic nature). Such properties are now well established for the case of small steps walks in the quarter-plane, meaning that the step set  $\mathfrak{S}$  is included in  $\{0, \pm 1\}^2$ . More qualitative properties of the  $f_\mathfrak{S}(x, y; n)$  were also investigated, such as the asymptotic behavior, as  $n \rightarrow \infty$ , of the number of excursions  $f_\mathfrak{S}(x, y; n)$  for fixed  $y$ , or that of the total number of walks,

$$(17) \quad f_\mathfrak{S}(x; n) = \sum_{y \in Q} f_\mathfrak{S}(x, y; n).$$

Concerning the excursions, several small steps cases have been treated by Bousquet-Mélou and Mishna [6] and by Fayolle and Raschel [15]. Later on, Denisov and Wachtel [9] obtained the very precise asymptotics of the excursions, for a quite large class of step sets and cones. As for the total number of walks (17), only very particular cases are solved, see again [6, 15]. In a most recent work [18], Johnson, Mishna and Yeats obtained an upper bound for the exponential growth constant, namely,

$$\limsup_{n \rightarrow \infty} f_\mathfrak{S}(x; n)^{1/n},$$

and proved by comparison with results of [15] that these bounds are tight for all small steps models in the quarter-plane. In the present article, we find the exponential growth constant of the total number of walks (17) in any dimension for any model such that:

- (H1') The step set  $\mathfrak{S}$  is not included in a linear hyperplane;  
(H2') The step set  $\mathfrak{S}$  is not included in a half-space  $u^-$ , with  $u \in Q \setminus \{0\}$ .

In the sequel we shall say that a step set  $\mathfrak{S}$  is proper if it satisfies to (H1') and (H2'). Note in particular that the well-known 79 models of walks in the quarter-plane studied in [6, 15] (including the so-called 5 singular walks) satisfy both hypotheses above.

**Corollary 9.** *Let  $\mathfrak{S}$  be any proper step set. The Laplace transform of  $\mathfrak{S}$ ,*

$$L_{\mathfrak{S}}(x) = \sum_{s \in \mathfrak{S}} e^{\langle x, s \rangle},$$

*reaches a global minimum on  $Q$  at a unique point  $x_0$ , and there exists  $\delta \geq 0$  such that for any starting point  $x \in Q_\delta$ ,*

$$\lim_{n \rightarrow \infty} f_{\mathfrak{S}}(x; n)^{1/n} = L_{\mathfrak{S}}(x_0).$$

Suppose that in addition:

- (H3') The step set allows a path staying in  $Q$  from the origin to some point in the interior of  $Q$ .

Then it follows from Proposition 2 that the result in Corollary 9 holds with  $\delta = 0$ , i.e., it is valid for all  $x \in Q$ . Note that this assumption is not restrictive from a combinatorial point of view, since if (H3') is not satisfied, the counting problem is obvious.

*Proof of Corollary 9.* Consider a random walk  $(S_n)_{n \geq 0}$  starting from  $x$  such that  $\mu$  is the uniform law on  $\mathfrak{S}$ . Let then  $\tau_Q$  denote the first exit time from  $Q$ . The enumeration problem is related to probabilities in a simple way:

$$(18) \quad \mathbb{P}_\mu^x[\tau_Q > n] = \frac{f_{\mathfrak{S}}(x; n)}{|\mathfrak{S}|^n}.$$

Further, it is immediate from our definitions that  $L_{\mathfrak{S}}(x) = |\mathfrak{S}| L_\mu(x)$ . Corollary 9 then follows from Theorem 1 and from the fact that  $Q^* = Q$ .  $\square$

As a consequence, we obtain the following result, which was conjectured in [18]:

