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Alfredas Račkauskas, Charles Suquet

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# FUNCTIONAL LAWS OF LARGE NUMBERS IN HÖLDER SPACES

ALFREDAS RAČKAUSKAS AND CHARLES SUQUET

ABSTRACT. Let  $S_n = X_1 + \dots + X_n$ ,  $n \geq 1$ , where  $(X_i)_{i \geq 1}$  are random variables. Let  $\mu$  be a constant and  $I$  be the identity function on  $[0, 1]$ . We study the almost sure convergence to  $\mu I$  of the two polygonal line partial sums processes  $\zeta_n$  and  $\zeta_n^{\text{ad}}$  with respective vertices  $(k/n, S_k)$  and  $(\tau_k, S_k)$ ,  $0 \leq k \leq n$ , where  $\tau_k = T_k/T_n$  and  $T_k = |X_1| + \dots + |X_k|$ . These convergences are considered in the space  $C[0, 1]$  or in the Hölder spaces  $\mathcal{H}_\alpha^o[0, 1]$ ,  $0 \leq \alpha < 1$ . In  $C[0, 1]$ , any strong law of large numbers satisfied by  $S_n$  is inherited by  $\zeta_n$ . In  $\mathcal{H}_\alpha^o[0, 1]$ , assuming moreover that the  $X_i$ 's are i.i.d.,  $n^{-1}\zeta_n$  converges almost surely to  $\mu I$  if and only if  $\mathbf{E}|X_1|^{1/(1-\alpha)} < \infty$  and  $\mu = \mathbf{E} X_1$ . In contrast, the same convergence for  $\zeta_n^{\text{ad}}$  is equivalent to  $\mathbf{E}|X_1| < \infty$  and  $\mu = \mathbf{E} X_1$ .

## 1. INTRODUCTION AND MAIN RESULTS

On the same probability space  $(\Omega, \mathcal{F}, P)$ , let us consider a sequence of real valued random variables  $(X_i)_{i \geq 1}$  together with its partial sums  $(S_n)_{n \geq 0}$

$$S_0 = 0, \quad S_n = X_1 + \dots + X_n, \quad n \geq 1,$$

and its polygonal line partial sums processes  $(\zeta_n)_{n \geq 1}$ , where

$$\zeta_n(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1}, \quad t \in [0, 1].$$

In the case when the  $X_i$ 's are i.i.d., limit theorems establish a strong relationship between the degree of integrability of  $X_1$  and the asymptotic behavior of  $S_n$  and of  $\zeta_n$ . To mention the two most famous examples, it is well known that the almost sure convergence of  $n^{-1}S_n$  to some constant  $\mu$  is equivalent to  $\mathbf{E}|X_1| < \infty$  and  $\mathbf{E} X_1 = \mu$ , by Kolmogorov's strong law of large numbers (SLLN) and its converse; similarly, the convergence in distribution of  $n^{-1/2}(S_n - n\mu)$  is equivalent to  $\mathbf{E} X_1^2 < \infty$  and  $\mathbf{E} X_1 = \mu$ , by central limit theorem (CLT) and its converse. One may look at the central limit theorem as a convergence rate result for the strong law of large numbers, showing that a convergence rate better or equal than  $n^{-1/2}$  cannot be obtained in the SLLN. Intermediate convergence rates of the form  $n^{1/p-1}$ ,  $0 < p < 2$  are obtained from Marcinkiewicz-Zygmund SLLN under the existence of  $p$ -th moment.

As for  $\zeta_n$ , by Donsker-Prohorov theorem or functional central limit theorem (FCLT), the process  $n^{1/2}(n^{-1}\zeta_n - \mu I)$  converges in distribution to the Brownian

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motion in the classical space of continuous functions  $C[0, 1]$  if and only if  $\mathbf{E} X_1^2 < \infty$  and  $\mu = \mathbf{E} X_1$ , where  $I$  is the identity function,

$$I : [0, 1] \rightarrow [0, 1], \quad t \mapsto I(t) = t.$$

When  $\mathbf{E} |X_1|^p < \infty$  for some  $p > 2$ , the convergence of  $n^{1/2}(n^{-1}\zeta_n - \mu I)$  can be strengthened in a convergence in some Hölder space  $\mathcal{H}_\alpha^o[0, 1]$ , giving a FCLT in  $\mathcal{H}_\alpha^o$ , see Račkauskas and Suquet (2004) for the precise connection between the degree of integrability of  $X_1$  and the strength of the relevant Hölder topology. Alternatively, one can also modify the construction of  $\zeta_n$  in an adaptive way to obtain Hölder convergence under mild integrability assumptions, see Račkauskas and Suquet (2001).

It is a natural question then, to ask whether all these functional central limit theorems in  $C[0, 1]$  or in Hölder spaces may be viewed as convergence rate results for some corresponding *functional* strong law of large numbers (FSLLN).

Our aim in this contribution is to discuss various functional laws of large numbers for  $\zeta_n$  or for some adaptive modification, in terms of the degree of integrability of  $X_1$ .

Throughout the paper,  $\xrightarrow[n \rightarrow \infty]{\text{a.s.}}$  denotes almost sure convergence and  $C[0, 1]$  is the Banach space of continuous functions  $f : [0, 1] \rightarrow \mathbb{R}$  endowed with the so-called supremum or uniform norm  $\|f\|_\infty = \sup_{t \in [0, 1]} |f(t)|$ ,  $f \in C[0, 1]$ .

The simplest law of large numbers for  $\zeta_n$  reads as follows.

**Theorem 1.1.** *Assume that the  $X_i$ 's are i.i.d. Then the convergence*

$$\frac{1}{n} \zeta_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \mu I \quad \text{in the space } C[0, 1] \quad (1.1)$$

*holds if and only if  $\mathbf{E} |X_1| < \infty$  and  $\mu = \mathbf{E} X_1$ .*

Since the supremum norm of a polygonal line is reached at some vertex, the above functional strong law of large numbers for  $\zeta_n$  can be viewed as an uniform law of large numbers for the partial sums as follows.

**Theorem 1.2.** *Assume that the  $X_i$ 's are i.i.d. Then the convergence*

$$\frac{1}{n} \max_{1 \leq k \leq n} |S_k - k\mu| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0, \quad (1.2)$$

*holds if and only if  $\mathbf{E} |X_1| < \infty$  and  $\mu = \mathbf{E} X_1$ .*

As a matter of fact, functional strong law of large numbers in  $C[0, 1]$  are easily inherited from the corresponding strong law for  $S_n$ , according to the following result whose proof is a simple exercise in analysis. It is worth noticing that no direct assumption on the dependence structure of the  $X_i$ 's is made here.

**Theorem 1.3.** *Let  $X_i$  be random variables with an arbitrary dependence structure. Let  $\mu$  be a real number and  $(b_n)_{n \geq 1}$  be a non-decreasing sequence of positive numbers going to infinity. Then the following three convergences are equivalent.*

- i)  $b_n^{-1}(S_n - n\mu) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0$ .
- ii)  $nb_n^{-1} \|n^{-1}\zeta_n - \mu I\|_\infty \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0$ .
- iii)  $b_n^{-1} \max_{1 \leq k \leq n} |S_k - k\mu| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0$ .

Combining classical Marcinkiewicz-Zygmund SLLN with Theorem 1.3 gives the following functional Marcinkiewicz-Zygmund strong law of large numbers for  $\zeta_n$  (whose special case  $p = 1$  is equivalent to Theorem 1.1).

