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Poisson approximation for some point processes in reliability

J.-B. Gravereaux and James Ledoux*

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Abstract

In this paper, we consider a failure point process related to the Markovian Arrival Process defined by Neuts. We show that it converges in distribution to a homogeneous Poisson process. This convergence takes place in context of rare occurrences of failures. We also provide a convergence rate of the convergence in total variation of this point process using an approach developed by Kabanov, Liptser and Shiryaev for the doubly-stochastic Poisson process driven by a finite Markov process.

KEYWORDS: Compensator, Software reliability, Markovian Arrival Process, Doubly Stochastic Process

AMS : 60G55; 60J25; 60J75; 90B25; 93E11

1 Introduction

This work originates in Littlewood papers [12],[13] on a Markov-type model for reliability assessment of a modular software. Basically, for a software with a finite number of modules :

- the structure of the software is represented by a finite continuous time Markov chain (CTMC) $(X_t)_t$ where X_t is the active module at time t ;
- when module i is active, failures times are part of a homogeneous Poisson Process (HPP) with intensity $\mu(i)$;
- when control switches from module i to module j a failure may happen with probability $\mu(i, j)$;

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- when any failure appears, it does not affect the software because the execution is assumed to be restarted instantaneously. Such event is referred as to a secondary failure in [9].

Extend of such a model is considered in [9], taking into account the influence of failures on the execution dynamic of the software and dealing with the delays in recovering an operational state. Transient analysis was provided by means of results from [10]. Roughly speaking, failure point process was a Markovian Arrival Process (MAP) as defined by Neuts (see e.g. [15]). Therefore, it is well known that we obtain as particular instances of our failure process: a phase-type renewal process, a doubly stochastic poisson process with a stochastic intensity driven by a CTMC (also called a Markov Modulated Poisson Process (MMPP) in queueing literature), etc.

An important issue in reliability theory, specifically for software systems, is what happens when the failure parameters tend to be smaller and smaller. Littlewood stated in [12] (or in [13] for a semi-Markov process (X_t))

As all failure parameters $\mu(i), \mu(i, j)$ tend to zero, the failure process described above is asymptotically an HPP with intensity

$$\lambda = \sum_i \pi(i) \left[\sum_{j \neq i} Q(i, j) \mu(i, j) + \mu(i) \right] \quad (1.1)$$

where π and Q are the stationary distribution and the generator of the CTMC (X_t) (assumed to be irreducible), respectively. This statement is well-known in the community of software reliability and has widely supported the *hierarchical approach* for modeling modular software (see e.g. [6] for details). However, to the best of our knowledge, no proof of this fact is reported in the applied probability literature. The aim of this note is to provide precise statements and proofs for the asymptotic of the general failure point process (pp) defined in [9] for which the Littlewood's model is a particular case. Specifically, we show that the counting process corresponding to this pp converges in distribution to the counting process of an HPP when failure parameters tend to zero but at a specific time scale. Roughly speaking, we introduce a small parameter ε in the failure parameters and the convergence takes place at time scale t/ε (in other words when failure parameters are small and on a large horizon time). Proving this result is easy using a criterion of convergence in distribution given in [8] for instance. It is based on the convergence in probability of compensators corresponding to the various counting processes. In fact, the counting process converges in variation to the counting process of an HPP and the convergence rate will be stated using a method developed in [8], for the MMPP.

Note that the class of MMPP is widely used as a model of traffic streams for communication systems. It is easily seen that dealing with the present issue is equivalent to consider asymptotic for MMPP with a fast modulating Markov chain. Thus we retrieve the more or less known fact that, when jitterness takes place, the arrival process tends to Poissonian (see [14, page 116] for a partial discussion).

Paper is organized as follows. Section 2 recalls some background on the pp studied here. Moreover, the compensator of the pp is derived in a straightforward manner. In Section 3, we report results about convergence in distribution of the pp to an HPP. Connection to problem of fast modulation in the case of an MMPP is briefly addressed in Subsection 3.3. Rate of convergence in total variation of the pp is stated in Section 4. Appendix A recalls some estimate of convergence rate of singularly perturbed generator provided in [18]. The derivation of an inequality used in the text is reported in Appendix B.

2 Definition of the counting process. Compensator

2.1 Definition of the model

We do not report the rationale underlying the definition of the reliability model discussed here. We refer to [9] for details. We just need of its mathematical formulation.

Parameters $\mu(\cdot, \cdot), \mu(\cdot)$ are as in Introduction. These failures was called secondary events in [9]. Process $X = (X_t)_t$ is an irreducible CTMC with infinitesimal generator $Q = (Q(i, j))_{i, j \in \mathcal{M}}$ where \mathcal{M} is the finite set $\{1, \dots, M\}$. X_t is the active component at time t for a failure-free system, that is X is the execution process. Vector $\alpha = (\alpha(i))_{i \in \mathcal{M}}$ denotes the distribution of random variable X_0 . New parameters $\lambda(i, j)$ and $\lambda(i)$ ($i, j \in \mathcal{M}$) are introduced with the same meaning than $\mu(i, j)$ and $\mu(i)$. But when such a type of failure happens in module i or during a transition from module i , there is a probability $p(i, k)$ that execution restarts in module k . So that, for each $i \in \mathcal{M}$, $(p(i, k))_{k \in \mathcal{M}}$ is a probability distribution. Such event was referred as to a primary failure in [9]. Simultaneous occurrence of a primary and secondary events is neglected. For simplicity, we do not consider delay in recovering an operational state as used in [9].

Then taking into account failure occurrences, random variable X_t^* gives the active component at time t . The random variable N_t counts the number of (primary and secondary) failures in interval $]0, t]$ ($N(0) = 0$). Thus, $N = (N_t)_t$ is the counting process of the failure pp. Under the various assumptions in [9], the bivariate process $Z := (N_t, X_t^*)_t$ can be considered as a CTMC over the state space $S = \mathbb{N} \times \mathcal{M}$. Its infinitesimal generator, denoted by G , has the special structure

$$G := \begin{pmatrix} D_0 & D_1 & 0 & \cdots \\ 0 & D_0 & D_1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

using a lexicographic order on state space S . Matrices D_0 and D_1 are defined by

$$\begin{aligned} D_0(i, j) &:= Q(i, j)(1 - \lambda(i, j))(1 - \mu(i, j)) \quad \text{if } i \neq j, \\ D_0(i, i) &:= - \sum_{j \neq i} Q(i, j) - \lambda(i) - \mu(i); \end{aligned}$$

$$D_1(i, j) := \left[\lambda(i) + \sum_{k \neq i} Q(i, k) \lambda(i, k) \right] p(i, j) + Q(i, j) [1 - \lambda(i, j)] \mu(i, j) \quad \text{if } i \neq j,$$

$$D_1(i, i) := \left[\lambda(i) + \sum_{k \neq i} Q(i, k) \lambda(i, k) \right] p(i, i) + \mu(i).$$

Note that $\max_x |G(x, x)| < +\infty$. The structure of the generator G shows that N is the counting process of a MAP. Finally, X^* is a CTMC with state space \mathcal{M} , initial distribution α and generator

$$Q^* := D_0 + D_1.$$

X^* is supposed to be right continuous with left limits (c.a.d-l.a.g.). If the failure parameters are assumed to be such that $\lambda(i, j) < 1$ for any (i, j) , then X^* is irreducible since X is. This assumption is not very stringent.

