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# Geometrically nonlinear higher-gradient elasticity with energetic boundaries

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## Abstract

The objective of this contribution is to formulate a geometrically nonlinear theory of higher-gradient elasticity accounting for boundary (surface and curve) energies. Surfaces and curves can significantly influence the overall response of a solid body. Such influences are becoming increasingly important when modeling the response of structures at the nanoscale. The behavior of the boundaries is well described by continuum theories that endow the surface and curve with their own energetic structures. Such theories often allow the boundary energy density to depend only on the superficial boundary deformation gradient. From a physical point of view though, it seems necessary to define the boundary deformation gradient as the evaluation of the deformation gradient at the boundary rather than its projection. This controversial issue is carefully studied and several conclusions are extracted from the rigorous mathematical framework presented.

In this manuscript the internal energy density of the bulk is a function of the deformation gradient and its first and second derivatives. The internal energy density of the surface is, consequently, a function of the deformation gradient at the surface and its first derivative. The internal energy density of a curve is, consequently, a function of the deformation gradient at the curve.

It is shown that in order to have a surface energy depending on the total (surface) deformation gradient, the bulk energy needs to be a function of at least the first derivative of the deformation gradient. Furthermore, in order to have a curve energy depending on the total (curve) deformation gradient, the bulk energy needs to be a function of at least the second derivative of the deformation gradient. Clearly, the theory of elasticity of Gurtin and Murdoch is intrinsically limited since it is associated with the classical (first-order) continuum theory of elasticity in the bulk. In this sense this contribution shall be also understood as a higher-gradient surface elasticity theory.

*Keywords:* Higher-gradient elasticity, Surface elasticity, Nano-materials, Cosserat continua, Generalized continua

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## 1. Introduction

The deep analysis performed by Mindlin (1965), was extremely original and in some aspects even too “premature” as proven by the fact that its importance is being appreciated only recently. As happens for only true fundamental papers, it includes the seeds of many ideas, and only few of them have been developed, up to now, as they deserve.

The aim of this contribution is to advance the seminal work of Mindlin in one among the many possible directions. In particular, we want to investigate how this conceptual frame can be enlarged to include surface and curve energies into the picture. This is motivated by the fact that boundary effects are becoming increasingly important when modeling the response of structures at the nanoscale. Furthermore this manuscript generalizes Mindlin’s work to a geometrically nonlinear framework. On the other hand this work extends the surface elasticity theory of Gurtin and Murdoch (1975) whereby not only the flexural resistance, but the thickness of the surface are accounted for.

### *1.1. State of the art review of continuum mechanics with energetic boundaries*

The widely-adopted surface elasticity model, proposed by Gurtin and Murdoch (1975, 1978), gives a surface its own tensorial stress measures (see e.g. Cammarata, 1994; Dingreville and Qu, 2005; He and Lilley, 2008; Duan et al., 2009, for applications in nano-materials). Murdoch (1976); Gurtin and Struthers (1990); Gurtin et al. (1998) extended this approach to consider interfaces within a solid. An important extension of the surface elasticity model to account for the flexural resistance of the surface was developed by Steigmann and Ogden (1999) and further studied in (Fried and Todres, 2005; Chhapadia et al., 2011). Moeckel (1975) followed a different approach to Gurtin and Murdoch (1975) for a moving interface within a thermomechanical solid. Daher and Maugin (1986) used the method of virtual power (Germain, 1973; Maugin, 1980) to derive the governing equations for an interface. An alternative approach to develop general governing equations for the interface is to integrate the known equations for the bulk over the thickness of the interfacial layers (see e.g. Gogosov et al., 1983). Murdoch (2005) addressed various aspects of surface modeling. Šilhavý (2011) proved the existence of equilibrium of a two-phase state with an elastic solid bulk and deformation dependent interfacial energy. Park et al. (2006); Park and Klein (2007, 2008) developed an alternative continuum framework based on the surface Cauchy–Born model, an extension of the classical Cauchy–Born model, to include surface stresses. The thermodynamic fundamentals of surface science was reviewed in (Rusanov, 1996, 2005). Müller and Saul (2004) presented a review on the importance of stress and strain effects on surface physics. The role of stress at solid surfaces was critically examined by Ibach (1997). Leo and Sekerka (1989) investigated the equilibrium conditions for interfaces using a variational approach wherein the excess energy associated with the interface is allowed to depend on both the deformation of the interface and the crystallographic normal to the interface. Cammarata (Cammarata, 1997; Cammarata et al., 2000; Cammarata, 2009) highlighted the surface and interface stress

effects on thin films and nano-scaled structures. Fischer et al. (2008) studied the role of surface energy and surface stress in phase-transforming nano-particles and reported on the thermodynamics of a moving surface. The effect of surface energetics, e.g. for inclusions, and the size-dependent elastic state of the material has been widely investigated recently e.g. in (Sharma et al., 2003; Sharma and Ganti, 2004; Sharma and Wheeler, 2007; Duan et al., 2005a,b; Duan and Karihaloo, 2007; Benveniste and Miloh, 2001; Huang and Sun, 2007; Fischer and Svoboda, 2010; Yvonnet et al., 2011). In other models (e.g. dell’Isola and Romano, 1986, 1987; dell’Isola and Kosinski, 1993; Vardoulakis et al., 1996; Exadaktylos et al., 1996; Exadaktylos, 1998; Exadaktylos and Vardoulakis, 1998; Danescu and Sidoroff, 1998; Exadaktylos, 1999; Steinmann, 2008) it is assumed that some surfaces may carry a finite amount of energy. Also some attention was attracted to phenomena of energy concentration on edges (see e.g. Simha and Bhattacharya (1998, 2000); Tsepoura and Pavlou (2006); Li et al. (2011)): clearly the description of these phenomena requires the introduction of the concept of curve deformation energy density. See Javili et al. (2013) for a unifying review of different approaches including surface, interface and curve energies.

### *1.2. State of the art review of higher-gradient elasticity*

The pioneering paper by Mindlin opened new perspectives in continuum mechanics: it represents one of the first works (after those of Cosserat and Cosserat (1908, 1909)) in which the Cauchy format is purposefully abandoned and the equilibrium equations for a continuum are obtained by means of a variational principle. The work of Mindlin stemmed from some papers by Toupin (1962, 1964); Green and Rivlin (1964b,a,c, 1965) and prepared those by Germain (1972, 1973).

Second and third gradient continuum theories, as formulated in the previously listed papers, can be used to describe some phenomena of deformation energy concentration in narrow “boundary layers”. In the aforementioned models, however, it is never assumed that regions with vanishing volume, but with nonvanishing area or length, may “carry” deformation energy.

To substantiate the previous statement a short description of the literature originated by Mindlin is here needed. Indeed the papers moving in the same conceptual framework as Mindlin’s work can be divided in several categories. We, simply for making the exposition easier, try to gather them as follows:

- The papers which are investigating the mathematical structure of “generalized” (with respect to Navier–Cauchy’s) continuum theories: Germain (1972, 1973); Seppecher (1987, 1989); dell’Isola and Seppecher (1995, 1997); Suiker and Chang (2000); Charlotte and Truskinovsky (2002); Polizzotto (2003); Mareno (2004); Kirchner and Steinmann (2005); Lazar and Maugin (2005); Rambert et al. (2007); Kirchner and Steinmann (2007); Van and Papenfuss (2008); Charlotte and Truskinovsky (2008); Agiasofitou and Lazar (2009); Ganghoffer

(2010); Yi et al. (2010); Ciarletta et al. (2012).

- The papers which try to establish which “microscopical” structure may produce at a macroscopic level generalized continuum theories: Forest et al. (2000, 2001); Tenek and Aifantis (2001); Sunyk and Steinmann (2003); Forest and Sievert (2003); Maranganti and Sharma (2007); Nikolov et al. (2007); Pozrikidis (2008); Sun et al. (2008); Cihan Tekoglu and Patrick R. Onck (2008); Weissmüller et al. (2010); Shodja and Tehranchi (2010); Seppecher et al. (2011); Temizer and Wriggers (2011); Danescu (2012); Mühlich et al. (2012); Polyzos and Fotiadis (2012); Tran et al. (2012); Zheng et al. (2012).
- The papers which try to use higher gradient models to cure “singularities” arising in already established field theories. Indeed higher gradient models allow for the description of many among those phenomena leading to the formation of various types of boundary layers: and indeed they are of use in plasticity, damage, in the formulation of models for the formation of shear bands, dilatancy regions and stress localization, or in the description of multiscale deformable bodies. To this class belong for instance the papers by Fleck and Hutchinson (1997); Fatemi et al. (2002); Chang et al. (2003); Lazar et al. (2005); Dobovsek (2006); Lazar et al. (2006); Fernandes et al. (2008); Aifantis (2009); Chen and Feng (2011); Forest et al. (2011).
- The papers which investigated how the mechanical behavior of higher gradient bodies differs from the one of Navier–Cauchy continua: Altan et al. (1996); Huang et al. (1997); Chambon et al. (1998); Anthoine (2000); Altan et al. (2003); Lam et al. (2003); Li et al. (2004); Gao and Park (2007); Kong et al. (2009); Gao and Ma (2010); Han (2010); Kahrobaiyan et al. (2011); Shvartsburg and Erokhin (2011); Akgoz and Civalek (2011); Askes and Aifantis (2011); Akgoz and Civalek (2012).
- The papers which are dealing with many interesting coupling phenomena, e.g. between 3D and 1D continua in static or dynamic conditions, that have never been modeled because of their complexity and the clear insufficiency of the available continuum models. These phenomena include e.g. buckling, galloping and internal resonance induced by the interaction of beams and cables with 3D bodies Luongo and Piccardo (see e.g. 1998, 2005); Paolone et al. (see e.g. 2006); Luongo et al. (see e.g. 2008, 2009). The main difficulties do not arise solely due to non-linearities of the material behavior but are related also to the complex mechanical interaction occurring at the interface between the involved bodies. Higher gradient continuum models are expected to supply another modeling tool for confronting these engineering challenges: for instance third gradient continua, studied here, are able to exert line forces, couples and double forces thus being able to interact with cables or beams, and are also able to interact with forces, couples, double couples, double forces and triple forces with shells, being able, for instance, to induce dilatational deformations along their thickness. Also very important

Romeo and Luongo (see 2003); Luongo and Romeo (see 2005) are the phenomena of wave propagation inside highly inhomogeneous bodies, which show very interesting features; already Toupin's and Mindlin's papers were motivated by the need of finding, in this context, suitable dispersion relations.

### *1.3. Key objectives and contributions of this work*

Of course, apart from the mathematical elegance of the proposed theory, there are several physical motivations for performing the current analysis. This section summarizes the key objectives and contributions of this work and more importantly the need for such a theory is motivated with recourse to the available literature on the subject.

The surface elasticity theory of Gurtin and Murdoch is based on the idea of a two-dimensional membrane bonded to the boundary of the bulk material whereby the surface possesses no thickness and consequently no flexural (curvature-dependent) resistance. An extension of the surface elasticity theory to account for flexural resistance was pioneered by Steigmann and Ogden (1999). The findings of Steigmann and Ogden (1999) revealed a marked departure from the earlier predictions including the presence of dispersion. This issue has been carefully studied by Fried and Todres (2005) where they obtain a quintic dispersion relation in contrast to the quadratic dispersion relation of Andreussi and Gurtin (1977). Their flexure-enhanced analysis always predicts an increased number of linearly stable wrinkled configurations compared to Andreussi and Gurtin (1977). Furthermore, Fried and Todres (2005) motivate in detail that from an experimental or design point of view the ability to attain linearly stable regions in thin films is desirable for semiconductors, mirrors, etc. Motivated by the flexure-enhanced surface elasticity theory, Chhapadia et al. (2011) studied the influence of curvature-dependence of surface energy on the effective elastic modulus of nanostructures. From atomistic calculations they show that the flexure-enhanced surface elasticity theory can better explain the behavior of bent nanostructures. Very recently, Olsson and Park (2012) have detailed on the importance of surface elastic contributions to the flexural rigidity of nanowires. They show that the classical surface elasticity theory lose accuracy for ultrasmall nanowires since at that scale the nature of the cross section plays an important role. Olsson and Park (2012) present that continuum mechanics can be utilized to study the elastic and mechanical behavior and properties of nanowires if surface elastic contributions to the flexural rigidity are accounted for (see also Bar On et al., 2010). Finally, in Javili et al. (2012) we have shown that, critically, the admissible range for the surface material properties differs for inherently non-local atomistic models and the surface elasticity theory, both of which are used to model surface effects at the nanometer-scale. Therefore, care should be taken when fitting material properties obtained from atomistic models to classical continuum formulations that inherently lack a length-scale.

Apart from the above-mentioned observations, from logical reasoning it is clear that assuming a zero-thickness surface could be a good approximation only if the actual thickness of the surface compared to the bulk is very small. Of course, this is not the case for ultra-small nanowires for instance. Therefore, the need to enhance the classical

surface elasticity theory is obvious. Steigmann and Ogden (1999) enhanced the surface elasticity theory with a flexural resistance. The present contribution advances the flexure-enhanced (zero-thickness) surface elasticity theory to the case where the surface is endowed with a finite thickness. We rigorously prove that endowing the surface with a finite thickness leads to several consequences one of which being the flexural (curvature-dependent) resistance of the surface. In fact, the curvature-dependent resistance is implied by the purely tangential part of the second gradient of the surface deformation map.

In order to carry out the analysis and to be able to see the hierarchical nature of the problem, the following key features need to be pointed out.

- In contrast to the classical surface elasticity theory and the flexure-enhanced surface elasticity theory, we allow the surface energy to depend on the total (first) gradient of the surface deformation map and not only the (superficial) projection of it. This assumption requires the bulk energy to depend on the second gradient of the deformation map, i.e. a second-order bulk. A first-order bulk though could only accommodate for the tangential part of the surface deformation gradient.
- The surface energy is allowed also to depend on the total second gradient of the surface deformation map. This requires a third-order bulk. Nevertheless, a first-order bulk could accommodate for the purely tangential part of the second surface deformation gradient which is dependent on the curvature. This finding explains the curvature-dependent resistance of the flexure-enhanced surface elasticity theory.
- The curve energy is also accounted for and depends on the total first gradient of the curve deformation map. This requires a third-order bulk and/or a second-order surface.

To the authors' knowledge the case of surface and curve deformation energy depending on the *total* (including orthogonal components) deformation gradient has not been yet considered. It has therefore to be established if it is possible to coherently formulate models in which the considered deformable body can be described by introducing at the same time i) bulk energies depending on higher deformation gradients and ii) surface and curve energies depending on the *total* deformation gradient and possibly on higher deformation gradients. The analysis which we will perform here shows that it is possible, generalizing what has been done by Mindlin, to formulate a continuum theory in which surface tension is not only related to the dependence of the volume deformation energy on the third gradient of placement, but it is also (which is closer to the original ideas of Gibbs) possible to relate coherently surface stress to a surface energy. Note that although the principle of virtual work is conventional our application of it is novel.

**Remark** The material modeling of bulk materials is a mature field with many standard references. This is not the case for the surface though. There have been many theoretical studies of surface elasticity, but there are very

few experiments for measuring the materials constants. Nevertheless, in the opinion of the authors without a clear theoretical understanding of suitable models no experimental evidence may be obtained. We recall that the controversy about the true number of coefficients in isotropic elasticity (Navier believed that only one was sufficient) was first clarified theoretically by Lamé and only then it was possible to measure the material constants (Benvenuto, 1991). We believe that sooner or later new coefficient introduced by well-posed theories will be measured.

The material parameters on the surface, and indeed in the bulk, can be obtained from fundamental reasoning or from atomistic modeling (see e.g. Haiss, 2001). A method for the determination of surface elastic properties of face centered cubic metals from atomistic simulations is developed by Shenoy (2005). He shows the importance of additional effects due to relaxations of atoms at the crystal surface (see also Miller and Shenoy, 2000). Dingreville and Qu (2007) developed a semi-analytic method to compute the surface elastic properties of crystalline materials. Moreover, a surface energy can be constructed using the surface Cauchy-Born hypothesis (Park and Klein, 2007). Yvonnet et al. (2011) extract the surface elastic parameters from ab-initio calculations. In our recent contribution (Davydov et al., 2013) we have shown the ability of the continuum formulation enhanced with a surface energy to model size effects, as observed in the atomistic simulations whereby molecular statics simulations are employed. Similar strategies can be employed to extract the material constants via an inverse parameter identification procedure.

□

#### *1.4. Organization of this manuscript*

The notation and certain key concepts are introduced in Section 2. The problem and its kinematics are defined. A brief overview of the differential geometry of surfaces and curves is given. The divergence theorems and integrations by parts are generalized to account for surfaces and curves and also tensors of higher order. In Section 3 governing equations for higher order continua are obtained. This is performed by setting the first variation of the total energy functional to zero. Contributions from the bulk, surface and curves are studied separately. Local balance equations of a third order bulk, second order surface and first order curve are gathered in Table 1. Finally in Section 4 the balance equations are simplified for some special cases to better understand the meaning of each of the equations. This procedure results in extracting several important logical conclusions about the compatibility of the bulk, surface and curve energies. These conclusions shape and rather justify the framework of this manuscript.

## **2. Preliminaries**

The purpose of this section is to summarize certain key concepts in nonlinear continuum mechanics and to introduce the notation adopted here. Detailed expositions on nonlinear continuum mechanics can be found in Marsden and

Hughes (1994), among others. For further details concerning the continuum mechanics of deformable surfaces the reader is referred to Javili and Steinmann (2010b) and the references therein.