**Corollary 10.** *Let  $\mathfrak{S} \subset \mathbb{Z}^d$  be a proper step set (hypotheses (H1') and (H2')), which besides satisfies (H3'), and let  $K_{\mathfrak{S}}$  be the growth constant for the total number of walks (17). Let  $\mathcal{P}$  be the set of hyperplanes through the origin in  $\mathbb{R}^d$  which do not meet the interior of the first orthant. Given  $p \in \mathcal{P}$ , let  $K_{\mathfrak{S}}(p)$  be the growth constant of the walks on  $\mathfrak{S}$  which are restricted to the side of  $p$  which includes the first orthant. Then  $K_{\mathfrak{S}} = \min_{p \in \mathcal{P}} K_{\mathfrak{S}}(p)$ .*

*Proof.* Let us first notice that  $\mathcal{P}$  can be described as the set of hyperplanes  $u^\perp$  such that  $u \in Q \cap \mathbb{S}^{d-1}$ , and that the side of  $p = u^\perp$  which includes the first orthant is then the half-space  $u^+ = \{x \in \mathbb{R}^d : \langle x, u \rangle \geq 0\}$ . By Theorem 1, the exponential rate for the random walk associated to the step set  $\mathfrak{S}$  and confined to  $u^+$  is the minimum of  $L_{\mathfrak{S}}/|\mathfrak{S}|$  on the dual cone  $(u^+)^* = \{tu : t \geq 0\}$ . Therefore, the growth constant  $K_{\mathfrak{S}}(p)$  equals  $\min_{t \geq 0} L_{\mathfrak{S}}(tu)$ , and the equality

$$\min_{x \in Q} L_{\mathfrak{S}}(x) = \min_{u \in Q \cap \mathbb{S}^{d-1}} \min_{t \geq 0} L_{\mathfrak{S}}(tu)$$

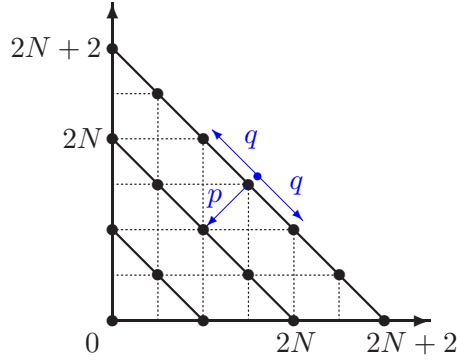


FIGURE 2. Random walks considered in the proof of Proposition 11 on the lines  $\{(i, j) \in Q : i + j = 2N\}$  for  $N \geq 0$

immediately translates into

$$K_{\mathfrak{C}} = \min_{p \in \mathcal{P}} K_{\mathfrak{C}}(p).$$

The proof of Corollary 10 is completed.  $\square$

Note the Corollary 10 admits a direct probabilistic analogue: In words, the exponential decay of the non-exit probability from the orthant is equal to the minimum of the exponential decay of the non-exit probability from the half-spaces containing the orthant.

#### 4. HALF-SPACE WALKS

In this section we illustrate the following phenomenon: If the support of the random walk is included in a certain half-space (chosen so as contradicting (H2)), such a universal result as Theorem 1 does not hold; in particular, the exponential decay of the non-exit probability may depend on the starting point.

Let  $(S_n)_{n \geq 0}$  be the random walk on  $Q$  starting at  $x$  and with transition probabilities to  $(1, -1)$ ,  $(-1, 1)$  and  $(-1, -1)$ , with respective probabilities  $q$ ,  $q$  and  $p$ , where  $p + 2q = 1$ , see Figures 1 and 2. Let  $\tau_Q$  be the exit time (1) of this random walk from the quarter-plane  $Q$ . Finally, define for fixed  $N$  the segment  $D_{2N} = \{(i, j) \in Q : i + j = 2N\}$ .

**Proposition 11.** *For any  $N \geq 1$  and any  $x \in D_{2N}$ , we have*

$$(19) \quad \lim_{n \rightarrow \infty} \mathbb{P}^x[\tau_Q > n]^{1/n} = 2q \cos\left(\frac{\pi}{2N+2}\right).$$

We shall need the following result on the simple symmetric random walk on  $\mathbb{Z}$  (we refer to [20, page 243] for a proof):