Example 2.1 (Littlewood's model)

Assume that there is no primary failures, that is, $\lambda(i) = 0$ and $\lambda(i, j) = 0$ for all $i, j \in \mathcal{M}$. We obtain the model of Littlewood. Then matrices D_0 and D_1 are given by

$$D_0(i, j) = \begin{cases} Q(i, j)(1 - \mu(i, j)) & \text{if } i \neq j, \\ -\sum_{j \neq i} Q(i, j) - \mu(i) & \text{if } i = j, \end{cases} \quad D_1(i, j) = \begin{cases} Q(i, j)\mu(i, j) & \text{if } i \neq j, \\ \mu(i) & \text{if } i = j. \end{cases}$$

Furthermore, we have $Q^* = Q$ and we retrieve the fact that failures do not affect the execution process.

Example 2.2 (Markov modulated Poisson Process)

Another interesting point process is the one obtained by assuming in Littlewood's model that the probability of a secondary failure during a control transfer is 0. In this case, setting $\mu(i, j) = 0$ for all $i, j \in \mathcal{M}$ in the previous expressions, we get $D_0 = Q - \text{diag}(\mu(i))$ and $D_1 = \text{diag}(\mu(i))$. This an MMPP.

Note that $Z' = (X_t^*, N_t)_t$ is also a Markov process with homogeneous second component as defined in [5], or a Markov-additive process of arrivals discussed in [16].

2.2 Compensator and intensity of the counting process

The basic facts on point processes, martingales, compensator and intensity used in this paper are reported in [2]. A nice survey on point processes is [17]. For any process $V = (V_t)_t$, $\mathcal{F}^V = (\mathcal{F}_t^V)_t$ will denote its internal history, i.e. $\mathcal{F}_t^V := \sigma(V_s, s \leq t)$.

Considering the bivariate process Z to analyze the counting process N allows to deal with a Markov process with discrete state space. Thus, we can take advantage of the powerful analytic theory and the computational material developed for such class of processes. This fact was exploited in [10],[9] to assess various reliability measures related to the transient behavior of the counting process. Due to the special structure of the

generator G of Z , N may be interpreted as the counter of specific transitions in Z . More precisely, if we are interested in the following pairs of states in S

$$T := \{((n, i); (n + 1, j)), \quad i, j \in \mathcal{M}, n \geq 0\},$$

then

$$\begin{aligned} N_t &= \sum_{(x,y) \in T} \sum_{0 < s \leq t} \mathbf{1}_{\{Z_{s-}=x, Z_s=y\}} \\ &= \sum_{(x,y) \in T} N_t(x, y). \end{aligned} \tag{2.1}$$

In other words, N_t counts the transitions from state x to y ($x, y \in T$) in interval $]0, t]$. It is well-known (see e.g. [2]) that for a counter $N(x, y) = (N_t(x, y))_t$ of transitions from state x to y in a Markov chain, we have

$$N_t(x, y) - \int_0^t \mathbf{1}_{\{Z_{s-}=x\}} G(x, y) ds$$

is a $\mathcal{F}^Z = (\mathcal{F}_t^Z)_t$ martingale. The random function

$$\lambda_t(x, y) := \mathbf{1}_{\{Z_{t-}=x\}} G(x, y) \tag{2.2}$$

is the \mathcal{F}^Z -intensity of $N(x, y)$ and

$$A_t(x, y) := \int_0^t \lambda_s(x, y) ds$$

is the \mathcal{F}^Z -compensator (\mathcal{F}_t^Z dual predictable-projection) of $N(x, y)$. Then it is easily seen that $M = (M_t)_t$ with $M_t = N_t - A_t$ and

$$A_t = \sum_{(x,y) \in T} A_t(x, y)$$

is a \mathcal{F}^Z -martingale. The \mathcal{F}^Z -intensity of N is

$$\begin{aligned} \lambda_t &:= \sum_{(x,y) \in T} \mathbf{1}_{\{Z_{s-}=x\}} G(x, y) \\ &= \sum_{n \geq 0} \sum_{i \in \mathcal{M}} \sum_{j \in \mathcal{M}} G((n, i); (n + 1, j)) \mathbf{1}_{\{(N_{t-}, X_{t-}^*)=(n, i)\}} \\ &= \sum_{n \geq 0} \sum_{i \in \mathcal{M}} \sum_{j \in \mathcal{M}} D_1(i, j) \mathbf{1}_{\{(N_{t-}, X_{t-}^*)=(n, i)\}} \\ &= \sum_{i \in \mathcal{M}} \mathbf{1}_{\{X_{t-}^*=i\}} \left[\sum_{j \in \mathcal{M}} D_1(i, j) \right] = \sum_{j \in \mathcal{M}} D_1(X_{t-}^*, j). \end{aligned}$$

In the case of Example 2.1, the intensity of the counting process corresponding to Littlewood's model is

$$\mu(X_{t-}) + \sum_{j \neq X_{t-}} Q(X_{t-}, j) \mu(X_{t-}, j).$$

In the case of Example 2.2, we retrieve the well-known expression for the $\mathcal{F}^{(N, X)}$ -intensity for an MMPP

$$\mu(X_{t-}).$$

3 Convergence of the counting process

3.1 What does it mean that failure parameters are small?

A basic way to represent smaller and smaller failure parameters is to multiply each of them by a scalar ε and to investigate the behavior of the counting process N as ε tends to 0. Thus, we consider the perturbed failure parameters

$$\varepsilon\lambda(i), \varepsilon\lambda(i, j), \varepsilon\mu(i), \varepsilon\mu(i, j), \quad i, j \in \mathcal{M}. \quad (3.1)$$

Let us consider the example of an MMPP (see Example 2.2) with associated matrices

$$Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \quad D_1 = \begin{pmatrix} \mu(1) & 0 \\ 0 & \mu(2) \end{pmatrix}$$

($\mu(1) \neq \mu(2)$) and with $\alpha = \pi = (1/2, 1/2)$ (i.e. with a stationary environment). Matrix $D_0^{(\varepsilon)}$ is

$$D_0^{(\varepsilon)} = Q - \begin{pmatrix} \mu(1)\varepsilon & 0 \\ 0 & \mu(2)\varepsilon \end{pmatrix}.$$

If T is the time to first failure, then

$$\mathbb{P}\{T > t\} = \mathbb{P}\{N_t = 0\} = \pi \exp(D_0^{(\varepsilon)} t) \mathbf{1}^t.$$

Setting $a = 4 + (\mu(1) - \mu(2))^2 \varepsilon^2$, we have

$$\begin{aligned} \pi \exp(D_0^{(\varepsilon)} t) \mathbf{1}^t &= \left[\cosh\left(\frac{t\sqrt{a}}{2}\right) + \frac{2}{\sqrt{a}} \sinh\left(\frac{t\sqrt{a}}{2}\right) \right] \exp(-t) \\ &\quad \times \exp(-(\mu(1)\pi(1) + \mu(2)\pi(2))\varepsilon t) \end{aligned}$$