**Theorem 1.4.** *Assume that the  $X_i$ 's are i.i.d. Then the following statements hold true.*

- a) *If  $\mathbf{E}|X_1|^p < \infty$  for some  $p \in (0, 1)$ , then  $n^{-1/p}\|\zeta_n\|_\infty$  goes to zero almost surely.*
- b) *If  $\mathbf{E}|X_1|^p < \infty$  for some  $p \in [1, 2)$ , then  $n^{1-1/p}\|n^{-1}\zeta_n - (\mathbf{E}X_1)I\|_\infty$  goes to zero almost surely.*
- c) *If  $n^{-1/p}\|\zeta_n - c_n I\|_\infty$  goes to zero almost surely for some  $p \in (0, 2)$  and some sequence  $(c_n)_{n \geq 1}$  of real numbers, then  $\mathbf{E}|X_1|^p < \infty$ .*

*Remark 1.5.* There is a large literature on the Marcinkiewicz-Zygmund strong law of large numbers for dependent variables, see e.g. Rio (1995), Fazekas and Klesov (2000), Fazekas (2006) and the references therein. From these results, FSLLN in  $C[0, 1]$  for dependent variables are easily inherited via Theorem 1.3.

*Remark 1.6.* When  $\mu = 0$ , any FSLLN of the form *ii*) in Theorem 1.3 verified by  $\zeta_n$  in  $C[0, 1]$  is satisfied also by the polygonal line  $\xi_n$  with vertices  $(\tau_{n,k}, S_k)$ ,  $0 \leq k \leq n$  where the  $\tau_{n,k}$  are deterministic or random in  $[0, 1]$ , with  $\min_{0 \leq k \leq n} \tau_{n,k} = 0$ ,  $\max_{0 \leq k \leq n} \tau_{n,k} = 1$  and  $\tau_{n,k} \neq \tau_{n,j}$  every time  $S_k \neq S_j$ . This results from *iii*) in Theorem 1.3. When  $\zeta_n$  satisfies *ii*) with  $\mu \neq 0$ ,  $\xi_n$  satisfies the same functional law of large numbers if and only if  $b_n^{-1} \max_{0 \leq k \leq n} |k - n\tau_{n,k}|$  converges almost surely to zero.

Intuitively, the more concentrated is the distribution of  $X_1$  (in the i.i.d. case), the closer to  $\mu I$  should be the paths of  $n^{-1}\zeta_n$ . In the functional framework of  $C[0, 1]$  this closeness is expressed by a convergence rate in the uniform norm, and there is no gain in assuming that  $\mathbf{E}|X_1|^p < \infty$  for  $p > 2$ . This can lead us to look for a different closeness “in shape”, more sensitive to the degree of integrability of  $X_1$ , by considering stronger norms than the uniform one. This question is natural since both functions  $\zeta_n$  and  $\mu I$  have a much stronger global regularity than the continuity.

For  $\alpha \in [0, 1)$  we consider the Hölderian modulus of smoothness of a function  $f : [0, 1] \rightarrow \mathbb{R}$  defined by

$$\omega_\alpha(f, \delta) := \sup_{\substack{s, t \in [0, 1] \\ 0 < t - s < \delta}} \frac{|f(t) - f(s)|}{|t - s|^\alpha}, \quad \delta \in (0, 1).$$

The Hölder space  $\mathcal{H}_\alpha^\circ[0, 1]$  is then the set of functions  $f \in C[0, 1]$  such that  $\omega_\alpha(f, \delta)$  converges to zero as  $\delta$  goes to zero, endowed with the norm  $\|f\|_\alpha = |f(0)| + \omega_\alpha(f, 1)$ . The following result gives a characterization of the functional strong law of large numbers for  $\zeta_n$  in the ladder of spaces  $\mathcal{H}_\alpha^\circ[0, 1]$ ,  $0 \leq \alpha < 1$ . As the space  $\mathcal{H}_0^\circ[0, 1]$  is isomorphic to  $C[0, 1]$ , the special case  $\alpha = 0$  is equivalent to Theorem 1.1.

**Theorem 1.7.** *Let  $0 \leq \alpha < 1$ . When the  $X_i$ 's are i.i.d., the following statements are equivalent:*

- a)  $\mathbf{E}|X_1|^{1/(1-\alpha)} < \infty$  and  $\mu = \mathbf{E}X_1$ ;
- b)  $n^{-1}\zeta_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \mu I$  in the space  $\mathcal{H}_\alpha^\circ[0, 1]$ ;
- c) for any  $\varepsilon > 0$ ,

$$\sum_{n=1}^{\infty} n^{-1} P(\|\zeta_n - n\mu I\|_\alpha > \varepsilon n) < \infty.$$

In terms of increments of partial sums, Theorem 1.7 can be stated in the following form.

**Theorem 1.8.** *Let  $0 \leq \alpha < 1$ . Then (a) of Theorem 1.7 is also equivalent with each of the following statements:*

$$b') \quad n^{-1+\alpha} \max_{0 \leq j < k \leq n} \frac{|S_k - S_j - \mu(k-j)|}{(k-j)^\alpha} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0;$$

*c') for any  $\varepsilon > 0$ ,*

$$\sum_{n=1}^{\infty} n^{-1} P\left(\sup_{0 \leq j < k \leq n} \frac{|S_k - S_j - \mu(k-j)|}{(k-j)^\alpha} > \varepsilon n^{1-\alpha}\right) < \infty.$$

*In the special case where  $\alpha = 0$ , condition b') is equivalent to (1.2), while c') is equivalent to*

*c'') for any  $\varepsilon > 0$ ,*

$$\sum_{n=1}^{\infty} n^{-1} P\left(\sup_{0 \leq k \leq n} |S_k - k\mu| > \varepsilon n\right) < \infty.$$

*Remark 1.9.* Assuming  $\mathbf{E}|X_1|^{1/(1-\alpha)} < \infty$  for some  $0 \leq \alpha < 1$  and  $\mathbf{E}X_1 = 0$  we have particularly

$$\|n^{-1}\zeta_n\|_\alpha \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0 \quad (1.3)$$

and one can ask what is the best possible rate of this convergence. From the classical Donsker-Prohorov invariance principle, we have that if  $\mathbf{E}X_1^2 < \infty$ , then

$$\sqrt{n}\|n^{-1}\zeta_n\|_\infty \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \|W\|_\infty,$$

where  $W = (W_t, t \in [0, 1])$  is a standard Wiener process and  $\xrightarrow[n \rightarrow \infty]{\mathcal{D}}$  stands for the convergence in distribution. For  $0 < \alpha < 1$  we observe different rates of convergence. For  $0 < \alpha < 1/2$  we have from Račkauskas and Suquet (2004) that

$$\sqrt{n}\|n^{-1}\zeta_n\|_\alpha \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \|W\|_\alpha$$

if and only if  $\lim_{t \rightarrow \infty} t^{1/(1/2-\alpha)} P(|X_1| > t) = 0$ , a condition which is stronger than  $\mathbf{E}|X_1|^{1/(1-\alpha)} < \infty$  which in turn is stronger than  $\mathbf{E}X_1^2 < \infty$ . If  $1/2 \leq \alpha < 1$  and the random variables  $X_k$ 's are regularly varying with exponent  $a > 2$ , then we have from Mikosch and Račkauskas (2010) that

$$n^{1-\alpha} a_n^{-1} \|n^{-1}\zeta_n\|_\alpha \xrightarrow[n \rightarrow \infty]{\mathcal{D}} Y, \quad (1.4)$$

where  $Y$  has Fréchet distribution with parameter  $a$ :  $P(Y \leq x) = \exp\{-x^{-a}\}$ ,  $x \geq 0$ , and  $a_n = \inf\{x \in \mathbb{R} : P(|X_1| \leq x) \geq 1 - 1/n\}$ . Since  $a_n = n^{1/a} \ell_n$  with slowly varying  $\ell_n$ , the normalization in (1.4) is  $n^{1-1/a-\alpha} \ell_n^{-1}$ . Let us observe that  $1-1/a-\alpha < 1/2$ , so the rate in (1.3) is slower for  $1/2 \leq \alpha < 1$  than for  $0 \leq \alpha < 1/2$ .

