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FUNCTIONAL LAWS OF LARGE NUMBERS IN HÖLDER SPACES

ALFREDAS RAČKAUSKAS AND CHARLES SUQUET

ABSTRACT. Let $S_n = X_1 + \dots + X_n$, $n \geq 1$, where $(X_i)_{i \geq 1}$ are random variables. Let μ be a constant and I be the identity function on $[0, 1]$. We study the almost sure convergence to μI of the two polygonal line partial sums processes ζ_n and ζ_n^{ad} with respective vertices $(k/n, S_k)$ and (τ_k, S_k) , $0 \leq k \leq n$, where $\tau_k = T_k/T_n$ and $T_k = |X_1| + \dots + |X_k|$. These convergences are considered in the space $C[0, 1]$ or in the Hölder spaces $\mathcal{H}_\alpha^o[0, 1]$, $0 \leq \alpha < 1$. In $C[0, 1]$, any strong law of large numbers satisfied by S_n is inherited by ζ_n . In $\mathcal{H}_\alpha^o[0, 1]$, assuming moreover that the X_i 's are i.i.d., $n^{-1}\zeta_n$ converges almost surely to μI if and only if $\mathbf{E}|X_1|^{1/(1-\alpha)} < \infty$ and $\mu = \mathbf{E}X_1$. In contrast, the same convergence for ζ_n^{ad} is equivalent to $\mathbf{E}|X_1| < \infty$ and $\mu = \mathbf{E}X_1$.

1. INTRODUCTION AND MAIN RESULTS

On the same probability space (Ω, \mathcal{F}, P) , let us consider a sequence of real valued random variables $(X_i)_{i \geq 1}$ together with its partial sums $(S_n)_{n \geq 0}$

$$S_0 = 0, \quad S_n = X_1 + \dots + X_n, \quad n \geq 1,$$

and its polygonal line partial sums processes $(\zeta_n)_{n \geq 1}$, where

$$\zeta_n(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1}, \quad t \in [0, 1].$$

In the case when the X_i 's are i.i.d., limit theorems establish a strong relationship between the degree of integrability of X_1 and the asymptotic behavior of S_n and of ζ_n . To mention the two most famous examples, it is well known that the almost sure convergence of $n^{-1}S_n$ to some constant μ is equivalent to $\mathbf{E}|X_1| < \infty$ and $\mathbf{E}X_1 = \mu$, by Kolmogorov's strong law of large numbers (SLLN) and its converse; similarly, the convergence in distribution of $n^{-1/2}(S_n - n\mu)$ is equivalent to $\mathbf{E}X_1^2 < \infty$ and $\mathbf{E}X_1 = \mu$, by central limit theorem (CLT) and its converse. One may look at the central limit theorem as a convergence rate result for the strong law of large numbers, showing that a convergence rate better or equal than $n^{-1/2}$ cannot be obtained in the SLLN. Intermediate convergence rates of the form $n^{1/p-1}$, $0 < p < 2$ are obtained from Marcinkiewicz-Zygmund SLLN under the existence of p -th moment.

As for ζ_n , by Donsker-Prohorov theorem or functional central limit theorem (FCLT), the process $n^{1/2}(n^{-1}\zeta_n - \mu I)$ converges in distribution to the Brownian

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motion in the classical space of continuous functions $C[0, 1]$ if and only if $\mathbf{E} X_1^2 < \infty$ and $\mu = \mathbf{E} X_1$, where I is the identity function,

$$I : [0, 1] \rightarrow [0, 1], \quad t \mapsto I(t) = t.$$

When $\mathbf{E} |X_1|^p < \infty$ for some $p > 2$, the convergence of $n^{1/2}(n^{-1}\zeta_n - \mu I)$ can be strengthened in a convergence in some Hölder space $\mathcal{H}_\alpha^o[0, 1]$, giving a FCLT in \mathcal{H}_α^o , see Račkauskas and Suquet (2004) for the precise connection between the degree of integrability of X_1 and the strength of the relevant Hölder topology. Alternatively, one can also modify the construction of ζ_n in an adaptive way to obtain Hölder convergence under mild integrability assumptions, see Račkauskas and Suquet (2001).

It is a natural question then, to ask whether all these functional central limit theorems in $C[0, 1]$ or in Hölder spaces may be viewed as convergence rate results for some corresponding *functional* strong law of large numbers (FSLLN).

