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# Clique-width and edge contraction

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**Abstract** : We prove that edge contractions do not preserve the property that a set of graphs has bounded clique-width. This property is preserved by contractions of edges, one end of which is a vertex of degree 2.

**Keywords** : Edge contraction; clique-width; rank-width; monadic second-order transduction.

## 1 Introduction

Clique-width is, like tree-width, an integer graph invariant that is an appropriate parameter for the construction of many FPT algorithms ([CMR, DF, FG, Hli+]). It is thus important to know that the graphs of a particular type have bounded tree-width or clique-width. See [KLM] for a survey. Gurski has reviewed in [Gur] how clique-width behaves under different graph operations. He asks whether, for each  $k$ , the class of graphs of clique-width at most  $k$  is stable under edge contractions. This is true for  $k = 2$ , i.e., for cographs and we prove that this is false for  $k = 3$ . (For each  $k$ , this stability property is true for the class of graphs of tree-width at most  $k$ . It is thus natural to ask the question for clique-width.)

Gurski proves that contracting one edge can at most double the clique-width. The conjecture is made in [Lac+] (Conjecture 8) that contracting several edges in a graph of clique-width  $k$  yields a graph of clique-width at most  $f(k)$  for some fixed function  $f$ . We disprove this conjecture and answer Gurski's question by proving the following proposition.

**Proposition 1** : The graphs obtained by edge contractions from graphs of clique-width 3 or of linear clique-width at most 4, have unbounded clique-width.

The validity of Conjecture 8 of [Lac+] would have implied that the *restricted vertex multicut problem* is FPT for the clique-width of a certain graph describing the input in a natural way. This problem consists in finding a set of vertices of given size that meets every path between the two vertices of each pair of a given set and does not contain any vertex of these pairs. Without Conjecture 8, this problem is FPT under the additional condition that no two vertices from different pairs are adjacent.

For sake of comparison, we also consider contractions of edges, one end of which has degree 2. We say in this case that we *erase a vertex*: we erase  $x$  if it has two neighbours,  $y$  and  $z$ ; to do so, we add an edge between  $y$  and  $z$  (unless they are adjacent, we only consider graphs without parallel edges) and we delete  $x$  and its two incident edges. The graphs obtained from a graph by erasing and deleting vertices are its *induced topological minors*.

**Proposition 2:** The induced topological minors of the graphs of clique-width  $k$  have clique-width at most  $2^{k+1} - 1$ .

## 2 Definitions and proofs

Graphs are finite, undirected, loop-free and without parallel edges.

To keep this note as short as possible, we refer the reader to any of [CouEng, Hli+, KLM, OumSey, Oum] for the definitions of *clique-width* and *rank-width*. Other references for clique-width are [Cor+, CouOla, GolRot, Lac+]. We denote by  $cwd(G)$  and  $rwd(G)$  the clique-width and, respectively, the rank-width of a graph  $G$ . Proving that  $cwd(G) > k$  for given  $G$  and  $k$  is rather difficult in most cases. (See for instance the computation of the exact clique-width of a square grid in [GolRot]). We overcome this difficulty in our proof of Proposition 1 by using *monadic second-order transductions* (MST): they are graph transformations specified by formulas of monadic second-order logic. The (technical) definition is in [CouEng] and in [Cou]. We will only need the fact that the graphs defined by a MST from graphs of clique-width at most  $k$  have clique-width at most  $f(k)$  for some computable function  $f$  that can be determined from the formulas forming the definition of the transduction (Corollary 7.38(2), [CouEng]).

### Definitions and notation

(a) We denote by  $H/F$  the graph obtained from a graph  $H$  by contracting the edges of a set  $F$ . (Parallel edges are fused, no loops are created.) If  $\mathcal{H}$  is a set of graphs, we denote by  $EC(\mathcal{H})$  the set of graphs  $H/F$  such that  $H \in \mathcal{H}$  and  $F$  is a set of edges of  $H$ .

(b) We denote by  $\mathcal{R}$  the set of 4-regular graphs having a *proper edge coloring* with colors in  $\{1, \dots, 4\}$ : each vertex has degree 4 and adjacent edges have different colors. These graphs have  $2p$  vertices and  $4p$  edges. They have unbounded tree-width and clique-width as they include the toroidal square grids with  $4r^2$  vertices and  $8r^2$  edges. (More precisely, the toroidal square grid with  $4r^2$  vertices has clique-width at least  $2r$ , this follows from [GolRot]).

(c) For  $n \geq 2$ , we define a graph  $G_n$ . Its vertices are  $x_1, \dots, x_n, y_1, \dots, y_n$  and its edges are  $x_i - y_i, y_i - y_j$  for all  $i, j \neq i$ . We let  $D$  consist of 4 vertices and no edge, and we let  $H_n$  be obtained from  $G_n$  by substituting disjoint copies of  $D$  to each vertex  $y_i$ . More precisely,  $H_n$  has the  $5n$  vertices  $x_1, \dots, x_n, y_1^1, y_1^2, y_1^3, y_1^4, y_2^1, \dots, y_n^4$  and the  $8n^2 - 4n$  edges  $x_i - y_i^c, y_i^c - y_j^d$  for all  $i, j \neq i$  and  $c, d = 1, \dots, 4$ . The graphs  $G_n$  and  $H_n$  have clique-width 3 and linear clique-width at most 4. We denote by  $\mathcal{H}$  the set of graphs  $H_n$ .