In the Gaussian case the limiting behavior of  $\|\zeta_n\|_{1/2}$  has been investigated by Siegmund and Venkatraman (1995) and Kabluchko (2008). It is proved that centered and normed sequence  $b_n(\|n^{-1}\zeta_n\|_{1/2} - a_n)$  with appropriately chosen sequences  $(b_n)$  and  $(a_n)$  converges in distribution to a double exponential random variable. In the non-Gaussian light-tailed case, the limiting distribution of  $\|\zeta_n\|_{1/2}$  has been obtained in Kabluchko and Wang (2012).

Theorem 1.7 indicates that the construction of polygonal line process  $\zeta_n$  is not adapted to the structure of Hölder topology with respect to the law of large numbers. Next we introduce another construction which we call *adaptive*. For this we use the random partition of the interval  $[0, 1]$  generated by the points  $\tau_k$  constructed as follows. For  $n \geq 1$ , put

$$T_n := \sum_{i=1}^n |X_i|.$$

Then we define the triangular array  $\{\tau_{n,k}, 0 \leq k \leq n\}$  by setting  $\tau_{n,0} := 0$ ,  $\tau_{n,n} := 1$  and for  $1 \leq k < n$ ,

$$\tau_{n,k} := \begin{cases} \frac{T_k}{T_n} & \text{on the event } \{T_n > 0\} \\ 0 & \text{on the event } \{T_n = 0\}. \end{cases}$$

For notational convenience, we write  $\tau_k$  for  $\tau_{n,k}$  every time there is no ambiguity on which  $n$  is involved by  $\tau_k$ . Now we denote by  $\zeta_n^{\text{ad}}$  the random polygonal line process with vertices the points  $V_k := (\tau_k, S_k)$ . To express this in explicit formulas, we can start with:

$$\zeta_n^{\text{ad}}(\tau_k) = S_k, \quad 0 \leq k \leq n.$$

We note in passing that this writing is consistent, since on the event  $\{\tau_k = \tau_{k-1}\}$ ,  $X_k = 0$  and  $S_{k-1} = S_k$ . Next we remark that for every  $t \in [0, 1] \setminus \{\tau_k, 0 \leq k \leq n\}$ , there is a unique (random) integer  $1 \leq j \leq n$  such that  $\tau_{j-1} < t < \tau_j$  and then

$$\zeta_n^{\text{ad}}(t) = \frac{\tau_j - t}{\tau_j - \tau_{j-1}} S_{j-1} + \frac{t - \tau_{j-1}}{\tau_j - \tau_{j-1}} S_j = S_{j-1} + \frac{t - \tau_{j-1}}{\tau_j - \tau_{j-1}} X_j.$$

It is worth noticing here that, due to the direct definition of  $\tau_0 = 0$  and  $\tau_n = 1$ , the above formulas are still valid on the event  $\{T_n = 0\}$ , with  $j = n$ .

Before stating the functional law of large numbers for  $\zeta_n^{\text{ad}}$ , one can get an intuition by a rough comparison of  $\zeta_n$  and  $\zeta_n^{\text{ad}}$ . For the first process, the increment  $S_k - S_{k-1}$  ( $1 \leq k \leq n$ ) is realized in the deterministic interval  $[(k-1)/n, k/n]$  with length  $1/n$ . For  $\zeta_n^{\text{ad}}$ , the same increment is realized in the random interval  $I_k = [\tau_{k-1}, \tau_k]$ . In mean, the length  $|I_k| = \tau_k - \tau_{k-1}$  is asymptotically equivalent to  $1/n$  (or equal if  $P(X_1 = 0) = 0$ ). Indeed it is easily seen that  $\mathbf{E}|I_k| = P(T_n > 0)/n$  for  $1 \leq k < n$  while  $\mathbf{E}|I_n| = P(T_n > 0)/n + P(T_n = 0)$ . As the  $X_k$ 's are i.i.d.,  $P(T_n = 0) = P(X_1 = 0)^n$  so, discarding the degenerated case where  $P(X_1 = 0) = 1$ , each  $|I_k|$  is equivalent to  $1/n$ .

This apparent similarity between  $\zeta_n$  and  $\zeta_n^{\text{ad}}$  may be misleading when working with Hölder topologies. In this context, what matters is the slope of the polygonal line between two consecutive vertices. Contrary to a fixed interval of length  $1/n$ ,  $I_k$  reacts to a value of  $X_k$  which can be big with big probability if  $\mathbf{E}|X_1|^{1/(1-\alpha)} = \infty$ . Roughly speaking, for the same big increment  $S_k - S_{k-1}$ , the slope is weaker for  $\zeta_n^{\text{ad}}$  than for  $\zeta_n$ . This is the key of the better Hölderian behavior of  $\zeta_n^{\text{ad}}$  as illustrated by the following result.

**Theorem 1.10.** *Let  $0 \leq \alpha < 1$ . The following statements are equivalent*

- a)  $\mathbf{E}|X_1| < \infty$  and  $\mathbf{E}X_1 = \mu$ ;
- b)  $n^{-1}\zeta_n^{\text{ad}} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \mu I$  in the space  $\mathcal{H}_\alpha^o[0, 1]$ ;

c) for any  $\varepsilon > 0$ ,

$$\sum_{n=1}^{\infty} n^{-1} P(\|\zeta_n^{\text{ad}} - n\mu I\|_{\alpha} > \varepsilon n) < \infty.$$

## 2. FSLLN IN $C[0, 1]$

Since Theorems 1.1 and 1.2 are contained in Theorem 1.4, we need only to prove Theorems 1.3 and 1.4.

*Proof of Theorem 1.3.* Observing that the supremum norm of a polygonal line is reached at some vertex gives immediately the equivalence between conditions *ii*) and *iii*) in Theorem 1.3. Obviously *iii*) implies *i*). The proof of the implication *i*)  $\Rightarrow$  *ii*) is purely analytical and is provided by the following lemma.  $\square$

**Lemma 2.1.** *Let  $\mu$  be a real number and  $(b_n)_{n \geq 1}$  be a non-decreasing sequence of positive numbers going to infinity. Let  $(s_n)_{n \geq 1}$  be a sequence of real numbers such that  $b_n^{-1}(s_n - n\mu)$  converges to zero. Then*

$$\max_{1 \leq k \leq n} \frac{|s_k - k\mu|}{b_n} \xrightarrow{n \rightarrow \infty} 0. \quad (2.1)$$

*Proof.* Let  $\varepsilon$  be an arbitrary positive number. By the convergence to zero of  $b_n^{-1}(s_n - n\mu)$ , there is some integer  $k_0 = k_0(\varepsilon)$  such that for every  $k \geq k_0$ ,  $b_k^{-1}|s_k - k\mu| < \varepsilon$ . Together with the non-decreasingness of the sequence  $(b_n)_{n \geq 1}$ , this leads to the upper bound

$$\begin{aligned} \max_{1 \leq k \leq n} \frac{|s_k - k\mu|}{b_n} &\leq \frac{1}{b_n} \max_{k < k_0} |s_k - k\mu| + \max_{k_0 \leq k \leq n} \frac{|s_k - k\mu|}{b_k} \frac{b_k}{b_n} \\ &\leq \frac{1}{b_n} \max_{k < k_0} |s_k - k\mu| + \varepsilon. \end{aligned}$$