### 2.1. Notation and definitions

Direct notation is adopted throughout. Occasional use is made of index notation, the summation convention for repeated indices being implied. The scalar product of two vectors  $\mathbf{a}$  and  $\mathbf{b}$ , two second-order tensors  $\mathbf{A}$  and  $\mathbf{B}$ , two third-order tensors  $\mathcal{A}$  and  $\mathcal{B}$  and two fourth-order tensors  $\mathbb{A}$  and  $\mathbb{B}$  are, respectively, denoted

$$\mathbf{a} \cdot \mathbf{b} = [\mathbf{a}]_i [\mathbf{b}]_i, \quad \mathbf{A} : \mathbf{B} = [A]_{ij} [B]_{ij}, \quad \mathcal{A} \cdot \mathcal{B} = [\mathbf{a}]_{ijk} [\mathbf{b}]_{ijk}, \quad \mathbb{A} :: \mathbb{B} = [\mathbb{A}]_{ijkl} [\mathbb{B}]_{ijkl}.$$

The composition of two second-order tensors  $\mathbf{A}$  and  $\mathbf{B}$ , denoted  $\mathbf{A} \cdot \mathbf{B}$ , is a second-order tensor with components  $[\mathbf{A} \cdot \mathbf{B}]_{ij} = [A]_{im} [B]_{mj}$ . The action of a second-order tensor  $\mathbf{A}$  on a vector  $\mathbf{a}$  is given by  $[\mathbf{A} \cdot \mathbf{a}]_i = [A]_{ij} [\mathbf{a}]_j$ . The tensor product of two vectors  $\mathbf{a}$  and  $\mathbf{b}$  is a second-order tensor  $\mathbf{D} = \mathbf{a} \otimes \mathbf{b}$  with  $[D]_{ij} = [\mathbf{a}]_i [\mathbf{b}]_j$ . The non-standard tensor product of two second-order tensors  $\mathbf{A}$  and  $\mathbf{B}$  is the fourth-order tensors  $[\mathbf{A} \bar{\otimes} \mathbf{B}]_{ijkl} = [A]_{ik} [B]_{jl}$ . The non-standard tensor product of a second-order tensors  $\mathbf{A}$  and a vector  $\mathbf{b}$  is the third-order tensor  $[\mathbf{A} \bar{\otimes} \mathbf{b}]_{ijk} = [A]_{ik} [\mathbf{b}]_j$ . Let  $\mathbb{E}^3$  denote the three-dimensional Euclidean space.

Quantities or operators evaluated at the bulk, a surface, a curve and a point are denoted as  $\{\bullet\}$ ,  $\{\widehat{\bullet}\}$ ,  $\{\bar{\bullet}\}$  and  $\{\bar{\bar{\bullet}}\}$ , respectively, unless specified otherwise.

For the integrals over the bulk, a surface and a curve the corresponding domain increments are omitted for the sake of space, i.e.

$$\int_{\text{volume}} \{\bullet\} dV \equiv \int_{\text{volume}} \{\bullet\}, \quad \int_{\text{surface}} \{\widehat{\bullet}\} dA \equiv \int_{\text{surface}} \{\bullet\}, \quad \int_{\text{curve}} \{\bar{\bullet}\} dL \equiv \int_{\text{curve}} \{\bullet\}.$$

### 2.2. Differential geometry of surfaces and curves

It is enlightening to briefly review some basic terminologies and results on surfaces and curves. For further details the reader is referred to (Bowen and Wang, 1976; Kreyszig, 1991; Ciarlet, 2005) among others. Here, some technicalities are borrowed from Steinmann (2008).

#### 2.2.1. Surfaces

A two-dimensional (smooth) surface  $\mathcal{S}$  in the three dimensional, embedding Euclidean space  $\mathbb{E}^3$  with coordinates  $\widehat{\mathbf{x}}$  is parameterized by two surface coordinates  $\eta^\alpha$  with  $\alpha = 1, 2$  as  $\widehat{\mathbf{x}} = \widehat{\mathbf{x}}(\eta^\alpha)$ . The corresponding tangent vectors  $\mathbf{a}_\alpha \in T\mathcal{S}$  to the surface coordinate lines  $\eta^\alpha$ , i.e. the covariant (natural) surface basis vectors are given by  $\mathbf{a}_\alpha = \partial_{\eta^\alpha} \widehat{\mathbf{x}}$ .

The associated contra-variant (dual) surface basis vectors  $\mathbf{a}^\alpha$  are defined by the Kronecker property  $\delta^\alpha_\beta = \mathbf{a}^\alpha \cdot \mathbf{a}_\beta$  and are explicitly related to the covariant surface basis vectors  $\mathbf{a}_\alpha$  by the co- and contra-variant surface metric coefficients  $a_{\alpha\beta}$  (first fundamental form of the surface) and  $a^{\alpha\beta}$ , respectively, as

$$\begin{aligned} \mathbf{a}_\alpha &= a_{\alpha\beta} \mathbf{a}^\beta & \text{with} & \quad a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta = [a^{\alpha\beta}]^{-1}, \\ \mathbf{a}^\alpha &= a^{\alpha\beta} \mathbf{a}_\beta & \text{with} & \quad a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta = [a_{\alpha\beta}]^{-1}. \end{aligned}$$

The contra- and covariant base vectors  $\mathbf{a}^3$  and  $\mathbf{a}_3$ , normal to  $T\mathcal{S}$ , are defined by

$$\mathbf{a}^3 := \mathbf{a}_1 \times \mathbf{a}_2 \quad \text{and} \quad \mathbf{a}_3 := [a^{33}]^{-1} \mathbf{a}^3 \quad \text{so that} \quad \mathbf{a}^3 \cdot \mathbf{a}_3 = 1.$$

Thereby, the corresponding contra- and covariant metric coefficients, respectively,  $[a^{33}]$  and  $[a_{33}]$  follow as

$$a^{33} = |\mathbf{a}_1 \times \mathbf{a}_2|^2 = \det[a_{\alpha\beta}] = [\det[a^{\alpha\beta}]]^{-1} = [a_{33}]^{-1}.$$

Accordingly, the surface area element  $ds$  and the surface unit normal vector  $\widehat{\mathbf{n}}$  are computed as

$$ds = |\mathbf{a}_1 \times \mathbf{a}_2| d\eta^1 d\eta^2 = [a^{33}]^{1/2} d\eta^1 d\eta^2 \quad \text{and} \quad \widehat{\mathbf{n}} = [a_{33}]^{1/2} \mathbf{a}^3 = [a^{33}]^{1/2} \mathbf{a}_3.$$

Moreover, with  $\widehat{\mathbf{i}}$  denoting the surface unit tensor, or rather the ordinary unit tensor in  $\mathbb{E}^3$  evaluated at the surface, the surface tangent unit tensor  $\widehat{\mathbf{i}}_{||}$  is defined by

$$\widehat{\mathbf{i}}_{||} := \delta^\alpha_\beta \mathbf{a}_\alpha \otimes \mathbf{a}^\beta = \mathbf{a}_\alpha \otimes \mathbf{a}^\alpha = \widehat{\mathbf{i}} - \mathbf{a}_3 \otimes \mathbf{a}^3 = \widehat{\mathbf{i}} - \widehat{\mathbf{n}} \otimes \widehat{\mathbf{n}}.$$

The surface tangent gradient and surface tangent divergence operators for vector fields  $\{\widehat{\bullet}\}$  are defined by

$$\widehat{\text{grad}}_{||} \{\widehat{\bullet}\} := \partial_{\eta^\alpha} \{\widehat{\bullet}\} \otimes \mathbf{a}^\alpha \quad \text{and} \quad \widehat{\text{div}}_{||} \{\widehat{\bullet}\} := \partial_{\eta^\alpha} \{\widehat{\bullet}\} \cdot \mathbf{a}^\alpha.$$

As a consequence, observe that  $\widehat{\text{grad}}_{||} \{\widehat{\bullet}\} \cdot \widehat{\mathbf{n}} = \mathbf{0}$  holds by definition. For fields that are smooth in a neighborhood of the surface, the surface gradient and surface divergence operators are alternatively defined as

$$\widehat{\text{grad}}_{||} \{\widehat{\bullet}\} := \widehat{\text{grad}} \{\widehat{\bullet}\} \cdot \widehat{\mathbf{i}}_{||} \quad \text{and} \quad \widehat{\text{div}}_{||} \{\widehat{\bullet}\} := \widehat{\text{grad}}_{||} \{\widehat{\bullet}\} : \widehat{\mathbf{i}}_{||} = \widehat{\text{grad}} \{\widehat{\bullet}\} : \widehat{\mathbf{i}}_{||}.$$

The last equality holds since  $\widehat{\mathbf{i}}_{\parallel}$  is idempotent, i.e.  $\widehat{\mathbf{i}}_{\parallel} \cdot \widehat{\mathbf{i}}_{\parallel} = \widehat{\mathbf{i}}_{\parallel}$ .

### 2.2.2. Curves

A one-dimensional (smooth) curve  $\mathcal{C}$  in the three-dimensional, embedding Euclidean space with coordinates  $\widetilde{\mathbf{x}}$  is parameterized by the arc-length  $\eta$  as  $\widetilde{\mathbf{x}} = \widetilde{\mathbf{x}}(\eta)$ . The corresponding tangent vector  $\widetilde{\mathbf{t}} \in T\mathcal{C}$  to the curve, together with the (principal) normal and bi-normal vectors  $\widetilde{\mathbf{n}}$  and  $\widetilde{\mathbf{m}}$  in the sense of Frénet–Serret, orthogonal to  $T\mathcal{C}$ , are defined by

$$\widetilde{\mathbf{t}} := \partial_{\eta} \widetilde{\mathbf{x}} \quad \text{and} \quad \widetilde{\mathbf{n}} := \partial_{\eta} \widetilde{\mathbf{t}} / |\partial_{\eta} \widetilde{\mathbf{t}}| \quad \text{and} \quad \widetilde{\mathbf{m}} := \widetilde{\mathbf{t}} \times \widetilde{\mathbf{n}}.$$

Due to the parametrization of the curve in its arc-length  $\eta$ , the tangent vector  $\widetilde{\mathbf{t}}$  has unit length and the curve line element  $dc$  is computed as

$$dc = |\partial_{\eta} \widetilde{\mathbf{x}}| d\eta = |\widetilde{\mathbf{t}}| d\eta = d\eta.$$

Moreover, we define the curve tangent unit tensor  $\widetilde{\mathbf{i}}_{\parallel}$  as

$$\widetilde{\mathbf{i}}_{\parallel} := \widetilde{\mathbf{t}} \otimes \widetilde{\mathbf{t}} = \widetilde{\mathbf{i}} - \widetilde{\mathbf{n}} \otimes \widetilde{\mathbf{n}} - \widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}.$$

The curve tangent gradient and curve tangent divergence operators for vector fields are defined by

$$\widetilde{\text{grad}}_{\parallel} \{\bullet\} := \partial_{\eta} \{\bullet\} \otimes \widetilde{\mathbf{t}} \quad \text{and} \quad \widetilde{\text{div}}_{\parallel} \{\bullet\} := \partial_{\eta} \{\bullet\} \cdot \widetilde{\mathbf{t}}.$$

As a consequence, observe that  $\widetilde{\text{grad}}_{\parallel} \{\bullet\} \cdot \widetilde{\mathbf{n}} = \mathbf{0}$  and  $\widetilde{\text{grad}}_{\parallel} \{\bullet\} \cdot \widetilde{\mathbf{m}} = \mathbf{0}$  hold by definition. For fields that are smooth in a neighborhood of the curve, the curve gradient and curve divergence operators are alternatively defined as

$$\widetilde{\text{grad}}_{\parallel} \{\bullet\} := \widetilde{\text{grad}} \{\bullet\} \cdot \widetilde{\mathbf{i}}_{\parallel} \quad \text{and} \quad \widetilde{\text{div}}_{\parallel} \{\bullet\} := \widetilde{\text{grad}}_{\parallel} \{\bullet\} : \widetilde{\mathbf{i}}_{\parallel} = \widetilde{\text{grad}} \{\bullet\} : \widetilde{\mathbf{i}}_{\parallel}.$$

The last equality holds since  $\widetilde{\mathbf{i}}_{\parallel}$  is idempotent, i.e.  $\widetilde{\mathbf{i}}_{\parallel} \cdot \widetilde{\mathbf{i}}_{\parallel} = \widetilde{\mathbf{i}}_{\parallel}$ .

### 2.3. Problem definition

Consider a continuum body that takes the material configuration  $\mathcal{B}_0 \subset \mathbb{E}^3$  at the time  $t = 0$  and the spatial configuration  $\mathcal{B}_t \subset \mathbb{E}^3$  at time  $t > 0$ <sup>1</sup>, as depicted in Fig. 1.<sup>2</sup>

The boundary of the continuum body in the material configuration  $\mathcal{S}_0$  is described by a patchwork of smooth two-dimensional surfaces  $\mathcal{S}_0^\xi \subset \mathbb{E}^3$  with  $\xi = 1, n_{\text{surf}}$ . That is

$$\mathcal{S}_0 = \cup \mathcal{S}_0^\xi := \partial \mathcal{B}_0. \quad (1)$$

The outward unit normal to  $\mathcal{S}_0^\xi$  is denoted  $\widehat{\mathbf{N}}_\xi$ .

Intersections of the  $n_{\text{surf}}$  individual boundary surface patches  $\mathcal{S}_0^\xi$  define a network of boundary curves  $C_0^\eta$  ( $\eta = 1, n_{\text{curv}}$ ) the union of which defines

$$C_0 = \cup C_0^\eta := \partial^2 \mathcal{B}_0. \quad (2)$$

The unit tangent to  $C_0^\eta$  is denoted  $\widetilde{\mathbf{T}}_\eta$ . The unit normal and binormal, in the sense of Frénet–Serret, to  $C_0^\eta$  are denoted  $\widetilde{\mathbf{N}}_\eta$  and  $\widetilde{\mathbf{M}}_\eta$ , respectively. The unit normal to  $\partial \mathcal{S}_0^\xi$  and tangential to  $\mathcal{S}_0^\xi$  is denoted  $\widehat{\mathbf{M}}_\xi$ . The bi-normal  $\widetilde{\mathbf{M}}_\eta$  is clearly normal to  $\widetilde{\mathbf{T}}_\eta$  and  $\widetilde{\mathbf{N}}_\eta$  and is, in general, neither normal nor tangent to the surface  $\mathcal{S}_0^\xi$  containing  $C_0^\eta$ .

**Remark** Let  $C_0^k$  be the conjunction of two surfaces  $\mathcal{S}_0^i$  and  $\mathcal{S}_0^j$  as illustrated in Fig. 1. The unit normals to  $C_0^k$  and tangential to  $\mathcal{S}_0^i$  and  $\mathcal{S}_0^j$  are denoted  $\widehat{\mathbf{M}}_i$  and  $\widehat{\mathbf{M}}_j$ , respectively. The unit normal and binormal to  $C_0^k$ , i.e.  $\widetilde{\mathbf{N}}_k$  and  $\widetilde{\mathbf{M}}_k$  respectively, span the same space that  $\widehat{\mathbf{M}}_i$  and  $\widehat{\mathbf{M}}_j$  span.  $\square$

Also,  $n_{\text{poin}}$  intersections of the  $n_{\text{curv}}$  individual boundary curves  $C_0^\eta$  define a set of boundary points  $\mathcal{P}_0^\pi$  ( $\pi = 1, n_{\text{poin}}$ )

$$\mathcal{P}_0 = \cup \mathcal{P}_0^\pi := \partial^3 \mathcal{B}_0. \quad (3)$$

In an identical fashion to the material configuration the patchwork of surfaces  $\mathcal{S}_t$ , the network of curves  $C_t$  and the set of points  $\mathcal{P}_t$  in the spatial configuration are defined.

We focus on the derivations in the material configuration. The repetition of the relations and definitions in the spatial configuration is superfluous and thus is avoided in what follows. Nevertheless, the extension of this work to the spatial configuration is straightforward.

<sup>1</sup>Here time is understood as a history parameter ordering the sequence of external loading. Quasi-static loading conditions are assumed for the sake of simplicity. Here and henceforth, the subscripts  $t$  and  $0$  shall designate spatial and material quantities, respectively, unless specified otherwise.

<sup>2</sup>The topological boundary is the support of the boundary  $\partial \mathcal{B}_0$  (in the sense made precise in the Poincaré theorem for exterior forms, generalizing the Gauss divergence theorem for manifolds, see e.g. Arnold (1989)). This boundary is constituted by regular parts (faces) being rectifiable and orientable smooth embedded manifolds, by edges, which are smooth curves on which faces are concurring and where normals to the faces suffer jumps, and by wedges, where a finite number of edges are concurring.