**Lemma 12** ([20]). *For the simple symmetric random walk  $(\tilde{S}_n)_{n \geq 0}$  on  $\mathbb{Z}$  (with jumps to the left and to the right with equal probabilities  $1/2$ ), we have, for any  $x \in \llbracket 0, 2N \rrbracket$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{P}^x[\tilde{S}_1, \dots, \tilde{S}_n \in \llbracket 0, 2N \rrbracket]^{1/n} = \cos\left(\frac{\pi}{2N+2}\right).$$

*Proof of Proposition 11.* We shall prove Proposition 11 by induction over  $N \geq 1$ . For  $N = 1$ , we have three choices for  $x$  (see Figure 2). We write the proof when  $x = (1, 1)$ , since the arguments for other values of  $x$  are quite similar. For this choice of  $x$ , the origin  $(0, 0)$  can be reached only at odd times  $n$ , and in that event, the random walk gets out of  $Q$  at time  $n + 1$ . From this simple remark we deduce that (below, we note  $(X_k)_{k \geq 1}$  the increments of the random walk  $(S_n)_{n \geq 0}$ )

$$\begin{aligned} \mathbb{P}^x[\tau_Q > 2n] &= \mathbb{P}^x[\tau_Q > 2n, X_k \neq (-1, -1), \forall k \in \llbracket 1, n \rrbracket] \\ &= \mathbb{P}^x[\tau_Q > 2n | X_k \neq (-1, -1), \forall k \in \llbracket 1, n \rrbracket](2q)^n. \end{aligned}$$

Further, the random walk conditioned on never making the jump  $(-1, -1)$  is a simple symmetric random walk on the segment  $D_2$ . Therefore,

$$\mathbb{P}^x[\tau_Q > 2n] = \mathbb{P}^1[\tilde{S}_1, \dots, \tilde{S}_n \in \llbracket 0, 2 \rrbracket](2q)^{2n}.$$

Using Lemma 12, we conclude that for  $x = (1, 1)$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}^x[\tau_Q > 2n]^{1/(2n)} = 2q \cos\left(\frac{\pi}{4}\right).$$

The fact that  $\mathbb{P}^x[\tau > n]$  is decreasing in  $n$  implies that the above equation holds with  $2n + 1$  instead of  $2n$ . This achieves the proof of Proposition 11 for  $N = 1$ .

Let us now assume that Equation (19) holds for a fixed value of  $N \geq 1$ . For  $x \in D_{2N+2}$ , introduce

$$H = \inf\{n > 0 : S_n \in D_{2N}\}$$

the hitting time of the set  $D_{2N}$ , see Figure 2. We can write

$$(20) \quad \mathbb{P}^x[\tau_Q > n] = \mathbb{P}^x[\tau_Q > n, H > n] + \sum_{k=1}^n \mathbb{P}^x[\tau_Q > n, H = k].$$

The first term in the right-hand side of (20) can be written as

$$\mathbb{P}^x[\tau_Q > n, H > n] = \mathbb{P}^x[\tau_Q > n | H > n](2q)^n,$$

where (for the same reasons as for the case  $N = 1$ )

$$\lim_{n \rightarrow \infty} \mathbb{P}^x[\tau_Q > n | H > n]^{1/n} = \cos\left(\frac{\pi}{2N+4}\right).$$

As for the second term in the right-hand side of (20),

$$\begin{aligned} \mathbb{P}^x[\tau_Q > n, H = k] &= \mathbb{E}^x[\tau_Q > k, K = k, \mathbb{P}^{S_k}[\tau_Q > n - k]] \\ &\leq C \cdot \mathbb{P}^x[\tau_Q > k, H = k] \mathbb{P}^{x_0}[\tau_Q > n - k] \\ &\leq C \cdot \mathbb{P}^x[\tau_Q > k - 1, H > k - 1] \mathbb{P}^{x_0}[\tau_Q > n - k] \\ &:= C a_k b_{n-k}. \end{aligned}$$

The first equality above comes from the strong Markov property. The first inequality follows from the fact that for any fixed  $x_0 \in D_{2N}$ , there exists a constant  $C > 0$  such that, for any  $n \geq 0$  and any  $y$ ,  $\mathbb{P}^y[\tau_Q > n] \leq C \mathbb{P}^{x_0}[\tau_Q > n]$ . The second inequality is obvious, and the last line has to be read as a definition.

