



**HAL**  
open science

## Connected covering numbers

Jonathan Chappelon, Kolja Knauer, Luis Pedro Montejano, Jorge Luis  
Ramírez Alfonsín

► **To cite this version:**

Jonathan Chappelon, Kolja Knauer, Luis Pedro Montejano, Jorge Luis Ramírez Alfonsín. Connected covering numbers. 2013. hal-00837648v1

**HAL Id: hal-00837648**

**<https://hal.science/hal-00837648v1>**

Preprint submitted on 24 Jun 2013 (v1), last revised 13 Jan 2015 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# CONNECTED COVERING NUMBERS

JONATHAN CHAPPELON, KOLJA KNAUER, LUIS PEDRO MONTEJANO,  
AND JORGE LUIS RAMÍREZ ALFONSÍN

*Dedicated to the memory of Michel Las Vergnas*

ABSTRACT. A connected covering is a design system in which the corresponding *block graph* is connected. The minimum size of such coverings are called *connected coverings numbers*. In this paper, we present various formulas and bounds for several parameter settings for these numbers. We also investigate results in connection with *Turán systems*. Finally, a new general upper bound, improving an earlier result, is given. The latter is used to improve upper bounds on a question concerning oriented matroid due to Las Vergnas.

## 1. INTRODUCTION

Let  $n, k, r$  be positive integers such that  $n \geq k \geq r \geq 1$ . A  $(n, k, r)$ -*covering* is a family  $\mathcal{B}$  of  $k$ -subsets of  $\{1, \dots, n\}$ , called *blocks*, such that each  $r$ -subset of  $\{1, \dots, n\}$  is contained in at least one of the blocks. The number of blocks is the covering's *size*. The minimum size of such a covering is called the *covering number* and is denoted by  $C(n, k, r)$ . Given a  $(n, k, r)$ -covering  $\mathcal{B}$ , its graph  $G(\mathcal{B})$  has as vertices  $\mathcal{B}$  and two vertices are joined if they have one  $r$ -subset in common. We say that a  $(n, k, r)$ -covering is *connected* if the graph  $G(\mathcal{B})$  is connected. The minimum size of a connected  $(n, k, r)$ -covering is called the *connected covering number* and is denoted by  $CC(n, k, r)$ .

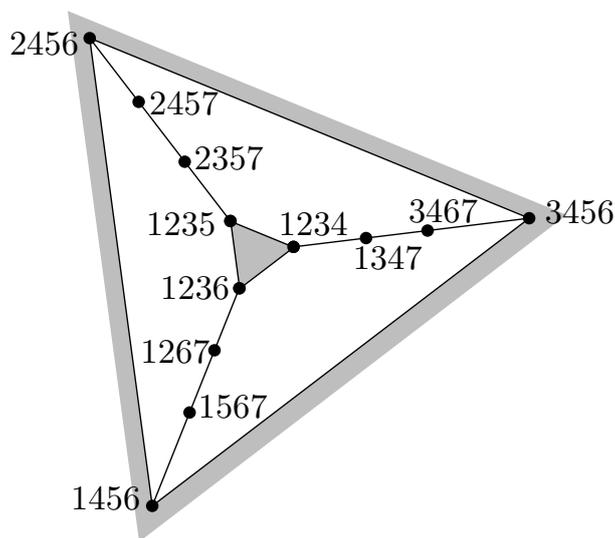


FIGURE 1. A connected  $(7, 4, 3)$ -covering with 12 blocks.

*Date:* June 23, 2013.

*2010 Mathematics Subject Classification.* 05B05, 51E05.

*Key words and phrases.* Covering design, Turán-system, uniform oriented matroid.

The second and fourth authors were supported by the ANR TEOMATRO grant ANR-10-BLAN 0207.

The third author was supported by CONACYT.

The graph corresponding to a connected  $(7, 4, 3)$ -covering is illustrated in Figure 1. In this paper, we mainly focus our attention to coverings when  $k = r + 1$  and thus, we will denote  $C(n, r + 1, r)$  (resp.  $CC(n, r + 1, r)$ ) by  $C(n, r)$  (resp. by  $CC(n, r)$ ) for short. The original motivation to study  $CC(n, r)$  comes from the following question posed by Michel Las Vergnas.

**Question 1.1.** Let  $U_{r,n}$  be the rank  $r$  uniform matroid on  $n$  elements. What is the smallest number  $s(n, r)$  of circuits of  $U_{r,n}$ , that uniquely determines all orientations of  $U_{r,n}$ ? This is, whenever two uniform oriented matroids coincide on these circuits they must be equal.

In [5], Forge and Ramírez Alfonsín introduced the notion of connected coverings and proved that

$$(1) \quad s(n, r) \leq CC(n, r).$$

The latter was then used to improve the best upper bound,  $s(n, r) \leq \binom{n-1}{r}$ , known at that time due to Hamidoune and Las Vergnas [9]; see also [6] for related results.

It turns out that  $s(n, r)$  is also closely related to  $C(n, r)$ .

**Proposition 1.2.** *Let  $n$  and  $r$  be positive integers such that  $n \geq r + 1$ . Then,*

$$C(n, r) \leq s(n, r).$$

The proof of Proposition 1.2, that is a bit technical (needing some oriented matroid notions and thus lying slightly out of scope of this paper), is given in the Appendix.

Although the study of covering designs have already a long history of research (see [8] for many upper bounds and [21] for a survey in the dual setting of Turán-systems), Proposition 1.2 motivates even further the study of  $C(n, r)$  and specially, in view of (1), its behavior in relation to  $CC(n, r)$ . This relationship was already remarked in [5] where it was proved that

$$(2) \quad CC(n, r) \leq 2C(n, r).$$

That was done by observing that the graph  $G$  associated to a covering with  $C(n, r)$  blocks (and thus with  $|V(G)| = C(n, r)$ ) can be made it connected by adding at most  $C(n, r)$  extra vertices (blocks), obtaining a graph corresponding to a  $(n, r + 1, r)$ -connected covering with at most  $2C(n, r)$  blocks.

Many interesting variants of Question 1.1 can be investigated. For instance, for non-uniform (oriented) matroids (graphic, representable, etc.) and by varying the notion of what *determine* means (up to orientations, bijections, etc.). These (and other) variants are treated in another paper (see [3]).

This paper is organized as follows. In the next section, we recall some basic definitions and results concerning (connected) coverings and its connection with *Turán systems* needed for the rest of the paper. In Section 3, we investigate connected covering numbers when the value  $r$  is either small or close to  $n$ . Among other results, we give the exact value for  $CC(n, 2)$  (Theorem 3.2), for  $CC(n, 3)$  for  $n \leq 12$  (Theorem 3.3) and for  $CC(n, n - 3)$  (Theorem 3.6). A famous conjecture of Turán and its connection with our results is also discussed. In Section 4, we present a general upper bound for  $CC(n, r)$  (Theorem 4.10) allowing us to improve the best known upper bound for  $s(n, r)$ . We end the paper by discussing some asymptotic results in Section 5.

## 2. BASIC RESULTS

Let  $n, m, p$  be positive integers such that  $n \geq m \geq p$ . A  $(n, m, p)$ -Turán-system is a family  $\mathcal{D}$  of  $p$ -subsets of  $\{1, \dots, n\}$ , called *blocks*, such that each  $m$ -subset of  $\{1, \dots, n\}$  contains at least one of the blocks. The number of blocks is the *size* of the Turán-system. The minimum size of such a covering is called the *Turán Number* and is denoted by  $T(n, m, p)$ . Given a  $(n, m, p)$ -Turán-system  $\mathcal{D}$ , with  $0 \leq 2p - m \leq p$ , its graph  $G(\mathcal{D})$  has as vertices  $\mathcal{D}$  and two vertices are joined if they have one  $2p - m$ -subset in common. We say that a  $(n, m, p)$ -Turán-system with  $0 \leq 2p - m \leq p$  is *connected* if  $G(\mathcal{D})$  is connected. The minimum size of a connected  $(n, m, p)$ -Turán-system is the *connected Turán Number* and is denoted by  $CT(n, m, p)$ . By applying set complement to blocks, it can be obtained that

$$(3) \quad C(n, k, r) = T(n, n - r, n - k).$$

Moreover, if  $0 \leq n - 2k + r \leq n - k$  then

$$(4) \quad CC(n, k, r) = CT(n, n - r, n - k).$$

Note that the precondition for (4) is fulfilled if  $k = r + 1$ .