As it is expected, $\mathbb{P}\{T > t\}$ converges to 1 as ε tends to 0. Therefore, convergence in distribution of the counting process to an HPP, i.e. weak convergence of the finite-dimensional distributions of N to those of an HPP, can not take place at the current time

scale. If we investigate the asymptotic distribution of T at time scale t/ε , we get from previous expression for $\mathbb{P}\{T > t\}$

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}\{T > t/\varepsilon\} = \exp\left(-(\mu(1)\pi(1) + \mu(2)\pi(2))t\right).$$

Therefore, we will deal with the counting process $N^{(\varepsilon)} = (N_t^{(\varepsilon)})_t$ defined by

$$N_t^{(\varepsilon)} = N_{\frac{t}{\varepsilon}}$$

where N_t counts the number of failures in interval $]0, t]$ for the reliability model of Section 2 with system (3.1) of perturbed failure parameters. $D_0^{(\varepsilon)}, D_1^{(\varepsilon)}$ are the matrices associated with model N . Note that

$$\begin{aligned} D_1^{(\varepsilon)} &= \varepsilon B + \varepsilon^2 L & (3.2) \\ \text{with } L(i, j) &= \begin{cases} -Q(i, j)\lambda(i, j)\mu(i, j) & \text{if } j \neq i \\ 0 & \text{if } j = i \end{cases} \\ B(i, j) &= \begin{cases} [\lambda(i) + \sum_{k \neq i} Q(i, k)\lambda(i, k)]p(i, j) + Q(i, j)\mu(i, j) & \text{if } j \neq i \\ D_1(i, i) & \text{if } j = i \end{cases} \end{aligned}$$

Note that B is a nonnegative matrix. The Markov processes $X^* := (X_t^*)_t$ and $X^{*,\varepsilon} := (X_{t/\varepsilon}^*)_t$ have as generator $Q^* = D_0^{(\varepsilon)} + D_1^{(\varepsilon)}$ and $Q^{*,\varepsilon} := Q^*/\varepsilon$ respectively.

3.2 Convergence of compensators

Using development of Subsection 2.2, the $\mathcal{F}^{(N^{(\varepsilon)}, X^{*,\varepsilon})}$ -compensator of $N^{(\varepsilon)}$ is

$$\begin{aligned} A_t^{(\varepsilon)} &= \int_0^{t/\varepsilon} \sum_{j \in \mathcal{M}} D_1^{(\varepsilon)}(X_{s-}^*, j) ds \\ &= \varepsilon \int_0^{t/\varepsilon} \sum_{j \in \mathcal{M}} B(X_{s-}^*, j) ds + \varepsilon^2 \int_0^{t/\varepsilon} \sum_{j \in \mathcal{M}} L(X_{s-}^*, j) ds \quad \text{from (3.2)} \\ &= \sum_{i \in \mathcal{M}} \int_0^{t/\varepsilon} \varepsilon \mathbf{1}_{\{X_{s-}^* = i\}} ds \sum_{j \in \mathcal{M}} B(i, j) + \sum_{i \in \mathcal{M}} \int_0^{t/\varepsilon} \varepsilon^2 \mathbf{1}_{\{X_{s-}^* = i\}} ds \sum_{j \in \mathcal{M}} L(i, j). \quad (3.3) \end{aligned}$$

Since X^* is c.a.d-l.a.g., we have for almost all $\omega \in \Omega$, $X_s^*(\omega) = X_{s-}^*(\omega)$ except for countably many s . Thus, we can replace X_{s-}^* by X_s^* in the above integrals. This observation will be used to equate similar integrals throughout this paper.

Now, let us consider the Littlewood's model corresponding to Example 2.1. Failures do not affect the execution process, that is $X^* = X$ and $Q^* = Q$. Then

$$A_t^{(\varepsilon)} = \sum_{i \in \mathcal{M}} \varepsilon \int_0^{t/\varepsilon} \mathbf{1}_{\{X_s = i\}} ds \sum_{j \in \mathcal{M}} B(i, j).$$

It follows from well-known time-average properties of cumulative process $\int_0^t f(X_s)ds$ for an irreducible Markov process X (see [3]) that $(\varepsilon/t) \int_0^{t/\varepsilon} \mathbf{1}_{\{X_s=i\}}ds$ converges a.s. to $\pi(i)$ where π is the stationary distribution of X . Thus, we derive that

$$A_t^{(\varepsilon)} \xrightarrow[\varepsilon \rightarrow 0]{a.s.} t \sum_{i \in \mathcal{M}} \pi(i) \sum_{j \in \mathcal{M}} B(i, j) = \lambda t$$

with λ as in (1). In particular this implies the probability convergence of $A_t^{(\varepsilon)}$ to λt . We recognize the compensator of an HPP with intensity λ . It follows from [7, Th 1] that

$$N^{(\varepsilon)} \xrightarrow[\varepsilon \rightarrow 0]{d} P$$

where $P = (P_t)_t$ is the counting process of an HPP with parameter λ . We have shown that Theorem 3.1 holds for an MMPP or the Littlewood's reliability model (Examples 2.1,2.2).

Since convergence in L^2 -norm implies convergence in probability, the following lemma will give the convergence in probability of compensator $A_t^{(\varepsilon)}$ to λt as ε tends to 0 for the general reliability model of Section 2.

Lemma 3.1 *If probability vector π is such that $\pi Q = 0$, then*

$$\lim_{\varepsilon \rightarrow 0} E \left(\varepsilon \int_0^{t/\varepsilon} (\mathbf{1}_{\{X_s^*=i\}} - \pi(i)) ds \right)^2 = 0.$$

Proof. The perturbed generator $Q^* = D_0^{(\varepsilon)} + D_1^{(\varepsilon)}$ of X^* can be decomposed under the following form

$$Q^* = Q + R\varepsilon \tag{3.4}$$

$$\text{with } R(i, j) = \begin{cases} \left[\lambda(i) + \sum_{k \neq i} Q(i, k)\lambda(i, k) \right] p(i, j) - Q(i, j)\lambda(i, j) & \text{if } j \neq i \\ \left[\lambda(i) + \sum_{k \neq i} Q(i, k)\lambda(i, k) \right] p(i, i) - \lambda(i) & \text{if } j = i \end{cases}$$

Changes of variables $u = s\varepsilon$ gives that $\varepsilon \int_0^{t/\varepsilon} \mathbf{1}_{\{X_s^*=i\}}ds = \int_0^t \mathbf{1}_{\{X_s^{*,\varepsilon}=i\}}ds$. Recall that $(X_t^{*,\varepsilon})_t$ has $Q^*/\varepsilon = Q/\varepsilon + R$ as generator. It is easily checked $R\mathbb{1}^t = 0$ where $\mathbb{1}^t$ is the M -dimensional column vector whose all entries are 1. In such a case, [18, Corollary C.1,p 349] gives the estimate

$$E \left| \int_0^t (\mathbf{1}_{\{X_s^{*,\varepsilon}=i\}} - \pi(i)) ds \right|^2 \leq C(1 + t^2) \varepsilon.$$

The convergence of $\int_0^t \mathbf{1}_{\{X_s^{*,\varepsilon}=i\}}ds$ to $\pi(i)$ in L^2 -norm as ε tends to 0, follows from the previous estimate. \square

The following theorem follows from [7, Th 1].

