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Geometric ρ -mixing property of the interarrival times of a stationary Markovian Arrival Process

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Abstract

In this note, the sequence of the interarrivals of a stationary Markovian Arrival process is shown to be ρ -mixing with a geometric rate of convergence when the driving process is ρ -mixing. This provides an answer to an issue raised in the recent paper [4] on the geometric convergence of the autocorrelation function of the stationary Markovian Arrival process.

KEYWORDS: Markov renewal process
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1 Introduction

We provide a positive answer to a question raised in [4] on the geometric convergence of the autocorrelation function associated with the interarrival times of a stationary m -state Markovian Arrival Process (MAP). Indeed, it is shown in [3, Prop. 3.1] that the increment sequence $\{T_n := S_n - S_{n-1}\}_{n \geq 1}$ associated with a discrete time stationary Markov additive process $\{(X_n, S_n)\}_{n \in \mathbb{N}} \in \mathbb{X} \times \mathbb{R}^d$ is ρ -mixing with a geometric rate provided that the driving stationary Markov chain $\{X_n\}_{n \in \mathbb{N}}$ is ρ -mixing. There, \mathbb{X} may be any measurable set. In the case where the increments $\{T_n\}_{n \geq 1}$ are non-negative random variables, $\{(X_n, S_n)\}_{n \in \mathbb{N}}$ is a Markov Renewal Process (MRP). Therefore, we obtain the expected answer to the question in [4] since such an MRP with $\{T_n\}_{n \geq 1}$ being the interarrival times can be associated with a m -state MAP and the ρ -mixing property of $\{T_n\}_{n \geq 1}$ with geometric rate ensures the geometric convergence of the autocorrelation function of $\{T_n\}_{n \geq 1}$. We refer to [1, Chap. XI] for basic properties of MAPs and Markov additive processes.

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2 Geometric ρ -mixing of the sequence of interarrivals of an MAP

Let us recall the definition of the ρ -mixing property of a (strictly) stationary sequence of random variables $\{T_n\}_{n \geq 1}$ (e.g. see [2]). The ρ -mixing coefficient with time lag $k > 0$, denoted usually by $\rho(k)$, is defined by

$$\rho(k) := \sup_{n \geq 1} \sup_{m \in \mathbb{N}} \sup \left\{ \left| \text{Corr}(f(T_1, \dots, T_n); h(T_{n+k}, \dots, T_{n+k+m})) \right|, \right. \\ \left. f, g \text{ } \mathbb{R}\text{-valued functions such that} \right. \\ \left. \mathbb{E}[|f(T_1, \dots, T_n)|^2] \text{ and } \mathbb{E}[|h(T_{n+k}, \dots, T_{n+k+m})|^2] \text{ are finite} \right\} \quad (1)$$

where $\text{Corr}(f(T_1, \dots, T_n); h(T_{n+k}, \dots, T_{n+k+m}))$ is the correlation coefficient of the two square-integrable random variables. Note that $\{\rho(k)\}_{k \geq 1}$ is a non-increasing sequence. Then $\{T_n\}_{n \geq 1}$ is said to be ρ -mixing if

$$\lim_{k \rightarrow +\infty} \rho(k) = 0.$$

When, for any $n \in \mathbb{N}$, the random variable T_n has a moment of order 2, the autocorrelation function of $\{T_n\}_{n \geq 1}$ as studied in [4], that is $\text{Corr}(T_1; T_{k+1})$ as a function of the time lag k , clearly satisfies

$$\forall k \geq 1, \quad |\text{Corr}(T_1; T_{k+1})| \leq \rho(k). \quad (2)$$

Therefore, any rate of convergence of the ρ -mixing coefficients $\{\rho(k)\}_{k \geq 1}$ is a rate of convergence for the autocorrelation function.