Most of the papers on coverings consider  $n$  large compared with  $k$  and  $r$ , while for Turán numbers it has frequently been considered  $n$  large compared with  $m$  and  $p$ , and often focusing on the quantity  $\lim_{n \rightarrow \infty} T(n, m, p) / \binom{n}{p}$  for fixed  $m$  and  $p$ . Thus, for Turán-type problems, the value  $C(n, k, r)$  has usually been studied in the case when  $k$  and  $r$  are not too far from  $n$ .

Forge and Ramírez Alfonsín [5] proved that

$$(5) \quad CC(n, r) \geq \frac{\binom{n}{r} - 1}{r} =: CC_1^*(n, r).$$

Moreover, Sidorenko [22] proved that  $T(n, r + 1, r) \geq \left(\frac{n-r}{n-r+1}\right) \frac{\binom{n}{r}}{r}$ , which by (3), we obtain that

$$(6) \quad CC(n, r) \geq C(n, r) = T(n, n - r, n - r - 1) \geq \left(\frac{r+1}{r+2}\right) \frac{\binom{n}{r+1}}{n - r - 1} =: CC_2^*(n, r).$$

Combining (5) and (6), together with a straight forward computation we have

$$(7) \quad CC(n, r) \geq \max\{CC_1^*(n, r), CC_2^*(n, r)\},$$

where the maximum is attained by the second term if and only if  $r \geq \frac{2}{3}(n - 1)$ .

The following recursive lower bound for covering numbers was obtained by Schönheim [20] and, independently, by Katona, Nemetz and Simonovits [14]

$$(8) \quad C(n, r) \geq \left\lceil \frac{n}{r+1} C(n-1, r-1) \right\rceil$$

which can be iterated yielding to

$$(9) \quad C(n, r) \geq \left\lceil \frac{n}{r+1} \left\lceil \frac{n-1}{r} \left\lceil \dots \left\lceil \frac{n-r+1}{2} \right\rceil \dots \right\rceil \right\rceil =: L(n, r).$$

The following recursive upper bound for  $CC(n, r)$  due to Forge and Ramírez Alfonsín [5] will be used later.

$$(10) \quad CC(n, r) \leq CC(n-1, r) + C(n-1, r-1).$$

### 3. RESULTS FOR SMALL AND LARGE $r$

In this section, we investigate connected covering numbers for *small* and *large*  $r$ , that is, when  $r$  is very close to either 1 or  $n$ . Let us start with the following observations.

*Remarks 3.1.*

- a)  $\text{CC}(n, 0) = 1$  since any 1-element set contains the empty set.
- b)  $\text{CC}(n, 1) = n - 1$  by taking the edges of a spanning tree of  $K_n$ .
- c)  $\text{CC}(n, n - 2) = n - 1$  by taking all but one  $(n - 1)$ -sets.
- d)  $\text{CC}(n, n - 1) = 1$  by taking the entire set.

All these values coincide with the corresponding covering numbers except in the case  $r = 1$ , where  $C(n, 1) = \lceil \frac{n}{2} \rceil$ .

#### 3.1. Results when $r$ is small.

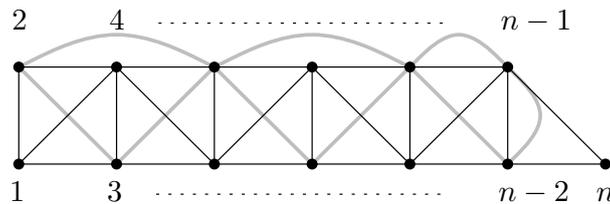
For ordinary covering numbers, Fort and Hedlund [7] have shown that  $C(n, 2) := \lceil \frac{n}{3} \lceil \frac{n}{2} \rceil \rceil$  that coincides with the lower bounds given in (9) when the case  $r = 2$ .

We also have the precise value for the connected case when  $r = 2$ .

**Theorem 3.2.** *Let  $n$  be a positive integer with  $n \geq 3$ . Then, we have*

$$\text{CC}(n, 2) = \left\lceil \frac{\binom{n}{2} - 1}{2} \right\rceil.$$

*Proof.* Note that the claimed value coincides with the lower bound  $\text{CC}_1^*(n, 2)$ . This lower bound comes from the fact that every connected covering has a construction sequence, where every new triangle shares at least one edge with an already constructed triangle. We present a construction sequence where indeed every new triangle (except possibly the last one) shares exactly one edge with the already constructed ones. Therefore, we attain the lower bound. Part of the construction is shown in Figure 2. We start presenting the



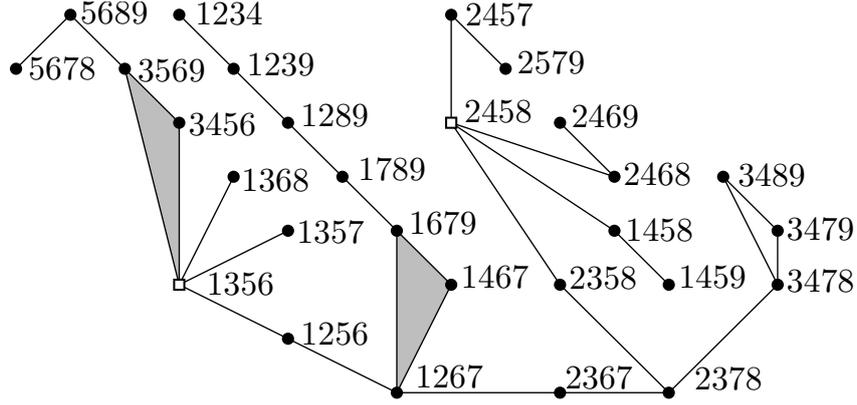


FIGURE 3. An example proving  $CC(9, 3) \leq 28$ . The circle-vertices are a covering.

**Theorem 3.3.** *Let  $n$  be a positive integer with  $4 \leq n \leq 12$ . Then, we have*

$$CC(n, 3) = \left\lceil \frac{\binom{n}{3} - 1}{3} \right\rceil.$$

*Proof.* Note that the claimed value coincides with the lower bound  $CC_1^*(n, 3)$ . For  $n \leq 6$  this is already checked in [5]. Figure 1 proves  $CC(7, 3) \leq 12 = CC_1^*(7, 3)$ . By using (10),  $C(7, 2) = 7$ , and  $CC(7, 3) = 12$ , we obtain that  $CC(8, 3) \leq 19 = CC_1^*(8, 3)$ . Figure 3 proves  $CC(9, 3) \leq 28 = CC_1^*(9, 3)$ . From equation (10) and the fact that  $CC(9, 3) = 28$  and  $C(9, 2) = 12$ , we conclude that  $CC(10, 3) \leq 40 = CC_1^*(10, 3)$ . Now, Figure 4 proves that  $CC(11, 3) \leq 55 = CC_1^*(11, 3)$ . Finally, to construct a connected covering witnessing  $CC_1^*(12, 3)$  we delete the block  $\{2, 4, 6, 8\}$  from the covering in Figure 4. One can check that this still leaves a covering  $\mathcal{B}$ , whose graph now has three components. Now, we take the following (disconnected)  $(11, 3, 2)$ -covering:

$$\left\{ \begin{array}{l} \{1, 3, 11\}, \{1, 4, 6\}, \{1, 2, 8\}, \{1, 5, 9\}, \{1, 7, 10\}, \{3, 4, 9\}, \{2, 3, 10\}, \{3, 5, 6\}, \\ \{3, 7, 8\}, \{2, 4, 6\}, \{4, 5, 7\}, \{4, 11, 10\}, \{4, 6, 8\}, \{2, 5, 11\}, \{2, 7, 9\}, \{5, 8, 10\}, \\ \{6, 7, 11\}, \{8, 9, 11\}, \{6, 9, 10\} \end{array} \right\}.$$