Theorem 3.1 *Probability vector π is such that $\pi Q = 0$. As ε tends to 0, the counting process $N^{(\varepsilon)} = (N_{t/\varepsilon})_t$ converges in distribution to the counting process of an HPP with intensity*

$$\lambda = \sum_{i \in \mathcal{M}} \pi(i) \left[\mu(i) + \lambda(i) + \sum_{j \neq i} Q(i, j) [\mu(i, j) + \lambda(i, j)] \right]. \quad (3.5)$$

3.3 Fast modulation in MMPP

When we consider an MMPP (see Example 2.2), the previous issue is equivalent to investigate the asymptotic of a Poisson process with an intensity modulated by a fast Markov chain. Indeed, the compensator of $N^{(\varepsilon)}$ is then

$$\begin{aligned} A_t^{(\varepsilon)} &= \varepsilon \int_0^{t/\varepsilon} \mu(X_{s-}) ds \\ &= \int_0^t \mu(X_{\frac{s}{\varepsilon}-}) ds \quad \text{with changes of variable } u = \varepsilon s \end{aligned}$$

The Markov chain $X^{(\varepsilon)} = (X_{t/\varepsilon})_t$ has the generator Q/ε . As ε tends to 0, it is now clear that introduction of small parameter ε speeds up the rate of switches between states. Theorem 3.1 states that the asymptotic process is an HPP with intensity

$$\lambda = \sum_{i \in \mathcal{M}} \pi(i) \mu(i).$$

This fact is known and has been investigated for various applications. The closest context of reliability theory is [1], where the underlying Poisson process is either homogeneous or nonhomogeneous. Proofs are based on asymptotic expansion of the transition semi-group of the bivariate Markov process $Z^{(\varepsilon)}$. Introduction of time-dependent intensity is not relevant in our context. Indeed, the decrease of failure parameters, or the reliability growth, is already taken into account by the small parameter ε .

The case where the modulating process is a finite nonhomogeneous Markov process X is addressed in [4]. The asymptotic process is a nonhomogeneous Poisson process with suitable ergodicity assumptions on X .

From a strictly mathematical point of view, the issue addressed in this paper is equivalent to both a fast modulating MP X^* (with generator Q^*/ε) and the introduction of a small scalar ε only in the failure parameters $\mu(\cdot, \cdot)$ corresponding to a transition between states. Speed up the modulating MP X^* at rate $1/\varepsilon$ implies a speeding up of the number of transitions between states of X^* . Therefore, we have to compensate this "explosion" by introducing a small factor ε in $\mu(\cdot, \cdot)$.

4 Convergence rate

We provide in this section an estimate of the convergence rate of the finite-dimensional distributions of $N^{(\varepsilon)}$ to those of an HPP with intensity as in Theorem 3.1. The counting process of the HPP is denoted by $P = (P_t)_t$. Note that λ in (3.5) is the scalar product $\langle \pi, B\mathbb{1}^t \rangle$, where $\mathbb{1}^t$ is the M -dimensional column vector whose all entries are 1. We can write the limit compensator as

$$A_t = \langle \pi, B\mathbb{1}^t \rangle t. \quad (4.1)$$

The $\mathcal{F}^{N^{(\varepsilon)}, X^{*,\varepsilon}}$ -compensator of $N^{(\varepsilon)}$ is from (3.3)

$$\begin{aligned} A_t^{(\varepsilon)} &= \sum_{i \in \mathcal{M}} (B + \varepsilon L)\mathbb{1}^t(i) \int_0^{t/\varepsilon} \varepsilon \mathbf{1}_{\{X_{s-}^* = i\}} ds \\ &= \sum_{i \in \mathcal{M}} (B + \varepsilon L)\mathbb{1}^t(i) \int_0^t \mathbf{1}_{\{X_{s-}^{*,\varepsilon} = i\}} ds \quad (\text{setting } u = s\varepsilon) \\ &= \int_0^t \langle Y_{s-}^{(\varepsilon)}, (B + \varepsilon L)\mathbb{1}^t \rangle ds \end{aligned} \quad (4.2)$$

where $Y_s^{(\varepsilon)} := (\mathbf{1}_{\{X_s^{*,\varepsilon} = i\}})_{i \in \mathcal{M}}$. Hence, the $\mathcal{F}^{N^{(\varepsilon)}, X^{*,\varepsilon}}$ -intensity of $N^{(\varepsilon)}$ is

$$\lambda_s^{(\varepsilon)} := \langle Y_{s-}^{(\varepsilon)}, (B + \varepsilon L)\mathbb{1}^t \rangle. \quad (4.3)$$

Let T be any positive scalar and $\mathbb{T} := \{t_0, t_1, \dots, t_n\}$ with $0 = t_0 < t_1 < \dots < t_n = T$. To evaluate proximity between the respective distributions $\mathcal{L}(N_{\mathbb{T}}^{(\varepsilon)})$ and $\mathcal{L}(P_{\mathbb{T}})$ of $N_{\mathbb{T}}^{(\varepsilon)} := (N_{t_1}^{(\varepsilon)}, \dots, N_{t_n}^{(\varepsilon)})$ and $P_{\mathbb{T}} := (P_{t_1}, \dots, P_{t_n})$, the distance in total variation, denoted by $d_{\text{TV}}(\mathcal{L}(N_{\mathbb{T}}^{(\varepsilon)}), \mathcal{L}(P_{\mathbb{T}}))$, may be used, that is [17]

$$\begin{aligned} d_{\text{TV}}(\mathcal{L}(N_{\mathbb{T}}^{(\varepsilon)}), \mathcal{L}(P_{\mathbb{T}})) &\stackrel{\text{def}}{=} \sup_{B \subset \mathbb{N}^n} \left| \mathbb{P}\{N_{\mathbb{T}}^{(\varepsilon)} \in B\} - \mathbb{P}\{P_{\mathbb{T}} \in B\} \right| \\ &= \frac{1}{2} \sum_{\mathbf{k} \in \mathbb{N}^n} \left| \mathbb{P}\{N_{\mathbb{T}}^{(\varepsilon)} = \mathbf{k}\} - \mathbb{P}\{P_{\mathbb{T}} = \mathbf{k}\} \right| \end{aligned}$$

For a locally bounded variation function $t \mapsto f(t)$, the total variation in the interval $[0, T]$ is

$$\text{Var}_{[0, T]}(f) \stackrel{\text{def}}{=} \sup_{\{t_1, \dots, t_n\} \in \mathcal{P}([0, T])} \sum_{i=1}^n |f(t_i) - f(t_{i-1})|$$

where $\mathcal{P}([0, T])$ is the set of all the finite subdivisions of the interval $[0, T]$.