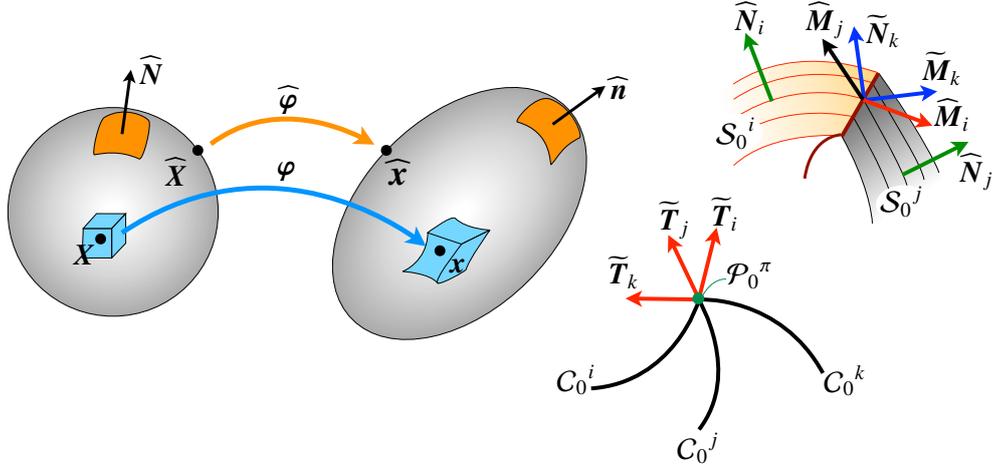


Figure 1: The material and spatial configurations of a continuum body, and the associated motions and deformation gradients.

Equipped with this prescriptions, recall that the bulk, surface, curve and point quantities or operators  $\{\bullet\}$ ,  $\{\widehat{\bullet}\}$ ,  $\{\bar{\bullet}\}$  and  $\{\bar{\bullet}\}$ , respectively, in the material configuration, can be expressed as

$$\{\widehat{\bullet}\} = \{\bullet\}|_{\partial\mathcal{B}_0}, \quad \{\bar{\bullet}\} = \{\bullet\}|_{\partial^2\mathcal{B}_0}, \quad \{\bar{\bullet}\} = \{\bullet\}|_{\partial^3\mathcal{B}_0}.$$

For instance,  $\{\bar{\bullet}\}$  denotes a surface-quantity which is not necessarily tangent to the surface.

In the sequel, for the sake of simplicity, the subscripts are dropped from the definitions of the normals, tangents and binormals (if possible) keeping in mind that throughout the derivations they are employed according to their precise aforementioned definitions. That is the following definitions hold henceforth:

$$\begin{aligned} \widehat{N} &: \text{normal to } \partial\mathcal{B}_0, & \widehat{M} &: \text{normal to } \partial^2\mathcal{B}_0 \text{ and tangent to } \partial\mathcal{B}_0, \\ \widetilde{N} &: \text{Frénet–Serret normal to } \partial^2\mathcal{B}_0, & \widetilde{M} &: \text{Frénet–Serret binormal to } \partial^2\mathcal{B}_0, & \widetilde{T} &: \text{tangent to } \partial^2\mathcal{B}_0. \end{aligned}$$

Let  $I$  denotes the identity tensor. The surface and curve identity tensors, respectively, are defined by

$$\widehat{I} := I|_{\partial\mathcal{B}_0}, \quad \widetilde{I} := I|_{\partial^2\mathcal{B}_0},$$

where each can be decomposed into its tangential and normal parts with

$$\widehat{I}_{\parallel} = \widehat{I} - \widehat{I}_{\perp}, \quad \widehat{I}_{\perp} = \widehat{N} \otimes \widehat{N}, \quad \widetilde{I}_{\parallel} = \widetilde{I} - \widetilde{I}_{\perp} = \widetilde{T} \otimes \widetilde{T}, \quad \widetilde{I}_{\perp} = \widetilde{N} \otimes \widetilde{N} + \widetilde{M} \otimes \widetilde{M}.$$

The total surface and curve gradient operators  $\widehat{\text{Grad}}$  and  $\widetilde{\text{Grad}}$ , respectively, are defined by

$$\widehat{\text{Grad}}\{\bullet\} := \text{Grad}\{\bullet\}|_{\partial\mathcal{B}_0}, \quad \widetilde{\text{Grad}}\{\bullet\} := \text{Grad}\{\bullet\}|_{\partial^2\mathcal{B}_0},$$

where each can be decomposed into its tangential and normal parts in the following sense

$$\begin{aligned} \widehat{\text{Grad}}\{\bullet\} &= \widehat{\text{Grad}}_{\perp}\{\bullet\} + \widehat{\text{Grad}}_{\parallel}\{\bullet\} \quad \text{with} \quad \widehat{\text{Grad}}_{\perp}\{\bullet\} = \widehat{\text{Grad}}\{\bullet\} \cdot \widehat{\mathbf{I}}_{\perp}, \quad \widehat{\text{Grad}}_{\parallel}\{\bullet\} = \widehat{\text{Grad}}\{\bullet\} \cdot \widehat{\mathbf{I}}_{\parallel}, \\ \widetilde{\text{Grad}}\{\bullet\} &= \widetilde{\text{Grad}}_{\perp}\{\bullet\} + \widetilde{\text{Grad}}_{\parallel}\{\bullet\} \quad \text{with} \quad \widetilde{\text{Grad}}_{\perp}\{\bullet\} = \widetilde{\text{Grad}}\{\bullet\} \cdot \widetilde{\mathbf{I}}_{\perp}, \quad \widetilde{\text{Grad}}_{\parallel}\{\bullet\} = \widetilde{\text{Grad}}\{\bullet\} \cdot \widetilde{\mathbf{I}}_{\parallel}. \end{aligned} \quad (4)$$

Furthermore, instead of the normal projection of the gradient in Eq. (4), it proves convenient to work with normal gradients  $\widehat{\text{Grad}}_N$ ,  $\widetilde{\text{Grad}}_N$  and  $\widetilde{\text{Grad}}_M$  defined by

$$\widehat{\text{Grad}}_N\{\bullet\} := \text{Grad}\{\bullet\} \cdot \widehat{\mathbf{N}}, \quad \widetilde{\text{Grad}}_N\{\bullet\} := \text{Grad}\{\bullet\} \cdot \widetilde{\mathbf{N}}, \quad \widetilde{\text{Grad}}_M\{\bullet\} := \text{Grad}\{\bullet\} \cdot \widetilde{\mathbf{M}}, \quad (5)$$

which can be related to the normal projection of the gradients in Eq. (4) according to

$$\widehat{\text{Grad}}_{\perp}\{\bullet\} = \widehat{\text{Grad}}_N\{\bullet\} \otimes \widehat{\mathbf{N}}, \quad \widetilde{\text{Grad}}_{\perp}\{\bullet\} = \widetilde{\text{Grad}}_N\{\bullet\} \otimes \widetilde{\mathbf{N}} + \widetilde{\text{Grad}}_M\{\bullet\} \otimes \widetilde{\mathbf{M}}. \quad (6)$$

The total bulk divergence operator  $\text{Div}$ , surface divergence operator  $\widehat{\text{Div}}$  and curve divergence operator  $\widetilde{\text{Div}}$  are defined by

$$\text{Div}\{\bullet\} = \text{Grad}\{\bullet\} : \mathbf{I}, \quad \widehat{\text{Div}}\{\bullet\} := \text{Div}\{\bullet\}|_{\partial\mathcal{B}_0} = \widehat{\text{Grad}}\{\bullet\} : \widehat{\mathbf{I}}, \quad \widetilde{\text{Div}}\{\bullet\} := \text{Div}\{\bullet\}|_{\partial^2\mathcal{B}_0} = \widetilde{\text{Grad}}\{\bullet\} : \widetilde{\mathbf{I}}.$$

In a similar fashion to the gradient operators, for tensorial arguments the total divergence operators can be decomposed into their tangential and normal parts as

$$\widehat{\text{Div}}\{\bullet\} = \widehat{\text{Div}}_{\perp}\{\bullet\} + \widehat{\text{Div}}_{\parallel}\{\bullet\} \quad \text{and} \quad \widetilde{\text{Div}}\{\bullet\} = \widetilde{\text{Div}}_{\perp}\{\bullet\} + \widetilde{\text{Div}}_{\parallel}\{\bullet\}.$$

according to the following definitions

$$\widehat{\text{Div}}_{\parallel}\{\bullet\} := \widehat{\text{Grad}}\{\bullet\} : \widehat{\mathbf{I}}_{\parallel}, \quad \widehat{\text{Div}}_{\perp}\{\bullet\} := \widehat{\text{Grad}}\{\bullet\} : \widehat{\mathbf{I}}_{\perp}, \quad \widetilde{\text{Div}}_{\parallel}\{\bullet\} := \widetilde{\text{Grad}}\{\bullet\} : \widetilde{\mathbf{I}}_{\parallel}, \quad \widetilde{\text{Div}}_{\perp}\{\bullet\} := \widetilde{\text{Grad}}\{\bullet\} : \widetilde{\mathbf{I}}_{\perp}.$$

#### 2.4. Kinematics

The material and spatial placement of particles are labeled  $X \in \mathcal{B}_0$  and  $\mathbf{x} \in \mathcal{B}_t$ , respectively. Let  $\mathcal{T} = [0, t_{\text{end}}] \subset \mathbb{R}_+$  denote the time domain. A motion of the material placement  $X$  for a time  $t \in \mathcal{T}$  is denoted by the orientation-preserving map  $\varphi$  via  $\mathbf{x} = \varphi(X)$ . The first, second and third deformation gradients of the deformation map  $\varphi$  are defined by

$$\begin{aligned} F_1(X) &:= \text{Grad}\varphi(X) , \\ F_2(X) &:= \text{Grad}(\text{Grad}\varphi(X)) = \text{Grad}^2\varphi(X) , \\ F_3(X) &:= \text{Grad}(\text{Grad}(\text{Grad}\varphi(X))) = \text{Grad}^3\varphi(X) . \end{aligned}$$

The surface is assumed to be material; that is, it does not move independently of the surrounded bulk material.<sup>3</sup> The boundary surface placements  $\widehat{X}$  and  $\widehat{\mathbf{x}}$ , respectively in the material and the spatial configurations, are related by the invertible (nonlinear) surface deformation map  $\widehat{\varphi}$  through

$$\widehat{\mathbf{x}} = \widehat{\varphi}(\widehat{X}) \quad \text{with} \quad \widehat{\varphi} = \varphi|_{\mathcal{S}_0}, \quad \widehat{X} = X|_{\mathcal{S}_0}, \quad \widehat{\mathbf{x}} = \mathbf{x}|_{\mathcal{S}_t}. \quad (7)$$

The total first and the second surface gradients of the surface deformation map  $\widehat{\varphi}$  are defined by

$$\begin{aligned} \widehat{F}_1(\widehat{X}) &:= \widehat{\text{Grad}}\widehat{\varphi}(\widehat{X}), \\ \widehat{F}_2(\widehat{X}) &:= \widehat{\text{Grad}}(\widehat{\text{Grad}}\widehat{\varphi}(\widehat{X})) = \widehat{\text{Grad}}^2\widehat{\varphi}(\widehat{X}). \end{aligned}$$

**Remark** It is enlightening to decompose the total second gradient of the surface deformation map into its tangential, normal and mixed parts as follows

$$\widehat{\text{Grad}}^2\widehat{\varphi} = \underbrace{\widehat{\text{Grad}}_{\parallel}\widehat{\text{Grad}}_{\parallel}\widehat{\varphi}}_{\widehat{\text{Grad}}_{\parallel}^2\widehat{\varphi}} + \underbrace{\widehat{\text{Grad}}_{\perp}\widehat{\text{Grad}}_{\parallel}\widehat{\varphi}}_{\widehat{\text{Grad}}_{\perp\parallel}^2\widehat{\varphi}} + \underbrace{\widehat{\text{Grad}}_{\parallel}\widehat{\text{Grad}}_{\perp}\widehat{\varphi}}_{\widehat{\text{Grad}}_{\parallel\perp}^2\widehat{\varphi}} + \underbrace{\widehat{\text{Grad}}_{\perp}\widehat{\text{Grad}}_{\perp}\widehat{\varphi}}_{\widehat{\text{Grad}}_{\perp}^2\widehat{\varphi}}. \quad (8)$$

Using the identity

$$\widehat{\text{Grad}}\widehat{\mathbf{I}}_{\parallel} = -\widehat{\text{Grad}}\widehat{\mathbf{I}}_{\perp} = -\widehat{\mathbf{K}} \otimes \widehat{\mathbf{N}} - \widehat{\mathbf{N}} \otimes \widehat{\mathbf{K}},$$

<sup>3</sup>Remark that in the present context we do not allow for surface or edge kinematical descriptors which are independent of the bulk corresponding descriptors. The deformation state of the boundary, i.e. of all its faces, edges and wedges, is univocally determined by the “limit values” on this boundary of bulk kinematical fields.

where  $\widehat{\mathbf{K}} := -\text{Grad}\widehat{\mathbf{N}} = -\widehat{\text{Grad}}_{\parallel}\widehat{\mathbf{N}}$  denotes the surface curvature tensor, it can be shown that

$$\widehat{\text{Grad}}^2\widehat{\boldsymbol{\varphi}} = \widehat{\text{Grad}}^2\widehat{\boldsymbol{\varphi}} : [\widehat{\mathbf{I}} \otimes \widehat{\mathbf{I}}], \quad (9a)$$

$$\widehat{\text{Grad}}_{\parallel}^2\widehat{\boldsymbol{\varphi}} = \widehat{\text{Grad}}^2\widehat{\boldsymbol{\varphi}} : [\widehat{\mathbf{I}}_{\parallel} \otimes \widehat{\mathbf{I}}_{\parallel}] - \widehat{\text{Grad}}\widehat{\boldsymbol{\varphi}} \cdot [\widehat{\mathbf{K}} \otimes \widehat{\mathbf{N}} + \widehat{\mathbf{N}} \otimes \widehat{\mathbf{K}}], \quad (9b)$$

$$\widehat{\text{Grad}}_{\perp}^2\widehat{\boldsymbol{\varphi}} = \widehat{\text{Grad}}^2\widehat{\boldsymbol{\varphi}} : [\widehat{\mathbf{I}}_{\perp} \otimes \widehat{\mathbf{I}}_{\perp}], \quad (9c)$$

$$\widehat{\text{Grad}}_{\parallel\perp}^2\widehat{\boldsymbol{\varphi}} = \widehat{\text{Grad}}^2\widehat{\boldsymbol{\varphi}} : [\widehat{\mathbf{I}}_{\perp} \otimes \widehat{\mathbf{I}}_{\parallel}] + \widehat{\text{Grad}}\widehat{\boldsymbol{\varphi}} \cdot [\widehat{\mathbf{K}} \otimes \widehat{\mathbf{N}} + \widehat{\mathbf{N}} \otimes \widehat{\mathbf{K}}], \quad (9d)$$

$$\widehat{\text{Grad}}_{\perp}^2\widehat{\boldsymbol{\varphi}} = \widehat{\text{Grad}}^2\widehat{\boldsymbol{\varphi}} : [\widehat{\mathbf{I}}_{\perp} \otimes \widehat{\mathbf{I}}_{\perp}]. \quad (9e)$$

Note that equations (9b) to (9e) add up to (9a) exactly. The total second gradient of the surface deformation map has more structure than its purely tangential part  $\widehat{\text{Grad}}_{\parallel}^2\widehat{\boldsymbol{\varphi}}$  according to Eq. (8). Nevertheless, it is clear from (9b) that even the purely tangential part depends on the curvature tensor. This shall be compared to the extensions of the surface elasticity theory to capture flexural resistance pioneered by Steigmann and Ogden (1999) and followed e.g. by Fried and Todres (2005); Chhapadia et al. (2011). One novel aspect of this manuscript is to allow for the total second gradient of the surface deformation map and not only its purely tangential part.  $\square$

The boundary curve placements  $\widetilde{\mathbf{X}}$  and  $\widetilde{\mathbf{x}}$ , respectively in the material and the spatial configurations, are related by the invertible (nonlinear) curve deformation map  $\widetilde{\boldsymbol{\varphi}}$  through

$$\widetilde{\mathbf{x}} = \widetilde{\boldsymbol{\varphi}}(\widetilde{\mathbf{X}}) \quad \text{with} \quad \widetilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi}|_{C_0}, \quad \widetilde{\mathbf{X}} = \mathbf{X}|_{C_0}, \quad \widetilde{\mathbf{x}} = \mathbf{x}|_{C_1}. \quad (10)$$

The total curve gradient of the curve deformation map  $\widetilde{\boldsymbol{\varphi}}$  is defined by

$$\widetilde{\mathbf{F}}_1(\widetilde{\mathbf{X}}) := \widetilde{\text{Grad}}\widetilde{\boldsymbol{\varphi}}(\widetilde{\mathbf{X}}).$$

**Remark** In the functional analytic framework here (and by taking into account the results obtained e.g. by Silhavy (1985, 1991) in generalising the divergence theorem), some regularity assumptions which are logically coherent and are able to allow for all differential and tensorial manipulations which will be presented in the sequel are made:

- inside  $\mathcal{B}_0$  the tensor fields  $\mathbf{F}_1$ ,  $\mathbf{F}_2$  and  $\mathbf{F}_3$  are (square) integrable (or shortly  $\boldsymbol{\varphi}$  belongs to the Sobolev space  $H^3$ ).
- on the faces of  $\partial\mathcal{B}_0$  it is possible to consider the restrictions of  $\mathbf{F}_1$  and  $\mathbf{F}_2$  and these restrictions are (square) integrable with respect to a Hausdorff bi-dimensional measure (therefore there exist suitable traces -in the sense of Sobolev- of  $\mathbf{F}_1$  and  $\mathbf{F}_2$  on every face).
- on the edges of  $\partial\mathcal{B}_0$  it is possible to consider the restrictions of  $\mathbf{F}_1$  and this restriction is (square) integrable with

respect to a Hausdorff unidimensional measure (therefore there exist suitable traces -in the sense of Sobolev- of  $F_1$  on every edge).