Our aim in this contribution is to discuss various functional laws of large numbers for ζ_n or for some adaptive modification, in terms of the degree of integrability of X_1 .

Throughout the paper, $\xrightarrow[n \rightarrow \infty]{\text{a.s.}}$ denotes almost sure convergence and $C[0, 1]$ is the Banach space of continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ endowed with the so-called supremum or uniform norm $\|f\|_\infty = \sup_{t \in [0, 1]} |f(t)|$, $f \in C[0, 1]$.

The simplest law of large numbers for ζ_n reads as follows.

Theorem 1.1. *Assume that the X_i 's are i.i.d. Then the convergence*

$$\frac{1}{n} \zeta_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \mu I \quad \text{in the space } C[0, 1] \quad (1.1)$$

holds if and only if $\mathbf{E} |X_1| < \infty$ and $\mu = \mathbf{E} X_1$.

Since the supremum norm of a polygonal line is reached at some vertex, the above functional strong law of large numbers for ζ_n can be viewed as an uniform law of large numbers for the partial sums as follows.

Theorem 1.2. *Assume that the X_i 's are i.i.d. Then the convergence*

$$\frac{1}{n} \max_{1 \leq k \leq n} |S_k - k\mu| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0, \quad (1.2)$$

holds if and only if $\mathbf{E} |X_1| < \infty$ and $\mu = \mathbf{E} X_1$.

As a matter of fact, functional strong law of large numbers in $C[0, 1]$ are easily inherited from the corresponding strong law for S_n , according to the following result whose proof is a simple exercise in analysis. It is worth noticing that no direct assumption on the dependence structure of the X_i 's is made here.

Theorem 1.3. *Let X_i be random variables with an arbitrary dependence structure. Let μ be a real number and $(b_n)_{n \geq 1}$ be a non-decreasing sequence of positive numbers going to infinity. Then the following three convergences are equivalent.*

- i) $b_n^{-1}(S_n - n\mu) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0$.
- ii) $nb_n^{-1} \|n^{-1}\zeta_n - \mu I\|_\infty \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0$.
- iii) $b_n^{-1} \max_{1 \leq k \leq n} |S_k - k\mu| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0$.

Combining classical Marcinkiewicz-Zygmund SLLN with Theorem 1.3 gives the following functional Marcinkiewicz-Zygmund strong law of large numbers for ζ_n (whose special case $p = 1$ is equivalent to Theorem 1.1).

Theorem 1.4. *Assume that the X_i 's are i.i.d. Then the following statements hold true.*

- a) *If $\mathbf{E}|X_1|^p < \infty$ for some $p \in (0, 1)$, then $n^{-1/p}\|\zeta_n\|_\infty$ goes to zero almost surely.*
- b) *If $\mathbf{E}|X_1|^p < \infty$ for some $p \in [1, 2)$, then $n^{1-1/p}\|n^{-1}\zeta_n - (\mathbf{E}X_1)I\|_\infty$ goes to zero almost surely.*
- c) *If $n^{-1/p}\|\zeta_n - c_n I\|_\infty$ goes to zero almost surely for some $p \in (0, 2)$ and some sequence $(c_n)_{n \geq 1}$ of real numbers, then $\mathbf{E}|X_1|^p < \infty$.*

Remark 1.5. There is a large literature on the Marcinkiewicz-Zygmund strong law of large numbers for dependent variables, see e.g. Rio (1995), Fazekas and Klesov (2000), Fazekas (2006) and the references therein. From these results, FSLLN in $C[0, 1]$ for dependent variables are easily inherited via Theorem 1.3.

Remark 1.6. When $\mu = 0$, any FSLLN of the form *ii*) in Theorem 1.3 verified by ζ_n in $C[0, 1]$ is satisfied also by the polygonal line ξ_n with vertices $(\tau_{n,k}, S_k)$, $0 \leq k \leq n$ where the $\tau_{n,k}$ are deterministic or random in $[0, 1]$, with $\min_{0 \leq k \leq n} \tau_{n,k} = 0$, $\max_{0 \leq k \leq n} \tau_{n,k} = 1$ and $\tau_{n,k} \neq \tau_{n,j}$ every time $S_k \neq S_j$. This results from *iii*) in Theorem 1.3. When ζ_n satisfies *ii*) with $\mu \neq 0$, ξ_n satisfies the same functional law of large numbers if and only if $b_n^{-1} \max_{0 \leq k \leq n} |k - n\tau_{n,k}|$ converges almost surely to zero.