We add to each of these block the element 12 and thus together with  $\mathcal{B}$  obtain a  $(12, 4, 3)$ -covering  $\mathcal{B}'$ . To see that  $\mathcal{B}'$  is connected, note that each of the blocks containing 12 is connected to a block from  $\mathcal{B}$ . Moreover, the blocks  $\{1, 4, 6, 12\}$ ,  $\{2, 4, 6, 12\}$ ,  $\{4, 6, 8, 12\}$  form a triangle and each of them has a neighbor in a different component of  $G(\mathcal{B})$ . Thus,  $G(\mathcal{B}')$  is connected and  $\mathcal{B}'$  has 73 blocks which coincides with  $CC_1^*(12, 3)$ .  $\square$

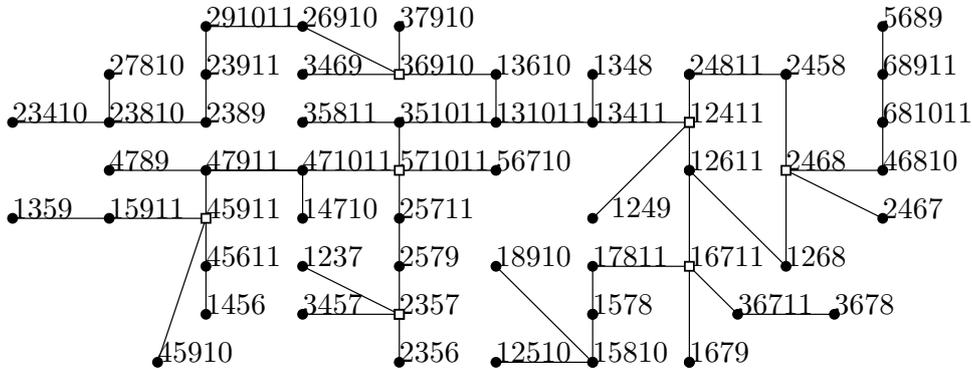


FIGURE 4. An example proving  $CC(11, 3) \leq 55$ . The circle-vertices are a covering.

Theorem 3.3 supports the following

**Conjecture 3.4.** For every positive integer  $n \geq 4$ , we have

$$\text{CC}(n, 3) = \text{CC}_1^*(n, 3).$$

Even, more ambitious,

**Question 3.5.** Let  $n$  and  $r$  be two positive integers such that  $n \geq r + 1 \geq 4$ . Is it true that if  $\text{CC}(n, r) = \text{CC}_1^*(n, r)$  then  $\text{CC}(n', r) = \text{CC}_1^*(n', r)$  for every integer  $n' \geq n$ ?

### 3.2. Results when $r$ is large.

**Theorem 3.6.** Let  $n$  be a positive integer with  $n \geq 3$ . Then, we have

$$\text{CC}(n, n - 3) = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2} + 1.$$

*Proof.* The parameter  $\text{C}(n, n - 3) = \text{T}(n, 3, 2)$  was determined already by Mantel in 1907 [16] and is  $\binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$ . Turán proved that the *unique* minimal configuration of sets of size 2 hitting all sets of size 3 of an  $n$ -set are the edges of two vertex-disjoint complete graphs  $K_{\lceil \frac{n}{2} \rceil}$  and  $K_{\lfloor \frac{n}{2} \rfloor}$ , see [24].

Now, by (3) and (4), the covering corresponding to the Turán-system is connected if and only if the graph whose edges correspond to the blocks of the Turán-system is connected. Thus, since the unique optimal construction by Turán is not connected but can be made connected by adding a single edge connecting the two complete graphs, this is optimal with respect to connectivity. Therefore,  $\text{CC}(n, n - 3) = \text{T}(n, 3, 2) + 1$ , giving the result.  $\square$

**Proposition 3.7.** Let  $n$  be a positive integer with  $n \geq 4$ . Then, we have

$$\text{CC}(n, n - 4) \leq \begin{cases} m(m - 1)(2m - 1) & \text{if } n = 3m, \\ m^2(2m - 1) & \text{if } n = 3m + 1, \\ m^2(2m + 1) & \text{if } n = 3m + 2. \end{cases}$$

*Proof.* As in Theorem 3.6 the construction comes from making one for a (well-known) Turán-system connected. Divide  $\{1, \dots, n\}$  into three sets  $A_0, A_1, A_2$  of as equal sizes as possible. The blocks  $\mathcal{D}$  of the  $(n, 4, 3)$ -Turán-system are the triples  $\{x, y, z\}$  with  $x, y \in A_i$  and  $z \in A_i \cup A_{(i+1) \bmod 3}$  for  $i = 0, 1, 2$ . It is easy to see, that this actually is a  $(n, 4, 3)$ -Turán-system of the claimed size. Now,  $G(\mathcal{D})$  has three connected components corresponding to the indices defining the triples. It can be turned into a connected  $(n, 4, 3)$ -Turán-system  $\mathcal{D}'$  by choosing one edge  $\{x, y\}$  in each  $A_i$  and replacing all triples of the form  $\{x, y, z\}$  with  $z \in A_{(i+1) \bmod 3}$  by all triples of the form  $\{x, y, z\}$  with  $z \in A_{(i-1) \bmod 3}$ . It is easy to check that  $|\mathcal{D}'| = |\mathcal{D}|$  and that  $G(\mathcal{D}')$  is connected for  $n \geq 8$ . See Figure 1 for proving our statement for  $n = 7$ , Theorem 3.2 for  $n = 6$ , and Remark 3.1 for  $n = 4, 5$ .  $\square$

A famous conjecture of Turán [25] states that the bounds in Proposition 3.7 are best possible for  $\text{C}(n, n - 4)$ . By combining (1) and Proposition 3.6, we have

$$(11) \quad \text{C}(n, n - 4) \leq \text{CC}(n, n - 4) \leq \begin{cases} m(m - 1)(2m - 1) & \text{if } n = 3m, \\ m^2(2m - 1) & \text{if } n = 3m + 1, \\ m^2(2m + 1) & \text{if } n = 3m + 2. \end{cases}$$

Turán's conjecture has been verified [22] for all  $n \leq 13$  and so, by (11), the connected covering number can also be determined for these same values.

Towards proving Turán's conjecture, it would be of interest to investigate the following.

**Question 3.8.** Is it true that one of the inequalities in (11) is actually an equality ?

Bounds and precise values for all  $CC(n, r)$  with  $n \leq 14$  are given in Table 1. All the exact values previously given in [5] for the same range have been improved by using our above results.