- on the wedges is possible to consider the trace of  $\varphi$ .
- for all previous surface and curve fields the regularity assumptions allowing for integration by parts hold.  $\square$

### 2.5. Divergence theorems

The extended form of the divergence theorem in the material configuration is now given. The (bulk) divergence theorem relates the material divergence of a quantity over the control volume  $\mathcal{B}_0$  to the flux of the quantity over the boundary  $\partial\mathcal{B}_0$ . For a tensor field  $\{\bullet\}$  it thus holds that

$$\int_{\mathcal{B}_0} \text{Div} \{\bullet\} = \int_{\partial\mathcal{B}_0} \{\bullet\} \cdot \widehat{N}. \quad (11a)$$

Similarly, the corresponding surface and curve divergence theorems for tensorial quantities on the surface  $\{\widehat{\bullet}\}$  and on the curve  $\{\widetilde{\bullet}\}$  are respectively given by

$$\int_{\partial\mathcal{B}_0} \widehat{\text{Div}}_{\parallel} \{\widehat{\bullet}\} = \int_{\partial^2\mathcal{B}_0} \{\widehat{\bullet}\} \cdot \widehat{M} - \int_{\partial\mathcal{B}_0} \widehat{K} \{\widehat{\bullet}\} \cdot \widehat{N}, \quad (11b)$$

$$\int_{\partial^2\mathcal{B}_0} \widetilde{\text{Div}}_{\parallel} \{\widetilde{\bullet}\} = \sum_{\partial^3\mathcal{B}_0} \{\widetilde{\bullet}\} \cdot \widetilde{T} - \int_{\partial^2\mathcal{B}_0} \widetilde{K} \{\widetilde{\bullet}\} \cdot \widetilde{N}, \quad (11c)$$

where  $\widehat{K}$  and  $\widetilde{K}$  denote twice the mean curvature of the surface and curve, respectively, defined by

$$\widehat{K} := -\widehat{\text{Div}}_{\parallel} \widehat{N}, \quad \text{and} \quad \widetilde{K} := -\widetilde{\text{Div}}_{\parallel} \widetilde{N}. \quad (12)$$

### 2.6. Key relations and identities

Various key relations and identities which are required in the remainder of the manuscript are now introduced with proof.

The tangent surface divergence of the tangent projection of a surface quantity  $\{\widehat{\bullet}\}$  is denoted as a surface differential operator  $\widehat{S}\{\widehat{\bullet}\}$ . In an identical fashion, a curve differential operator  $\widetilde{S}\{\widetilde{\bullet}\}$  is defined. That is

$$\widehat{S}\{\widehat{\bullet}\} := \widehat{\text{Div}}_{\parallel}(\{\widehat{\bullet}\} \cdot \widehat{I}_{\parallel}) = \widehat{K} \{\widehat{\bullet}\} \cdot \widehat{N} + \widehat{\text{Div}}_{\parallel} \{\widehat{\bullet}\}, \quad (13a)$$

$$\widetilde{S}\{\widetilde{\bullet}\} := \widetilde{\text{Div}}_{\parallel}(\{\widetilde{\bullet}\} \cdot \widetilde{I}_{\parallel}) = \widetilde{K} \{\widetilde{\bullet}\} \cdot \widetilde{N} + \widetilde{\text{Div}}_{\parallel} \{\widetilde{\bullet}\}. \quad (13b)$$

Let  $\mathbf{v}$ ,  $\mathbf{A}$ ,  $\mathcal{B}$  and  $\mathbb{C}$  denote a first, second, third and fourth order tensor, respectively. Note that  $\mathbf{v}$ ,  $\mathbf{A}$ ,  $\mathcal{B}$  and  $\mathbb{C}$  can represent surface, curve or point quantities. The following relations hold

$$\mathbf{A} : \text{Grad} \mathbf{v} = \text{Div}(\mathbf{v} \cdot \mathbf{A}) - \mathbf{v} \cdot \text{Div} \mathbf{A}, \quad (14a)$$

$$\mathcal{B} :: \text{Grad} \mathbf{A} = \text{Div}(\mathbf{A} : \mathcal{B}) - \mathbf{A} : \text{Div} \mathcal{B}, \quad (14b)$$

$$\mathbb{C} :: \text{Grad} \mathcal{B} = \text{Div}(\mathcal{B} :: \mathbb{C}) - \mathcal{B} :: \text{Div} \mathbb{C}. \quad (14c)$$

Now, integrations by parts in the bulk, on the surface and curve are obtained by combining equations (11a)-(11c) and (14a)-(14c). Inserting equations (14a), (14b) and (14c), respectively, in the bulk divergence theorem (11a) yields

$$\int_{\mathcal{B}_0} \mathbf{A} : \text{Grad} \mathbf{v} = \int_{\partial \mathcal{B}_0} \mathbf{v} \cdot \mathbf{A} \cdot \widehat{\mathbf{N}} - \int_{\mathcal{B}_0} \mathbf{v} \cdot \text{Div} \mathbf{A}, \quad (15a)$$

$$\int_{\mathcal{B}_0} \mathcal{B} :: \text{Grad} \mathbf{A} = \int_{\partial \mathcal{B}_0} \mathbf{A} : \mathcal{B} \cdot \widehat{\mathbf{N}} - \int_{\mathcal{B}_0} \mathbf{A} : \text{Div} \mathcal{B}, \quad (15b)$$

$$\int_{\mathcal{B}_0} \mathbb{C} :: \text{Grad} \mathcal{B} = \int_{\partial \mathcal{B}_0} \mathcal{B} :: \mathbb{C} \cdot \widehat{\mathbf{N}} - \int_{\mathcal{B}_0} \mathcal{B} :: \text{Div} \mathbb{C}. \quad (15c)$$

Next, integrating equation (14a) yields

$$\int_{\partial \mathcal{B}_0} \mathbf{A} : \text{Grad} \mathbf{v} = \int_{\partial \mathcal{B}_0} \widehat{\text{Div}}_{\parallel}(\mathbf{v} \cdot \mathbf{A}) + \widehat{\text{Div}}_{\perp}(\mathbf{v} \cdot \mathbf{A}) - \mathbf{v} \cdot \widehat{\text{Div}}_{\parallel} \mathbf{A} - \mathbf{v} \cdot \widehat{\text{Div}}_{\perp} \mathbf{A},$$

using the surface divergence theorem (11b) and the identity  $\widehat{\text{Div}}_{\perp}(\mathbf{v} \cdot \mathbf{A}) - \mathbf{v} \cdot \widehat{\text{Div}}_{\perp} \mathbf{A} = \widehat{\text{Grad}}_{\perp} \mathbf{v} : \mathbf{A}$ , renders further

$$\begin{aligned} &= \int_{\partial^2 \mathcal{B}_0} \mathbf{v} \cdot \mathbf{A} \cdot \widehat{\mathbf{M}} - \int_{\partial \mathcal{B}_0} \mathbf{v} \cdot \underbrace{[\widehat{\mathbf{K}} \mathbf{A} \cdot \widehat{\mathbf{N}} + \widehat{\text{Div}}_{\parallel} \mathbf{A}]}_{\widehat{\mathbf{S}}(\mathbf{A})} - \widehat{\text{Grad}}_{\perp} \mathbf{v} : \mathbf{A}, \\ &= \int_{\partial^2 \mathcal{B}_0} \mathbf{v} \cdot \mathbf{A} \cdot \widehat{\mathbf{M}} - \int_{\partial \mathcal{B}_0} \mathbf{v} \cdot \widehat{\mathbf{S}}(\mathbf{A}) - \widehat{\text{Grad}}_N \mathbf{v} \cdot [\mathbf{A} \cdot \widehat{\mathbf{N}}], \end{aligned} \quad (16)$$

Likewise starting from equation (14b), we obtain

$$\begin{aligned} \int_{\partial \mathcal{B}_0} \mathcal{B} :: \text{Grad} \mathbf{A} &= \int_{\partial^2 \mathcal{B}_0} \mathbf{A} : \mathcal{B} \cdot \widehat{\mathbf{M}} - \int_{\partial \mathcal{B}_0} \mathbf{A} : \underbrace{[\widehat{\mathbf{K}} \mathcal{B} \cdot \widehat{\mathbf{N}} + \widehat{\text{Div}}_{\parallel} \mathcal{B}]}_{\widehat{\mathbf{S}}(\mathcal{B})} - \widehat{\text{Grad}}_{\perp} \mathbf{A} : \mathcal{B}, \\ &= \int_{\partial^2 \mathcal{B}_0} \mathbf{A} : \mathcal{B} \cdot \widehat{\mathbf{M}} - \int_{\partial \mathcal{B}_0} \mathbf{A} : \widehat{\mathbf{S}}(\mathcal{B}) - \widehat{\text{Grad}}_N \mathbf{A} : [\mathcal{B} \cdot \widehat{\mathbf{N}}], \end{aligned} \quad (17)$$

Finally, inserting equations (14a) in the curve divergence theorem (11c) results in

$$\begin{aligned} \int_{\partial^2 \mathcal{B}_0} \mathbf{A} : \text{Grad} \mathbf{v} &= \sum_{\partial^3 \mathcal{B}_0} \mathbf{v} \cdot \mathbf{A} \cdot \tilde{\mathbf{T}} - \int_{\partial^2 \mathcal{B}_0} \mathbf{v} \cdot \underbrace{[\tilde{K} \mathbf{A} \cdot \tilde{\mathbf{N}} + \tilde{\text{Div}}_{\parallel} \mathbf{A}]}_{\tilde{\mathbf{S}}(\mathbf{A})} - \widetilde{\text{Grad}}_{\perp} \mathbf{v} : \mathbf{A}, \\ &= \sum_{\partial^3 \mathcal{B}_0} \mathbf{v} \cdot \mathbf{A} \cdot \tilde{\mathbf{T}} - \int_{\partial^2 \mathcal{B}_0} \mathbf{v} \cdot \tilde{\mathbf{S}}(\mathbf{A}) - \widetilde{\text{Grad}}_N \mathbf{v} \cdot [\mathbf{A} \cdot \tilde{\mathbf{N}}] - \widetilde{\text{Grad}}_M \mathbf{v} \cdot [\mathbf{A} \cdot \tilde{\mathbf{M}}]. \end{aligned} \quad (18)$$

The techniques we use in this manuscript are indeed a reformulation of those used by Mindlin (1965). In the appendix of that paper the results of integration by parts on surfaces and curves are also shown leading to edge and point boundary conditions. Our reformulation is, however, intended to make the hierarchical structures of the procedure used evident, relative to the dimension of considered support of deformation energy, and to clarify the role of the newly introduced surface and curve deformation energy densities.

### 3. Governing equations

In order to obtain the governing equations for higher order continua, the total energy functional is minimized. This is done by setting the first variation of the total energy functional to zero. The total energy functional  $\Psi^{\text{tot}}$  consists of the internal and external contributions denoted as  $\Psi_{\text{int}}^{\text{tot}}$  and  $\Psi_{\text{ext}}^{\text{tot}}$ , respectively. That is

$$\Psi^{\text{tot}} = \Psi_{\text{int}}^{\text{tot}} + \Psi_{\text{ext}}^{\text{tot}}. \quad (19)$$

The contributions from the internal and external energies are detailed in Section 3.1 and Section 3.2, respectively. In Section 3.3 local balance equations are obtained from the results discussed in Section 3.1 and Section 3.2. In Section 3.4 the local balance equations obtained in Section 3.3 are discussed. In particular, the balance equations are simplified for some special cases to better understand the meaning of each of the equations. This procedure results in extracting several important logical conclusions on the compatibility of bulk, surface and curve energies.

#### 3.1. Internal energy

Let  $\Psi_{\text{int}}^{\text{tot}}$  be the total internal energy functional consisting of the stored energy in the bulk, surface and curve due to the deformation. That is

$$\Psi_{\text{int}}^{\text{tot}} = \int_{\mathcal{B}_0} \psi_{\text{int}} + \int_{\partial \mathcal{B}_0} \widehat{\psi}_{\text{int}} + \int_{\partial^2 \mathcal{B}_0} \widetilde{\psi}_{\text{int}}, \quad (20)$$

where  $\psi_{\text{int}}$ ,  $\widehat{\psi}_{\text{int}}$  and  $\widetilde{\psi}_{\text{int}}$  are, respectively, the bulk, surface and curve internal energy densities depending on the local state of the deformation.

Following Mindlin and Germain we depart from the Navier–Cauchy conceptual framework by assuming that the set of tensors needed to measure deformation at any given point  $\mathbf{X}$  does not reduce to the deformation gradient  $\mathbf{F}_1$ . Instead we assume that, e.g. because of length scale phenomena and microscopic inhomogeneities, actually the deformation must be measured also by introducing at each  $\mathbf{X}$  the two higher order gradients of the deformation map tensors  $\mathbf{F}_2$  and  $\mathbf{F}_3$ . Therefore, in this spirit, to get a complete deformation measure (i.e. to get a suitable set of independent variables for the deformation energy density) we need to consider the triple  $\mathbf{F}_1$ ,  $\mathbf{F}_2$  and  $\mathbf{F}_3$ . We will assume that inside  $\mathcal{B}_0$  these three fields are suitably regular and, most importantly, there exist the limits of these fields on any face, any edge and any wedge of the boundary. The reader interested into mathematical details may think to the obtained fields on faces, edges and wedges as the restrictions of bulk fields in the sense specified in Kellogg (1929).

The arguments of the energy densities are assumed to be (see dell’Isola et al., 2012, for further reasoning)

$$\psi_{\text{int}} = \psi_{\text{int}}(\text{Grad}\boldsymbol{\varphi}, \text{Grad}^2\boldsymbol{\varphi}, \text{Grad}^3\boldsymbol{\varphi}), \quad \widehat{\psi}_{\text{int}} = \widehat{\psi}_{\text{int}}(\widehat{\text{Grad}}\widehat{\boldsymbol{\varphi}}, \widehat{\text{Grad}}^2\widehat{\boldsymbol{\varphi}}), \quad \widetilde{\psi}_{\text{int}} = \widetilde{\psi}_{\text{int}}(\widetilde{\text{Grad}}\widetilde{\boldsymbol{\varphi}}). \quad (21)$$

Note that  $\widehat{\text{Grad}}\widehat{\boldsymbol{\varphi}}$  and  $\widehat{\text{Grad}}^2\widehat{\boldsymbol{\varphi}}$  denote the restriction of  $\text{Grad}\boldsymbol{\varphi}$  and  $\text{Grad}^2\boldsymbol{\varphi}$  to  $\partial\mathcal{B}_0$ . Similarly,  $\widetilde{\text{Grad}}\widetilde{\boldsymbol{\varphi}}$  denotes the restriction of  $\text{Grad}\boldsymbol{\varphi}$  to  $\partial^2\mathcal{B}_0$ . The total internal energy functional  $\Psi_{\text{int}}^{\text{tot}}$  can be expressed in terms of the placement field  $\boldsymbol{\varphi}$  as

$$\Psi_{\text{int}}^{\text{tot}}(\boldsymbol{\varphi}) = \int_{\mathcal{B}_0} \psi_{\text{int}}(\text{Grad}\boldsymbol{\varphi}, \text{Grad}^2\boldsymbol{\varphi}, \text{Grad}^3\boldsymbol{\varphi}) + \int_{\partial\mathcal{B}_0} \widehat{\psi}_{\text{int}}(\widehat{\text{Grad}}\widehat{\boldsymbol{\varphi}}, \widehat{\text{Grad}}^2\widehat{\boldsymbol{\varphi}}) + \int_{\partial^2\mathcal{B}_0} \widetilde{\psi}_{\text{int}}(\widetilde{\text{Grad}}\widetilde{\boldsymbol{\varphi}}). \quad (22)$$

Next, the first variation of the total energy functional  $\delta\Psi_{\text{int}}^{\text{tot}}(\boldsymbol{\varphi})$  is calculated. In order to calculate the first variation of  $\Psi_{\text{int}}^{\text{tot}}(\boldsymbol{\varphi})$ , we split the derivations into three parts, i.e. for the bulk, the surface and the curve detailed in Section 3.1.1, Section 3.1.2 and Section 3.1.3, respectively. The derivations for each part is carried out independently from the others.