Intuitively, the more concentrated is the distribution of X_1 (in the i.i.d. case), the closer to μI should be the paths of $n^{-1}\zeta_n$. In the functional framework of $C[0, 1]$ this closeness is expressed by a convergence rate in the uniform norm, and there is no gain in assuming that $\mathbf{E}|X_1|^p < \infty$ for $p > 2$. This can lead us to look for a different closeness “in shape”, more sensitive to the degree of integrability of X_1 , by considering stronger norms than the uniform one. This question is natural since both functions ζ_n and μI have a much stronger global regularity than the continuity.

For $\alpha \in [0, 1)$ we consider the Hölderian modulus of smoothness of a function $f : [0, 1] \rightarrow \mathbb{R}$ defined by

$$\omega_\alpha(f, \delta) := \sup_{\substack{s, t \in [0, 1] \\ 0 < t - s < \delta}} \frac{|f(t) - f(s)|}{|t - s|^\alpha}, \quad \delta \in (0, 1).$$

The Hölder space $\mathcal{H}_\alpha^\circ[0, 1]$ is then the set of functions $f \in C[0, 1]$ such that $\omega_\alpha(f, \delta)$ converges to zero as δ goes to zero, endowed with the norm $\|f\|_\alpha = |f(0)| + \omega_\alpha(f, 1)$. The following result gives a characterization of the functional strong law of large numbers for ζ_n in the ladder of spaces $\mathcal{H}_\alpha^\circ[0, 1]$, $0 \leq \alpha < 1$. As the space $\mathcal{H}_0^\circ[0, 1]$ is isomorphic to $C[0, 1]$, the special case $\alpha = 0$ is equivalent to Theorem 1.1.

Theorem 1.7. *Let $0 \leq \alpha < 1$. When the X_i 's are i.i.d., the following statements are equivalent:*

- a) $\mathbf{E}|X_1|^{1/(1-\alpha)} < \infty$ and $\mu = \mathbf{E}X_1$;
- b) $n^{-1}\zeta_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \mu I$ in the space $\mathcal{H}_\alpha^\circ[0, 1]$;
- c) for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P(\|\zeta_n - n\mu I\|_\alpha > \varepsilon n) < \infty.$$

In terms of increments of partial sums, Theorem 1.7 can be stated in the following form.

Theorem 1.8. *Let $0 \leq \alpha < 1$. Then (a) of Theorem 1.7 is also equivalent with each of the following statements:*

$$b') \quad n^{-1+\alpha} \max_{0 \leq j < k \leq n} \frac{|S_k - S_j - \mu(k-j)|}{(k-j)^\alpha} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0;$$

c') for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\sup_{0 \leq j < k \leq n} \frac{|S_k - S_j - \mu(k-j)|}{(k-j)^\alpha} > \varepsilon n^{1-\alpha}\right) < \infty.$$

In the special case where $\alpha = 0$, condition b') is equivalent to (1.2), while c') is equivalent to

c'') for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\sup_{0 \leq k \leq n} |S_k - k\mu| > \varepsilon n\right) < \infty.$$

Remark 1.9. Assuming $\mathbf{E}|X_1|^{1/(1-\alpha)} < \infty$ for some $0 \leq \alpha < 1$ and $\mathbf{E}X_1 = 0$ we have particularly

$$\|n^{-1}\zeta_n\|_\alpha \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0 \quad (1.3)$$

and one can ask what is the best possible rate of this convergence. From the classical Donsker-Prohorov invariance principle, we have that if $\mathbf{E}X_1^2 < \infty$, then

$$\sqrt{n}\|n^{-1}\zeta_n\|_\infty \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \|W\|_\infty,$$

where $W = (W_t, t \in [0, 1])$ is a standard Wiener process and $\xrightarrow[n \rightarrow \infty]{\mathcal{D}}$ stands for the convergence in distribution. For $0 < \alpha < 1$ we observe different rates of convergence. For $0 < \alpha < 1/2$ we have from Račkauskas and Suquet (2004) that

$$\sqrt{n}\|n^{-1}\zeta_n\|_\alpha \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \|W\|_\alpha$$