Using on the one hand the same reasoning as for the case  $N = 1$ , and on the other hand the induction hypothesis, we obtain

$$\lim_{n \rightarrow \infty} a_n^{1/n} = 2q \cos\left(\frac{\pi}{2N+4}\right) > 2q \cos\left(\frac{\pi}{2N+2}\right) = \lim_{n \rightarrow \infty} b_n^{1/n}.$$

Standard properties of the Cauchy product then lead to

$$\limsup_{n \rightarrow \infty} \left( \sum_{k=1}^n a_k b_{n-k} \right)^{1/n} \leq 2q \cos\left(\frac{\pi}{2N+4}\right).$$

To summarize, with the help of (20) we have written  $\mathbb{P}^x[\tau_Q > n] = A_n + B_n$ , where

$$\lim_{n \rightarrow \infty} A_n^{1/n} = 2q \cos\left(\frac{\pi}{2N+4}\right), \quad \limsup_{n \rightarrow \infty} B_n^{1/n} \leq 2q \cos\left(\frac{\pi}{2N+4}\right).$$

The formula (19) therefore holds for  $N + 1$ , and Proposition 11 is proved.  $\square$

#### APPENDIX A. NON-EXPONENTIAL DECAY OF THE NON-EXIT PROBABILITY FOR RANDOM WALKS WITH DRIFT IN THE CONE

In this appendix we refine a result announced in [16] concerning the non-exponential decay of the non-exit probability (2) from a cone for a centered, square-integrable and non-degenerate multi-dimensional random walk. We prove that the result in [16] still holds if the random walk has a drift in the cone and if the hypothesis that its variance-covariance matrix is non-degenerate is weakened to (H1). In the sequel, we only assume that  $K$  is a closed convex cone with non-empty interior  $K^\circ$ . As before, we fix some  $v \in K^\circ$  and define  $K_\delta = K + \delta v$ . We shall prove the following result:

**Theorem 13.** *Assume that the distribution of the random walk increments is square-integrable and satisfies (H1). Let  $I$  be the set of indices  $i \in \llbracket 1, d \rrbracket$  such that  $\mathbb{E}^0[S_1^{(i)}] = 0$ . If the drift  $m = \mathbb{E}^0[S_1]$  belongs to the cone  $K$ , there exists  $\delta \geq 0$  such that for all  $x \in K_\delta$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{P}^x[\tau_K > n, |S_n^{(i)}| \leq \sqrt{n}, \forall i \in I]^{1/n} = 1.$$

If  $m = 0$ , as we have already explained, (H1) is equivalent to the fact that the variance-covariance matrix of the increments distribution is non-degenerate. Hence, we exactly recover [16, Theorem]. However, if  $m \neq 0$ , Theorem 13 can not be derived from [16, Theorem]. Indeed, under the hypothesis of Theorem 13, it is clear that

$$\mathbb{P}^x[\tau_K > n, |S_n^{(i)}| \leq \sqrt{n}, \forall i \in I] \geq \mathbb{P}^x[\tau_K(\tilde{S}) > n, |\tilde{S}_n^{(i)}| \leq \sqrt{n}, \forall i \in I],$$

where  $(\tilde{S}_n = S_n - nm)_{n \geq 0}$  is the centered random walk associated to  $(S_n)_{n \geq 0}$ . But the variance-covariance matrix of  $\tilde{S}_1$  is equal to that of  $S_1$  and might be degenerate, so that [16, Theorem] would not apply to the walk  $(\tilde{S}_n)_{n \geq 0}$ . This is for example the case when  $(S_n)_{n \geq 0}$  is the uniform two-dimensional random walk with step set  $\mathfrak{S} = \{(0, 1), (1, 0)\}$ .

In order to prove Theorem 13, we will need a series of lemmas. We begin with some geometric considerations.