### 3.1.1. Bulk

The first variation of the bulk contributions to the total internal energy functional is derived in this section. That is

$$\begin{aligned} \delta \int_{\mathcal{B}_0} \psi_{\text{int}}(\text{Grad}\boldsymbol{\varphi}, \text{Grad}^2\boldsymbol{\varphi}, \text{Grad}^3\boldsymbol{\varphi}) \\ = \int_{\mathcal{B}_0} \mathbf{P}_1 : \text{Grad}\delta\boldsymbol{\varphi} + \int_{\mathcal{B}_0} \mathbf{P}_2 :: \text{Grad}^2\delta\boldsymbol{\varphi} + \int_{\mathcal{B}_0} \mathbf{P}_3 ::: \text{Grad}^3\delta\boldsymbol{\varphi}, \end{aligned} \quad (23)$$

with the Piola-type stress, double stress and triple stress denoted as  $\mathbf{P}_1$ ,  $\mathbf{P}_2$  and  $\mathbf{P}_3$ , respectively, defined by:

$$\mathbf{P}_1 := \frac{\partial \psi_{\text{int}}}{\partial \text{Grad} \boldsymbol{\varphi}} = \frac{\partial \psi_{\text{int}}}{\partial \mathbf{F}_1}, \quad \mathbf{P}_2 := \frac{\partial \psi_{\text{int}}}{\partial \text{Grad}^2 \boldsymbol{\varphi}} = \frac{\partial \psi_{\text{int}}}{\partial \mathbf{F}_2}, \quad \mathbf{P}_3 := \frac{\partial \psi_{\text{int}}}{\partial \text{Grad}^3 \boldsymbol{\varphi}} = \frac{\partial \psi_{\text{int}}}{\partial \mathbf{F}_3}.$$

The three terms on the right-hand side of the Eq. (23) are studied separately.

**Remark** In the same sense that the Piola stress  $\mathbf{P}_1$  is the energy conjugate to  $\mathbf{F}_1$ , the (first) gradient of the deformation map  $\boldsymbol{\varphi}$ , the Piola-type double stress  $\mathbf{P}_2$  is the energy conjugate to  $\mathbf{F}_2$ , the second gradient of the deformation map and also, the Piola-type triple stress  $\mathbf{P}_3$  is the energy conjugate to  $\mathbf{F}_3$ , the third gradient of the deformation map.

The first term on the right-hand side of the Eq. (23) can be simplified by employing Eq. (15a) as

$$\int_{\mathcal{B}_0} \mathbf{P}_1 : \text{Grad} \delta \boldsymbol{\varphi} = \int_{\partial \mathcal{B}_0} \delta \boldsymbol{\varphi} \cdot \mathbf{P}_1 \cdot \widehat{\mathbf{N}} - \int_{\mathcal{B}_0} \delta \boldsymbol{\varphi} \cdot \text{Div} \mathbf{P}_1. \quad (24)$$

The second term on the right-hand side of the Eq. (23) can be reformulated using Eq. (15b) as

$$\int_{\mathcal{B}_0} \mathbf{P}_2 :: \text{Grad}^2 \delta \boldsymbol{\varphi} = \int_{\partial \mathcal{B}_0} \text{Grad} \delta \boldsymbol{\varphi} : [\mathbf{P}_2 \cdot \widehat{\mathbf{N}}] - \int_{\mathcal{B}_0} \text{Grad} \delta \boldsymbol{\varphi} : \text{Div} \mathbf{P}_2$$

applying Eq. (16) on the first and Eq. (15a) on the second term,

$$\begin{aligned} &= \int_{\partial^2 \mathcal{B}_0} \delta \boldsymbol{\varphi} \cdot [\mathbf{P}_2 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]] - \int_{\partial \mathcal{B}_0} \delta \boldsymbol{\varphi} \cdot \widehat{\mathbf{S}}(\mathbf{P}_2 \cdot \widehat{\mathbf{N}}) + \int_{\partial \mathcal{B}_0} \widehat{\text{Grad}}_N \delta \boldsymbol{\varphi} \cdot [\mathbf{P}_2 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] \\ &\quad - \int_{\partial \mathcal{B}_0} \delta \boldsymbol{\varphi} \cdot \text{Div} \mathbf{P}_2 \cdot \widehat{\mathbf{N}} + \int_{\mathcal{B}_0} \delta \boldsymbol{\varphi} \cdot \text{Div}^2 \mathbf{P}_2 \end{aligned} \quad (25)$$

The third term on the right-hand side of the Eq. (23) is written by using Eq. (15c) as

$$\int_{\mathcal{B}_0} \mathbf{P}_3 :: \text{Grad}^3 \delta \boldsymbol{\varphi} = \int_{\partial \mathcal{B}_0} \text{Grad}^2 \delta \boldsymbol{\varphi} : [\mathbf{P}_3 \cdot \widehat{\mathbf{N}}] - \int_{\mathcal{B}_0} \text{Grad}^2 \delta \boldsymbol{\varphi} : \text{Div} \mathbf{P}_3$$

applying Eq. (17) on the first term and Eq. (15b) on the second term,

$$\begin{aligned} &= \int_{\partial^2 \mathcal{B}_0} \text{Grad} \delta \boldsymbol{\varphi} : [\mathbf{P}_3 \cdot \widehat{\mathbf{N}}] \cdot \widehat{\mathbf{M}} - \int_{\partial \mathcal{B}_0} \text{Grad} \delta \boldsymbol{\varphi} : \widehat{\mathbf{S}}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}}) + \int_{\partial \mathcal{B}_0} \widehat{\text{Grad}}_N \text{Grad} \delta \boldsymbol{\varphi} : [\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] \\ &\quad - \int_{\partial \mathcal{B}_0} \text{Grad} \delta \boldsymbol{\varphi} : [\text{Div} \mathbf{P}_3 \cdot \widehat{\mathbf{N}}] + \int_{\mathcal{B}_0} \text{Grad} \delta \boldsymbol{\varphi} : \text{Div}^2 \mathbf{P}_3 \end{aligned}$$

applying Eq. (18) on the first term, Eq. (16) on the second term, the identity  $\widehat{\text{Grad}}_N \text{Grad} \{\bullet\} = \text{Grad} \widehat{\text{Grad}}_N \{\bullet\} + \text{Grad} \{\bullet\} \cdot \widehat{\mathbf{K}}$  on the third term, Eq. (16) on the fourth term and Eq. (15a) on the last term,

$$\begin{aligned}
&= \sum_{\partial^3 \mathcal{B}_0} \delta \varphi \cdot [[\mathbf{P}_3 \cdot \widehat{\mathbf{N}}] \cdot \widehat{\mathbf{M}}] \cdot \widetilde{\mathbf{T}} - \int_{\partial^2 \mathcal{B}_0} \delta \varphi \cdot \widetilde{\mathbf{S}}([\mathbf{P}_3 \cdot \widehat{\mathbf{N}}] \cdot \widehat{\mathbf{M}}) + \int_{\partial^2 \mathcal{B}_0} \widehat{\text{Grad}}_N \delta \varphi \cdot [[\mathbf{P}_3 \cdot \widehat{\mathbf{N}}] : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{M}}]] \\
&\quad + \int_{\partial^2 \mathcal{B}_0} \widehat{\text{Grad}}_M \delta \varphi \cdot [[\mathbf{P}_3 \cdot \widehat{\mathbf{N}}] : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{M}}]] - \int_{\partial^2 \mathcal{B}_0} \delta \varphi \cdot \widetilde{\mathbf{S}}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}}) \cdot \widehat{\mathbf{M}} + \int_{\partial \mathcal{B}_0} \delta \varphi \cdot \widetilde{\mathbf{S}}(\widehat{\mathbf{S}}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}})) \\
&\quad - \int_{\partial \mathcal{B}_0} \widehat{\text{Grad}}_N \delta \varphi \cdot [\widehat{\mathbf{S}}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}}) \cdot \widehat{\mathbf{N}}] + \int_{\partial \mathcal{B}_0} \widehat{\text{Grad}} \widehat{\text{Grad}}_N \delta \varphi : [\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] + \int_{\partial \mathcal{B}_0} [\text{Grad} \delta \varphi \cdot \widehat{\mathbf{K}}] : [\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] \\
&\quad - \int_{\partial^2 \mathcal{B}_0} \delta \varphi \cdot [\text{Div} \mathbf{P}_3 \cdot \widehat{\mathbf{N}}] \cdot \widehat{\mathbf{M}} + \int_{\partial \mathcal{B}_0} \delta \varphi \cdot \widetilde{\mathbf{S}}(\text{Div} \mathbf{P}_3 \cdot \widehat{\mathbf{N}}) \\
&\quad - \int_{\partial \mathcal{B}_0} \widehat{\text{Grad}}_N \delta \varphi \cdot [\text{Div} \mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] + \int_{\partial \mathcal{B}_0} \delta \varphi \cdot \text{Div}^2 \mathbf{P}_3 \cdot \widehat{\mathbf{N}} - \int_{\mathcal{B}_0} \delta \varphi \cdot \text{Div}^3 \mathbf{P}_3
\end{aligned}$$

applying Eq. (16) on the last two terms of the third line and using the relation  $[\text{Grad} \delta \varphi \cdot \widehat{\mathbf{K}}] : [\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] = \text{Grad} \delta \varphi : [[\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] \cdot \widehat{\mathbf{K}}]$  which is valid due to the symmetry of the curvature tensor renders finally

$$\begin{aligned}
&= \sum_{\partial^3 \mathcal{B}_0} \delta \varphi \cdot [\mathbf{P}_3 : [\widetilde{\mathbf{T}} \otimes \widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]] - \int_{\partial^2 \mathcal{B}_0} \delta \varphi \cdot \widetilde{\mathbf{S}}([\mathbf{P}_3 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]]) + \int_{\partial^2 \mathcal{B}_0} \widehat{\text{Grad}}_N \delta \varphi \cdot [\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]] \\
&\quad + \int_{\partial^2 \mathcal{B}_0} \widehat{\text{Grad}}_M \delta \varphi \cdot [\mathbf{P}_3 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]] - \int_{\partial^2 \mathcal{B}_0} \delta \varphi \cdot \widetilde{\mathbf{S}}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}}) \cdot \widehat{\mathbf{M}} + \int_{\partial \mathcal{B}_0} \delta \varphi \cdot \widetilde{\mathbf{S}}(\widehat{\mathbf{S}}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}})) \\
&\quad - \int_{\partial \mathcal{B}_0} \widehat{\text{Grad}}_N \delta \varphi \cdot [\widehat{\mathbf{S}}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}}) \cdot \widehat{\mathbf{N}}] + \int_{\partial^2 \mathcal{B}_0} \widehat{\text{Grad}}_N \delta \varphi \cdot [\mathbf{P}_3 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] - \int_{\partial \mathcal{B}_0} \widehat{\text{Grad}}_N \delta \varphi \cdot \widetilde{\mathbf{S}}([\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]]) \\
&\quad + \int_{\partial \mathcal{B}_0} \widehat{\text{Grad}}_N^2 \delta \varphi \cdot [\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] + \int_{\partial^2 \mathcal{B}_0} \delta \varphi \cdot [[\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] \cdot \widehat{\mathbf{K}}] \cdot \widehat{\mathbf{M}} - \int_{\partial \mathcal{B}_0} \delta \varphi \cdot \widetilde{\mathbf{S}}([\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] \cdot \widehat{\mathbf{K}}) \\
&\quad + \int_{\partial \mathcal{B}_0} \underline{\widehat{\text{Grad}}_N \delta \varphi \cdot [[\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] \cdot \widehat{\mathbf{K}}] \cdot \widehat{\mathbf{N}}} - \int_{\partial^2 \mathcal{B}_0} \delta \varphi \cdot [\text{Div} \mathbf{P}_3 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]] + \int_{\partial \mathcal{B}_0} \delta \varphi \cdot \widetilde{\mathbf{S}}(\text{Div} \mathbf{P}_3 \cdot \widehat{\mathbf{N}}) \\
&\quad - \int_{\partial \mathcal{B}_0} \widehat{\text{Grad}}_N \delta \varphi \cdot [\text{Div} \mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] + \int_{\partial \mathcal{B}_0} \delta \varphi \cdot \text{Div}^2 \mathbf{P}_3 \cdot \widehat{\mathbf{N}} - \int_{\mathcal{B}_0} \delta \varphi \cdot \text{Div}^3 \mathbf{P}_3 .
\end{aligned}$$

Note that the first term of the fifth line (underlined) vanishes due to the fact that the curvature tensor is tangential.

**Remark** The contributions for the total bulk internal energy functional consist of not only integrals in the bulk  $\mathcal{B}_0$  but also integrals on the surface  $\partial \mathcal{B}_0$  and the curve  $\partial^2 \mathcal{B}_0$ , and the sum over points  $\partial^3 \mathcal{B}_0$ . For instance, the sum over the points is containing the fourth-order tensor  $\mathbf{P}_3$ . That is, in order to capture the internal energy contributions at points, at least a third-order bulk gradient is needed.  $\square$

### 3.1.2. Surface

The first variation of the surface contributions to the total internal energy functional is

$$\delta \int_{\partial \mathcal{B}_0} \widehat{\psi}_{\text{int}}(\widehat{\text{Grad}}\widehat{\boldsymbol{\varphi}}, \widehat{\text{Grad}}^2\widehat{\boldsymbol{\varphi}}) = \int_{\partial \mathcal{B}_0} \widehat{\mathbf{P}}_1 : \widehat{\text{Grad}}\delta\widehat{\boldsymbol{\varphi}} + \int_{\partial \mathcal{B}_0} \widehat{\mathbf{P}}_2 : \widehat{\text{Grad}}^2\delta\widehat{\boldsymbol{\varphi}}, \quad (26)$$

with the Piola-type surface stress and double-stress denoted as  $\widehat{\mathbf{P}}_1$  and  $\widehat{\mathbf{P}}_2$ , respectively, defined by:

$$\widehat{\mathbf{P}}_1 := \frac{\partial \widehat{\psi}_{\text{int}}}{\partial \widehat{\text{Grad}}\widehat{\boldsymbol{\varphi}}} = \frac{\partial \widehat{\psi}_{\text{int}}}{\partial \widehat{\mathbf{F}}_1}, \quad \widehat{\mathbf{P}}_2 := \frac{\partial \widehat{\psi}_{\text{int}}}{\partial \widehat{\text{Grad}}^2\widehat{\boldsymbol{\varphi}}} = \frac{\partial \widehat{\psi}_{\text{int}}}{\partial \widehat{\mathbf{F}}_2}.$$

Note that  $\widehat{\mathbf{P}}_1$  and  $\widehat{\mathbf{P}}_2$  are *not* superficial tensors in the sense of Gurtin and Murdoch (1975).<sup>4</sup>

**Remark** In the same sense that the surface Piola stress  $\widehat{\mathbf{P}}_1$  is the energy conjugate to  $\widehat{\mathbf{F}}_1$ , the (first) gradient of the surface deformation map  $\widehat{\boldsymbol{\varphi}}$ , the surface Piola-type double stress  $\widehat{\mathbf{P}}_2$  is the energy conjugate to  $\widehat{\mathbf{F}}_2$  the second gradient of the deformation map.

The right-hand side of the Eq. (26) can be simplified by employing Eq. (16) and Eq. (17) as

$$\begin{aligned} & \int_{\partial \mathcal{B}_0} \widehat{\mathbf{P}}_1 : \widehat{\text{Grad}}\delta\widehat{\boldsymbol{\varphi}} + \widehat{\mathbf{P}}_2 : \widehat{\text{Grad}}^2\delta\widehat{\boldsymbol{\varphi}} \\ &= \int_{\partial^2 \mathcal{B}_0} \delta\widehat{\boldsymbol{\varphi}} \cdot \widehat{\mathbf{P}}_1 \cdot \widehat{\mathbf{M}} - \int_{\partial \mathcal{B}_0} \delta\widehat{\boldsymbol{\varphi}} \cdot \widehat{\mathbf{S}}(\widehat{\mathbf{P}}_1) + \int_{\partial \mathcal{B}_0} \widehat{\text{Grad}}_N \delta\widehat{\boldsymbol{\varphi}} \cdot [\widehat{\mathbf{P}}_1 \cdot \widehat{\mathbf{N}}] \\ & \quad + \int_{\partial^2 \mathcal{B}_0} \widehat{\text{Grad}}\delta\widehat{\boldsymbol{\varphi}} : [\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{M}}] - \int_{\partial \mathcal{B}_0} \widehat{\text{Grad}}\delta\widehat{\boldsymbol{\varphi}} : \widehat{\mathbf{S}}(\widehat{\mathbf{P}}_2) + \int_{\partial \mathcal{B}_0} \widehat{\text{Grad}}_N \widehat{\text{Grad}}\delta\widehat{\boldsymbol{\varphi}} : [\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{N}}] \end{aligned}$$

applying Eq. (18) on the fourth term and Eq. (16) on the fifth term and the identity  $\widehat{\text{Grad}}_N \widehat{\text{Grad}}\{\bullet\} = \widehat{\text{Grad}}\widehat{\text{Grad}}_N\{\bullet\} + \widehat{\text{Grad}}\{\bullet\} \cdot \widehat{\mathbf{K}}$  on the last term,

$$\begin{aligned} &= \int_{\partial^2 \mathcal{B}_0} \delta\widehat{\boldsymbol{\varphi}} \cdot \widehat{\mathbf{P}}_1 \cdot \widehat{\mathbf{M}} - \int_{\partial \mathcal{B}_0} \delta\widehat{\boldsymbol{\varphi}} \cdot \widehat{\mathbf{S}}(\widehat{\mathbf{P}}_1) + \int_{\partial \mathcal{B}_0} \widehat{\text{Grad}}_N \delta\widehat{\boldsymbol{\varphi}} \cdot [\widehat{\mathbf{P}}_1 \cdot \widehat{\mathbf{N}}] \\ & \quad + \sum_{\partial^3 \mathcal{B}_0} \delta\widehat{\boldsymbol{\varphi}} \cdot [\widehat{\mathbf{P}}_2 : [\widehat{\mathbf{T}} \otimes \widehat{\mathbf{M}}]] - \int_{\partial^2 \mathcal{B}_0} \delta\widehat{\boldsymbol{\varphi}} \cdot \widehat{\mathbf{S}}(\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{M}}) + \int_{\partial^2 \mathcal{B}_0} \widehat{\text{Grad}}_N \delta\widehat{\boldsymbol{\varphi}} \cdot [\widehat{\mathbf{P}}_2 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{M}}]] \\ & \quad + \int_{\partial^2 \mathcal{B}_0} \widehat{\text{Grad}}_M \delta\widehat{\boldsymbol{\varphi}} \cdot [\widehat{\mathbf{P}}_2 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{M}}]] - \int_{\partial^2 \mathcal{B}_0} \delta\widehat{\boldsymbol{\varphi}} \cdot \widehat{\mathbf{S}}(\widehat{\mathbf{P}}_2) \cdot \widehat{\mathbf{M}} + \int_{\partial \mathcal{B}_0} \delta\widehat{\boldsymbol{\varphi}} \cdot \widehat{\mathbf{S}}(\widehat{\mathbf{S}}(\widehat{\mathbf{P}}_2)) \\ & \quad - \int_{\partial \mathcal{B}_0} \widehat{\text{Grad}}_N \delta\widehat{\boldsymbol{\varphi}} \cdot [\widehat{\mathbf{S}}(\widehat{\mathbf{P}}_2) \cdot \widehat{\mathbf{N}}] + \int_{\partial \mathcal{B}_0} \widehat{\text{Grad}}\widehat{\text{Grad}}_N \delta\widehat{\boldsymbol{\varphi}} : [\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{N}}] + \int_{\partial \mathcal{B}_0} [\widehat{\text{Grad}}\delta\widehat{\boldsymbol{\varphi}} \cdot \widehat{\mathbf{K}}] : [\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{N}}] \end{aligned}$$