$r \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1		1	2	3	4	5	6	7	8	9	10	11	12	13
2			1	3	$5^{e,t}$	$7^{e,u}$	$10^e$	$14^e$	$18^e$	$22^e$	$27^e$	$33^e$	$39^e$	$45^e$
3				1	4	$7^{p,t}$	$12^{p,u}$	$19^p$	$28^p$	$40^p$	$55^p$	$73^p$	$[95^l, 97^r]$	$[121^l, 123^r]$
4					1	5	$10^t$	$20^u$	$[32^l, 34^r]$	$[53^l, 59^r]$	$[83^l, 89^r]$	$[124^l, 136^r]$	$[179^l, 193^r]$	$[250^l, 271^r]$
5						1	6	$13^t$	$30^u$	$[51^l, 60^r]$	$[96^a, 111^r]$	$[159^l, 177^r]$	$[258^l, 290^r]$	$[401^l, 447^r]$
6							1	7	$17^t$	$45^u$	$[84^a, 95^r]$	$[165^a, 195^r]$	$[286^l, 327^r]$	$[501^l, 572^r]$
7								1	8	$21^t$	$63^u$	$[126^a, 147^r]$	$[269^a, 323^r]$	$[491^l, 587^r]$
8									1	9	$26^t$	$84^u$	$[185^a, 210^r]$	$[419^a, 505^r]$
9										1	10	$31^t$	$112^u$	$[259^s, 297^r]$
10											1	11	$37^t$	$[143^s, 144^u]$
11												1	12	$43^t$
12													1	13
13														1

TABLE 1. Bounds and values of  $CC(n, r)$  for  $n \leq 14$ .

Key of Table 1 :

- $r$  — Upper bound for  $CC(n, r)$ (from (10))
- $e$  — Exact values for  $CC(n, 2)$  (Theorem 3.2)
- $t$  — Exact values for  $CC(n, n - 3)$  (Theorem 3.6)
- $l$  — Lower bound  $CC_1^*(n, r)$
- $p$  — Some exact values for  $CC(n, 3)$  (Theorem 3.3)
- $u$  — Upper bound for  $CC(n, n - 4)$  (Proposition 3.7)
- $s$  — Lower bound for  $C(n, r)$  (from (8))
- $a$  — Lower bounds for  $C(n, r)$  (from [1])

Table 1 led us to consider the following.

**Question 3.9.** Is the sequence  $(CC(n, i))_{0 \leq i \leq n-1}$  *unimodal* for every  $n$  ? or perhaps *logarithmically concave*<sup>1</sup> ?

#### 4. A GENERAL UPPER BOUND

Let  $n$  and  $r$  be positive integers such that  $n \geq r + 1 \geq 3$ . Forge and Ramírez Alfonsín [5] obtained the following general upper bound

$$(12) \quad CC(n, r) \leq \sum_{i=1}^{\lfloor \frac{n-r+1}{2} \rfloor} \binom{n-2i}{r-1} + \left\lfloor \frac{n-r}{2} \right\rfloor =: S(n, r).$$

<sup>1</sup>A finite sequence of real numbers  $\{a_1, a_2, \dots, a_n\}$  is said to be *unimodal* (resp. *logarithmically concave* or *log-concave*) if there exists a  $t$  such that  $s_1 \leq s_2 \leq \dots \leq s_t$  and  $s_t \geq s_{t+1} \geq \dots \geq s_n$  (resp. if  $a_i^2 \geq a_{i-1}a_{i+1}$  holds for every  $a_i$  with  $1 \leq i \leq n-1$ ). Notice that a log-concave sequence is unimodal.

Let us notice that the upper bounds obtained by applying the recursive equation (10), that were used in Table 1, are better than the one given by (12). Moreover, by iterating (10) it can be obtained

$$(13) \quad \text{CC}(n, r) \leq \sum_{i=r}^{n-1} \text{C}(i, r-1).$$

Although (13) might be used to get an *explicit* upper bound for  $s(n, r)$ , it is not clear how good it would be since that would depend on the known exact values and the upper bounds of  $\text{C}(n, r)$  used in the recurrence (and thus intrinsically difficult to compute). On the contrary, in [5] was used (12) to give the best known (to our knowledge) explicit upper bound for  $s(n, r)$ .

In this section, we will construct a connected  $(n, r+1, r)$ -covering giving an upper bound for  $\text{CC}(n, r)$  better than  $\text{S}(n, r)$  and so, yielding to a better upper bound for  $s(n, r)$  than that given in [5].

**Theorem 4.1.** *Let  $n$  and  $r$  be positive integers such that  $n \geq r+1 \geq 3$ . Then,*

$$\text{CC}(n, r) \leq \text{N}(n, r),$$

with

$$(14) \quad \text{N}(n, r) := \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil - 1} (n-r-2i) \binom{r-2+2i}{r-2} + \left\lceil \frac{n-r}{2} \right\rceil - 1 + \delta_0 \text{C}(n-2, r-2),$$

where  $\delta_0$  is the parity function of  $n-r$ , that is,

$$\delta_0 = \begin{cases} 0 & \text{if } n-r \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$$

*Proof.* From now, for any positive integer  $s$ , we denote by  $[s]$  the set of the first  $s$  positive integers, that is,  $[s] = \{1, \dots, s\}$  and we denote by  $U_{s,t}$  the set of all  $t$ -subsets of  $[s]$ . Moreover, suppose that for any subset of integers  $\{b_1, \dots, b_s\}$ , we have  $b_i < b_j$  for all  $1 \leq i < j \leq s$ . We distinguish different cases depending on the parity of  $n-r$ .

*Case 1.* Suppose that  $n-r$  is an odd number and let  $m$  such that  $n-r = 2m+1$ . We will construct a connected  $(r+2m+1, r+1, r)$ -covering of size

$$\sum_{i=0}^m \binom{r-2+2i}{r-2} (2m+1-2i) + m.$$

First, we consider a particular  $(r+2m+1, r+1, r)$ -covering, which is constituted by a large number of blocks but whose associated graph has a small number of connected components. For any  $i \in \{0, \dots, m\}$ , let  $\mathcal{N}_i$  be the following subset of  $(r+1)$ -subsets of  $[r+2m+1]$ :

$$\mathcal{N}_i := \left\{ \{b_1, \dots, b_{r+1}\} \left| \begin{array}{l} \{b_1, \dots, b_{r-2}\} \in U_{r+2i-2, r-2} \\ b_{r-1} = r+2i-1, \quad b_r = r+2i \\ b_{r+1} \in \{r+2i+1, \dots, r+2m+1\} \end{array} \right. \right\}.$$

**Claim 4.2.** *The set  $\mathcal{A} = \bigcup_{i=0}^m \mathcal{N}_i$  is an  $(r+2m+1, r+1, r)$ -covering.*

Let  $S = \{s_1, \dots, s_r\}$  be an  $r$ -subset of  $[r + 2m + 1]$ . If  $s_{r-1} = r + 2i - 1$  for some  $i \in \{0, \dots, m\}$ , then

$$\begin{aligned} S \subset (S \cup \{r + 2i + 1\}) &\in \mathcal{N}_i, \quad \text{for } s_r = r + 2i, \\ S \subset (S \cup \{r + 2i\}) &\in \mathcal{N}_i, \quad \text{for } s_r > r + 2i. \end{aligned}$$

Either, if  $s_{r-1} = r + 2i$  for some  $i \in \{0, \dots, m\}$ , then

$$\begin{aligned} S \subset (S \cup \{r + 2i - 1\}) &\in \mathcal{N}_i, \quad \text{for } s_{r-2} < r + 2i - 1, \\ S \subset (S \cup \{\alpha\}) &\in \mathcal{N}_i, \quad \text{for } s_{r-2} = r + 2i - 1, \end{aligned}$$

where  $\alpha \in [r + 2i - 2] \setminus \{s_1, \dots, s_{r-3}\}$ . In every case, there exists a block in  $\mathcal{N}_i$ , for some  $i \in \{0, \dots, m\}$ , containing the  $r$ -subset  $S$ . This concludes the proof of Claim 4.2.