We only outline the main steps to obtain from [3, Prop. 3.1] a geometric convergence rate of $\{\rho(k)\}_{k \geq 1}$ for the m -state MRP $\{(X_n, S_n)\}_{n \in \mathbb{N}}$ associated with a m -state MAP. In [4, Section 2], the analysis of the autocorrelation function in the two-states case is based on such an MRP (notation and background in [4] are that of [5]). Recall that a m -state MAP is a bivariate continuous-time Markov process $\{(J_t, N_t)\}_{t \geq 0}$ on $\{1, \dots, m\} \times \mathbb{N}$ where N_t represents the number of arrivals up to time t , while the states of the driving Markov process $\{J_t\}_{t \geq 0}$ are called phases. Let S_n be the time at the n th arrival ($S_0 = 0$ a.s.) and let X_n be the state of the driving process just after the n th arrival. Then $\{(X_n, S_n)\}_{n \in \mathbb{N}}$ is known to be an MRP with the following semi-Markov kernel Q on $\{1, \dots, m\} \times [0, \infty)$

$$\forall (x_1, x_2) \in \{1, \dots, m\}^2, \quad Q(x_1; \{x_2\} \times dy) := (e^{D_0 y} D_1)(x_1, x_2) dy \quad (3)$$

parametrized by a pair of $m \times m$ -matrices usually denoted by D_0 and D_1 . The matrix $D_0 + D_1$ is the infinitesimal generator of the background Markov process $\{J_t\}_{t \geq 0}$ which is always assumed to be irreducible, and D_0 is stable. The process $\{X_n\}_{n \in \mathbb{N}}$ is a Markov chain with state space $\mathbb{X} := \{1, \dots, m\}$ and transition probability matrix P :

$$\forall (x_1, x_2) \in \mathbb{X}^2, \quad P(x_1, x_2) = Q(x_1; \{x_2\} \times [0, \infty)) = ((-D_0)^{-1} D_1)(x_1, x_2). \quad (4)$$

$\{X_n\}_{n \in \mathbb{N}}$ has an invariant probability measure ϕ (i.e. $\phi P = \phi$). It is well-known that, for $n \geq 1$, the interarrival time $T_n := S_n - S_{n-1}$ has a moment of order 2 (whatever the probability distribution of X_0). We refer to [1] for details about the above basic facts on an MAP and its associated MRP.

Let us introduce the $m \times m$ -matrix

$$\Phi := e^\top \phi \tag{5}$$

when e is the m -dimensional row-vector with all components equal to 1. Any \mathbb{R} -valued function v on \mathbb{X} may be identified to a \mathbb{R}^m -dimensional vector. We use the subordinate matrix norm induced by $\ell^2(\phi)$ -norm $\|v\|_2 := \sqrt{\sum_{x \in \mathbb{X}} |v(x)|^2 \phi(x)}$ on \mathbb{R}^m

$$\|M\|_2 := \sup_{v: \|v\|_2=1} \|Mv\|_2.$$

Let \mathbb{E}_ϕ be the expectation with respect to the initial conditions $(X_0, S_0) \sim (\phi, \delta_0)$. Recall that $T_n := S_n - S_{n-1}$ for $n \geq 1$. When $X_0 \sim \phi$, we have (see [3, Section 3]):

1. if g is a \mathbb{R} -valued function such that $\mathbb{E}[|g(X_1, T_1, \dots, X_n, T_n)|] < \infty$, then $\forall k \geq 0, \forall n \geq 1$

$$\begin{aligned} & \mathbb{E}[g(X_{k+1}, T_{k+1}, \dots, X_{k+n}, T_{k+n}) \mid \sigma(X_l, T_l : l \leq k)] \\ &= \int_{(\mathbb{X} \times [0, \infty))^n} Q(X_s; dx_1 \times dz_1) \prod_{i=2}^n Q(x_{i-1}; dx_i \times dz_i) g(x_1, z_1, \dots, x_n, z_n) \\ &= (Q^{\otimes n})(g)(X_k) \end{aligned} \tag{6}$$

where $Q^{\otimes n}$ denotes the n -fold kernel product $\bigotimes_{i=1}^n Q$ of Q defined in (3).