In this section, we state that the finite-dimensional distributions of $N^{(\varepsilon)}$ converges in total variation to those of an HPP with intensity λ at rate ε . Proof is borrowed from [8, Th 6.1] where a similar result is stated for an MMPP. This is heavily based on the

following estimate of the total variation between finite-dimensional distributions of $N^{(\varepsilon)}$ and P [8, Th 3.1]

$$d_{\text{TV}}(\mathcal{L}(N_{\mathbb{T}}^{(\varepsilon)}), \mathcal{L}(P_{\mathbb{T}})) \leq E \text{Var}_{[0, T]}(\widehat{A}^{(\varepsilon)} - A).$$

where $\widehat{A}^{(\varepsilon)}$ is the $\mathcal{F}^{N^{(\varepsilon)}}$ -compensator of $N^{(\varepsilon)}$. $\widehat{A}^{(\varepsilon)}$ is from (4.2) and [11, Th 18.3]

$$\widehat{A}_t^{(\varepsilon)} = \int_0^t \langle \widehat{Y}_{s-}^{(\varepsilon)}, (B + \varepsilon L) \mathbb{1}^t \rangle ds$$

with $\widehat{Y}_t^{(\varepsilon)} := (\mathbb{P}\{X_t^{*,\varepsilon} = i \mid \mathcal{F}_t^{N^{(\varepsilon)}}\})_{i \in \mathcal{M}}$. Hence, the $\mathcal{F}^{N^{(\varepsilon)}}$ -intensity of $N^{(\varepsilon)}$ is

$$\widehat{\lambda}_s^{(\varepsilon)} := \langle \widehat{Y}_{s-}^{(\varepsilon)}, (B + \varepsilon L) \mathbb{1}^t \rangle. \quad (4.4)$$

Therefore, we obtain from (4.1) and (4.2) that

$$\begin{aligned} d_{\text{TV}}(\mathcal{L}(N_{\mathbb{T}}^{(\varepsilon)}), \mathcal{L}(P_{\mathbb{T}})) &\leq E \text{Var}_{[0, T]} \left(\int_0^t \langle \widehat{Y}_{s-}^{(\varepsilon)} - \pi, B \mathbb{1}^t \rangle + \varepsilon \langle \widehat{Y}_{s-}^{(\varepsilon)}, L \mathbb{1}^t \rangle ds \right) \\ &= E \int_0^T |\langle \widehat{Y}_{s-}^{(\varepsilon)} - \pi, B \mathbb{1}^t \rangle + \varepsilon \langle \widehat{Y}_{s-}^{(\varepsilon)}, L \mathbb{1}^t \rangle| ds \\ &\leq E \int_0^T |\langle \widehat{Y}_{s-}^{(\varepsilon)} - \pi, B \mathbb{1}^t \rangle| ds + \varepsilon E \int_0^T |\langle \widehat{Y}_{s-}^{(\varepsilon)}, L \mathbb{1}^t \rangle| ds \\ &\leq E \int_0^T |\langle \widehat{Y}_{s-}^{(\varepsilon)} - \pi, B \mathbb{1}^t \rangle| ds + \varepsilon C_T \\ &\quad (\text{since } \widehat{Y}^{(\varepsilon)} \text{ is bounded}). \end{aligned} \quad (4.5)$$

Since $\widehat{Y}_s^{(\varepsilon)}$ and π are stochastic vectors, it follows from (.19)

$$|\langle \widehat{Y}_s^{(\varepsilon)} - \pi, B \mathbb{1}^t \rangle| \leq \frac{\delta}{2} \|\widehat{Y}_s^{(\varepsilon)} - \pi\|_1 \quad (4.6)$$

where $\delta := \max(B \mathbb{1}^t(i)) - \min(B \mathbb{1}^t(i))$ and $\|\cdot\|_1$ is the l_1 -norm. Hence, it remains to estimate in (4.5) the convergence rate of $\|\widehat{Y}_s^{(\varepsilon)} - \pi\|_1$ to 0 when $\varepsilon \rightarrow 0$. The first step consists in writing a filtering equation for vector $Y_t^{(\varepsilon)}$.

4.1 Filtering equation for $Y^{(\varepsilon)}$

Let us recall that $Y_t^{(\varepsilon)} = (\mathbf{1}_{\{X_t^{*,\varepsilon} = i\}})_{i \in \mathcal{M}}$. Note that each component $Y_t^{(\varepsilon)}(i)$ of vector $Y_t^{(\varepsilon)}$ is a bounded random variable. We basically follows [2, Ch IV].

Lemma 4.1 *Define $\widehat{Y}_t^{(\varepsilon)} := E[Y_t^{(\varepsilon)} \mid \mathcal{F}_t^{N^{(\varepsilon)}}]$. Let α be the probability distribution of $X_0^{*\varepsilon}$. We have for all $t \geq 0$*

$$\widehat{Y}_t^{(\varepsilon)} = \alpha + \frac{1}{\varepsilon} \int_0^t \widehat{Y}_{s-}^{(\varepsilon)} Q^* ds + \int_0^t v_{s-}^{(\varepsilon)} (dN_s^{(\varepsilon)} - \widehat{\lambda}_s^{(\varepsilon)} ds) \quad (4.7)$$

where $\widehat{\lambda}^{(\varepsilon)}$ is the $\mathcal{F}^{N^{(\varepsilon)}}$ -intensity of $N^{(\varepsilon)}$ given in (4.4) and

$$v_{s-}^{(\varepsilon)} := \frac{\widehat{Y}_{s-}^{(\varepsilon)}(B + \varepsilon L)}{\widehat{\lambda}_s^{(\varepsilon)}} - \widehat{Y}_{s-}^{(\varepsilon)}. \quad (4.8)$$

Proof. We recall that the MP $X^{*,\varepsilon}$ has the generator $Q^{*,\varepsilon} = Q^*/\varepsilon$. It follows from Dynkin formula that

$$Y_t^{(\varepsilon)} = Y_0^{(\varepsilon)} + \frac{1}{\varepsilon} \int_0^t Y_{s-}^{(\varepsilon)} Q^* ds + M_t \quad (4.9)$$

where $M = (M_t)$ is a $\mathcal{F}^{X^{*,\varepsilon}}$ -martingale. Then applying [2, Ch IV, Th 1] to the representation (4.9) of the bounded process $Y^{(\varepsilon)}$, we get for $\widehat{Y}_t^{(\varepsilon)}$

$$\widehat{Y}_t^{(\varepsilon)} = \widehat{Y}_0^{(\varepsilon)} + \frac{1}{\varepsilon} \int_0^t \widehat{Y}_{s-}^{(\varepsilon)} Q^* ds + \widehat{M}_t$$

where $\widehat{M} = (\widehat{M}_t)$ is a $\mathcal{F}^{N^{(\varepsilon)}}$ -martingale. Now, [2, Ch III, Th 17] gives us the following representation of the $\mathcal{F}^{N^{(\varepsilon)}}$ -martingale \widehat{M}