Since  $b_n$  goes to infinity, it follows that

$$\limsup_{n \rightarrow \infty} \max_{1 \leq k \leq n} \frac{|s_k - k\mu|}{b_n} \leq \varepsilon,$$

which gives (2.1) by arbitrariness of  $\varepsilon$ .  $\square$

*Proof of Theorem 1.4.* Let  $p \in (0, 2)$ . From the classical Marcinkiewicz-Zygmund SLLN, we know that if  $\mathbf{E}|X_1|^p < \infty$ , then  $n^{-1/p}S_n$  goes a.s. to zero in the case  $0 < p < 1$ , while  $n^{-1/p}(S_n - n\mathbf{E}X_1)$  goes a.s. to zero in the case  $1 \leq p < 2$ . Accounting the equivalence of *i*) and *ii*) in Theorem 1.3, this immediately gives the statements *a*) by choosing  $\mu = 0$  and *b*) with  $\mu = \mathbf{E}X_1$ .

By the converse part in classical Marcinkiewicz-Zygmund SLLN, if  $n^{-1/p}(S_n - c_n)$  goes to zero almost surely for some  $p \in (0, 2)$  and some sequence  $(c_n)_{n \geq 1}$  of real numbers, then  $\mathbf{E}|X_1|^p < \infty$ . Together with Theorem 1.3, this gives the statement *c*), noting that  $\|\zeta_n - c_n I\|_{\infty} \geq |\zeta_n(1) - c_n I(1)| = |S_n - c_n|$ .  $\square$

## 3. SOME HÖLDERIAN TOOLS

The Hölder norm of a polygonal line function is very easy to compute according to the following lemma for which we refer e.g. to Markevičiūtė et al. (2012) Lemma 3, where it is proved in a more general setting.

**Lemma 3.1.** *Let  $t_0 = 0 < t_1 < \dots < t_n = 1$  be a partition of  $[0, 1]$  and  $f$  be a real-valued polygonal line function on  $[0, 1]$  with vertices at the  $t_i$ 's, i.e.  $f$  is continuous on  $[0, 1]$  and its restriction to each interval  $[t_i, t_{i+1}]$  is an affine function. Then for any  $0 \leq \alpha < 1$ ,*

$$\sup_{0 \leq s < t \leq 1} \frac{|f(t) - f(s)|}{(t - s)^\alpha} = \max_{0 \leq i < j \leq n} \frac{|f(t_j) - f(t_i)|}{(t_j - t_i)^\alpha}.$$

Let  $D_j$  denotes the set of dyadic numbers of level  $j$  in  $[0, 1]$ , that is  $D_0 := \{0, 1\}$  and for  $j \geq 1$ ,  $D_j := \{(2l - 1)2^{-j}; 1 \leq l \leq 2^{j-1}\}$ . For  $r \in D_j$  set  $r^- := r - 2^{-j}$ ,  $r^+ := r + 2^{-j}$ ,  $j \geq 0$ . For  $f : [0, 1] \rightarrow \mathbb{R}$  and  $r \in D_j$  let us define

$$\lambda_r(f) := \begin{cases} f(r^+) + f(r^-) - 2f(r) & \text{if } j \geq 1, \\ f(r) & \text{if } j = 0. \end{cases}$$

The following sequential norm defined on  $\mathcal{H}_\alpha^o[0, 1]$  by

$$\|f\|_\alpha^{\text{seq}} := \sup_{j \geq 0} 2^{\alpha j} \max_{r \in D_j} |\lambda_r(f)|,$$

is equivalent to the natural norm  $\|f\|_\alpha$ , see [Ciesielski \(1960\)](#). Let us define also  $\mathbb{D}_j := \{k2^{-j}, 0 \leq k < 2^j\}$ , so that  $\mathbb{D}_j = \{0\} \cup \bigcup_{1 \leq i \leq j} D_i$ . In what follows, we denote by  $\log$  the logarithm with basis 2 ( $\log 2 = 1$ ).

**Lemma 3.2.** *For  $0 \leq \alpha < 1$ ,*

$$\|\zeta_n - \mu I\|_\alpha^{\text{seq}} \leq 2 \max_{0 \leq j \leq \log n} 2^{\alpha j} \max_{r \in \mathbb{D}_j} \left| \sum_{nr < i \leq n(r+2^{-j})} (X_i - \mu) \right| + 4n^\alpha \max_{1 \leq i \leq n} |X_i - \mu|.$$

*Proof.* First we remark that for  $j \geq 1$ ,

$$\max_{r \in D_j} |\lambda_r(f)| \leq \max_{r \in D_j} |f(r^+) - f(r)| + \max_{r \in D_j} |f(r) - f(r^-)|.$$

As  $r^+$  and  $r^-$  belong to  $\mathbb{D}_j$ , this gives:

$$\sup_{j \geq 1} 2^{\alpha j} \max_{r \in D_j} |\lambda_r(f)| \leq 2 \sup_{j \geq 1} 2^{\alpha j} \max_{r \in \mathbb{D}_j} |f(r + 2^{-j}) - f(r)|.$$

It follows that if  $f \in \mathcal{H}_\alpha^o[0, 1]$  and  $f(0) = 0$ ,

$$\|f\|_\alpha^{\text{seq}} \leq 2 \sup_{j \geq 0} 2^{\alpha j} \max_{r \in \mathbb{D}_j} |f(r + 2^{-j}) - f(r)|.$$

This inequality can be applied to the random polygonal line  $\zeta_n - \mu I$ . Moreover, it is clear that there is no loss of generality in assuming  $\mu = 0$  for notational simplicity. We claim that for  $r \in \mathbb{D}_j$ ,

$$|\zeta_n(r + 2^{-j}) - \zeta_n(r)| \leq \begin{cases} |S_{[nr+n2^{-j}]} - S_{[nr]}| + 2 \max_{1 \leq i \leq n} |X_i| & \text{if } j \leq \log n, \\ 2n2^{-j} \max_{1 \leq i \leq n} |X_i| & \text{if } j > \log n. \end{cases}$$

Indeed, if  $j \leq \log n$ , this follows immediately by triangular inequality. If  $j > \log n$ ,  $2^{-j} < 1/n$  and then with  $r$  in say  $[i/n, (i+1)/n]$ , either  $r + 2^{-j}$  is in  $(i/n, (i+1)/n]$  or belongs to  $((i+1)/n, (i+2)/n]$ . In the first case, noting that the slope of  $\zeta_n$  on  $[i/n, (i+1)/n]$  is exactly  $nX_{i+1}$ , we have

$$|\zeta_n(r + 2^{-j}) - \zeta_n(r)| = n|X_{i+1}|2^{-j} \leq 2^{-j}n \max_{1 \leq i \leq n} |X_i|.$$



If  $r$  and  $r + 2^{-j}$  are in consecutive intervals, the same argument applies after chaining, so

$$\begin{aligned} |\zeta_n(r + 2^{-j}) - \zeta_n(r)| &\leq |\zeta_n(r) - \zeta_n((i+1)/n)| + |\zeta_n((i+1)/n) - \zeta_n(r + 2^{-j})| \\ &\leq 2^{-j+1} n \max_{1 \leq i \leq n} |X_i|. \end{aligned}$$