<sup>4</sup>Second-order tensors and vectors on the surface can be classified as superficial (in their tangent spaces) or tangential. For instance, a superficial material second-order tensor  $\{\bullet\}$  on the surface possesses the orthogonality property  $\{\bullet\} \cdot \widehat{\mathbf{N}} = \mathbf{0}$ . If the arbitrary quantity in the preceding relations is a vector, it is termed tangential. A tangential material second-order tensor  $\{\bullet\}$  on the surface possesses both orthogonality properties  $\{\bullet\} \cdot \widehat{\mathbf{N}} = \mathbf{0}$  and  $\widehat{\mathbf{N}} \cdot \{\bullet\} = \mathbf{0}$ .

applying Eq. (16) on the last two terms and using the relation  $[\widehat{\text{Grad}}\delta\widehat{\varphi} \cdot \widehat{\mathbf{K}}] : [\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{N}}] = \widehat{\text{Grad}}\delta\widehat{\varphi} : [[\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{N}}] \cdot \widehat{\mathbf{K}}]$  which is valid due to the symmetry of the curvature tensor finally renders

$$\begin{aligned}
&= \int_{\partial^2\mathcal{B}_0} \delta\widehat{\varphi} \cdot \widehat{\mathbf{P}}_1 \cdot \widehat{\mathbf{M}} - \int_{\partial\mathcal{B}_0} \delta\widehat{\varphi} \cdot \widehat{\mathbf{S}}(\widehat{\mathbf{P}}_1) + \int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\widehat{\varphi} \cdot [\widehat{\mathbf{P}}_1 \cdot \widehat{\mathbf{N}}] + \sum_{\partial^3\mathcal{B}_0} \delta\widehat{\varphi} \cdot [\widehat{\mathbf{P}}_2 : [\widehat{\mathbf{T}} \otimes \widehat{\mathbf{M}}]] \\
&\quad - \int_{\partial^2\mathcal{B}_0} \delta\widehat{\varphi} \cdot \widehat{\mathbf{S}}(\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{M}}) + \int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\widehat{\varphi} \cdot [\widehat{\mathbf{P}}_2 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{M}}]] + \int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_M\delta\widehat{\varphi} \cdot [\widehat{\mathbf{P}}_2 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{M}}]] \\
&\quad - \int_{\partial^2\mathcal{B}_0} \delta\widehat{\varphi} \cdot \widehat{\mathbf{S}}(\widehat{\mathbf{P}}_2) \cdot \widehat{\mathbf{M}} + \int_{\partial\mathcal{B}_0} \delta\widehat{\varphi} \cdot \widehat{\mathbf{S}}(\widehat{\mathbf{S}}(\widehat{\mathbf{P}}_2)) - \int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\widehat{\varphi} \cdot [\widehat{\mathbf{S}}(\widehat{\mathbf{P}}_2) \cdot \widehat{\mathbf{N}}] \\
&\quad + \int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\widehat{\varphi} \cdot [\widehat{\mathbf{P}}_2 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]] - \int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\widehat{\varphi} \cdot \widehat{\mathbf{S}}(\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{N}}) + \int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N^2\delta\widehat{\varphi} \cdot [\widehat{\mathbf{P}}_2 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] \\
&\quad + \int_{\partial^2\mathcal{B}_0} \delta\widehat{\varphi} \cdot [[\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{N}}] \cdot \widehat{\mathbf{K}}] \cdot \widehat{\mathbf{M}} - \int_{\partial\mathcal{B}_0} \delta\widehat{\varphi} \cdot \widehat{\mathbf{S}}([\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{N}}] \cdot \widehat{\mathbf{K}}) + \int_{\partial\mathcal{B}_0} \underline{\widehat{\text{Grad}}_N\delta\widehat{\varphi} \cdot [[\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{N}}] \cdot \widehat{\mathbf{K}}] \cdot \widehat{\mathbf{N}}}.
\end{aligned}$$

Note that the last term (underlined) vanishes due to the fact that the curvature tensor is tangential.

**Remark** The contributions for the total surface internal energy functional consist of not only integrals on the surface  $\partial\mathcal{B}_0$  but also, integrals on the curve  $\partial^2\mathcal{B}_0$ , and the sum over points  $\partial^3\mathcal{B}_0$ . For instance, the sum over the points is containing the third-order tensor  $\widehat{\mathbf{P}}_2$ . That is, in order to capture the internal energy contributions at points, at least a second-order surface gradient is needed.  $\square$

### 3.1.3. Curve

The first variation of the curve contributions to the total internal energy functional is

$$\delta \int_{\partial^2\mathcal{B}_0} \widetilde{\psi}_{\text{int}}(\widehat{\text{Grad}}\delta\widehat{\varphi}) = \int_{\partial^2\mathcal{B}_0} \widetilde{\mathbf{P}}_1 : \widehat{\text{Grad}}\delta\widehat{\varphi}, \quad (27)$$

with the Piola-type curve stress  $\widetilde{\mathbf{P}}_1$  defined as follows:

$$\widetilde{\mathbf{P}}_1 := \frac{\partial\widetilde{\psi}_{\text{int}}}{\partial\widehat{\text{Grad}}\delta\widehat{\varphi}} = \frac{\partial\widetilde{\psi}_{\text{int}}}{\partial\mathbf{F}_1}.$$

The right-hand side of the Eq. (27) can be simplified by employing Eq. (18) as

$$\begin{aligned}
&\int_{\partial^2\mathcal{B}_0} \widetilde{\mathbf{P}}_1 : \widehat{\text{Grad}}\delta\widehat{\varphi} \\
&= \sum_{\partial^3\mathcal{B}_0} \delta\widehat{\varphi} \cdot \widetilde{\mathbf{P}}_1 \cdot \widetilde{\mathbf{T}} - \int_{\partial^2\mathcal{B}_0} \delta\widehat{\varphi} \cdot \widetilde{\mathbf{S}}(\widetilde{\mathbf{P}}_1) + \int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\widehat{\varphi} \cdot [\widetilde{\mathbf{P}}_1 \cdot \widehat{\mathbf{N}}] + \int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_M\delta\widehat{\varphi} \cdot [\widetilde{\mathbf{P}}_1 \cdot \widehat{\mathbf{M}}].
\end{aligned}$$

**Remark** The contributions for the total curve internal energy functional consist of not only integrals on the curve

$\partial^2 \mathcal{B}_0$  but also the sum over points  $\partial^3 \mathcal{B}_0$ . For instance, the sum over the points is containing the second-order tensor  $\widetilde{\mathbf{P}}_1$ . That is, in order to capture the internal energy contributions at points, at least a first-order curve gradient is needed.  $\square$

### 3.1.4. Bulk, surface and curve

In this section we add all the contributions to the internal energy functional from the bulk, surface and curve resulting from Section 3.1.1, Section 3.1.2 and Section 3.1.3. That is

$$\begin{aligned} \delta \Psi_{\text{int}}^{\text{tot}}(\boldsymbol{\varphi}) &= \delta \int_{\mathcal{B}_0} \psi_{\text{int}}(\text{Grad} \boldsymbol{\varphi}, \text{Grad}^2 \boldsymbol{\varphi}, \text{Grad}^3 \boldsymbol{\varphi}) + \delta \int_{\partial \mathcal{B}_0} \widehat{\psi}_{\text{int}}(\widehat{\text{Grad}} \boldsymbol{\varphi}, \widehat{\text{Grad}}^2 \boldsymbol{\varphi}) + \delta \int_{\partial^2 \mathcal{B}_0} \widetilde{\psi}_{\text{int}}(\widetilde{\text{Grad}} \boldsymbol{\varphi}) \\ &= \int_{\mathcal{B}_0} \mathbf{P}_1 : \text{Grad} \delta \boldsymbol{\varphi} + \mathbf{P}_2 :: \text{Grad}^2 \delta \boldsymbol{\varphi} + \mathbf{P}_3 ::: \text{Grad}^3 \delta \boldsymbol{\varphi} \\ &\quad + \int_{\partial \mathcal{B}_0} \widehat{\mathbf{P}}_1 : \widehat{\text{Grad}} \delta \boldsymbol{\varphi} + \widehat{\mathbf{P}}_2 :: \widehat{\text{Grad}}^2 \delta \boldsymbol{\varphi} + \int_{\partial^2 \mathcal{B}_0} \widetilde{\mathbf{P}}_1 : \widetilde{\text{Grad}} \delta \boldsymbol{\varphi}, \end{aligned}$$

whereby, recalling that we impose kinematic slavery

$$\delta \widehat{\boldsymbol{\varphi}} = \delta \boldsymbol{\varphi}|_{\partial \mathcal{B}_0}, \quad \delta \widetilde{\boldsymbol{\varphi}} = \delta \widehat{\boldsymbol{\varphi}}|_{\partial^2 \mathcal{B}_0} = \delta \boldsymbol{\varphi}|_{\partial^2 \mathcal{B}_0}, \quad \delta \overline{\boldsymbol{\varphi}} = \delta \widetilde{\boldsymbol{\varphi}}|_{\partial^3 \mathcal{B}_0} = \delta \widehat{\boldsymbol{\varphi}}|_{\partial^3 \mathcal{B}_0} = \delta \boldsymbol{\varphi}|_{\partial^3 \mathcal{B}_0},$$

nine types of integrals (see Appendix A for the intermediate steps) are distinguishable and are formally expressed as

$$\begin{aligned} &= \int_{\mathcal{B}_0} \delta \boldsymbol{\varphi} \cdot \{\bullet\} + \int_{\partial \mathcal{B}_0} \delta \boldsymbol{\varphi} \cdot \{\bullet\} + \int_{\partial^2 \mathcal{B}_0} \delta \boldsymbol{\varphi} \cdot \{\bullet\} + \sum_{\partial^3 \mathcal{B}_0} \delta \boldsymbol{\varphi} \cdot \{\bullet\} \\ &\quad + \int_{\partial \mathcal{B}_0} \widehat{\text{Grad}}_N \delta \boldsymbol{\varphi} \cdot \{\bullet\} + \int_{\partial^2 \mathcal{B}_0} \widehat{\text{Grad}}_N \delta \boldsymbol{\varphi} \cdot \{\bullet\} + \int_{\partial^2 \mathcal{B}_0} \widetilde{\text{Grad}}_N \delta \boldsymbol{\varphi} \cdot \{\bullet\} \\ &\quad + \int_{\partial^2 \mathcal{B}_0} \widetilde{\text{Grad}}_M \delta \boldsymbol{\varphi} \cdot \{\bullet\} + \int_{\partial \mathcal{B}_0} \widehat{\text{Grad}}_N^2 \delta \boldsymbol{\varphi} \cdot \{\bullet\}. \end{aligned} \tag{28}$$

### 3.2. External energy

Let  $\Psi_{\text{ext}}^{\text{tot}}$  be the total external energy functional consisting of the contributions from the externally prescribed forces in the bulk and (generalized) tractions on the surface, the curve and the points. That is

$$\Psi_{\text{ext}}^{\text{tot}} = \int_{\mathcal{B}_0} \psi_{\text{ext}} + \int_{\partial \mathcal{B}_0} \widehat{\psi}_{\text{ext}} + \int_{\partial^2 \mathcal{B}_0} \widetilde{\psi}_{\text{ext}} + \sum_{\partial^3 \mathcal{B}_0} \overline{\psi}_{\text{ext}}, \tag{29}$$

where  $\psi_{\text{ext}}$ ,  $\widehat{\psi}_{\text{ext}}$ ,  $\widetilde{\psi}_{\text{ext}}$  and  $\overline{\psi}_{\text{ext}}$  are, respectively, the bulk, surface and curve internal energy densities depending on the local state of the deformation. We assume, without loss of generality, that the first variation of the total external

energy functional  $\Psi_{\text{ext}}^{\text{tot}}$  is of the form

$$\begin{aligned}
-\delta\Psi_{\text{ext}}^{\text{tot}} = & \int_{\mathcal{B}_0} \delta\boldsymbol{\varphi} \cdot \mathbf{b}_1^{\text{p}} + \int_{\partial\mathcal{B}_0} \delta\boldsymbol{\varphi} \cdot \widehat{\mathbf{b}}_1^{\text{p}} + \int_{\partial^2\mathcal{B}_0} \delta\boldsymbol{\varphi} \cdot \widetilde{\mathbf{b}}_1^{\text{p}} + \sum_{\partial^3\mathcal{B}_0} \delta\boldsymbol{\varphi} \cdot \overline{\mathbf{b}}_1^{\text{p}} \\
& + \int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N \delta\boldsymbol{\varphi} \cdot \widehat{\mathbf{b}}_2^{\text{p}} + \int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_N \delta\boldsymbol{\varphi} \cdot \widetilde{\mathbf{b}}_{2_{\widetilde{N}}}^{\text{p}} + \int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_N \delta\boldsymbol{\varphi} \cdot \widetilde{\mathbf{b}}_{2_{\widetilde{N}}}^{\text{p}} \\
& + \int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_M \delta\boldsymbol{\varphi} \cdot \widetilde{\mathbf{b}}_{2_M}^{\text{p}} + \int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N^2 \delta\boldsymbol{\varphi} \cdot \widehat{\mathbf{b}}_3^{\text{p}},
\end{aligned} \tag{30}$$

where  $\mathbf{b}_1^{\text{p}}$  is the common body force per unit volume in the material configuration and  $\widehat{\mathbf{b}}_1^{\text{p}}, \widetilde{\mathbf{b}}_1^{\text{p}}, \overline{\mathbf{b}}_1^{\text{p}}, \widehat{\mathbf{b}}_2^{\text{p}}, \widetilde{\mathbf{b}}_{2_{\widetilde{N}}}^{\text{p}}, \widetilde{\mathbf{b}}_{2_{\widetilde{N}}}^{\text{p}}, \widetilde{\mathbf{b}}_{2_M}^{\text{p}}$  and  $\widehat{\mathbf{b}}_3^{\text{p}}$  are (generalized) tractions on the surface, the curve and the points. As an example,  $\overline{\mathbf{b}}_1^{\text{p}}$  denotes a single-force acting on a point. The generalized tractions are the work conjugates to  $\delta\boldsymbol{\varphi}$  (indicated by the index 1), its normal gradients (indicated by the index 2) and its normal curvature (indicated by the index 3).