**Lemma 14.** *Let  $m$  be a point in  $K$  and  $V$  be a linear subspace of  $\mathbb{R}^d$ . If  $(m+V) \cap K^\circ = \emptyset$ , then  $m+V$  is included in a hyperplane.*



*Proof.* Clearly,  $\dim V < d$ , for else  $m + V = \mathbb{R}^d$  would intersect  $K^o$ . Since  $m + V$  is included in the linear subspace generated by  $m$  and  $V$ , whose dimension is less or equal to  $\dim V + 1$ , the affine space  $m + V$  is always included in a hyperplane if  $\dim V < d - 1$ . Thus, it remains to consider the case where  $V$  is a hyperplane, that is,

$$V = u^\perp = \{x \in \mathbb{R}^d : \langle u, x \rangle = 0\}$$

for some  $u \neq 0$ . Assume  $m \notin V$ . Possibly changing  $u$  to  $-u$ , we can assume in addition that  $\langle m, u \rangle > 0$ . Then,

$$\bigcup_{\lambda > 0} (\lambda m + V) = \bigcup_{\lambda > 0} (\lambda u + V) = \{x \in \mathbb{R}^d : \langle u, x \rangle > 0\} =: u_*^+.$$

Further, by homogeneity of  $V$  and  $K^o$ , we have  $(\lambda m + V) \cap K^o = \emptyset$  for every  $\lambda > 0$ . Hence,  $K^o$  does not intersect  $u_*^+$ , and is therefore included in  $u^- = \{x \in \mathbb{R}^d : \langle u, x \rangle \leq 0\}$ . This is a contradiction since  $m \in K = \overline{(K^o)}$  (this equality holds for any convex set with non-empty interior) and  $\langle m, u \rangle > 0$ . Thus  $m$  belongs to  $V$  and  $m + V = V$  is a hyperplane.  $\square$

**Lemma 15.** *Let  $Y \in \mathbb{R}^d$  be a random vector with gaussian distribution  $\mathcal{N}(m, \Gamma)$ . If  $m$  belongs to  $K$  and if  $m + (\ker \Gamma)^\perp$  is not included in a hyperplane, then*

$$\mathbb{P}[Y \in K^o] > 0.$$

*Proof.* It is well known that the gaussian distribution  $\mathcal{N}(m, \Gamma)$  admits a positive density with respect to Lebesgue measure on the affine space  $m + (\ker \Gamma)^\perp$ . Thus, it suffices to show that  $(m + (\ker \Gamma)^\perp) \cap K^o$  is a non-empty open set in  $m + (\ker \Gamma)^\perp$ . This follows from Lemma 14 since  $m + (\ker \Gamma)^\perp$  is not contained in a hyperplane.  $\square$

**Lemma 16.** *Assume that the distribution  $\mu$  of the random walk increments is square-integrable with mean  $m \in K$ , and satisfies (H1). Then there exist  $k \geq 1$  and  $\delta \geq 0$  such that*

$$\mathbb{P}[\tau_{K_{-\delta}} > k, S_k \in K^o] > 0.$$

*Proof.* For  $n \geq 1$  define  $Z_n = (S_n - nm)/\sqrt{n}$ . The homogeneity and convexity properties of  $K$  ensure that

$$\{S_n \in K^o\} = \{Z_n \in K^o - \sqrt{n}m\} \supset \{Z_n \in K^o - m\}$$

Let  $\Gamma$  denote the variance-covariance matrix of  $\mu$ . We notice that  $m + (\ker \Gamma)^\perp$  is not included in a hyperplane since  $\mu$  satisfies (H1). The central limit theorem asserts that  $(Z_n)_{n \geq 1}$  converges in distribution to a random vector  $Y$  with gaussian distribution  $\mathcal{N}(0, \Gamma)$ . Hence, applying the Portmanteau theorem [5, Theorem 2.1], we obtain the inequality

$$\liminf_{n \rightarrow \infty} \mathbb{P}[S_n \in K^o] \geq \mathbb{P}[Y \in K^o - m],$$

where the right-hand side is positive according to Lemma 15. Now, to conclude it suffices to fix  $k$  so that  $\mathbb{P}[S_k \in K^o] = 2\epsilon > 0$ , and then choose  $\delta$  so large that  $\mathbb{P}[\tau_{K_{-\delta}} > k] \geq 1 - \epsilon$  (this is possible since  $K_{-\delta} \uparrow \mathbb{R}^d$  as  $\delta \uparrow \infty$ ).  $\square$

*Proof of Theorem 13.* The proof follows the same arguments as in [16]. We shall first use Lemma 16 in order to push the random walk inside the cone at a distance  $\sqrt{n}$  from the boundary, and then apply the functional central limit theorem.