$$\int_0^t G_s (dN_s^{(\varepsilon)} - \widehat{\lambda}_s^{(\varepsilon)} ds)$$

where $\widehat{\lambda}^{(\varepsilon)}$ is the $\mathcal{F}^{N^{(\varepsilon)}}$ -intensity of $N^{(\varepsilon)}$ and $G = (G_t)$ is a $\mathcal{F}^{N^{(\varepsilon)}}$ -predictable process called the innovations process. We also know from [2, Ch IV, Th 2] that $G_t := G_{1,t} - \widehat{Y}_{t-}^{(\varepsilon)} + G_{3,t}$, where the entry i of vectors $G_{1,s}$ and $G_{3,s}$ must be computed from

$$E \int_0^t C_s Y_s(i) \lambda_s^{(\varepsilon)} ds = E \int_0^t C_s G_{1,s}(i) \widehat{\lambda}_s^{(\varepsilon)} ds \quad (4.10)$$

$$E \sum_{0 < s \leq t} C_s \Delta M_s(i) \Delta N_s^{(\varepsilon)} = E \int_0^t C_s G_{3,s}(i) \widehat{\lambda}_s^{(\varepsilon)} ds \quad (4.11)$$

where $C := (C_s)_s$ is any nonnegative $\mathcal{F}^{N^{(\varepsilon)}}$ -predictable process, $\Delta M_s(i) = M_s(i) - M_{s-}(i)$ and $\Delta N_s^{(\varepsilon)} = N_s^{(\varepsilon)} - N_{s-}^{(\varepsilon)}$ are the jumps of martingale $M(i)$ and counting process $N^{(\varepsilon)}$, respectively. Random processes $\lambda^{(\varepsilon)}$ and $\widehat{\lambda}^{(\varepsilon)}$ are given by (4.3) and (4.4). We determine an explicit expression for $G_{1,t}$ and $G_{3,t}$. This is similar to the MMPP case (see [2, page 98]) else but $X^{*,\varepsilon}$ and $N^{(\varepsilon)}$ may have common jumps.

The left hand-side of Equality (4.10) can be rewritten as

$$\begin{aligned}
E \int_0^t C_s Y_s^{(\varepsilon)}(i) \lambda_s^{(\varepsilon)} ds &= E \int_0^t C_s Y_{s-}^{(\varepsilon)}(i) \lambda_s^{(\varepsilon)} ds \\
&= E \int_0^t C_s Y_{s-}^{(\varepsilon)}(i) \langle Y_{s-}^{(\varepsilon)}, (B + \varepsilon L) \mathbb{1}^t \rangle ds \quad \text{from (4.3)} \\
&= E \int_0^t C_s Y_{s-}^{(\varepsilon)}(i) ((B + \varepsilon L) \mathbb{1}^t)(i) ds \\
&= E \int_0^t C_s \widehat{Y}_{s-}^{(\varepsilon)}(i) ((B + \varepsilon L) \mathbb{1}^t)(i) ds \quad \text{since } C_s \text{ is } \mathcal{F}_s^{N^{(\varepsilon)}}\text{-measurable.}
\end{aligned}$$

Then, we deduce from (4.10) that

$$G_{1,s} = \left(\frac{\widehat{Y}_{s-}^{(\varepsilon)}(i) ((B + \varepsilon L) \mathbb{1}^t)(i)}{\widehat{\lambda}_s^{(\varepsilon)}} \right)_{i \in \mathcal{M}}.$$

The counting process $N^{(\varepsilon)}$ has the following representation (see (2.1))

$$N_t^{(\varepsilon)} = \sum_{n=0}^{+\infty} \sum_{j \in \mathcal{M}} \sum_{k \in \mathcal{M}} N_t^{(\varepsilon)}((n, j), (n+1, k)). \quad (4.12)$$

where $N_t^{(\varepsilon)}(x, y)$ is the cumulative number of transitions of the bivariate process $(N^{(\varepsilon)}, X^{*, \varepsilon})$ up to time t . Since $\Delta M_s(i) = \Delta Y_s^{(\varepsilon)}(i)$, we have

$$\sum_{0 < s \leq t} C_s \Delta M_s(i) \Delta N_s^{(\varepsilon)} = \sum_{0 < s \leq t} C_s Y_s^{(\varepsilon)}(i) \Delta N_s^{(\varepsilon)} - \sum_{0 < s \leq t} C_s Y_{s-}^{(\varepsilon)}(i) \Delta N_s^{(\varepsilon)}. \quad (4.13)$$

The first term in the right-hand side of the last equality can be rewritten from (4.12)

$$\sum_{0 < s \leq t} C_s Y_s^{(\varepsilon)}(i) \Delta N_s^{(\varepsilon)} = \sum_{0 < s \leq t} C_s Y_s^{(\varepsilon)}(i) \sum_{n=0}^{+\infty} \sum_{j \in \mathcal{M}} \sum_{k \in \mathcal{M}} \Delta N_s^{(\varepsilon)}((n, j), (n+1, k)).$$

It is easily seen that the right-hand side of the last equality is also

$$\sum_{0 < s \leq t} C_s \sum_{n=0}^{+\infty} \sum_{j \in \mathcal{M}} Y_{s-}^{(\varepsilon)}(j) \Delta N_s^{(\varepsilon)}((n, j), (n+1, i)).$$

So, we have from (4.13) and the previous equality

$$\sum_{0 < s \leq t} C_s \Delta M_s(i) \Delta N_s^{(\varepsilon)} = \int_0^t C_s \sum_{n=0}^{+\infty} \sum_{j \in \mathcal{M}} Y_{s-}^{(\varepsilon)}(j) dN_s^{(\varepsilon)}((n, j), (n+1, i)) - \int_0^t C_s Y_{s-}^{(\varepsilon)}(i) dN_s^{(\varepsilon)}$$

Since the processes C and $Y^{(\varepsilon)}$ are $\mathcal{F}^{N^{(\varepsilon)}, X^{*, \varepsilon}}$ -predictable, we get from the definition of the compensator, the formulae (2.2) and (4.3)

$$\begin{aligned}
& E \sum_{0 < s \leq t} C_s \Delta M_s(i) \Delta N_s^{(\varepsilon)} \\
&= E \int_0^t C_s \left[\sum_{j \in \mathcal{M}} Y_{s-}^{(\varepsilon)}(j) (B(j, i) + \varepsilon L(j, i)) - Y_{s-}^{(\varepsilon)}(i) \langle Y_{s-}^{(\varepsilon)}, (B + \varepsilon L) \mathbf{\Pi}^t \rangle \right] ds \\
&= E \int_0^t C_s \left[\sum_{j \in \mathcal{M}} Y_{s-}^{(\varepsilon)}(j) (B(j, i) + \varepsilon L(j, i)) - Y_{s-}^{(\varepsilon)}(i) \sum_{j \in \mathcal{M}} (B(i, j) + \varepsilon L(i, j)) \right] ds \\
&= E \int_0^t C_s \left[\sum_{j \neq i} Y_{s-}^{(\varepsilon)}(j) (B(j, i) + \varepsilon L(j, i)) - Y_{s-}^{(\varepsilon)}(i) \sum_{j \neq i} (B(i, j) + \varepsilon L(i, j)) \right] ds \\
&= E \int_0^t C_s \left[\sum_{j \neq i} \widehat{Y}_{s-}^{(\varepsilon)}(j) (B(j, i) + \varepsilon L(j, i)) - \widehat{Y}_{s-}^{(\varepsilon)}(i) \sum_{j \neq i} (B(i, j) + \varepsilon L(i, j)) \right] ds \\
&\quad \text{since } C_s \text{ is } \mathcal{F}_s^{N^{(\varepsilon)}}\text{-measurable.}
\end{aligned}$$