To complete the proof of the lemma, it remains to note that if  $j \leq \log n$ ,  $2^{j\alpha} \leq n^\alpha$  and if  $j > \log n$ ,  $2^{j\alpha} 2^{-j} n = (2^{-j})^{1-\alpha} n \leq (n^{-1})^{1-\alpha} n = n^\alpha$ .  $\square$

#### 4. PROOF OF THEOREMS 1.7 AND 1.8

Theorem 1.8 follows immediately from Theorem 1.7 because the Hölder norm of a polygonal line is reached at two vertices (Lemma 3.1). We shall prove Theorem 1.7 following the scheme:

$$(a) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a).$$

*Proof of (a)  $\Rightarrow$  (c).* Since (a) yields  $\mathbf{E}|X_1| < \infty$ , we can assume without loss of generality that  $\mathbf{E}X_1 = 0$ . Define for every positive  $\varepsilon$ ,

$$P_1(n, \varepsilon) := P \left( \max_{0 \leq j \leq \log n} 2^{\alpha j} \max_{r \in \mathbb{D}_j} \left| \sum_{nr < k \leq n(r+2^{-j})} X_k \right| > \varepsilon n \right)$$

and

$$P_2(n, \varepsilon) := P \left( n^{-1+\alpha} \max_{1 \leq k \leq n} |X_k| > \varepsilon \right).$$

According to Lemma 3.2 it is enough to prove for  $i = 1, 2$  and for each  $\varepsilon > 0$  that

$$\sum_{n=1}^{\infty} n^{-1} P_i(n, \varepsilon) < \infty. \quad (4.1)$$

First let us check (4.1) with  $i = 2$ . Since  $P_2(n, \varepsilon) \leq \sum_{k=1}^n P(|X_k| > \varepsilon n^{1-\alpha})$  and the  $X_k$ 's are identically distributed, we get

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1} P_2(n, \varepsilon) &\leq \sum_{n=1}^{\infty} P(|X_1| > \varepsilon n^{1-\alpha}) \leq \sum_{n=1}^{\infty} \int_{n-1}^n P(|X_1| > \varepsilon t^{1-\alpha}) dt \\ &= \int_0^{\infty} P(|X_1| > \varepsilon t^{1-\alpha}) dt \\ &= \int_0^{\infty} P(|X_1|^{1/(1-\alpha)} > \varepsilon^{1/(1-\alpha)} t) dt \\ &= \varepsilon^{-1/(1-\alpha)} \mathbf{E}|X_1|^{1/(1-\alpha)} < \infty, \end{aligned}$$

by (a).

Next we prove (4.1) for  $i = 1$ . Introducing

$$\begin{aligned} X'_i &= X_i \mathbf{1}_{\{|X_i| \leq n^{1-\alpha}\}} - \mathbf{E} X_i \mathbf{1}_{\{|X_i| \leq n^{1-\alpha}\}}, \\ X''_i &= X_i \mathbf{1}_{\{|X_i| > n^{1-\alpha}\}} - \mathbf{E} X_i \mathbf{1}_{\{|X_i| > n^{1-\alpha}\}}, \end{aligned}$$

and recalling that  $\mathbf{E}X_k = 0$ , we decompose  $X_i$  in the sum  $X'_i + X''_i$  of two truncated and centered random variables. We split  $P_1(n, \varepsilon) \leq P'_1(n, \varepsilon/2) + P''_1(n, \varepsilon/2)$  where in the definition of  $P_1(n, \varepsilon)$  one has to substitute respectively  $X_i$  by  $X'_i$  to define  $P'_1(n, \varepsilon)$  and by  $X''_i$  to define  $P''_1(n, \varepsilon)$ .

First we estimate  $P_1''(n, \varepsilon)$ . By Markov inequality we have

$$\begin{aligned} P_1''(n, \varepsilon) &\leq \varepsilon^{-1} n^{-1} \sum_{1 \leq j \leq \log n} 2^{\alpha j} \mathbf{E} \max_{r \in \mathbb{D}_j} \left| \sum_{nr < k \leq n(r+2^{-j})} X_k'' \right| \\ &\leq \varepsilon^{-1} \mathbf{E} |X_1''| \sum_{j=0}^{\log n} 2^{\alpha j} \\ &\leq \frac{2^\alpha n^\alpha}{\varepsilon(2^\alpha - 1)} \mathbf{E} |X_1''|. \end{aligned} \quad (4.2)$$

Next we note that  $\mathbf{E} |X_1''| \leq 2\mathbf{E} |X_1| \mathbf{1}_{\{|X_1| > n^{1-\alpha}\}}$  and

$$\begin{aligned} \mathbf{E} |X_1| \mathbf{1}_{\{|X_1| > n^{1-\alpha}\}} &= \int_0^\infty P(|X_1| \mathbf{1}_{\{|X_1| > n^{1-\alpha}\}} > s) ds \\ &= \int_0^{n^{1-\alpha}} P(|X_1| \mathbf{1}_{\{|X_1| > n^{1-\alpha}\}} > s) ds + \int_{n^{1-\alpha}}^\infty P(|X_1| > s) ds \\ &= n^{1-\alpha} P(|X_1| > n^{1-\alpha}) + \int_{n^{1-\alpha}}^\infty P(|X_1| > s) ds. \end{aligned} \quad (4.3)$$

Now from (4.2) and (4.3), we obtain

$$\sum_{n=1}^\infty n^{-1} P_1''(n, \varepsilon) \leq \frac{c(\alpha)}{\varepsilon} \left( \sum_{n=1}^\infty P(|X_1| > n^{1-\alpha}) + \sum_{n=1}^\infty n^{-1+\alpha} \int_{n^{1-\alpha}}^\infty P(|X_1| > s) ds \right).$$

The first series in the right hand side converges since

$$\sum_{n=1}^\infty P(|X_1| > n^{1-\alpha}) \leq \mathbf{E} |X_1|^{1/(1-\alpha)},$$

as already seen above when bounding  $\sum_{n=1}^\infty n^{-1} P_2(n, \varepsilon)$ . For the second series, we can write

$$\begin{aligned} \sum_{n=1}^\infty n^{-1+\alpha} \int_{n^{1-\alpha}}^\infty P(|X_1| > s) ds &= \sum_{n=1}^\infty n^{-1+\alpha} \sum_{k=n}^\infty \int_{k^{1-\alpha}}^{(k+1)^{1-\alpha}} P(|X_1| > s) ds \\ &= \sum_{k=1}^\infty \int_{k^{1-\alpha}}^{(k+1)^{1-\alpha}} P(|X_1| > s) ds \sum_{1 \leq n \leq k} n^{-1+\alpha} \\ &\leq c'(\alpha) \sum_{k=1}^\infty \int_{k^{1-\alpha}}^{(k+1)^{1-\alpha}} k^\alpha P(|X_1| > s) ds. \end{aligned}$$

Noting that  $k^\alpha = (k^{1-\alpha})^{\alpha/(1-\alpha)} \leq s^{\alpha/(1-\alpha)}$  for every  $s \geq k^{1-\alpha}$ , we get

$$\begin{aligned} \sum_{n=1}^\infty n^{-1+\alpha} \int_{n^{1-\alpha}}^\infty P(|X_1| > s) ds &\leq c'(\alpha) \sum_{k=1}^\infty \int_{k^{1-\alpha}}^{(k+1)^{1-\alpha}} s^{\alpha/(1-\alpha)} P(|X_1| > s) ds \\ &\leq c'(\alpha) \int_0^\infty s^{\alpha/(1-\alpha)} P(|X_1| > s) ds \\ &= \frac{c'(\alpha)(1-\alpha)}{\alpha} \mathbf{E} |X_1|^{1/(1-\alpha)} < \infty. \end{aligned}$$

This achieves the verification of the convergence of  $\sum_{n=1}^\infty n^{-1} P_1''(n, \varepsilon)$ .