Recall that  $v \in K^\circ$  is fixed and that  $K_\delta = K + \delta v$ . We know by Lemma 16 that there exist  $\delta \geq 0$  and  $k \geq 1$  such that

$$\mathbb{P}[\tau_{K-\delta} > k, S_k \in K^\circ] > 0.$$

Therefore, we can find a closed ball  $B = \overline{B(z, \epsilon)} \subset K^\circ$ , with center at  $z \in K^\circ$  and radius  $\epsilon > 0$ , such that

$$\mathbb{P}[\tau_{K-\delta} > k, S_k \in B] = \gamma > 0.$$

Since  $K$  is a convex cone, it satisfies the relation  $K + K \subset K$ , thus

$$\mathbb{P}^x[\tau_K > k, S_k - x \in B] \geq \gamma,$$

for all  $x \in K_\delta$  (by inclusion of events). From this, we shall deduce by induction that

$$(21) \quad p_\ell := \mathbb{P}^x[\tau_K > \ell k, S_{\ell k} - x \in \ell B] \geq \gamma^\ell,$$

for all  $\ell \geq 1$  and  $x \in K_\delta$ . Indeed, by the Markov property of the random walk,

$$\begin{aligned} p_{\ell+1} &\geq \mathbb{E}^x[\tau_K > \ell k, S_{\ell k} - x \in \ell B, \mathbb{P}^{S_{\ell k}}[\tau_K > k, S_k - x \in (\ell+1)B]] \\ &\geq p_\ell \cdot \inf_{\{y \in K: y-x \in \ell B\}} \mathbb{P}^y[\tau_K > k, S_k - x \in (\ell+1)B]. \end{aligned}$$

Noting that  $y - x \in \ell B$  and  $S_k - y \in B$  yields  $S_k - x \in (\ell+1)B$ , we obtain

$$p_{\ell+1} \geq p_\ell \cdot \inf_{\{y \in K: y-x \in \ell B\}} \mathbb{P}^y[\tau_K > k, S_k - y \in B].$$

But the facts that  $x \in K_\delta$  and  $y - x \in \ell B \subset K$  imply  $y \in K_\delta$ . Hence  $p_{\ell+1} \geq p_\ell \cdot \gamma$  and (21) is proved.

For  $x \in K_\delta$ , define

$$p_n = \mathbb{P}^x[\tau_K > n, |S_n^{(i)}| \leq \sqrt{n}, \forall i \in I],$$

and set  $\ell = \lfloor \sqrt{n} \rfloor$  for the lower integer part of  $\sqrt{n}$ . Using the Markov property at time  $\ell k$  and the estimate in (21) leads to

$$\begin{aligned} p_n &\geq \mathbb{P}^x[\tau_K > n, S_{\ell k} - x \in \ell B, |S_n^{(i)}| \leq \sqrt{n}, \forall i \in I] \\ &\geq \mathbb{E}^x[\tau_K > \ell k, S_{\ell k} - x \in \ell B, \mathbb{P}^{S_{\ell k}}[\tau_K > n - \ell k, |S_{n-\ell k}^{(i)}| \leq \sqrt{n}, \forall i \in I]] \\ &\geq \gamma^\ell \cdot \inf_{\{y \in K: y-x \in \ell B\}} \mathbb{P}^y[\tau_K > n - \ell k, |S_{n-\ell k}^{(i)}| \leq \sqrt{n}, \forall i \in I]. \end{aligned}$$