(d) We define an MST transduction  $\alpha$  with one parameter  $X$ . If  $G$  is a graph and  $X$  a set of vertices, then the graph  $\alpha(G, X)$  is defined if  $X$  is stable (no two vertices are adjacent), its vertex set is then  $X$  and it has an edge between  $x$  and  $y$  if and only if these vertices are at distance 2 in  $G$ . We denote by  $\alpha(G)$  the set of all such graphs, and by  $\alpha(\mathcal{G})$  the union of the sets  $\alpha(G)$  for  $G$  in a set of graphs  $\mathcal{G}$ .

**Lemma 3 :** We have  $\alpha(EC(\mathcal{H})) \supseteq \mathcal{R}$ .

**Proof:** Let  $R$  be a graph in  $\mathcal{R}$  with vertices  $x_1, \dots, x_n$  and a proper edge coloring with colors 1 to 4. The set  $X = \{x_1, \dots, x_n\}$  is also a subset of the vertex set of  $H_n$ . The four neighbours of  $x_i$  in  $H_n$  are  $y_i^1, y_i^2, y_i^3, y_i^4$ .

We let  $F$  be the set of edges  $y_i^c - y_j^c$  such that  $x_i - x_j$  is an edge of  $R$  colored by  $c$ . The graph  $K = H_n/F$  belongs to  $EC(\mathcal{H})$  and  $X$  is stable in this graph (the vertices  $x_1, \dots, x_n$  are not affected by the contractions of edges). It is clear that  $x_i - x_j$  is an edge of  $R$  if and only if there is in  $K$  a path  $x_i - z - x_j$  where  $z$  results from the contraction of  $y_i^c - y_j^c$  and  $c$  is the color of  $x_i - x_j$  in  $R$ . It follows that  $R = \alpha(K, X)$ .  $\square$

**Proof of Proposition 1:** By Lemma 3, the set  $\alpha(EC(\mathcal{H}))$  has unbounded clique-width. Hence, so has  $EC(\mathcal{H})$  by Corollary 7.38(2) of [CouEng] recalled above. This concludes the proof because the graphs  $H_n$  have clique-width 3 and linear clique-width at most 4.  $\square$

*Remarks :* 1) In this proof, no two edges of a set  $F$  are adjacent because the edge coloring of  $R$  is proper.

2) It is not hard to check that cographs (the graphs of clique-width at most 2) are preserved under edge contractions.

3) *NLC-width* and clique-width are linearly related (see [Gur]). Hence, the graphs obtained by edge contractions from graphs of NLC-width at most 3 have unbounded NLC-width. Edge contractions can also increase rank-width because

the same sets of graphs have bounded rank-width and bounded clique-width [OumSey].

**Corollary 4** : One can determine a graph of clique-width 3 that yields a graph of clique-width more than 3 by the contraction of a single edge.

**Proof** : By exhaustive search until some graph is obtained: for each  $n = 2, 3, \dots$  we consider the finitely many sets  $F$  of pairwise nonadjacent edges of  $H_n$ . By using the polynomial-time algorithm of [Cor+] to check if a graph has clique-width at most 3, we can look for a set  $F$  and an edge  $f \in F$  such that  $H_n/(F - \{f\})$  has clique-width 3 and  $H_n/F$  has clique-width more than 3 (actually 4, 5 or 6 by Theorem 4.8 of [Gur]). By Proposition 1, one must find some  $n$  and such  $F$  and  $f$ .  $\square$

We have not implemented this algorithm, hence, we do not know the resulting graph.

Gurski has proved that erasing a vertex of degree 2 can increase (or decrease) the clique-width by at most 2. In Proposition 2, we consider the effect of erasing several vertices and taking induced subgraphs.

**Proof of Proposition 2** : Deleting vertices does not increase rank-width. We prove the same for erasing a degree 2 vertex  $x$ . Let  $y$  and  $z$  be its neighbours. If they are adjacent, erasing  $x$  is the same as deleting it because we fuse parallel edges. If they are not, erasing  $x$  is the same as performing first a local complementation at  $x$ , which creates an edge between  $y$  and  $z$ , and then deleting  $x$ . As local complementation preserves rank-width [Oum], erasing a degree 2 vertex cannot increase rank-width. Hence, by transitivity, the same holds for taking an induced topological minor. The result follows since, for every graph  $G$ , we have  $rdw(G) \leq cwd(G) \leq 2^{rdw(G)+1} - 1$  by [OumSey].  $\square$

This proof leaves open the question of improving the upper bound  $2^{k+1} - 1$ , possibly to a polynomial in  $k$  or even to  $k$ .

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