if and only if $\lim_{t \rightarrow \infty} t^{1/(1/2-\alpha)} P(|X_1| > t) = 0$, a condition which is stronger than $\mathbf{E}|X_1|^{1/(1-\alpha)} < \infty$ which in turn is stronger than $\mathbf{E}X_1^2 < \infty$. If $1/2 \leq \alpha < 1$ and the random variables X_k 's are regularly varying with exponent $a > 2$, then we have from Mikosch and Račkauskas (2010) that

$$n^{1-\alpha} a_n^{-1} \|n^{-1}\zeta_n\|_\alpha \xrightarrow[n \rightarrow \infty]{\mathcal{D}} Y, \quad (1.4)$$

where Y has Fréchet distribution with parameter a : $P(Y \leq x) = \exp\{-x^{-a}\}$, $x \geq 0$, and $a_n = \inf\{x \in \mathbb{R} : P(|X_1| \leq x) \geq 1 - 1/n\}$. Since $a_n = n^{1/a} \ell_n$ with slowly varying ℓ_n , the normalization in (1.4) is $n^{1-1/a-\alpha} \ell_n^{-1}$. Let us observe that $1 - 1/a - \alpha < 1/2$, so the rate in (1.3) is slower for $1/2 \leq \alpha < 1$ than for $0 \leq \alpha < 1/2$.

In the Gaussian case the limiting behavior of $\|\zeta_n\|_{1/2}$ has been investigated by Siegmund and Venkatraman (1995) and Kabluchko (2008). It is proved that centered and normed sequence $b_n(\|n^{-1}\zeta_n\|_{1/2} - a_n)$ with appropriately chosen sequences (b_n) and (a_n) converges in distribution to a double exponential random variable. In the non-Gaussian light-tailed case, the limiting distribution of $\|\zeta_n\|_{1/2}$ has been obtained in Kabluchko and Wang (2012).

Theorem 1.7 indicates that the construction of polygonal line process ζ_n is not adapted to the structure of Hölder topology with respect to the law of large numbers. Next we introduce another construction which we call *adaptive*. For this we use the random partition of the interval $[0, 1]$ generated by the points τ_k constructed as follows. For $n \geq 1$, put

$$T_n := \sum_{i=1}^n |X_i|.$$

Then we define the triangular array $\{\tau_{n,k}, 0 \leq k \leq n\}$ by setting $\tau_{n,0} := 0$, $\tau_{n,n} := 1$ and for $1 \leq k < n$,

$$\tau_{n,k} := \begin{cases} \frac{T_k}{T_n} & \text{on the event } \{T_n > 0\} \\ 0 & \text{on the event } \{T_n = 0\}. \end{cases}$$

For notational convenience, we write τ_k for $\tau_{n,k}$ every time there is no ambiguity on which n is involved by τ_k . Now we denote by ζ_n^{ad} the random polygonal line process with vertices the points $V_k := (\tau_k, S_k)$. To express this in explicit formulas, we can start with:

$$\zeta_n^{\text{ad}}(\tau_k) = S_k, \quad 0 \leq k \leq n.$$

We note in passing that this writing is consistent, since on the event $\{\tau_k = \tau_{k-1}\}$, $X_k = 0$ and $S_{k-1} = S_k$. Next we remark that for every $t \in [0, 1] \setminus \{\tau_k, 0 \leq k \leq n\}$, there is a unique (random) integer $1 \leq j \leq n$ such that $\tau_{j-1} < t < \tau_j$ and then

$$\zeta_n^{\text{ad}}(t) = \frac{\tau_j - t}{\tau_j - \tau_{j-1}} S_{j-1} + \frac{t - \tau_{j-1}}{\tau_j - \tau_{j-1}} S_j = S_{j-1} + \frac{t - \tau_{j-1}}{\tau_j - \tau_{j-1}} X_j.$$

It is worth noticing here that, due to the direct definition of $\tau_0 = 0$ and $\tau_n = 1$, the above formulas are still valid on the event $\{T_n = 0\}$, with $j = n$.