Now consider the series  $\sum_{n=1}^{\infty} n^{-1} P'_1(n, \varepsilon)$ . We claim that one can find some real  $q > 1/(1 - \alpha)$  and some constant  $c(\alpha, q)$  such that

$$P'_1(n, \varepsilon) \leq \frac{c(\alpha, q)}{\varepsilon^q} n^{1-q(1-\alpha)} \mathbf{E} |X'_1|^q. \quad (4.4)$$

Postponing the verification of (4.4), let us see how it gives the convergence of  $\sum_{n=1}^{\infty} n^{-1} P'_1(n, \varepsilon)$ .

First we note that  $\mathbf{E} |X'_1|^q \leq 2^q \mathbf{E} |X_1|^q \mathbf{1}_{\{|X_i| \leq n^{1-\alpha}\}}$  and

$$\begin{aligned} \mathbf{E} |X_1|^q \mathbf{1}_{\{|X_i| \leq n^{1-\alpha}\}} &= \int_0^{\infty} P(|X_1|^q \mathbf{1}_{\{|X_i| \leq n^{1-\alpha}\}} > t) dt \\ &= \int_0^{n^{q(1-\alpha)}} P(|X_1|^q \mathbf{1}_{\{|X_i| \leq n^{1-\alpha}\}} > t) dt \\ &= \int_0^{n^{q(1-\alpha)}} P(n^{q(1-\alpha)} \geq |X_1|^q > t) dt \\ &\leq \int_0^{n^{q(1-\alpha)}} P(|X_1|^q > t) dt = \int_0^{n^{1-\alpha}} q s^{q-1} P(|X_1| > s) ds. \end{aligned}$$

Combining this estimate with (4.4) reduces the problem to the convergence of the series

$$\Sigma(q, \alpha) := \sum_{n=1}^{\infty} n^{-q(1-\alpha)} \int_0^{n^{1-\alpha}} s^{q-1} P(|X_1| > s) ds. \quad (4.5)$$

For this aim, it is convenient to exchange the summations as follows:

$$\begin{aligned} \Sigma(q, \alpha) &= \sum_{n=1}^{\infty} n^{-q(1-\alpha)} \sum_{k=1}^n \int_{(k-1)^{(1-\alpha)}}^{k^{(1-\alpha)}} s^{q-1} P(|X_1| > s) ds \\ &= \sum_{k=1}^{\infty} \int_{(k-1)^{(1-\alpha)}}^{k^{(1-\alpha)}} s^{q-1} P(|X_1| > s) ds \sum_{n=k}^{\infty} n^{-q(1-\alpha)} \\ &\leq c'(q, \alpha) \sum_{k=1}^{\infty} \int_{(k-1)^{(1-\alpha)}}^{k^{(1-\alpha)}} k^{1-q(1-\alpha)} s^{q-1} P(|X_1| > s) ds. \end{aligned}$$

Now we note that in the last integral above,  $s \leq k^{(1-\alpha)}$ , so  $k \geq s^{1/(1-\alpha)}$  and as  $1 - q(1 - \alpha) < 0$ ,  $k^{1-q(1-\alpha)} \leq (s^{1/(1-\alpha)})^{1-q(1-\alpha)} = s^{1/(1-\alpha)-q}$  and finally  $k^{1-q(1-\alpha)} s^{q-1} \leq s^{1/(1-\alpha)-1}$ . This leads to

$$\Sigma(q, \alpha) \leq c'(q, \alpha) \int_0^{\infty} s^{1/(1-\alpha)-1} P(|X_1| > s) ds = c''(q, \alpha) \mathbf{E} |X_1|^{1/(1-\alpha)} < \infty.$$

To complete the proof, it only remains to check (4.4). In the case where  $0 \leq \alpha < 1/2$ , we can choose  $q = 2$  and then (4.4) is obtained by Chebyshev inequality:

$$\begin{aligned}
P'_1(n, \varepsilon) &\leq \sum_{j=0}^{\log n} \sum_{r \in \mathbb{D}_j} P \left( \left| \sum_{nr < k \leq n(r+2^{-j})} X'_k \right| > \varepsilon n 2^{-j\alpha} \right) \\
&\leq \frac{1}{n^2 \varepsilon^2} \sum_{j=0}^{\log n} 2^{2j\alpha} \sum_{r \in \mathbb{D}_j} \mathbf{E} \left| \sum_{nr < k \leq n(r+2^{-j})} X'_k \right|^2 \\
&= \frac{1}{n^2 \varepsilon^2} \sum_{j=0}^{\log n} 2^{2j\alpha} 2^j \mathbf{E} X_1'^2 \\
&\leq \frac{2^{1+2\alpha}}{\varepsilon^2 (2^{1+2\alpha} - 1)} n^{-1+2\alpha} \mathbf{E} X_1'^2.
\end{aligned}$$

If  $\alpha \geq 1/2$ , then by Markov and Rosenthal (1970) inequalities we obtain for any  $q > 1/(1 - \alpha) \geq 2$ ,

$$\begin{aligned}
P'_1(n, \varepsilon) &\leq \frac{1}{n^q \varepsilon^q} \sum_{j=0}^{\log n} 2^{jq\alpha} \sum_{r \in \mathbb{D}_j} \mathbf{E} \left| \sum_{nr < k \leq n(r+2^{-j})} X'_k \right|^q \\
&\leq \frac{C_q}{n^q \varepsilon^q} \sum_{j=0}^{\log n} 2^{jq\alpha} 2^j \left( (n 2^{-j} \mathbf{E} (X_1')^2)^{q/2} + n 2^{-j} \mathbf{E} |X_1'|^q \right),
\end{aligned}$$

denoting by  $C_q$  the universal constant in Rosenthal inequality. Since  $q > 2$ ,  $(\mathbf{E} (X_1')^2)^{q/2} = ((\mathbf{E} |X_1'|^2)^{1/2})^q \leq ((\mathbf{E} |X_1'|^q)^{1/q})^q = \mathbf{E} |X_1'|^q$ . Moreover, in the range of summation,  $n 2^{-j} \geq 1$  and as  $q/2 > 1$ ,  $(n 2^{-j})^{q/2} \geq n 2^{-j}$ . This gives

$$P'_1(n, \varepsilon) \leq \frac{2C_q \mathbf{E} |X_1'|^q}{\varepsilon^q n^{q/2}} \sum_{j=0}^{\log n} 2^{j(q(\alpha-1/2)+1)} \leq \frac{4C_q}{(2^{q(\alpha-1/2)+1} - 1) \varepsilon^q} n^{1-q(1-\alpha)} \mathbf{E} |X_1'|^q,$$

so (4.4) is verified.  $\square$

*Proof of (c)  $\Rightarrow$  (b).* Since  $\zeta_n(0) = 0$ , we have

$$\|\zeta_n - n\mu I\|_\alpha = \omega_\alpha(\zeta_n - n\mu I, 1) = \sup_{0 \leq s < t \leq 1} \frac{|\zeta_n(t) - \zeta_n(s) - n\mu(t-s)|}{|t-s|^\alpha}.$$

Applying Lemma 3.1 to the polygonal line  $\zeta_n - \mu I$  gives

$$\|\zeta_n - \mu I\|_\alpha = n^\alpha \max_{1 \leq i < j \leq n} \frac{|S_j - S_i - (j-i)\mu|}{|j-i|^\alpha}. \quad (4.6)$$