We get from (4.11) that

$$G_{3,s}(i) = \frac{\sum_{j \neq i} \widehat{Y}_{s-}^{(\varepsilon)}(j) (B(j, i) + \varepsilon L(j, i)) - \widehat{Y}_{s-}^{(\varepsilon)}(i) \sum_{j \neq i} (B(i, j) + \varepsilon L(i, j))}{\widehat{\lambda}_s^{(\varepsilon)}}.$$

Consequently, the gain G_s has the form reported in Lemma 4.1. \square

Lemma 4.2 *Let α be the probability distribution of $X_0^{*\varepsilon}$. Then Equation (4.7) has the unique solution*

$$\widehat{Y}_t^{(\varepsilon)} = \alpha \exp(Q^* t / \varepsilon) + \int_0^t v_{s-}^{(\varepsilon)} \exp(Q^*(t-s)/\varepsilon) (dN_s^{(\varepsilon)} - \widehat{\lambda}_s^{(\varepsilon)} ds). \quad (4.14)$$

Proof. First, let us check that the right-hand side term of (4.14) is a solution of (4.7). It has the form $U_t V_t$ where

$$\begin{aligned}
U_t &= \alpha + \int_0^t v_{s-}^{(\varepsilon)} \exp(-Q^* s / \varepsilon) (dN_s^{(\varepsilon)} - \widehat{\lambda}_s^{(\varepsilon)} ds) \\
V_t &= \exp(Q^* t / \varepsilon).
\end{aligned}$$

Using an integration by parts [2, T2, p 336], we obtain

$$\begin{aligned}
U_t V_t &= U_0 V_0 + \int_0^t U_s dV_s + \int_0^t dU_s V_s \\
&= \alpha + \frac{1}{\varepsilon} \int_0^t (U_s V_s) Q^* ds + \int_0^t v_{s-}^{(\varepsilon)} (dN_s^{(\varepsilon)} - \widehat{\lambda}_s^{(\varepsilon)} ds).
\end{aligned}$$

Second, note that $(\widehat{Y}_t^{(\varepsilon)} - U_t V_t)_t$ is a solution of the homogeneous linear differential equation

$$y(t) = \frac{1}{\varepsilon} \int_0^t y(s) Q^* ds$$

with initial condition $y(0) = 0$. Then, $\widehat{Y}_t^{(\varepsilon)} - U_t V_t \equiv 0$. \square

4.2 Convergence rate

Theorem 4.1 $P = (P_t)$ is the counting process of an HPP with intensity

$$\lambda = \langle \pi, B \mathbb{1}^t \rangle = \sum_{i \in \mathcal{M}} \pi(i) \left[\mu(i) + \lambda(i) + \sum_{j \neq i} Q(i, j) (\lambda(i, j) + \mu(i, j)) \right]$$

where π is the probability distribution such that $\pi Q = 0$. For any $T > 0$, there exists a constant C_T such that

$$d_{\text{TV}}(\mathcal{L}(N_{\mathbb{T}}^{(\varepsilon)}), \mathcal{L}(P_{\mathbb{T}})) \leq C_T \varepsilon.$$

Proof. Let us recall that (see (4.5) and (4.6))

$$d_{\text{TV}}(\mathcal{L}(N_{\mathbb{T}}^{(\varepsilon)}), \mathcal{L}(P_{\mathbb{T}})) \leq \frac{\delta}{2} E \int_0^T \|\widehat{Y}_t^{(\varepsilon)} - \pi\|_1 dt + C_{1,T} \varepsilon.$$

We just have to control the first term in the right-hand side of the inequality. Since $v_{s-}^{(\varepsilon)} \mathbb{1}^t = 0$, we can write from (4.14)

$$\begin{aligned} \widehat{Y}_t^{(\varepsilon)} - \pi &= \alpha \exp(Q^* t / \varepsilon) - \pi \\ &\quad + \int_0^t v_{s-}^{(\varepsilon)} [\exp(Q^*(t-s)/\varepsilon) - \mathbb{1}^t \pi] (dN_s^{(\varepsilon)} - \widehat{\lambda}_s^{(\varepsilon)} ds). \end{aligned}$$

Using the triangle inequality, it is easily seen that

$$\begin{aligned} E \int_0^T \|\widehat{Y}_t^{(\varepsilon)} - \pi\|_1 dt &\leq \int_0^T \|\alpha \exp(Q^* t / \varepsilon) - \pi\|_1 dt \\ &\quad + E \int_0^T \left\| \int_0^t v_{s-}^{(\varepsilon)} [\exp(Q^*(t-s)/\varepsilon) - \mathbb{1}^t \pi] (dN_s^{(\varepsilon)} - \widehat{\lambda}_s^{(\varepsilon)} ds) \right\|_1 dt. \end{aligned} \tag{4.15}$$

In a first step, let us consider the former term in the right hand side of the previous inequality. We have from (.18) with $Q^{(\varepsilon)} = Q^*/\varepsilon = Q/\varepsilon + R$ (see (3.4)) that for all $s \geq 0$

$$\|\alpha \exp(Q^* t / \varepsilon) - \pi\|_1 \leq C_1 (\varepsilon + \exp(-\rho t / \varepsilon)).$$

Then

$$\int_0^T \|\alpha \exp(Q^* t / \varepsilon) - \pi\|_1 dt \leq (C_1 T + \frac{1}{\rho}) \varepsilon = C_{2,T} \varepsilon. \tag{4.16}$$

In a second step, we have

$$\begin{aligned}
& E \left\| \int_0^t v_{s-}^{(\varepsilon)} [\exp(Q^*(t-s)/\varepsilon) - \mathbb{1}^t \pi] (dN_s^{(\varepsilon)} - \widehat{\lambda}_s^{(\varepsilon)} ds) \right\|_1 \\
& \leq E \int_0^t \|v_{s-}^{(\varepsilon)} [\exp(Q^*(t-s)/\varepsilon) - \mathbb{1}^t \pi]\|_1 (dN_s^{(\varepsilon)} + \widehat{\lambda}_s^{(\varepsilon)} ds) \\
& = 2E \int_0^t \|v_{s-}^{(\varepsilon)} [\exp(Q^*(t-s)/\varepsilon) - \mathbb{1}^t \pi]\|_1 \widehat{\lambda}_s^{(\varepsilon)} ds
\end{aligned}$$

since $\widehat{\lambda}_s^{(\varepsilon)}$ is the $\mathcal{F}^{N^{(\varepsilon)}}$ -intensity of $N^{(\varepsilon)}$ and $v_{s-}^{(\varepsilon)}$ is $\mathcal{F}^{N^{(\varepsilon)}}$ -predictable. We recall that $\|xM\|_1 \leq \|x\|_1 \|M\|_1$ for any vector x and matrix M , where $\|M\|_1 := \max_i (\sum_j |M(i, j)|)$ is the 1-matrix norm. Now, we get

$$\|v_{s-}^{(\varepsilon)} [\exp(Q^*(t-s)/\varepsilon) - \mathbb{1}^t \pi]\|_1 \leq \|v_{s-}^{(\varepsilon)}\|_1 \|\exp(Q^*(t-s)/\varepsilon) - \mathbb{1}^t \pi\|_1.$$