Before stating the functional law of large numbers for ζ_n^{ad} , one can get an intuition by a rough comparison of ζ_n and ζ_n^{ad} . For the first process, the increment $S_k - S_{k-1}$ ($1 \leq k \leq n$) is realized in the deterministic interval $[(k-1)/n, k/n]$ with length $1/n$. For ζ_n^{ad} , the same increment is realized in the random interval $I_k = [\tau_{k-1}, \tau_k]$. In mean, the length $|I_k| = \tau_k - \tau_{k-1}$ is asymptotically equivalent to $1/n$ (or equal if $P(X_1 = 0) = 0$). Indeed it is easily seen that $\mathbf{E}|I_k| = P(T_n > 0)/n$ for $1 \leq k < n$ while $\mathbf{E}|I_n| = P(T_n > 0)/n + P(T_n = 0)$. As the X_k 's are i.i.d., $P(T_n = 0) = P(X_1 = 0)^n$ so, discarding the degenerated case where $P(X_1 = 0) = 1$, each $|I_k|$ is equivalent to $1/n$.

This apparent similarity between ζ_n and ζ_n^{ad} may be misleading when working with Hölder topologies. In this context, what matters is the slope of the polygonal line between two consecutive vertices. Contrary to a fixed interval of length $1/n$, I_k reacts to a value of X_k which can be big with big probability if $\mathbf{E}|X_1|^{1/(1-\alpha)} = \infty$. Roughly speaking, for the same big increment $S_k - S_{k-1}$, the slope is weaker for ζ_n^{ad} than for ζ_n . This is the key of the better Hölderian behavior of ζ_n^{ad} as illustrated by the following result.

Theorem 1.10. *Let $0 \leq \alpha < 1$. The following statements are equivalent*

- a) $\mathbf{E}|X_1| < \infty$ and $\mathbf{E}X_1 = \mu$;
- b) $n^{-1}\zeta_n^{\text{ad}} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \mu I$ in the space $\mathcal{H}_\alpha^o[0, 1]$;

c) for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P(\|\zeta_n^{\text{ad}} - n\mu I\|_{\alpha} > \varepsilon n) < \infty.$$

2. FSLLN IN $C[0, 1]$

Since Theorems 1.1 and 1.2 are contained in Theorem 1.4, we need only to prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Observing that the supremum norm of a polygonal line is reached at some vertex gives immediately the equivalence between conditions *ii*) and *iii*) in Theorem 1.3. Obviously *iii*) implies *i*). The proof of the implication *i*) \Rightarrow *ii*) is purely analytical and is provided by the following lemma. \square

Lemma 2.1. *Let μ be a real number and $(b_n)_{n \geq 1}$ be a non-decreasing sequence of positive numbers going to infinity. Let $(s_n)_{n \geq 1}$ be a sequence of real numbers such that $b_n^{-1}(s_n - n\mu)$ converges to zero. Then*

$$\max_{1 \leq k \leq n} \frac{|s_k - k\mu|}{b_n} \xrightarrow{n \rightarrow \infty} 0. \quad (2.1)$$

Proof. Let ε be an arbitrary positive number. By the convergence to zero of $b_n^{-1}(s_n - n\mu)$, there is some integer $k_0 = k_0(\varepsilon)$ such that for every $k \geq k_0$, $b_k^{-1}|s_k - k\mu| < \varepsilon$. Together with the non-decreasingness of the sequence $(b_n)_{n \geq 1}$, this leads to the upper bound

$$\begin{aligned} \max_{1 \leq k \leq n} \frac{|s_k - k\mu|}{b_n} &\leq \frac{1}{b_n} \max_{k < k_0} |s_k - k\mu| + \max_{k_0 \leq k \leq n} \frac{|s_k - k\mu|}{b_k} \frac{b_k}{b_n} \\ &\leq \frac{1}{b_n} \max_{k < k_0} |s_k - k\mu| + \varepsilon. \end{aligned}$$

Since b_n goes to infinity, it follows that

$$\limsup_{n \rightarrow \infty} \max_{1 \leq k \leq n} \frac{|s_k - k\mu|}{b_n} \leq \varepsilon,$$

which gives (2.1) by arbitrariness of ε . \square

Proof of Theorem 1.4. Let $p \in (0, 2)$. From the classical Marcinkiewicz-Zygmund SLLN, we know that if $\mathbf{E}|X_1|^p < \infty$, then $n^{-1/p}S_n$ goes a.s. to zero in the case $0 < p < 1$, while $n^{-1/p}(S_n - n\mathbf{E}X_1)$ goes a.s. to zero in the case $1 \leq p < 2$. Accounting the equivalence of *i*) and *ii*) in Theorem 1.3, this immediately gives the statements *a*) by choosing $\mu = 0$ and *b*) with $\mu = \mathbf{E}X_1$.