**Claim 4.3.** *The graph  $G(\mathcal{N}_i)$  is connected, for every  $i \in \{0, \dots, m\}$ .*

Let  $i \in \{0, \dots, m\}$  and let  $S = \{s_1, \dots, s_{r-2}, r + 2i - 1, r + 2i, s_{r+1}\}$  be a block in  $\mathcal{N}_i$ . For every  $j \in \{0, \dots, r - 2\}$ , we consider the following  $(r + 1)$ -subsets  $S_j \in U_{r+2i+1, r+1}$ :

$$\begin{aligned} S_0 &= \{s_1, \dots, s_{r-2}, r + 2i - 1, r + 2i, r + 2i + 1\}, \\ S_j &= \{1, \dots, j, s_{j+1}, \dots, s_{r-2}, r + 2i - 1, r + 2i, r + 2i + 1\}, \quad \text{for } 1 \leq j \leq r - 3, \\ S_{r-2} &= \{1, \dots, r - 2, r + 2i - 1, r + 2i, r + 2i + 1\}. \end{aligned}$$

By definitions, it is easy to see that  $S_j \in \mathcal{N}_i$  for all  $j \in \{0, \dots, r - 2\}$ . Now, we consider the  $r$ -subsets  $T_j \in U_{r+2i-1, r}$  defined by

$$\begin{aligned} T_0 &= \{s_1, \dots, s_{r-2}, r + 2i - 1, r + 2i\}, \\ T_1 &= \{s_2, \dots, s_{r-2}, r + 2i - 1, r + 2i, r + 2i + 1\}, \\ T_j &= \{1, \dots, j - 1, s_{j+1}, \dots, s_{r-2}, r + 2i - 1, r + 2i, r + 2i + 1\}, \quad \text{for } 1 \leq j \leq r - 3, \\ T_{r-2} &= \{1, \dots, r - 3, r + 2i - 1, r + 2i, r + 2i + 1\}. \end{aligned}$$

Since  $T_0 \subset S$  and  $T_0 \subset S_0$ , we know that  $S$  and  $S_0$  are adjacent in the graph  $G(\mathcal{N}_i)$ . By the same way, since  $T_j \subset S_{j-1}$  and  $T_j \subset S_j$ , the blocks  $S_{j-1}$  and  $S_j$  are adjacent in  $G(\mathcal{N}_i)$  for all  $j \in \{1, \dots, r - 2\}$ . It follows that  $(S, S_0, S_1, \dots, S_{r-2})$  is a path in  $G(\mathcal{N}_i)$ . So we have proved that there always exists a path between any block  $S$  of  $\mathcal{N}_i$  and the fixed block  $S_{r-2}$ . This concludes the proof of Claim 4.3.

**Claim 4.4.** *For any integer  $i \in \{0, \dots, m - 1\}$ , there exists an  $(r + 1)$ -subset  $C_i$  such that the graph  $G(\mathcal{N}_i \cup C_i \cup \mathcal{N}_{i+1})$  is connected.*

Let  $i \in \{0, \dots, m - 1\}$  and let

$$C_i := \{1, \dots, r - 2, r + 2i, r + 2i + 1, r + 2i + 2\}.$$

Then, the block  $C_i$  is adjacent to

$$B_i = \{1, \dots, r - 2, r + 2i - 1, r + 2i, r + 2i + 1\} \in \mathcal{N}_i$$

and is adjacent to

$$B_{i+1} = \{1, \dots, r - 2, r + 1 + 2i, r + 2 + 2i, r + 3 + 2i\} \in \mathcal{N}_{i+1}$$

in  $G(\mathcal{N}_i \cup C_i \cup \mathcal{N}_{i+1})$ . Since  $G(\mathcal{N}_i)$  and  $G(\mathcal{N}_{i+1})$  are connected by Claim 4.3, we obtain that  $G(\mathcal{N}_i \cup C_i \cup \mathcal{N}_{i+1})$  is connected. This concludes the proof of Claim 4.4.

By combining results of Claims 4.2, 4.3 and 4.4, we know that the set

$$\mathcal{B} = \left( \bigcup_{i=0}^m \mathcal{N}_i \right) \cup \left( \bigcup_{i=0}^{m-1} C_i \right)$$

is a connected  $(r + 2m + 1, r + 1, r)$ -covering. Finally, since

$$|\mathcal{N}_i| = \binom{r - 2 + 2i}{r - 2} (2m + 1 - 2i),$$

for all  $i \in \{0, \dots, m\}$ , we deduce that

$$|\mathcal{B}| = \sum_{i=0}^m \binom{r - 2 + 2i}{r - 2} (2m + 1 - 2i) + m.$$

This completes the proof in this case.

*Case 2.* Suppose that  $n - r$  is an even number and let  $m$  such that  $n - r = 2m$ . We will construct a connected  $(r + 2m, r + 1, r)$ -covering of size

$$\sum_{i=0}^{m-1} \binom{r - 2 + 2i}{r - 2} (2m - 2i) + C(r - 2 + 2m, r - 2) + m - 1.$$

As already defined in Case 1, for any integer  $i \in \{0, \dots, m - 1\}$ , we consider the collection  $\mathcal{N}_i$  of  $(r + 1)$ -subsets of  $[r + 2m]$  defined by

$$\mathcal{N}_i := U_{r+2i-2, r-2} \times \{r + 2i - 1\} \times \{r + 2i\} \times \{r + 2i + 1, \dots, r + 2m\}.$$

Let

$$\mathcal{A} := \bigcup_{i=0}^{m-1} \mathcal{N}_i.$$

Observe that  $\mathcal{A}$  is not an  $(r + 2m, r + 1, r)$ -covering. In fact, from Claim 4.2, we know that the  $r$ -subsets of  $[r + 2m]$  that are not in  $\mathcal{A}$  are just the subsets of the form

$$\{b_1, \dots, b_{r-2}, r + 2m - 1, r + 2m\},$$

where  $\{b_1, \dots, b_{r-2}\} \in U_{r+2m-2, r-2}$ . Now, we complete  $\mathcal{A}$  in order to obtain an  $(r + 2m, r + 1, r)$ -covering. Let  $\mathcal{C}$  be an  $(r + 2m - 2, r - 1, r - 2)$ -covering of size  $C(r + 2m - 2, r - 2)$ . We consider the set

$$\mathcal{N}_m := \{B \cup \{r + 2m - 1, r + 2m\} \mid B \in \mathcal{C}\}$$

**Claim 4.5.** *The set  $\mathcal{A} \cup \mathcal{N}_m$  is an  $(r + 2m, r + 1, r)$ -covering.*

Since the  $r$ -subsets of  $[r + 2m]$  that are not in  $\mathcal{A}$  are elements of  $U_{r+2m-2, r-2} \times \{r + 2m - 1\} \times \{r + 2m\}$  and since every element of  $U_{r+2m-2, r-2}$  is contained in a block of  $\mathcal{C}$ , the result of Claim 4.5 follows.

For every  $i \in \{0, \dots, m - 1\}$ , we already know, from Claim 4.3, that the graph  $G(\mathcal{N}_i)$  is connected. Moreover, for every  $i \in \{0, \dots, m - 2\}$ , Claim 4.4 implies that the graph  $G(\mathcal{N}_i \cup C_i \cup \mathcal{N}_{i+1})$  is connected, where

$$C_i := \{1, \dots, r - 2, r + 2i, r + 2i + 1, r + 2i + 2\} \in U_{r+2m, r+1}.$$

We end this proof by showing that the graph  $G(\mathcal{A} \cup \mathcal{N}_m \cup C_0 \cup \dots \cup C_{m-2})$  is connected.