### 3.3. Local balance equations

The total energy functional  $\Psi^{\text{tot}}$  that we seek to minimize with respect to all admissible (spatial) variations  $\delta\boldsymbol{\varphi}$  at fixed material placement is

$$\Psi^{\text{tot}} = \Psi_{\text{ext}}^{\text{tot}} + \Psi_{\text{int}}^{\text{tot}}.$$

The minimisation of the  $\Psi^{\text{tot}}$  renders

$$\delta\Psi^{\text{tot}} = \delta\Psi_{\text{ext}}^{\text{tot}} + \delta\Psi_{\text{int}}^{\text{tot}} = 0.$$

Using Eq. (28) and Eq. (30) renders the strong form of balance equations as follows:

$$\text{Div}(\mathbf{P}_1 - \text{Div}(\mathbf{P}_2 - \text{Div}\mathbf{P}_3)) + \mathbf{b}_1^p = \mathbf{0} \quad \text{in } \mathcal{B}_0, \quad (31a)$$

$$\begin{aligned} & [\mathbf{P}_1 - \text{Div}\mathbf{P}_2 + \text{Div}^2\mathbf{P}_3] \cdot \widehat{\mathbf{N}} - \widehat{\mathbf{S}}([\mathbf{P}_2 - \text{Div}\mathbf{P}_3] \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_1) \\ & - \widehat{\mathbf{S}}([\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] \cdot \widehat{\mathbf{N}}) \cdot \widehat{\mathbf{K}} + \widehat{\mathbf{S}}^2(\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2) - \widehat{\mathbf{b}}_1^p = \mathbf{0} \end{aligned} \quad \text{on } \partial\mathcal{B}_0, \quad (31b)$$

$$\begin{aligned} & \sum_{\xi} \left[ [(\mathbf{P}_2 - \text{Div}\mathbf{P}_3) \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_1] \cdot \widehat{\mathbf{M}} + [([\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] \cdot \widehat{\mathbf{N}}) \cdot \widehat{\mathbf{K}}] \cdot \widehat{\mathbf{M}} \right. \\ & \left. - \widehat{\mathbf{S}}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2) \cdot \widehat{\mathbf{M}} - \widehat{\mathbf{S}}([\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] \cdot \widehat{\mathbf{M}} + \widehat{\mathbf{P}}_1) \right]_{\xi} - \widetilde{\mathbf{b}}_1^p = \mathbf{0} \end{aligned} \quad \text{on } \partial^2\mathcal{B}_0, \quad (31c)$$

$$\sum_{\eta} \sum_{\xi} [([\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] \cdot \widehat{\mathbf{M}} + \widehat{\mathbf{P}}_1) \cdot \widetilde{\mathbf{T}}]_{\xi\eta} - \overline{\mathbf{b}}_1^p = \mathbf{0} \quad \text{on } \partial^3\mathcal{B}_0, \quad (31d)$$

$$[(\mathbf{P}_2 - \text{Div}\mathbf{P}_3) \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_1] \cdot \widehat{\mathbf{N}} - \widehat{\mathbf{S}}([\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] \cdot \widehat{\mathbf{N}}) - \widehat{\mathbf{S}}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2) \cdot \widehat{\mathbf{N}} - \widehat{\mathbf{b}}_2^p = \mathbf{0} \quad \text{on } \partial\mathcal{B}_0, \quad (31e)$$

$$\sum_{\xi} [([\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}])_{\xi} - \widetilde{\mathbf{b}}_{2\widehat{\mathbf{N}}}^p = \mathbf{0} \quad \text{on } \partial^2\mathcal{B}_0, \quad (31f)$$

$$\sum_{\xi} [([\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] \cdot \widehat{\mathbf{M}} + \widehat{\mathbf{P}}_1) \cdot \widehat{\mathbf{N}}]_{\xi} - \widetilde{\mathbf{b}}_{2\widehat{\mathbf{N}}}^p = \mathbf{0} \quad \text{on } \partial^2\mathcal{B}_0, \quad (31g)$$

$$\sum_{\xi} [([\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] \cdot \widehat{\mathbf{M}} + \widehat{\mathbf{P}}_1) \cdot \widehat{\mathbf{M}}]_{\xi} - \widetilde{\mathbf{b}}_{2\widehat{\mathbf{M}}}^p = \mathbf{0} \quad \text{on } \partial^2\mathcal{B}_0, \quad (31h)$$

$$[\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}] - \widehat{\mathbf{b}}_3^p = \mathbf{0} \quad \text{on } \partial\mathcal{B}_0. \quad (31i)$$

In Eq. (31) the integrals over the curves  $\partial^2\mathcal{B}_0$  in Eq. (28) and Eq. (30) are replaced by a sum over  $\xi$  where  $\xi$  indicates the surfaces whose junction is the curve for which the local balance equations (31c,f,g,h) holds. In a similar fashion, the sum over the points  $\partial^3\mathcal{B}_0$  is replaced by a (double) sum over  $\xi$  and  $\eta$  where  $\eta$  indicates the curves whose junction is the point for which the local balance equation (31d) holds and  $\xi$  indicates the surfaces whose junctions are those curves.

Table 1: Local balance equations of a third order bulk, second order surface and first order curve. Sum of all entities in each row vanishes. Recall the following definitions:  $\widehat{S}(\{\bullet\}) := \widehat{\text{Div}}_{\parallel}(\{\widehat{\bullet}\} \cdot \widehat{\mathbf{T}}_{\parallel}) = \widehat{K} \{\bullet\} \cdot \widehat{\mathbf{N}} + \widehat{\text{Div}}_{\parallel} \{\bullet\}$  and  $\widetilde{S}(\{\bullet\}) := \widetilde{\text{Div}}_{\parallel}(\{\widetilde{\bullet}\} \cdot \widetilde{\mathbf{T}}_{\parallel}) = \widetilde{K} \{\bullet\} \cdot \widetilde{\mathbf{N}} + \widetilde{\text{Div}}_{\parallel} \{\bullet\}$ .

in	integral type	Bulk (1st. o.)	Bulk (2nd. o.)	Bulk (3rd. o.)	Surface (1st. o.)	Surface (2nd. o.)	Curve (1st. o.)	Tractions
$\mathcal{B}_0$	$\int_{\mathcal{B}_0} \delta\varphi \cdot \{\bullet\}$	$\text{Div} \mathbf{P}_1$	$+\text{Div}^2 \mathbf{P}_2$	$+\text{Div}^3 \mathbf{P}_3$	—	—	—	$+\mathbf{b}_1^p$
$\partial\mathcal{B}_0$	$\int_{\partial\mathcal{B}_0} \delta\varphi \cdot \{\bullet\}$	$\mathbf{P}_1 \cdot \widehat{\mathbf{N}}$	$-\text{Div} \mathbf{P}_2 \cdot \widehat{\mathbf{N}}$ $-\widehat{S}(\mathbf{P}_2 \cdot \widehat{\mathbf{N}})$	$+\text{Div}^2 \mathbf{P}_3 \cdot \widehat{\mathbf{N}}$ $-\widehat{S}([-\text{Div} \mathbf{P}_3] \cdot \widehat{\mathbf{N}})$ $-\widehat{S}([\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] \cdot \widehat{\mathbf{K}})$ $+\widehat{S}^2(\mathbf{P}_3 \cdot \widehat{\mathbf{N}})$	$-\widehat{S}(\widehat{\mathbf{P}}_1)$	$-\widehat{S}([\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{N}}] \cdot \widehat{\mathbf{K}})$ $+\widehat{S}^2(\widehat{\mathbf{P}}_2)$	—	$-\widehat{\mathbf{b}}_1^p$
$\partial\mathcal{B}_0$	$\int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N \delta\varphi \cdot \{\bullet\}$	—	$\mathbf{P}_2 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]$	$-\text{Div} \mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]$ $-\widehat{S}(\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}])$ $-\widehat{S}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}}) \cdot \widehat{\mathbf{N}}$	$+\widehat{\mathbf{P}}_1 \cdot \widehat{\mathbf{N}}$	$-\widehat{S}(\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{N}})$ $-\widehat{S}(\widehat{\mathbf{P}}_2) \cdot \widehat{\mathbf{N}}$	—	$-\widehat{\mathbf{b}}_2^p$
$\partial\mathcal{B}_0$	$\int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N^2 \delta\varphi \cdot \{\bullet\}$	—	—	$+\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]$	—	$+\widehat{\mathbf{P}}_2 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]$	—	$-\widehat{\mathbf{b}}_3^p$
$\partial^2\mathcal{B}_0$	$\int_{\partial^2\mathcal{B}_0} \delta\varphi \cdot \{\bullet\}$	$\sum_{\xi}$	—	$\mathbf{P}_2 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]$ $+\text{Div} \mathbf{P}_3 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]$ $+\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}] \cdot \widehat{\mathbf{K}} \cdot \widehat{\mathbf{M}}$ $-\widehat{S}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}}) \cdot \widehat{\mathbf{M}}$ $-\widehat{S}(\mathbf{P}_3 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}])$	$+\widehat{\mathbf{P}}_1 \cdot \widehat{\mathbf{M}}$	$+\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{N}} \cdot \widehat{\mathbf{K}} \cdot \widehat{\mathbf{M}}$ $-\widehat{S}(\widehat{\mathbf{P}}_2) \cdot \widehat{\mathbf{M}}$ $-\widehat{S}(\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{M}})$	$-\widetilde{S}(\widetilde{\mathbf{P}}_1)$	$-\widetilde{\mathbf{b}}_1^p$
$\partial^2\mathcal{B}_0$	$\int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_N \delta\varphi \cdot \{\bullet\}$	$\sum_{\xi}$	—	$\mathbf{P}_3 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]$	—	$+\widehat{\mathbf{P}}_2 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]$	—	$-\widetilde{\mathbf{b}}_{2_{\widehat{\mathbf{N}}}}^p$
$\partial^2\mathcal{B}_0$	$\int_{\partial^2\mathcal{B}_0} \widetilde{\text{Grad}}_N \delta\varphi \cdot \{\bullet\}$	$\sum_{\xi}$	—	$\mathbf{P}_3 : [\widetilde{\mathbf{N}} \otimes \widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]$	—	$+\widehat{\mathbf{P}}_2 : [\widetilde{\mathbf{N}} \otimes \widehat{\mathbf{M}}]$	$+\widetilde{\mathbf{P}}_1 \cdot \widetilde{\mathbf{N}}$	$-\widetilde{\mathbf{b}}_{2_{\widetilde{\mathbf{N}}}}^p$
$\partial^2\mathcal{B}_0$	$\int_{\partial^2\mathcal{B}_0} \widetilde{\text{Grad}}_M \delta\varphi \cdot \{\bullet\}$	$\sum_{\xi}$	—	$\mathbf{P}_3 : [\widetilde{\mathbf{M}} \otimes \widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]$	—	$+\widehat{\mathbf{P}}_2 : [\widetilde{\mathbf{M}} \otimes \widehat{\mathbf{M}}]$	$+\widetilde{\mathbf{P}}_1 \cdot \widetilde{\mathbf{M}}$	$-\widetilde{\mathbf{b}}_{2_{\widetilde{\mathbf{M}}}}^p$
$\partial^3\mathcal{B}_0$	$\sum_{\partial^3\mathcal{B}_0} \delta\varphi \cdot \{\bullet\}$	$\sum_{\eta} \sum_{\xi}$	—	$\mathbf{P}_3 : [\widetilde{\mathbf{T}} \otimes \widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]$	—	$+\widehat{\mathbf{P}}_2 : [\widetilde{\mathbf{T}} \otimes \widehat{\mathbf{M}}]$	$+\widetilde{\mathbf{P}}_1 \cdot \widetilde{\mathbf{T}}$	$-\widetilde{\mathbf{b}}_1^p$

### 3.4. Discussion and logical consequences of the balance equations

In what follows, the balance equations (31) (that are tabulated in Table 1 according to their various contributions from the bulk, surface and curve, respectively) are studied for some special cases.

The special cases of interest are

- First order continua - without surface and curve energies

For a first order continuum which does not allow for surface and curve energies, the classical balance equations

$$\text{Div} \mathbf{P}_1 + \mathbf{b}_1^p = \mathbf{0} \quad \text{in } \mathcal{B}_0, \quad \mathbf{P}_1 \cdot \widehat{\mathbf{N}} - \widehat{\mathbf{b}}_1^p = \mathbf{0} \quad \text{in } \partial \mathcal{B}_0,$$

are recovered together with the conditions that the (generalized) tractions  $\widehat{\mathbf{b}}_1^p, \bar{\mathbf{b}}_1^p, \widehat{\mathbf{b}}_2^p, \widehat{\mathbf{b}}_{2_{\widehat{\mathbf{N}}}}^p, \widehat{\mathbf{b}}_{2_{\bar{\mathbf{N}}}}^p, \widehat{\mathbf{b}}_{2_{\bar{\mathbf{M}}}}^p$  and  $\widehat{\mathbf{b}}_3^p$  are not admissible.

- Third order continua - without surface and curve energies

For a third order continuum which does not allow for surface and curve energies, the geometrically nonlinear extensions of the balance equations of Mindlin (1965) equations (17,18a-c)

$$\begin{aligned} \text{Div}(\mathbf{P}_1 - \text{Div}(\mathbf{P}_2 - \text{Div} \mathbf{P}_3)) + \mathbf{b}_1^p &= \mathbf{0} && \text{in } \mathcal{B}_0, \\ [\mathbf{P}_1 - \text{Div}(\mathbf{P}_2 - \text{Div} \mathbf{P}_3)] \cdot \widehat{\mathbf{N}} - \widehat{\mathbf{S}}([\mathbf{P}_2 - \text{Div} \mathbf{P}_3] \cdot \widehat{\mathbf{N}}) - \widehat{\mathbf{S}}([\mathbf{P}_3 \cdot \widehat{\mathbf{N}}] \cdot \widehat{\mathbf{N}}) \cdot \widehat{\mathbf{K}} + \widehat{\mathbf{S}}^2(\mathbf{P}_3 \cdot \widehat{\mathbf{N}}) - \widehat{\mathbf{b}}_1^p &= \mathbf{0} && \text{in } \partial \mathcal{B}_0, \\ [\mathbf{P}_2 - \text{Div} \mathbf{P}_3] : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}] - \widehat{\mathbf{S}}(\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]) - \widehat{\mathbf{S}}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}}) \cdot \widehat{\mathbf{N}} - \widehat{\mathbf{b}}_2^p &= \mathbf{0} && \text{in } \partial \mathcal{B}_0, \\ \mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}] - \widehat{\mathbf{b}}_3^p &= \mathbf{0} && \text{in } \partial \mathcal{B}_0. \end{aligned}$$

are recovered together with additional balance equations

$$\sum_{\xi} \left[ [\mathbf{P}_2 - \text{Div} \mathbf{P}_3] : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}] + [[\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] \cdot \widehat{\mathbf{K}} \cdot \widehat{\mathbf{M}} - \widehat{\mathbf{S}}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}}) \cdot \widehat{\mathbf{M}} - \widetilde{\mathbf{S}}(\mathbf{P}_3 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}])] \Big]_{\xi} - \widetilde{\mathbf{b}}_1^p = \mathbf{0} \quad \text{in } \partial^2 \mathcal{B}_0, \quad (32a)$$

$$\sum_{\eta} \sum_{\xi} [\mathbf{P}_3 : [\widetilde{\mathbf{T}} \otimes \widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]]_{\xi \eta} - \widetilde{\mathbf{b}}_1^p = \mathbf{0} \quad \text{in } \partial^3 \mathcal{B}_0, \quad (32b)$$

$$\sum_{\xi} [\mathbf{P}_3 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]]_{\xi} - \widetilde{\mathbf{b}}_{2_{\widehat{\mathbf{N}}}}^p = \mathbf{0} \quad \text{in } \partial^2 \mathcal{B}_0, \quad (32c)$$

$$\sum_{\xi} [\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]]_{\xi} - \widetilde{\mathbf{b}}_{2_{\widehat{\mathbf{N}}}}^p = \mathbf{0} \quad \text{in } \partial^2 \mathcal{B}_0, \quad (32d)$$

$$\sum_{\xi} [\mathbf{P}_3 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]]_{\xi} - \widetilde{\mathbf{b}}_{2_{\widehat{\mathbf{M}}}}^p = \mathbf{0} \quad \text{in } \partial^2 \mathcal{B}_0, \quad (32e)$$

which are in fact also included in (Mindlin, 1965), however, in the rarely appreciated Appendix of that article and not in the main text. From equations (32a-e) several logical consequences can be drawn. For instance, according to equation (32b) a concentrated point traction  $\widetilde{\mathbf{b}}_1^p$  is only admissible if the bulk energy is a function of the third gradient of the deformation field. Similar conclusions can be made from other equations, as well.

- First order continua - with surface and curve energies

For a first order continuum which allows for surface and curve energies to depend on the (first) gradient of the deformation, one equation of interest is

$$\widehat{\mathbf{P}}_1 \cdot \widehat{\mathbf{N}} - \widehat{\mathbf{K}} \widehat{\mathbf{P}}_1 \cdot \widehat{\mathbf{N}} - \widehat{\text{Div}}_{\parallel} \widehat{\mathbf{P}}_1 - \widehat{\mathbf{b}}_1^p = \mathbf{0} \quad \text{with} \quad \widehat{\mathbf{P}}_1 \cdot \widehat{\mathbf{N}} - \widehat{\mathbf{b}}_2^p = \mathbf{0} \quad \text{in } \partial \mathcal{B}_0. \quad (33)$$

It is thus clear that in the absence of  $\widehat{\mathbf{b}}_2^p$  the term  $\widehat{\mathbf{P}}_1 \cdot \widehat{\mathbf{N}}$  vanishes and consequently  $\partial \widehat{\psi}_{\text{int}} / \partial \widehat{\mathbf{F}}_{1\perp} = \mathbf{0}$ . Therefore,

$$\widehat{\mathbf{P}}_1 = \widehat{\mathbf{P}}_1(\widehat{\mathbf{F}}_{1\parallel}) = \frac{\partial \widehat{\psi}_{\text{int}}}{\partial \widehat{\mathbf{F}}_{1\parallel}}.$$

which proves that the surface energy can only depend on the tangential part of the surface deformation gradient and  $\widehat{\mathbf{P}}_1$  is superficial. In other words, *in general, in the absence of higher order gradients in the bulk, the surface energy cannot depend on the normal part of the surface deformation gradient*. This, rather controversial, assumption is very common in the context of surface elasticity theory (see e.g. Javili and Steinmann, 2010a, and references therein). One novel aspect of this work is to justify this assumption and eventually study its validity at all.