It follows from (.17) that for all $s \leq t$

$$\|\exp(Q^*(t-s)/\varepsilon) - \mathbb{1}^t \pi\|_1 \leq C_2 (\varepsilon + \exp(-(t-s)\rho/\varepsilon)).$$

Since $v_{s-}^{(\varepsilon)}$ is the difference between two probability vectors, we have $\|v_{s-}^{(\varepsilon)}\|_1 \leq 2$. Using (4.4) and $\|\widehat{Y}_{s-}^{(\varepsilon)}\|_1 = 1$, we get $\|v_{s-}^{(\varepsilon)}\|_1 \widehat{\lambda}_s^{(\varepsilon)} \leq 2 \|(B + \varepsilon L)\mathbb{1}^t\|_\infty$. Then

$$\|v_{s-}^{(\varepsilon)}\|_1 \widehat{\lambda}_s^{(\varepsilon)} \leq C_3 + C_4 \varepsilon$$

is uniformly bounded in s . Therefore, for all $t \geq 0$, we can write

$$\begin{aligned}
E \int_0^t \|v_{s-}^{(\varepsilon)} [\exp(Q^*(t-s)/\varepsilon) - \mathbb{1}^t \pi]\|_1 \widehat{\lambda}_s^{(\varepsilon)} ds & \leq (C_5 + \varepsilon C_6) \int_0^t (\varepsilon + \exp(-(t-s)\rho/\varepsilon)) ds \\
& \leq (C_5 + \varepsilon C_6) (t\varepsilon + C_7 \varepsilon) \\
& \leq (C_8 t + C_9) \varepsilon. \quad (\text{for } \varepsilon \leq 1)
\end{aligned}$$

We deduce from the previous estimate (and Fubini's theorem) that the second term in the right-hand side of Inequality (4.15) is such that

$$\begin{aligned}
& E \int_0^T \left\| \int_0^t v_{s-}^{(\varepsilon)} (\exp(Q^*(t-s)/\varepsilon) - \mathbb{1}^t \pi) (dN_s^{(\varepsilon)} - \widehat{\lambda}_s^{(\varepsilon)} ds) \right\|_1 dt \\
& \leq \int_0^T (C_8 t + C_9) \varepsilon dt = C_{3,T} \varepsilon \quad (\text{for } \varepsilon \leq 1).
\end{aligned}$$

Theorem 4.1 follows from (4.15) with the last inequality and (4.16). \square

Remark 4.1 *With respect to Theorem 3.1, note that [7, Th 1] would give, in fact, convergence in distribution of the counting process $N^{(\varepsilon)}$ to Poisson process P in the space of all counting processes, equipped with the Skorokhod topology. Moreover, convergence in variation also takes place in this space. Indeed, the distance in total variation over interval $[0, T]$ between distributions of $N^{(\varepsilon)}$ and P is also bounded from above by $E \text{Var}_{[0, T]}(\widehat{A}^{(\varepsilon)} - A)$ (see [8, Th 4.1]). Thus, it follows from Theorem 4.1 that the rate of convergence is ε .*

Remark 4.2 *The order of the convergence rate in Theorem 4.1 cannot be improved in general. This follows from [4, Section 5, Example 1], where the authors report a lower bound for the distance in variation, that has order 1 in ε for a Poisson process modulated by a 2-states Markov process.*

Estimate of convergence rate

Let us consider a generator $Q^{(\varepsilon)} = Q_1 + Q_2/\varepsilon$ where Q_2 is assumed to be an irreducible generator. There exists a probability vector π such that $\pi Q_2 = 0$. The following estimates are provided in [18, Lemma C.3, p 346]: for all $t \geq 0$ and any probability vector α , there exists constants K and $\rho > 0$ such that

$$\|\exp(Q^{(\varepsilon)}t) - \mathbb{1}^t \pi\|_1 \leq K(\varepsilon + \exp(-\rho t/\varepsilon)) \quad (.17)$$

$$\|\alpha \exp(Q^{(\varepsilon)}t) - \pi\|_1 \leq K(\varepsilon + \exp(-\rho t/\varepsilon)) \quad (.18)$$

and ρ only depends upon Q_2

A simple inequality

Let v be any M -dimensional vector. Any convex combinations $\langle u_1, v \rangle$ and $\langle u_2, v \rangle$ of scalars $(v(i), i = 1, \dots, M)$ are in the interval $[\min(v(i)), \max(v(i))]$, so that $|\langle u_1 - u_2, v \rangle| \leq \delta$ with $\delta := \max(v(i)) - \min(v(i))$.

Let α_1 and α_2 two stochastic vectors. Write $\alpha_1 - \alpha_2 = (\alpha_1 - \alpha_2)^+ - (\alpha_1 - \alpha_2)^-$ where $w^+ = (\max(w(i), 0))_{i=1, \dots, M}$, $w^- = -(\min(w(i), 0))_{i=1, \dots, M}$. Since $(\alpha_1 - \alpha_2) \mathbb{1}^t = 0$, we have $\|\alpha_1 - \alpha_2\|_1 = (\alpha_1 - \alpha_2)^+ \mathbb{1}^t + (\alpha_1 - \alpha_2)^- \mathbb{1}^t = 2(\alpha_1 - \alpha_2)^+ \mathbb{1}^t = 2(\alpha_1 - \alpha_2)^- \mathbb{1}^t$, where $\|\cdot\|_1$ is the l_1 -norm. Then, we may write the vector $\alpha_1 - \alpha_2$ as $(u_1 - u_2) \|\alpha_1 - \alpha_2\|_1/2$ where u_1, u_2 are the stochastic vectors

$$u_1 = \frac{(\alpha_1 - \alpha_2)^+}{(\alpha_1 - \alpha_2)^+ \mathbb{1}^t} \quad u_2 = \frac{(\alpha_1 - \alpha_2)^-}{(\alpha_1 - \alpha_2)^- \mathbb{1}^t}.$$

Next, it follows from the first part that

$$|\langle \alpha_1 - \alpha_2, v \rangle| = \frac{\|\alpha_1 - \alpha_2\|_1}{2} |\langle u_1 - u_2, v \rangle| \leq \frac{\delta}{2} \|\alpha_1 - \alpha_2\|_1. \quad (.19)$$

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