From (4.6), it is clear that the sequence  $(Z_n)_{n \geq 1}$  defined by

$$Z_n = n^{-\alpha} \|\zeta_n - n\mu I\|_\alpha, \quad n \geq 1,$$

is non-decreasing. Now by (c) we have for every positive  $\varepsilon$ ,

$$\sum_{n=1}^{\infty} n^{-1} P(Z_n > \varepsilon n^{1-\alpha}) < \infty.$$

Exploiting the monotonicity of  $(Z_n)_{n \geq 1}$ , we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1} P(Z_n > \varepsilon n^{1-\alpha}) &= \sum_{i=0}^{\infty} \sum_{2^i \leq n < 2^{i+1}} n^{-1} P(Z_n > \varepsilon n^{1-\alpha}) \\ &\geq \sum_{i=0}^{\infty} \sum_{2^i \leq n < 2^{i+1}} 2^{-i-1} P(Z_{2^i} > \varepsilon 2^{(i+1)(1-\alpha)}) \\ &= \sum_{i=0}^{\infty} 2^{-1} P(Z_{2^i} > \varepsilon 2^{(i+1)(1-\alpha)}). \end{aligned}$$

It follows that for every positive  $\varepsilon$ ,

$$\sum_{i=0}^{\infty} P(Z_{2^i} > \varepsilon 2^{i(1-\alpha)}) < \infty. \quad (4.7)$$

For  $N > 1$  let  $J \geq 0$  be such that  $2^{J-1} \leq N < 2^J$ . Then

$$\begin{aligned} P\left(\sup_{n \geq N} \|n^{-1}\zeta_n - \mu I\|_{\alpha} > \varepsilon\right) &\leq P\left(\sup_{n \geq 2^{J-1}} n^{-1+\alpha} Z_n > \varepsilon\right) \\ &= P\left(\sup_{i \geq J} \max_{2^{i-1} \leq n < 2^i} n^{-1+\alpha} Z_n > \varepsilon\right) \\ &\leq P\left(\sup_{i \geq J} 2^{(-1+\alpha)(i-1)} Z_{2^i} > \varepsilon\right) \\ &\leq \sum_{i=J}^{\infty} P(Z_{2^i} > \varepsilon 2^{(1-\alpha)(i-1)}). \end{aligned}$$

This upper bound goes to zero when  $J \rightarrow \infty$  by (4.7). Hence  $\sup_{n \geq N} \|n^{-1}\zeta_n - \mu I\|_{\alpha}$  converges in probability to zero as  $N \rightarrow \infty$ . From this it is easily deduced that  $\|n^{-1}\zeta_n - \mu I\|_{\alpha}$  converges almost surely to zero as  $n \rightarrow \infty$ , which gives (b).  $\square$

*Proof of (b)  $\Rightarrow$  (a).* Putting  $\zeta'_n := \zeta_n - n\mu I$ , we note that

$$n^{\alpha} |X_n - \mu| = \frac{|\zeta'_n(1) - \zeta'_n(1 - 1/n)|}{(\frac{1}{n})^{\alpha}} \leq \|\zeta'_n\|_{\alpha},$$

so (b) implies that

$$n^{-1+\alpha} |X_n - \mu| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0.$$

Then, by independence of the  $X_n$ 's, the second Borel-Cantelli lemma yields

$$\sum_{n=1}^{\infty} P(|X_n - \mu| \geq n^{1-\alpha}) < \infty,$$

which by identical distribution of the  $X_n$ 's can be recast as

$$\sum_{n=1}^{\infty} P(|X_1 - \mu| \geq n^{1-\alpha}) < \infty,$$

what gives  $\mathbf{E}|X_1 - \mu|^{1/(1-\alpha)} < \infty$ . It follows that  $\mathbf{E}|X_1|^{1/(1-\alpha)} < \infty$  and as  $(1-\alpha)^{-1} \geq 1$ ,  $\mathbf{E}|X_1| < \infty$ . Finally  $n^{-1}S_n = n^{-1}\zeta_n(1)$  converges almost surely to  $\mu$  by (b) and to  $\mathbf{E}X_1$  by the classical strong law of large numbers. Hence  $\mu = \mathbf{E}X_1$  and the proof of Theorem 1.7 is complete.  $\square$

## 5. PROOF OF THEOREM 1.10

As a preliminary remark, it seems worth noticing that the proof of Theorem 1.10 cannot be easily reduced to the case where  $\mu = 0$ . Indeed the centering which substitutes  $X_i$  by  $X'_i = X_i - \mu$  changes also the random partition of  $[0, 1]$  in another one, built on the  $X'_i$ , and it seems difficult to find a simple relationship between the two corresponding polygonal lines.

In what follows, we put

$$\nu := \mathbf{E} |X_1|$$

and discard the trivial case where  $\nu = 0$  since then all the  $X_i$ 's would be almost surely null.

We shall prove Theorem 1.10 following the scheme:

$$(a) \Rightarrow (b) \Rightarrow (a) \Rightarrow (c) \Rightarrow (a).$$

*Proof of (a)  $\Rightarrow$  (b).* The Hölder norm of the polygonal line  $\zeta_n^{\text{ad}}$  is reached at two vertices. Accounting the possibility of several consecutive equal  $\tau_i$ 's, this property can be translated here by

$$\begin{aligned} \Delta_n &:= \|n^{-1}\zeta_n^{\text{ad}} - \mu I\|_\alpha \\ &= \frac{1}{n} \max_{0 \leq \tau_j < \tau_k \leq 1} \frac{|S_k - S_j - n\mu(\tau_k - \tau_j)|}{(\tau_k - \tau_j)^\alpha} \mathbf{1}_{\{T_n > 0\}} + \mu \mathbf{1}_{\{T_n = 0\}} \\ &= \frac{T_n^\alpha}{n} \max_{0 \leq \tau_j < \tau_k \leq 1} \frac{|S_k - S_j - \frac{n\mu}{T_n}(T_k - T_j)|}{(T_k - T_j)^\alpha} \mathbf{1}_{\{T_n > 0\}} + \mu \mathbf{1}_{\{T_n = 0\}}. \end{aligned} \quad (5.1)$$

By triangle inequality, on the event  $\{T_n > 0\}$ ,

$$\left| \sum_{j < i \leq k} \left( X_i - \frac{n\mu}{T_n} |X_i| \right) \right| \leq \left( 1 + \frac{n|\mu|}{T_n} \right) \sum_{j < i \leq k} |X_i|$$

whence

$$(T_k - T_j)^\alpha \geq \left( 1 + \frac{n|\mu|}{T_n} \right)^{-\alpha} \left| S_k - S_j - \frac{n\mu}{T_n}(T_k - T_j) \right|^\alpha.$$

With this lower bound for the denominator in (5.1), we obtain

$$\begin{aligned} \Delta_n &\leq \left( \frac{T_n}{n} + |\mu| \right)^\alpha \left( \frac{1}{n} \max_{0 \leq \tau_j < \tau_k \leq 1} \left| S_k - S_j - \frac{n\mu}{T_n}(T_k - T_j) \right| \right)^{1-\alpha} \mathbf{1}_{\{T_n > 0\}} \\ &\quad + \mu \mathbf{1}_{\{T_n = 0\}} \\ &\leq \left( \frac{T_n}{n} + |\mu| \right)^\alpha \left( \frac{2}{n} \max_{1 \leq k \leq n} \left| S_k - \frac{n\mu}{T_n} T_k \right| \right)^{1-\alpha} \mathbf{1}_{\{T_n > 0\}} + \mu \mathbf{1}_{\{T_n = 0\}}. \end{aligned}$$