Another equation of interest, and indeed having the same spirit as Eq. (33), is

$$\widehat{\mathbf{P}}_1 \cdot \widehat{\mathbf{M}} - \widetilde{\mathbf{K}} \widetilde{\mathbf{P}}_1 \cdot \widetilde{\mathbf{N}} - \widetilde{\text{Div}}_{\parallel} \widetilde{\mathbf{P}}_1 - \widetilde{\mathbf{b}}_1^p = \mathbf{0} \quad \text{with} \quad \widetilde{\mathbf{P}}_1 \cdot \widetilde{\mathbf{N}} - \widetilde{\mathbf{b}}_{2\widetilde{\mathbf{N}}}^p = \mathbf{0} \quad \text{in } \partial^2 \mathcal{B}_0.$$

Analogous to the previous discussion, it is obvious that *in general, in the absence of higher order gradients in the bulk or surface, the curve energy cannot depend on the normal part of the curve deformation gradient*. This can be understood as a degeneration of the surface elasticity theory for curves (see e.g. Javili and Steinmann, 2009, and references therein).

This procedure can be extended by considering several possible cases. Nevertheless, it is straightforward to carry out such reasonings and draw corresponding conclusions. We limit the discussion here to the aforementioned cases and leave further reasoning to the interested reader. In summary, the consequences of the governing equations include the following somehow vague but suggestive statement: *Richer energetic “structures” on the boundary (i.e. surface or curve) require higher gradients in the bulk*. To be more precise, by simple inspection of the pertinent boundary conditions, we can conclude that e.g.

- The dependence of  $\widehat{\psi}_{\text{int}}$  on the normal component of  $\widehat{\mathbf{F}}_1$  is possible only if  $\psi_{\text{int}}$  correspondingly depends on  $\mathbf{F}_2$ .
- The dependence of  $\widetilde{\psi}_{\text{int}}$  on the normal component of  $\widetilde{\mathbf{F}}_1$  is possible only if  $\widetilde{\psi}_{\text{int}}$  correspondingly depends on  $\widetilde{\mathbf{F}}_2$ .
- The dependence of  $\widehat{\psi}_{\text{int}}$  on the normal component of  $\widehat{\mathbf{F}}_2$  is possible only if  $\psi_{\text{int}}$  depends on  $\mathbf{F}_3$ .
- If  $\widehat{\psi}_{\text{int}}$  does not depend on  $\widehat{\mathbf{F}}_2$  then it can depend on the normal component of  $\widehat{\mathbf{F}}_1$  only if  $\psi_{\text{int}}$  depend on  $\mathbf{F}_2$  or  $\mathbf{F}_3$ .

The previously listed requirements on coherent choices for deformation energy densities are suitable for the class of deformation energy functional which we have chosen in this paper.

#### 4. Conclusions

The work by Mindlin (1965) seems to be a source of constant inspiration of further improvements of continuum theory. The integration by parts technique presented there allowed Mindlin to get also those boundary conditions which are needed at parts of the boundary on which edges and points (i.e. where normals of regular faces may suffer discontinuities) are present.

In the present paper we generalize the results of Mindlin to the case of geometrically nonlinear deformable bodies whose boundaries are constituted by surfaces and curves which are “energetic”, on which respectively a surface and curve density of deformation energy must be introduced. We prove that

- volume, surface and curve deformation energy densities cannot depend on deformation measures in an arbitrary and unrelated way: actually it is possible to individuate a hierarchical pattern on their dependence of deformation gradients. If one considers curve energy densities depending on the first placement normal gradient (as evaluated on the edge with a limiting process from the bulk) then the surface energy density must depend on second gradient of placement. If the surface density depends on purely normal second gradient, then volume densities must depend on third gradient of placement,
- points in the considered case of third gradient theories cannot carry concentrated measures of deformation energy. This is, however, possible in models with fourth placements gradients in bulk energy densities,
- surface energy density may contribute to contact interactions on faces and edges of Cauchy cuts together with volume density,
- curve deformation energy density may contribute to contact interactions on edges and points of Cauchy cuts.

The surface elasticity theory pioneered by Gurtin and Murdoch endows the surface with its own energetic structures whereby the surface energy density depends only on the superficial surface deformation gradient. Nevertheless, motivated from physical reasoning and atomistic calculations, the surface energy density should also depend on the normal part of the surface deformation gradient which cannot be captured by the classical surface elasticity theory. This contribution generalizes the classical (first-order) surface elasticity theory in a sense that the surface energy density depends on the total surface deformation gradient. It is shown that such a surface energy requires that the bulk energy density depends not only on the deformation gradient but also on its derivative, i.e. a second-order bulk. Therefore, it is clear that the classical surface elasticity theory is intrinsically limited since it is associated with the classical (first-order) continuum theory of elasticity in the bulk.

Furthermore, a second order surface elasticity theory is introduced whereby the normal parts of the surface deformation gradient and its derivative are allowed. This generalized second-order surface elasticity theory requires a third-order bulk. For a first-order bulk though, it is possible to have a limited second-order surface elasticity theory, similar to the classical theory of surface elasticity, in which the surface energy can only depend on the purely tangential part of the first and second gradient of the deformation. It is important to note that even the purely tangential surface second gradient of the deformation accounts for the curvature which shall be compared to (Steigmann and Ogden, 1999; Fried and Todres, 2005; Chhapadia et al., 2011; Olsson and Park, 2012).

Finally, in analogy to the surface elasticity theory, the classical curve elasticity theory is generalized. The classical curve elasticity theory can be understood as a degeneration of the surface elasticity theory of Gurtin and Murdoch. It is proven that if the curve energy depends on the normal part of the deformation gradient, a third-order bulk and/or a

second-order surface are required.

## Appendix A. Further details on the variation of the internal energy

$$\begin{aligned}
\delta\Psi_{\text{int}}^{\text{tot}}(\varphi) &= \delta \int_{\mathcal{B}_0} \psi_{\text{int}}(\text{Grad}\varphi, \text{Grad}^2\varphi, \text{Grad}^3\varphi) + \delta \int_{\partial\mathcal{B}_0} \widehat{\psi}_{\text{int}}(\widehat{\text{Grad}}\varphi, \widehat{\text{Grad}}^2\varphi) + \delta \int_{\partial^2\mathcal{B}_0} \widetilde{\psi}_{\text{int}}(\widehat{\text{Grad}}\varphi) \\
&= \int_{\mathcal{B}_0} \mathbf{P}_1 : \text{Grad}\delta\varphi + \mathbf{P}_2 :: \text{Grad}^2\delta\varphi + \mathbf{P}_3 ::: \text{Grad}^3\delta\varphi \\
&\quad + \int_{\partial\mathcal{B}_0} \widehat{\mathbf{P}}_1 : \widehat{\text{Grad}}\delta\varphi + \widehat{\mathbf{P}}_2 :: \widehat{\text{Grad}}^2\delta\varphi + \int_{\partial^2\mathcal{B}_0} \widetilde{\mathbf{P}}_1 : \widetilde{\text{Grad}}\delta\varphi \\
&= - \int_{\mathcal{B}_0} \delta\varphi \cdot \text{Div}\mathbf{P}_1 + \int_{\mathcal{B}_0} \delta\varphi \cdot \text{Div}^2\mathbf{P}_2 - \int_{\mathcal{B}_0} \delta\varphi \cdot \text{Div}^3\mathbf{P}_3 \\
&\quad + \int_{\partial\mathcal{B}_0} \delta\varphi \cdot \mathbf{P}_1 \cdot \widehat{\mathbf{N}} - \int_{\partial\mathcal{B}_0} \delta\varphi \cdot \text{Div}\mathbf{P}_2 \cdot \widehat{\mathbf{N}} + \int_{\partial\mathcal{B}_0} \delta\varphi \cdot \text{Div}^2\mathbf{P}_3 \cdot \widehat{\mathbf{N}} \\
&\quad - \int_{\partial\mathcal{B}_0} \delta\varphi \cdot \widehat{\mathbf{S}}(\mathbf{P}_2 \cdot \widehat{\mathbf{N}}) - \int_{\partial\mathcal{B}_0} \delta\widehat{\varphi} \cdot \widehat{\mathbf{S}}([\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{N}}] \cdot \widehat{\mathbf{K}}) + \int_{\partial\mathcal{B}_0} \delta\widehat{\varphi} \cdot \widehat{\mathbf{S}}(\widehat{\mathbf{S}}(\widehat{\mathbf{P}}_2)) + \int_{\partial\mathcal{B}_0} \delta\varphi \cdot \widehat{\mathbf{S}}(\widehat{\mathbf{S}}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}})) \\
&\quad + \int_{\partial\mathcal{B}_0} \delta\varphi \cdot \widehat{\mathbf{S}}(\text{Div}\mathbf{P}_3 \cdot \widehat{\mathbf{N}}) - \int_{\partial\mathcal{B}_0} \delta\varphi \cdot \widehat{\mathbf{S}}([\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] \cdot \widehat{\mathbf{K}}) - \int_{\partial\mathcal{B}_0} \delta\widehat{\varphi} \cdot \widehat{\mathbf{S}}(\widetilde{\mathbf{P}}_1) \\
&\quad + \int_{\partial^2\mathcal{B}_0} \delta\varphi \cdot [\mathbf{P}_2 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]] - \int_{\partial^2\mathcal{B}_0} \delta\widehat{\varphi} \cdot \widetilde{\mathbf{S}}(\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{M}}) - \int_{\partial^2\mathcal{B}_0} \delta\widehat{\varphi} \cdot \widetilde{\mathbf{S}}(\widehat{\mathbf{P}}_2) \cdot \widehat{\mathbf{M}} + \int_{\partial^2\mathcal{B}_0} \delta\widehat{\varphi} \cdot [[\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{N}}] \cdot \widehat{\mathbf{K}}] \cdot \widehat{\mathbf{M}} \\
&\quad - \int_{\partial^2\mathcal{B}_0} \delta\varphi \cdot \widetilde{\mathbf{S}}([\mathbf{P}_3 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]]) - \int_{\partial^2\mathcal{B}_0} \delta\varphi \cdot \widehat{\mathbf{S}}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}}) \cdot \widehat{\mathbf{M}} - \int_{\partial^2\mathcal{B}_0} \delta\varphi \cdot [\text{Div}\mathbf{P}_3 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]] \\
&\quad + \int_{\partial^2\mathcal{B}_0} \delta\widehat{\varphi} \cdot \widetilde{\mathbf{P}}_1 \cdot \widehat{\mathbf{M}} + \int_{\partial^2\mathcal{B}_0} \delta\varphi \cdot [[\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] \cdot \widehat{\mathbf{K}}] \cdot \widehat{\mathbf{M}} - \int_{\partial^2\mathcal{B}_0} \delta\widehat{\varphi} \cdot \widetilde{\mathbf{S}}(\widetilde{\mathbf{P}}_1) \tag{A.1} \\
&\quad + \sum_{\partial^3\mathcal{B}_0} \delta\varphi \cdot [\mathbf{P}_3 : [\widetilde{\mathbf{T}} \otimes \widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]] + \sum_{\partial^3\mathcal{B}_0} \delta\widehat{\varphi} \cdot \widetilde{\mathbf{P}}_1 \cdot \widetilde{\mathbf{T}} + \sum_{\partial^3\mathcal{B}_0} \delta\widehat{\varphi} \cdot [\widetilde{\mathbf{P}}_2 : [\widetilde{\mathbf{T}} \otimes \widehat{\mathbf{M}}]] \\
&\quad + \int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\varphi \cdot [\mathbf{P}_2 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] - \int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\widehat{\varphi} \cdot \widehat{\mathbf{S}}(\widehat{\mathbf{P}}_2) \cdot \widehat{\mathbf{N}} + \int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\widehat{\varphi} \cdot [[\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{N}}] \cdot \widehat{\mathbf{K}}] \cdot \widehat{\mathbf{N}} \\
&\quad - \int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\varphi \cdot [\text{Div}\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] - \int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\widehat{\varphi} \cdot \widehat{\mathbf{S}}(\widehat{\mathbf{P}}_2 \cdot \widehat{\mathbf{N}}) - \int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\varphi \cdot \widehat{\mathbf{S}}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}}) \cdot \widehat{\mathbf{N}} \\
&\quad + \int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\widehat{\varphi} \cdot [\widetilde{\mathbf{P}}_1 \cdot \widehat{\mathbf{N}}] - \int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\varphi \cdot \widehat{\mathbf{S}}([\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]]) \\
&\quad + \int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\varphi \cdot [\mathbf{P}_3 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] + \int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\widehat{\varphi} \cdot [\widetilde{\mathbf{P}}_2 : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]] \\
&\quad + \int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\widehat{\varphi} \cdot [\widetilde{\mathbf{P}}_1 \cdot \widetilde{\mathbf{N}}] + \int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\varphi \cdot [\mathbf{P}_3 : [\widetilde{\mathbf{N}} \otimes \widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]] + \int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_N\delta\widehat{\varphi} \cdot [\widetilde{\mathbf{P}}_2 : [\widetilde{\mathbf{N}} \otimes \widehat{\mathbf{M}}]] \\
&\quad + \int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_M\delta\widehat{\varphi} \cdot [\widetilde{\mathbf{P}}_1 \cdot \widetilde{\mathbf{M}}] + \int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_M\delta\varphi \cdot [\mathbf{P}_3 : [\widetilde{\mathbf{M}} \otimes \widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}]] + \int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_M\delta\widehat{\varphi} \cdot [\widetilde{\mathbf{P}}_2 : [\widetilde{\mathbf{M}} \otimes \widehat{\mathbf{M}}]] \\
&\quad + \int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N^2\delta\varphi \cdot [\mathbf{P}_3 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]] + \int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N^2\delta\widehat{\varphi} \cdot [\widetilde{\mathbf{P}}_2 : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}]]
\end{aligned}$$

The contributions to the internal energy functional can be organised into nine integral types listed below:

$$\begin{aligned}
\int_{\mathcal{B}_0} \delta\varphi \cdot \{\bullet\}, & \quad \{\bullet\} \equiv -\text{Div}(\mathbf{P}_1 - \text{Div}(\mathbf{P}_2 - \text{Div}\mathbf{P}_3)), \\
\int_{\partial\mathcal{B}_0} \delta\varphi \cdot \{\bullet\}, & \quad \{\bullet\} \equiv [\mathbf{P}_1 - \text{Div}\mathbf{P}_2 + \text{Div}^2\mathbf{P}_3] \cdot \widehat{\mathbf{N}} - \widehat{\mathbf{S}}([\mathbf{P}_2 - \text{Div}\mathbf{P}_3] \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_1) \\
& \quad - \widehat{\mathbf{S}}([\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] \cdot \widehat{\mathbf{N}}) \cdot \widehat{\mathbf{K}} + \widehat{\mathbf{S}}^2(\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2), \\
\int_{\partial^2\mathcal{B}_0} \delta\varphi \cdot \{\bullet\}, & \quad \{\bullet\} \equiv \sum_{\xi} \left[ [\mathbf{P}_2 - \text{Div}\mathbf{P}_3] \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_1 \right] \cdot \widehat{\mathbf{M}} + \left[ [\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] \cdot \widehat{\mathbf{N}} \right] \cdot \widehat{\mathbf{K}} \cdot \widehat{\mathbf{M}} \\
& \quad - \widehat{\mathbf{S}}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2) \cdot \widehat{\mathbf{M}} - \widehat{\mathbf{S}}([\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] \cdot \widehat{\mathbf{M}} + \widehat{\mathbf{P}}_1) \Big]_{\xi}, \\
\sum_{\partial^3\mathcal{B}_0} \delta\varphi \cdot \{\bullet\}, & \quad \{\bullet\} \equiv \sum_{\eta} \sum_{\xi} \left[ [\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] \cdot \widehat{\mathbf{M}} + \widehat{\mathbf{P}}_1 \right] \cdot \widehat{\mathbf{T}}_{\xi\eta}, \\
\int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N \delta\varphi \cdot \{\bullet\}, & \quad \{\bullet\} \equiv [[\mathbf{P}_2 - \text{Div}\mathbf{P}_3] \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_1] \cdot \widehat{\mathbf{N}} - \widehat{\mathbf{S}}([\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] \cdot \widehat{\mathbf{N}}) - \widehat{\mathbf{S}}(\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2) \cdot \widehat{\mathbf{N}}, \\
\int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_N \delta\varphi \cdot \{\bullet\}, & \quad \{\bullet\} \equiv \sum_{\xi} \left[ [\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] : [\widehat{\mathbf{M}} \otimes \widehat{\mathbf{N}}] \right]_{\xi}, \\
\int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_N \delta\varphi \cdot \{\bullet\}, & \quad \{\bullet\} \equiv \sum_{\xi} \left[ [\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] \cdot \widehat{\mathbf{M}} + \widehat{\mathbf{P}}_1 \right] \cdot \widehat{\mathbf{N}} \Big]_{\xi}, \\
\int_{\partial^2\mathcal{B}_0} \widehat{\text{Grad}}_M \delta\varphi \cdot \{\bullet\}, & \quad \{\bullet\} \equiv \sum_{\xi} \left[ [\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] \cdot \widehat{\mathbf{M}} + \widehat{\mathbf{P}}_1 \right] \cdot \widehat{\mathbf{M}} \Big]_{\xi}, \\
\int_{\partial\mathcal{B}_0} \widehat{\text{Grad}}_N^2 \delta\varphi \cdot \{\bullet\}, & \quad \{\bullet\} \equiv [\mathbf{P}_3 \cdot \widehat{\mathbf{N}} + \widehat{\mathbf{P}}_2] : [\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}].
\end{aligned}$$

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