**Claim 4.6.** *For any  $B \in \mathcal{N}_m$ , there exist  $i \in \{0, \dots, m - 1\}$  and  $C \in \mathcal{N}_i$  such that  $B$  is adjacent to  $C$  in the graph  $G(\mathcal{N}_i \cup \mathcal{N}_m)$ .*

Let  $B = \{b_1, \dots, b_{r-1}, r + 2m - 1, r + 2m\} \in \mathcal{N}_m$ . If  $b_{r-1} = r + 2i - 1$  for some  $i \in \{0, \dots, m - 1\}$ , then we have  $\{b_1, \dots, b_{r-2}\} \in U_{r+2i-2, r-2}$ . Let

$$C := \{b_1, \dots, b_{r-2}, r + 2i - 1, r + 2i, r + 2m\}.$$

By definition,  $C$  is in  $\mathcal{N}_i$ . Moreover, since  $\{b_1, \dots, b_{r-2}, r + 2i - 1, r + 2m\} \subset B$  and  $\{b_1, \dots, b_{r-2}, r + 2i - 1, r + 2m\} \subset C$ , we deduce that  $B$  and  $C$  are adjacent in the

graph  $G(\mathcal{N}_i \cup \mathcal{N}_m)$ . Either, if  $b_{r-1} = r + 2i$  for some  $i \in \{0, \dots, m-1\}$ , then we have  $\{b_1, \dots, b_{r-2}\} \in U_{r+2i-1, r-2}$ . We distinguish two cases on the value of  $b_{r-2}$ . First, if  $b_{r-2} < r + 2i - 1$ , then  $\{b_1, \dots, b_{r-2}\} \in U_{r+2i-2, r-2}$ . Let

$$C := \{b_1, \dots, b_{r-2}, r + 2i - 1, r + 2i, r + 2m\}.$$

As above, since  $\{b_1, \dots, b_{r-2}, r + 2i, r + 2m\} \subset B$  and  $\{b_1, \dots, b_{r-2}, r + 2i, r + 2m\} \subset C$ , we deduce that  $B$  and  $C$  are adjacent in the graph  $G(\mathcal{N}_i \cup \mathcal{N}_m)$ . Finally, suppose that  $b_{r-2} = r + 2i - 1$ . Let  $\alpha \in [r + 2i - 2] \setminus \{b_1, \dots, b_{r-3}\}$  and let

$$C := \{b_1, \dots, b_{r-3}, r + 2i - 1, r + 2i, r + 2m\} \cup \{\alpha\} \in U_{r+2m, r+1}.$$

Since  $\{b_1, \dots, b_{r-3}, r + 2i - 1, r + 2i, r + 2m\} \subset B$  and  $\{b_1, \dots, b_{r-3}, r + 2i - 1, r + 2i, r + 2m\} \subset C$ , we deduce that  $B$  and  $C$  are adjacent in the graph  $G(\mathcal{N}_i \cup \mathcal{N}_m)$ . This concludes the proof of Claim 4.6.

By combining results of Claims 4.4, 4.5 and 4.6, we know that the set

$$\mathcal{B} = \left( \bigcup_{i=0}^m \mathcal{N}_i \right) \cup \left( \bigcup_{i=0}^{m-2} C_i \right)$$

is a connected  $(r + 2m, r + 1, r)$ -covering. Finally, since

$$|\mathcal{N}_i| = \binom{r-2+2i}{r-2} (2m+1-2i),$$

for all  $i \in \{0, \dots, m-1\}$ , and

$$|\mathcal{N}_m| = C(r + 2m - 2, r - 2),$$

we deduce that

$$|\mathcal{B}| = \sum_{i=0}^{m-1} \binom{r-2+2i}{r-2} (2m+1-2i) + C(r + 2m - 2, r - 2) + m - 1.$$

This completes the proof of Theorem 4.1. □

Let us illustrate the construction given in the above theorem.

**Example 4.7.**  $N(7, 4) = 10$ . We consider

$$\mathcal{N}_0 = \{12345, 12346, 12347\} \text{ and } \mathcal{N}_1 = \{12567, 13567, 14567, 23567, 24567, 34567\}.$$

It can be checked that  $\mathcal{N}_0 \cup \mathcal{N}_1$  is a  $(7, 5, 4)$ -covering and  $G(\mathcal{N}_0)$  and  $G(\mathcal{N}_1)$  are connected. Now, by taking  $C_0 = 12456$ , it follows that  $G(\mathcal{N}_0 \cup C_0 \cup \mathcal{N}_1)$  is connected.

We may now show that  $S(n, r) > N(n, r)$ . For this we need first the following Theorem and Proposition.

**Theorem 4.8.** *Let  $r$  and  $n$  be positive integers such that  $n \geq r + 1 \geq 3$ . Then,*

$$S(n, r) = N(n, r) + \sum_{i=0}^{\lfloor \frac{n-r}{2} \rfloor - 1} \left( \left\lfloor \frac{n-r}{2} \right\rfloor - i \right) \binom{r-2+2i}{r-3} + \delta_0 (1 - C(n-2, r-2)),$$

where  $\delta_0$  is the parity function of  $n - r$ .

*Proof.* By induction on  $n > r$ . For  $n = r + 1$  and  $n = r + 2$  we have from (12) and (14) that

$$S(r + 1, r) = \binom{r - 1}{r - 1} = 1, \quad N(r + 1, r) = 1 \cdot \binom{r - 2}{r - 2} + 1 - 1 = 1,$$

and

$$\begin{aligned} S(r + 2, r) &= \binom{r}{r - 1} + 1 = r + 1, \quad N(r + 2, r) = 2 \cdot \binom{r - 2}{r - 2} + 1 - 1 + C(r, r - 2) \\ &= 2 + 1 - 1 + r - 1 = r + 1. \end{aligned}$$

Thus the identity is verified for  $n = r + 1$  and  $n = r + 2$ . Suppose now that for a certain value of  $n$ , we have

$$S(n, r) = N(n, r) + \sum_{i=0}^{\lfloor \frac{n-r}{2} \rfloor - 1} \left( \left\lfloor \frac{n-r}{2} \right\rfloor - i \right) \binom{r - 2 + 2i}{r - 3} + \delta_0 (1 - C(n - 2, r - 2)),$$

and let  $D$  be the difference

$$D := (S(n + 2, r) - N(n + 2, r)) - (S(n, r) - N(n, r)).$$

Then,

$$S(n + 2, r) = N(n + 2, r) + (S(n, r) - N(n, r)) + D.$$

By using (12), we obtain

$$\begin{aligned} S(n + 2, r) - S(n, r) &= \sum_{i=1}^{\lfloor \frac{n-r+1}{2} \rfloor + 1} \binom{n + 2 - 2i}{r - 1} + \left\lfloor \frac{n-r}{2} \right\rfloor + 1 - \sum_{i=1}^{\lfloor \frac{n-r+1}{2} \rfloor} \binom{n - 2i}{r - 1} - \left\lfloor \frac{n-r}{2} \right\rfloor \\ &= \sum_{i=0}^{\lfloor \frac{n-r+1}{2} \rfloor} \binom{n - 2i}{r - 1} + 1 - \sum_{i=1}^{\lfloor \frac{n-r+1}{2} \rfloor} \binom{n - 2i}{r - 1} \\ &= \binom{n}{r - 1} + 1. \end{aligned}$$

By using (14), we have

$$\begin{aligned} N(n + 2, r) - N(n, r) &= \sum_{i=0}^{\lfloor \frac{n-r}{2} \rfloor} (n + 2 - r - 2i) \binom{r - 2 + 2i}{r - 2} + \left\lfloor \frac{n-r}{2} \right\rfloor + \delta_0 C(n, r - 2) \\ &\quad - \sum_{i=0}^{\lfloor \frac{n-r}{2} \rfloor - 1} (n - r - 2i) \binom{r - 2 + 2i}{r - 2} - \left\lfloor \frac{n-r}{2} \right\rfloor + 1 - \delta_0 C(n - 2, r - 2) \\ &= \sum_{i=0}^{\lfloor \frac{n-r}{2} \rfloor - 1} 2 \binom{r - 2 + 2i}{r - 2} + \left( n + 2 - r - 2 \left\lfloor \frac{n-r}{2} \right\rfloor \right) \binom{r - 2 + 2 \lfloor \frac{n-r}{2} \rfloor}{r - 2} \\ &\quad + \delta_0 (C(n, r - 2) - C(n - 2, r - 2)) + 1. \end{aligned}$$