Therefore, Theorem 13 will follow from the fact that

$$(22) \quad \liminf_{n \rightarrow \infty} \inf_{\{y \in K: y-x \in \ell B\}} \mathbb{P}^y[\tau_K > n - \ell k, |S_{n-\ell k}^{(i)}| \leq \sqrt{n}, \forall i \in I] > 0,$$

which we shall prove now. Since  $\ell k \ll n$  does not play any significant role in the last probability, we will neglect it in order to simplify notations. Also, for any  $\epsilon' > \epsilon$ , we have

$$x + \ell B = \ell \left( \frac{x}{\ell} + \overline{B(z, \epsilon)} \right) \subset \ell \overline{B(z, \epsilon')}$$

for all large enough  $n$ . Since an  $\epsilon' > \epsilon$  can be found so that  $\overline{B(z, \epsilon')} \subset K^\circ$ , we may replace  $x + \ell B$  by  $\ell B$  without loss of generality. Finally, since  $\ell \leq \sqrt{n}$ , we may replace  $\ell B$  by  $\sqrt{n}B$ . We now introduce

$$q_n = \inf_{\{y \in K: y \in \sqrt{n}B\}} \mathbb{P}^y[\tau_K > n, |S_n^{(i)}| \leq \sqrt{n}, \forall i \in I].$$

By mapping  $y$  to  $y/\sqrt{n}$ , we may write

$$q_n = \inf_{\{y \in K : y \in B\}} q_n(y)$$

where

$$q_n(y) = \mathbb{P}^0[\tau_K(y\sqrt{n} + S) > n, |y\sqrt{n} + S_n^{(i)}| \leq \sqrt{n}, \forall i \in I].$$

Let  $\tilde{S} = (\tilde{S}_n = S_n - nm)_{n \geq 0}$  denote the centered random walk associated with  $S = (S_n)_{n \geq 0}$ . By inclusion of events we get the lower bound

$$q_n(y) \geq \mathbb{P}^0[\tau_K(y\sqrt{n} + \tilde{S}) > n, |y\sqrt{n} + \tilde{S}_n^{(i)}| \leq \sqrt{n}, \forall i \in I],$$

where we have used the facts that  $m \in K$  and  $m^{(i)} = \mathbb{E}[S_1^{(i)}] = 0$  for all  $i \in I$ . Finally, let us denote by  $Z_n = (Z_n(t))_t$  the random process with continuous paths that coincides with  $\tilde{S}_k/\sqrt{n}$  for  $t = k/n$  and which is linearly interpolated elsewhere. By definition of  $Z_n$  and convexity of  $K$ , the last inequality immediately rewrites

$$q_n(y) \geq \mathbb{P}^0[\tau_K(y + Z_n(t)) > 1, |y + Z_n^{(i)}(1)| \leq 1, \forall i \in I].$$

The functional central limit theorem ensures that  $Z_n$  converges in distribution to a Brownian motion  $B$  with variance-covariance matrix  $\Gamma$ . If  $(y_n)_{n \geq 0}$  converges to some  $y \in K^\circ$ , then  $y_n + Z_n$  converges in distribution to  $y + B$ , and it follows from the Portmanteau theorem that

$$\liminf_{n \rightarrow \infty} q_n(y_n) \geq \mathbb{P}[\tau_{K^\circ}(y + B(t)) > 1, |y + B^{(i)}(1)| < 1, \forall i \in I].$$

Now, the condition in the last probability is fulfilled as soon as the Brownian motion  $B(t)$  stays near the origin for all  $t \in [0, 1]$ , and this happens with positive probability, even if  $\Gamma$  is only semi-positive definite.

To summarize, we have proved that

$$\liminf_{n \rightarrow \infty} q_n(y_n) > 0,$$

for any sequence  $(y_n)_{n \geq 0}$  that converges to some  $y \in K^\circ$ . Thus, by standard arguments using the compactness of  $K \cap B$ , we reach the conclusion that

$$\liminf_{n \rightarrow \infty} q_n = \liminf_{n \rightarrow \infty} \inf_{y \in K \cap B} q_n(y) > 0,$$

and Theorem 13 is proved.  $\square$

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