By the converse part in classical Marcinkiewicz-Zygmund SLLN, if $n^{-1/p}(S_n - c_n)$ goes to zero almost surely for some $p \in (0, 2)$ and some sequence $(c_n)_{n \geq 1}$ of real numbers, then $\mathbf{E}|X_1|^p < \infty$. Together with Theorem 1.3, this gives the statement *c*), noting that $\|\zeta_n - c_n I\|_{\infty} \geq |\zeta_n(1) - c_n I(1)| = |S_n - c_n|$. \square

3. SOME HÖLDERIAN TOOLS

The Hölder norm of a polygonal line function is very easy to compute according to the following lemma for which we refer e.g. to Markevičiūtė et al. (2012) Lemma 3, where it is proved in a more general setting.

Lemma 3.1. *Let $t_0 = 0 < t_1 < \dots < t_n = 1$ be a partition of $[0, 1]$ and f be a real-valued polygonal line function on $[0, 1]$ with vertices at the t_i 's, i.e. f is continuous on $[0, 1]$ and its restriction to each interval $[t_i, t_{i+1}]$ is an affine function. Then for any $0 \leq \alpha < 1$,*

$$\sup_{0 \leq s < t \leq 1} \frac{|f(t) - f(s)|}{(t - s)^\alpha} = \max_{0 \leq i < j \leq n} \frac{|f(t_j) - f(t_i)|}{(t_j - t_i)^\alpha}.$$

Let D_j denotes the set of dyadic numbers of level j in $[0, 1]$, that is $D_0 := \{0, 1\}$ and for $j \geq 1$, $D_j := \{(2l - 1)2^{-j}; 1 \leq l \leq 2^{j-1}\}$. For $r \in D_j$ set $r^- := r - 2^{-j}$, $r^+ := r + 2^{-j}$, $j \geq 0$. For $f : [0, 1] \rightarrow \mathbb{R}$ and $r \in D_j$ let us define

$$\lambda_r(f) := \begin{cases} f(r^+) + f(r^-) - 2f(r) & \text{if } j \geq 1, \\ f(r) & \text{if } j = 0. \end{cases}$$

The following sequential norm defined on $\mathcal{H}_\alpha^o[0, 1]$ by

$$\|f\|_\alpha^{\text{seq}} := \sup_{j \geq 0} 2^{\alpha j} \max_{r \in D_j} |\lambda_r(f)|,$$

is equivalent to the natural norm $\|f\|_\alpha$, see [Ciesielski \(1960\)](#). Let us define also $\mathbb{D}_j := \{k2^{-j}, 0 \leq k < 2^j\}$, so that $\mathbb{D}_j = \{0\} \cup \bigcup_{1 \leq i \leq j} D_i$. In what follows, we denote by \log the logarithm with basis 2 ($\log 2 = 1$).

Lemma 3.2. *For $0 \leq \alpha < 1$,*

$$\|\zeta_n - \mu I\|_\alpha^{\text{seq}} \leq 2 \max_{0 \leq j \leq \log n} 2^{\alpha j} \max_{r \in \mathbb{D}_j} \left| \sum_{nr < i \leq n(r+2^{-j})} (X_i - \mu) \right| + 4n^\alpha \max_{1 \leq i \leq n} |X_i - \mu|.$$

Proof. First we remark that for $j \geq 1$,

$$\max_{r \in D_j} |\lambda_r(f)| \leq \max_{r \in D_j} |f(r^+) - f(r)| + \max_{r \in D_j} |f(r) - f(r^-)|.$$

As r^+ and r^- belong to \mathbb{D}_j , this gives:

$$\sup_{j \geq 1} 2^{\alpha j} \max_{r \in D_j} |\lambda_r(f)| \leq 2 \sup_{j \geq 1} 2^{\alpha j} \max_{r \in \mathbb{D}_j} |f(r + 2^{-j}) - f(r)|.$$