Moreover, since

$$n - r - 2 \left\lfloor \frac{n-r}{2} \right\rfloor = \begin{cases} 0 & \text{if } n - r \text{ is even,} \\ -1 & \text{otherwise,} \end{cases}$$

and

$$\binom{r-2+2\lceil \frac{n-r}{2} \rceil}{r-2} = \binom{n-1}{r-2}$$

for  $n-r$  odd, it follows that

$$\begin{aligned} N(n+2, r) - N(n, r) &= \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} 2 \binom{r-2+2i}{r-2} + (\delta_0 - 1) \binom{n-1}{r-2} \\ &\quad + \delta_0 (C(n, r-2) - C(n-2, r-2)) + 1 \end{aligned}$$

Therefore

$$D = \binom{n}{r-1} - \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} 2 \binom{r-2+2i}{r-2} + (1-\delta_0) \binom{n-1}{r-2} + \delta_0 (C(n-2, r-2) - C(n, r-2)).$$

Moreover, from the identity

$$\binom{r-2+2i}{r-2} = \binom{r-1+2i}{r-2} - \binom{r-2+2i}{r-3},$$

we obtain that

$$\begin{aligned} \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} 2 \binom{r-2+2i}{r-2} &= \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} \binom{r-2+2i}{r-2} + \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} \binom{r-1+2i}{r-2} - \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} \binom{r-2+2i}{r-3} \\ &= \sum_{i=r-2}^{r-1+2\lceil \frac{n-r}{2} \rceil} \binom{i}{r-2} - \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} \binom{r-2+2i}{r-3} \\ &= \binom{r+2\lceil \frac{n-r}{2} \rceil}{r-1} - \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} \binom{r-2+2i}{r-3}. \end{aligned}$$

Thus,

$$\begin{aligned} D &= \binom{n}{r-1} - \binom{r+2\lceil \frac{n-r}{2} \rceil}{r-1} + (1-\delta_0) \binom{n-1}{r-2} + \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} \binom{r-2+2i}{r-3} \\ &\quad + \delta_0 (C(n-2, r-2) - C(n, r-2)). \end{aligned}$$

If  $n-r$  is even, then  $\delta_0 = 1$  and

$$\binom{n}{r-1} - \binom{r+2\lceil \frac{n-r}{2} \rceil}{r-1} + (1-\delta_0) \binom{n-1}{r-2} = \binom{n}{r-1} - \binom{n}{r-1} = 0.$$

Either, if  $n - r$  is odd, then  $\delta_0 = 0$  and

$$\begin{aligned} \binom{n}{r-1} - \binom{r+2 \lceil \frac{n-r}{2} \rceil}{r-1} + (1 - \delta_0) \binom{n-1}{r-2} &= \binom{n}{r-1} - \binom{n+1}{r-1} + \binom{n-1}{r-2} \\ &= -\binom{n}{r-2} + \binom{n-1}{r-2} \\ &= -\binom{n-1}{r-3}. \end{aligned}$$

It follows that

$$\begin{aligned} D &= \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} \binom{r-2+2i}{r-3} + (\delta_0 - 1) \binom{n-1}{r-3} + \delta_0 (C(n-2, r-2) - C(n, r-2)) \\ &= \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} \binom{r-2+2i}{r-3} + \delta_0 (C(n-2, r-2) - C(n, r-2)). \end{aligned}$$

Now, with the induction hypothesis, we obtain

$$\begin{aligned} S(n+2, r) - N(n+2, r) &= (S(n, r) - N(n, r)) + D \\ &= \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil - 1} \left( \binom{\lfloor \frac{n-r}{2} \rfloor - i}{r-3} \right) \binom{r-2+2i}{r-3} + \delta_0 (1 - C(n-2, r-2)) \\ &\quad + \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} \binom{r-2+2i}{r-3} + \delta_0 (C(n-2, r-2) - C(n, r-2)) \\ &= \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} \left( \binom{\lfloor \frac{n-r}{2} \rfloor + 1 - i}{r-3} \right) \binom{r-2+2i}{r-3} + \delta_0 (1 - C(n, r-2)). \end{aligned}$$

□

**Proposition 4.9.** *Let  $n$  and  $r$  be positive integers such that  $n \geq r + 3 \geq 5$ . Then,*

$$C(n, r) \leq \binom{n-2}{r-1} + C(n-2, r).$$

*Proof.* Let  $S_1 = \{A_1, \dots, A_s\}$  be a  $(n-2, r+1, r)$ -covering such that  $s = C(n-2, r+1, r)$  and let  $U_{n-2, r-1}$  be the set of all the  $(r-1)$ -subsets of  $\{1, \dots, n-2\}$ . We denote by  $U_{n-2, r-1} = \{B_1, \dots, B_t\}$  where  $t = \binom{n-2}{r-1}$  these elements. We will prove that the collection

$$S_2 = \{A_1, \dots, A_s, B_1 \cup \{n-1, n\}, \dots, B_t \cup \{n-1, n\}\}$$

is a  $(n, r+1, r)$ -covering. Obviously, all the blocks of the form  $A_i$  or  $B_i \cup \{n-1, n\}$  are  $(r+1)$ -subsets of  $\{1, \dots, n\}$ . Let  $C$  be an  $r$ -subset of  $\{1, \dots, n\}$ . First, if  $C \cap \{n-1, n\} = \emptyset$ , by definition of  $S_1$ , there exists at least one block  $A_i$  containing  $C$ . Now, if  $C \cap \{n-1, n\} = \{x\}$ , with either  $x = n-1$  or  $x = n$ , then  $C \setminus \{x\}$  is an  $(r-1)$ -subset of  $\{1, \dots, n-2\}$ , thus  $C \setminus \{x\} = B_i$  for a certain  $1 \leq i \leq t$ . It follows that the  $r$ -subset  $C$  is contained in  $B_i \cup \{n-1, n\}$ . Finally, if  $\{n-1, n\} \subset C$ , then  $C \setminus \{n-1, n\}$  is an  $(r-2)$ -subset

of  $\{1, \dots, n-2\}$  and thus there exist exactly  $n-r$  blocks  $B_i$  of  $U_{n-2, r-1}$  containing  $C \setminus \{n-1, n\}$ . It follows that there exist  $n-r$  blocks of the form  $B_i \cup \{n-1, n\}$  containing  $C$ . Therefore  $S_2$  is a  $(n, r+1, r)$ -covering and

$$C(n, r) \leq |S_2| = |S_1| + |U_{n-2, r-1}| = C(n-2, r) + \binom{n-2}{r-1}.$$

□

**Theorem 4.10.** *Let  $r$  and  $n$  be positive integers such that  $n-r$  is an even number. Then,*

$$S(n, r) \geq N(n, r) + \sum_{i=0}^{\frac{n-r}{2}-2} \binom{\frac{n-r}{2} - i - 1}{r-3} \binom{r-2+2i}{r-3}.$$

*Proof.* By applying repeatedly Proposition 4.9, we have

$$C(n-2, r-2) \leq \sum_{i=0}^{\frac{n-r}{2}-1} \binom{r-2+2i}{r-3} + 1.$$

Then, we deduce from Theorem 4.8 that

$$\begin{aligned} S(n, r) &= N(n, r) + \sum_{i=0}^{\frac{n-r}{2}-1} \binom{\frac{n-r}{2} - i}{r-3} \binom{r-2+2i}{r-3} + 1 - C(n-2, r-2) \\ &\geq N(n, r) + \sum_{i=0}^{\frac{n-r}{2}-1} \binom{\frac{n-r}{2} - i}{r-3} \binom{r-2+2i}{r-3} - \sum_{i=0}^{\frac{n-r}{2}-1} \binom{r-2+2i}{r-3} \\ &= N(n, r) + \sum_{i=0}^{\frac{n-r}{2}-2} \binom{\frac{n-r}{2} - i - 1}{r-3} \binom{r-2+2i}{r-3}. \end{aligned}$$

□

## 5. ASYMPTOTICS

In [19] Rödl uses the probabilistic method to show the existence of *asymptotically good coverings*. Restricted to our case this means that

$$\frac{C(n, r)}{\binom{n}{r}} \rightarrow \frac{1}{r+1} \text{ as } n \rightarrow \infty.$$

Since  $CC(n, r) \leq 2C(n, r)$  (see [5]) we immediately obtain:

$$\frac{CC(n, r)}{\binom{n}{r}} \rightarrow \frac{2}{r+1} \text{ as } n \rightarrow \infty.$$

This is asymptotically, i.e. for large  $n$  compared to  $r$ , better than the results of the previous section: In [5] it was shown that

$$\frac{S(n, r)}{\binom{n}{r}} \rightarrow \frac{1}{2} \text{ as } n \rightarrow \infty$$

and since by Theorem 4.10 the difference  $N(n, r) - S(n, r)$  is in  $\mathcal{O}(n^{r-1})$  we have the same asymptotic behavior for  $N(n, r)$ .