2. Let f and h be two \mathbb{R} -valued functions such that $\mathbb{E}_\phi[|f(T_1, \dots, T_n)|^2] < \infty$ and $\mathbb{E}_\phi[|h(T_{n+k}, \dots, T_{n+k+m})|^2] < \infty$ for $(k, n) \in (\mathbb{N}^*)^2, m \in \mathbb{N}$. From (6) with $g(x_1, z_1, \dots, x_{n+k+m}, z_{n+k+m}) \equiv f(z_1, \dots, z_n)h(z_{n+k}, \dots, z_{n+k+m})$, the process $\{T_n\}_{n \geq 1}$ is stationary and the following covariance formula holds (see [3, Lem. 3.3] for details)

$$\begin{aligned} & \text{Cov}(f(T_1, \dots, T_n); h(T_{n+k}, \dots, T_{n+k+m})) \\ &= \mathbb{E}_\phi[f(T_1, \dots, T_n) (P^{k-1} - \Phi)(Q^{\otimes m+1}(h))(X_n)]. \end{aligned} \tag{7}$$

where matrices P, Φ are defined in (4) and (5).

First, note that the random variables $f(\cdot)$ and $h(\cdot)$ in (1) may be assumed to be of \mathbb{L}^2 -norm 1. Thus we just have to deal with covariances. Second, the Cauchy-Schwarz inequality

and Formula (7) allow us to write

$$\begin{aligned}
& \text{Cov}\left(f(T_1, \dots, T_n); h(T_{n+k}, \dots, T_{n+k+m})\right)^2 \\
& \leq \mathbb{E}_\phi \left[|f(T_1, \dots, T_n)|^2 \right] \mathbb{E}_\phi \left[|(P^{k-1} - \Phi)(Q^{\otimes m+1}(h))(X_n)|^2 \right] \\
& = \mathbb{E}_\phi \left[|(P^{k-1} - \Phi)(Q^{\otimes m+1}(h))(X_0)|^2 \right] \quad (\phi \text{ is } P\text{-invariant}) \\
& = \|(P^{k-1} - \Phi)(Q^{\otimes m+1}(h))\|_2^2 \\
& \leq \|P^{k-1} - \Phi\|_2^2 \|Q^{\otimes m+1}(h)\|_2^2 \\
& \leq \|P^{k-1} - \Phi\|_2^2 \quad (\text{since } \|Q^{\otimes m+1}(h)\|_2 \leq 1).
\end{aligned}$$

Therefore, we obtain from (1) and (2) that the autocorrelation coefficient $\text{Corr}(T_1; T_{k+1})$ as studied in [4], satisfies

$$\forall k \geq 1, \quad |\text{Corr}(T_1; T_{k+1})| \leq \rho(k) \leq \|P^{k-1} - \Phi\|_2^2. \quad (8)$$

The convergence rate to 0 of the sequence $\{\text{Corr}(T_1; T_{k+1})\}_{n \geq 1}$ is bounded from above by that of $\{\|P^{k-1} - \Phi\|_2\}_{k \geq 1}$. Under usual assumptions on the MAP, $\{X_n\}_{n \in \mathbb{N}}$ is irreducible and aperiodic so that there exists $r \in (0, 1)$ such that

$$\|P^k - \Phi\|_2 = O(r^k) \quad (9)$$

with $r = \max(|\lambda|, \lambda \text{ is an eigenvalue of } P \text{ such that } |\lambda| < 1)$. For a stationary Markov chain $\{X_n\}_{n \in \mathbb{N}}$ with general state space, we know from [6, p 200,207] that Property (9) is equivalent to the ρ -mixing property of $\{X_n\}_{n \in \mathbb{N}}$.

3 Comments on [4]

In [4], the analysis is based on a known explicit formula of the correlation function in terms of the parameters of the m -state MRP (see [4, (2.6)]). Note that this formula can be obtained using $n = 1, m = 0$ and $f(T_1) = T_1, h(T_{1+k}) = T_{1+k}$ in (7). When $m := 2$ and under standard assumptions on MAPs, matrix P is diagonalizable with two distinct real eigenvalues, 1 and $0 < \lambda < 1$ which has an explicit form in terms of entries of P . Then, the authors can analyze the correlation function with respect to the entries of matrix P [4, (3.4)-(3.7)]. As quoted by the authors, such an analysis would be tedious and difficult with $m > 2$ due to the increasing number of parameters defining an m -state MAP. Note that Inequality (8) and Estimate (9) when $m := 2$ provide the same convergence rate as in [4], that is λ the second eigenvalue of matrix P .

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