It follows that if $f \in \mathcal{H}_\alpha^o[0, 1]$ and $f(0) = 0$,

$$\|f\|_\alpha^{\text{seq}} \leq 2 \sup_{j \geq 0} 2^{\alpha j} \max_{r \in \mathbb{D}_j} |f(r + 2^{-j}) - f(r)|.$$

This inequality can be applied to the random polygonal line $\zeta_n - \mu I$. Moreover, it is clear that there is no loss of generality in assuming $\mu = 0$ for notational simplicity. We claim that for $r \in \mathbb{D}_j$,

$$|\zeta_n(r + 2^{-j}) - \zeta_n(r)| \leq \begin{cases} |S_{[nr+n2^{-j}]} - S_{[nr]}| + 2 \max_{1 \leq i \leq n} |X_i| & \text{if } j \leq \log n, \\ 2n2^{-j} \max_{1 \leq i \leq n} |X_i| & \text{if } j > \log n. \end{cases}$$

Indeed, if $j \leq \log n$, this follows immediately by triangular inequality. If $j > \log n$, $2^{-j} < 1/n$ and then with r in say $[i/n, (i+1)/n]$, either $r + 2^{-j}$ is in $(i/n, (i+1)/n]$ or belongs to $((i+1)/n, (i+2)/n]$. In the first case, noting that the slope of ζ_n on $[i/n, (i+1)/n]$ is exactly nX_{i+1} , we have

$$|\zeta_n(r + 2^{-j}) - \zeta_n(r)| = n|X_{i+1}|2^{-j} \leq 2^{-j}n \max_{1 \leq i \leq n} |X_i|.$$

If r and $r + 2^{-j}$ are in consecutive intervals, the same argument applies after chaining, so

$$\begin{aligned} |\zeta_n(r + 2^{-j}) - \zeta_n(r)| &\leq |\zeta_n(r) - \zeta_n((i+1)/n)| + |\zeta_n((i+1)/n) - \zeta_n(r + 2^{-j})| \\ &\leq 2^{-j+1} n \max_{1 \leq i \leq n} |X_i|. \end{aligned}$$

To complete the proof of the lemma, it remains to note that if $j \leq \log n$, $2^{j\alpha} \leq n^\alpha$ and if $j > \log n$, $2^{j\alpha} 2^{-j} n = (2^{-j})^{1-\alpha} n \leq (n^{-1})^{1-\alpha} n = n^\alpha$. \square

4. PROOF OF THEOREMS 1.7 AND 1.8

Theorem 1.8 follows immediately from Theorem 1.7 because the Hölder norm of a polygonal line is reached at two vertices (Lemma 3.1). We shall prove Theorem 1.7 following the scheme:

$$(a) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a).$$

Proof of (a) \Rightarrow (c). Since (a) yields $\mathbf{E}|X_1| < \infty$, we can assume without loss of generality that $\mathbf{E}X_1 = 0$. Define for every positive ε ,

$$P_1(n, \varepsilon) := P \left(\max_{0 \leq j \leq \log n} 2^{\alpha j} \max_{r \in \mathbb{D}_j} \left| \sum_{nr < k \leq n(r+2^{-j})} X_k \right| > \varepsilon n \right)$$

and

$$P_2(n, \varepsilon) := P \left(n^{-1+\alpha} \max_{1 \leq k \leq n} |X_k| > \varepsilon \right).$$

According to Lemma 3.2 it is enough to prove for $i = 1, 2$ and for each $\varepsilon > 0$ that

$$\sum_{n=1}^{\infty} n^{-1} P_i(n, \varepsilon) < \infty. \quad (4.1)$$

First let us check (4.1) with $i = 2$. Since $P_2(n, \varepsilon) \leq \sum_{k=1}^n P(|X_k| > \varepsilon n^{1-\alpha})$ and the X_k 's are identically distributed, we get

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1} P_2(n, \varepsilon) &\leq \sum_{n=1}^{\infty} P(|X_1| > \varepsilon n^{1-\alpha}) \leq \sum_{n=1}^{\infty} \int_{n-1}^n P(|X_1| > \varepsilon t^{1-\alpha}) dt \\ &= \int_0^{\infty} P(|X_1| > \varepsilon t^{1-\alpha}) dt \\ &= \int_0^{\infty} P(|X_1|^{1/(1-\alpha)} > \varepsilon^{1/(1-\alpha)} t) dt \\ &= \varepsilon^{-1/(1-\alpha)} \mathbf{E}|X_1|^{1/(1-\alpha)} < \infty, \end{aligned}$$

by (a).