It is however still a topic of research to find explicit constructions witnessing the bound of Rödl, see [15].

*Acknowledgments.* Much of this work in particular for the construction of the connected covering designs in Figures 1, 3, 4 strongly benefited from the La Jolla Covering Repository (<http://www.ccrwest.org/cover.html>) maintained by Dan Gordon.

## REFERENCES

- [1] D. Applegate, E.M. Rains, and N.J.A. Sloane. On asymmetric coverings and covering numbers. *J. Combin. Des.*, 11(3):218–228, 2003.
- [2] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G.M. Ziegler. *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1999.
- [3] J. Chappelon, K. Knauer, L.P. Montejano, and J.L. Ramírez Alfonsín. How many circuits determine a (oriented) matroid? (in preparation).
- [4] J. Folkman and J. Lawrence. Oriented matroids. *J. Combin. Theory Ser. B*, 25(2):199–236, 1978.
- [5] D. Forge and J.L. Ramírez Alfonsín. Connected coverings and an application to oriented matroids. *Discrete Math.*, 187(1-3):109–121, 1998.
- [6] D. Forge, J.L. Ramírez Alfonsín, and H. Yeun. Disconnected coverings for oriented matroids via simultaneous mutations. *Discrete Math.*, 258(1-3):353–359, 2002.
- [7] M.K. Fort, Jr. and G.A. Hedlund. Minimal coverings of pairs by triples. *Pacific J. Math.*, 8:709–719, 1958.
- [8] D.M. Gordon, O. Patashnik, and G. Kuperberg. New constructions for covering designs. *J. Combin. Des.*, 3(4):269–284, 1995.
- [9] Y.O. Hamidoune and M. Las Vergnas. Directed switching games on graphs and matroids. *J. Combin. Theory Ser. B*, 40(3):237–269, 1986.
- [10] H. Hanani. On quadruple systems. *Canad. J. Math.*, 12:145–157, 1960.
- [11] A. Hartman, W.H. Mills, and R.C. Mullin. Covering triples by quadruples: an asymptotic solution. *J. Combin. Theory Ser. A*, 41(1):117–138, 1986.
- [12] L. Ji. An improvement on covering triples by quadruples. *J. Combin. Des.*, 16(3):231–243, 2008.
- [13] J.G. Kalbfleisch and R.G. Stanton. Maximal and minimal coverings of  $(k - 1)$ -tuples by  $k$ -tuples. *Pacific J. Math.*, 26:131–140, 1968.
- [14] G. Katona, T. Nemetz, and M. Simonovits. On a problem of Turán in the theory of graphs. *Mat. Lapok*, 15:228–238, 1964.
- [15] N.N. Kuzjurin. Explicit constructions of Rödl’s asymptotically good packings and coverings. *Combin. Probab. Comput.*, 9(3):265–276, 2000.
- [16] W. Mantel. Problem 28. *Wiskundige Opgaven*, 10(60-61):320, 1907.
- [17] W.H. Mills. On the covering of triples by quadruples. In *Proceedings of the Fifth Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1974)*, pages 563–581. Congressus Numerantium, No. X, Winnipeg, Man., 1974. Utilitas Math.
- [18] W.H. Mills. A covering of triples by quadruples. In *Proceedings of the Twelfth Southeastern Conference on Combinatorics, Graph Theory and Computing, Vol. II (Baton Rouge, La., 1981)*, volume 33, pages 253–260, 1981.
- [19] V. Rödl. On a packing and covering problem. *European J. Combin.*, 6(1):69–78, 1985.
- [20] J. Schönheim. On coverings. *Pacific J. Math.*, 14:1405–1411, 1964.
- [21] A. Sidorenko. What we know and what we do not know about Turán numbers. *Graphs Combin.*, 11(2):179–199, 1995.
- [22] A.F. Sidorenko. The method of quadratic forms in a combinatorial problem of Turán. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, (1):3–6, 76, 1982.
- [23] J.D. Swift. A generalized Steiner problem. *Rend. Mat. (6)*, 2:563–569, 1969.
- [24] P. Turán. Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok*, 48:436–452, 1941.
- [25] P. Turán. Research problems. *Magyar Tud. Akad. Mat. Kut. Int. Közl.*, 6:417–423, 1961.

## APPENDIX - PROOF OF PROPOSITION 1.2

For basic notions of oriented matroids we refer the reader to [2].

Let  $S$  be a set of circuits of  $U_{r,n}$  with  $|S| < C(n,r)$ , that is, a set of circuits not covering all the bases, and let  $B$  be a base not covered by  $S$ , that is,  $B$  is not contained in any circuit of  $S$ .

Let us consider  $U_{r,n}^* = U_{n-r,n}$  the dual of  $U_{r,n}$ . By the well-known Topological Representation Theorem due to Folkman and Lawrence [4], there is a correspondence between oriented matroids whose *adjacent* non-oriented matroid is  $U_{r,n}^*$  and  $(n-r)$ -dimensional oriented simple pseudo-sphere arrangements. Moreover, we can choose an oriented matroid  $\mathcal{M}$  such that the elements of  $B$  correspond to pseudo-spheres not bounding a simplicial region of  $\mathcal{A}$  corresponding to  $\mathcal{M}$ . Let us call the simplex  $R$ .

By duality, the circuits of  $U_{r,n}$  correspond to the cocircuits of  $U_{n-r,n}$ . Their orientations in  $\mathcal{M}$  in turn are in bijection with the vertices of the arrangement  $\mathcal{A}$ . We notice that among the circuits of  $U_{r,n}$  containing  $B$ , there are those corresponding to the vertices of simplex  $R$ .

Now, we apply a *mutation* to simplex  $R$  obtaining from  $\mathcal{M}$  another oriented matroid  $\mathcal{M}'$  adjacent to  $U_{r,n}^*$ . All cocircuits of  $\mathcal{M}'$  have the same orientation as those of  $\mathcal{M}$  except for the cocircuits corresponding to vertices of  $R$  that have changed after the mutation. Since none of the cocircuits corresponding to the vertices of  $R$  belong to  $S$  (that does not cover  $B$ )  $S$  does not determine uniquely  $\mathcal{M}$  of  $U_{r,n}^*$  and dualizing back neither determines uniquely the dual  $\mathcal{M}^*$  which is adjacent to  $U_{r,n}$ .  $\square$

UNIVERSITÉ MONTPELLIER 2, INSTITUT DE MATHÉMATIQUES ET DE MODÉLISATION DE MONTPELLIER, CASE COURRIER 051, PLACE EUGÈNE BATAILLON, 34095 MONTPELLIER CEDEX 05, FRANCE.

*E-mail address:* jonathan.chappelon@um2.fr

*E-mail address:* kolja.knauer@googlemail.com

*E-mail address:* luispedro81@yahoo.com.mx

*E-mail address:* jramirez@math.univ-montp2.fr