We introduce some centering by writing, on the event  $\{T_n > 0\}$ ,

$$S_k - \frac{n\mu}{T_n} T_k = S_k - k\mu - \frac{n\mu}{T_n}(T_k - k\nu) + \frac{\mu k}{T_n}(T_n - n\nu),$$

whence

$$\frac{1}{n} \max_{1 \leq k \leq n} \left| S_k - \frac{n\mu}{T_n} T_k \right| \leq \frac{1}{n} \max_{1 \leq k \leq n} |S_k - k\mu| + \frac{2|\mu|}{T_n} \max_{1 \leq k \leq n} |T_k - k\nu|.$$

Finally

$$\begin{aligned} \Delta_n \leq & \left( \frac{T_n}{n} + |\mu| \right)^\alpha \left( \frac{2}{n} \max_{1 \leq k \leq n} |S_k - k\mu| + \frac{4|\mu|}{T_n} \max_{1 \leq k \leq n} |T_k - k\nu| \right)^{1-\alpha} \mathbf{1}_{\{T_n > 0\}} \\ & + \mu \mathbf{1}_{\{T_n = 0\}} \end{aligned} \quad (5.2)$$

Now we recall that  $P(T_n = 0) = P(X_1 = 0)^n$  which goes to zero, since we discarded the trivial case where  $X_1 = 0$  almost surely. As the sequence  $T_n$  is non-decreasing, this implies that

$$\mathbf{1}_{\{T_n > 0\}} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 1. \quad (5.3)$$

As  $\mathbf{E}|X_1| < \infty$ , we have the following convergences

$$\frac{T_n}{n} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \nu > 0, \quad (5.4)$$

by classical SLLN applied to the  $|X_i|$ 's,

$$\frac{1}{n} \max_{1 \leq k \leq n} |S_k - k\mu| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0, \quad (5.5)$$

by Theorem 1.2,

$$\frac{1}{T_n} \max_{1 \leq k \leq n} |T_k - k\nu| \mathbf{1}_{\{T_n > 0\}} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0, \quad (5.6)$$

by (5.3), (5.4) and Theorem 1.2 applied to the  $|X_i|$ 's.

Then the almost sure convergence of  $\|n^{-1}\zeta_n^{\text{ad}} - \mu I\|_\alpha$  to zero results from (5.2) to (5.6).  $\square$

*Proof of (b)  $\Rightarrow$  (a).* We note that

$$\|n^{-1}\zeta_n^{\text{ad}} - \mu I\|_\alpha = \omega_\alpha(n^{-1}\zeta_n^{\text{ad}} - \mu I, 1) \geq |n^{-1}(\zeta^{\text{ad}}(1) - \zeta^{\text{ad}}(0)) - \mu| = \left| \frac{S_n}{n} - \mu \right|.$$

Then (b) implies that  $n^{-1}S_n$  converges almost surely to  $\mu$ , which gives (a) by the converse part in the classical Kolmogorov-Khintchine strong law of large numbers for i.i.d. random variables.  $\square$

*Proof of (a)  $\Rightarrow$  (c).* Writing  $E_n := \{\Delta_n > \varepsilon\}$ , we have to prove that for every  $\varepsilon > 0$ ,

$$\sum_{n=1}^{\infty} n^{-1} P(E_n) < \infty. \quad (5.7)$$

Recalling that  $\nu = \mathbf{E}|X_1|$  is assumed to be positive, we split

$$P(E_n) = P(E'_n) + P(E''_n),$$

where

$$E'_n := E_n \cap \left\{ \left| \frac{T_n}{n} - \nu \right| \leq \frac{\nu}{2} \right\}, \quad E''_n := E_n \cap \left\{ \left| \frac{T_n}{n} - \nu \right| > \frac{\nu}{2} \right\}.$$

On the event  $E'_n$ , as  $0 < n\nu/2 \leq T_n \leq 3n\nu/2$ , we deduce from (5.2) the following upper bound.

$$\begin{aligned} \Delta_n &\leq \left(\frac{5\nu}{2}\right)^\alpha \left(\frac{2}{n} \max_{1 \leq k \leq n} |S_k - k\mu| + \frac{8}{n} \max_{1 \leq k \leq n} |T_k - k\nu|\right)^{1-\alpha} \\ &\leq a \left(\frac{1}{n} \max_{1 \leq k \leq n} |S_k - k\mu|\right)^{1-\alpha} + b \left(\frac{1}{n} \max_{1 \leq k \leq n} |T_k - k\nu|\right)^{1-\alpha}, \end{aligned}$$

with  $a = 2(5\nu)^\alpha$  and  $b = 2^{3-4\alpha}(5\nu)^\alpha$ .

It follows that

$$\begin{aligned} P(E'_n) &\leq P\left(\left(\frac{1}{n} \max_{1 \leq k \leq n} |S_k - k\mu|\right)^{1-\alpha} > \frac{\varepsilon}{2a}\right) + P\left(\left(\frac{1}{n} \max_{1 \leq k \leq n} |T_k - k\nu|\right)^{1-\alpha} > \frac{\varepsilon}{2b}\right) \\ &= P\left(\max_{1 \leq k \leq n} |S_k - k\mu| > n\varepsilon_a\right) + P\left(\max_{1 \leq k \leq n} |T_k - k\nu| > n\varepsilon_b\right) \end{aligned}$$

where  $\varepsilon_a = (\varepsilon/2a)^{1/(1-\alpha)}$  and  $\varepsilon_b = (\varepsilon/2b)^{1/(1-\alpha)}$ . Therefore

$$\sum_{n=1}^{\infty} n^{-1} P(E'_n) < \infty, \quad (5.8)$$

by Theorem 1.8 (c'') applied to the sequences  $(X_i)_{i \geq 1}$  and  $(|X_i|)_{i \geq 1}$ .

Next we note that

$$\sum_{n=1}^{\infty} n^{-1} P(E''_n) \leq \sum_{n=1}^{\infty} n^{-1} P\left(|T_n - n\nu| > \frac{\nu}{2}n\right) < \infty, \quad (5.9)$$

by Theorem 1.8 (c'') applied to the random variables  $|X_i|$ .

Gathering (5.8) and (5.9) gives (5.7), establishing (c).  $\square$

*Proof of (c)  $\Rightarrow$  (a).* As already observed above,

$$\|n^{-1}\zeta_n^{\text{ad}} - \mu I\|_\alpha \geq \left|\frac{S_n}{n} - \mu\right|.$$

Then (c) implies that for every  $\varepsilon > 0$ ,

$$\sum_{n=1}^{\infty} n^{-1} P(|S_n - n\nu| > n\varepsilon) < \infty,$$

whence (a) follows by the part (c'')  $\Rightarrow$  (a) in Theorem 1.8.  $\square$

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DEPARTMENT OF MATHEMATICS AND INFORMATICS, VILNIUS UNIVERSITY, NAUGARDUKO 24, LT-2006 VILNIUS, LITHUANIA

LABORATOIRE P. PAINLEVÉ (UMR 8524 CNRS), BÂT. M2 U.F.R. DE MATHÉMATIQUES, UNIVERSITÉ LILLE 1 SCIENCES ET TECHNOLOGIES, F-59655 VILLENEUVE D'ASCQ CEDEX FRANCE

*E-mail address:* `alfredas.rackauskas@maf.vu.lt`, `Charles.Suquet@univ-lille1.fr`

*URL:* <http://math.univ-lille1.fr/~suquet/>