Next we prove (4.1) for $i = 1$. Introducing

$$\begin{aligned} X'_i &= X_i \mathbf{1}_{\{|X_i| \leq n^{1-\alpha}\}} - \mathbf{E}X_i \mathbf{1}_{\{|X_i| \leq n^{1-\alpha}\}}, \\ X''_i &= X_i \mathbf{1}_{\{|X_i| > n^{1-\alpha}\}} - \mathbf{E}X_i \mathbf{1}_{\{|X_i| > n^{1-\alpha}\}}, \end{aligned}$$

and recalling that $\mathbf{E}X_k = 0$, we decompose X_i in the sum $X'_i + X''_i$ of two truncated and centered random variables. We split $P_1(n, \varepsilon) \leq P'_1(n, \varepsilon/2) + P''_1(n, \varepsilon/2)$ where in the definition of $P_1(n, \varepsilon)$ one has to substitute respectively X_i by X'_i to define $P'_1(n, \varepsilon)$ and by X''_i to define $P''_1(n, \varepsilon)$.

First we estimate $P_1''(n, \varepsilon)$. By Markov inequality we have

$$\begin{aligned} P_1''(n, \varepsilon) &\leq \varepsilon^{-1} n^{-1} \sum_{1 \leq j \leq \log n} 2^{\alpha j} \mathbf{E} \max_{r \in \mathbb{D}_j} \left| \sum_{nr < k \leq n(r+2^{-j})} X_k'' \right| \\ &\leq \varepsilon^{-1} \mathbf{E} |X_1''| \sum_{j=0}^{\log n} 2^{\alpha j} \\ &\leq \frac{2^\alpha n^\alpha}{\varepsilon(2^\alpha - 1)} \mathbf{E} |X_1''|. \end{aligned} \quad (4.2)$$

Next we note that $\mathbf{E} |X_1''| \leq 2\mathbf{E} |X_1| \mathbf{1}_{\{|X_1| > n^{1-\alpha}\}}$ and

$$\begin{aligned} \mathbf{E} |X_1| \mathbf{1}_{\{|X_1| > n^{1-\alpha}\}} &= \int_0^\infty P(|X_1| \mathbf{1}_{\{|X_1| > n^{1-\alpha}\}} > s) ds \\ &= \int_0^{n^{1-\alpha}} P(|X_1| \mathbf{1}_{\{|X_1| > n^{1-\alpha}\}} > s) ds + \int_{n^{1-\alpha}}^\infty P(|X_1| > s) ds \\ &= n^{1-\alpha} P(|X_1| > n^{1-\alpha}) + \int_{n^{1-\alpha}}^\infty P(|X_1| > s) ds. \end{aligned} \quad (4.3)$$

Now from (4.2) and (4.3), we obtain

$$\sum_{n=1}^\infty n^{-1} P_1''(n, \varepsilon) \leq \frac{c(\alpha)}{\varepsilon} \left(\sum_{n=1}^\infty P(|X_1| > n^{1-\alpha}) + \sum_{n=1}^\infty n^{-1+\alpha} \int_{n^{1-\alpha}}^\infty P(|X_1| > s) ds \right).$$

The first series in the right hand side converges since

$$\sum_{n=1}^\infty P(|X_1| > n^{1-\alpha}) \leq \mathbf{E} |X_1|^{1/(1-\alpha)},$$

as already seen above when bounding $\sum_{n=1}^\infty n^{-1} P_2(n, \varepsilon)$. For the second series, we can write

$$\begin{aligned} \sum_{n=1}^\infty n^{-1+\alpha} \int_{n^{1-\alpha}}^\infty P(|X_1| > s) ds &= \sum_{n=1}^\infty n^{-1+\alpha} \sum_{k=n}^\infty \int_{k^{1-\alpha}}^{(k+1)^{1-\alpha}} P(|X_1| > s) ds \\ &= \sum_{k=1}^\infty \int_{k^{1-\alpha}}^{(k+1)^{1-\alpha}} P(|X_1| > s) ds \sum_{1 \leq n \leq k} n^{-1+\alpha} \\ &\leq c'(\alpha) \sum_{k=1}^\infty \int_{k^{1-\alpha}}^{(k+1)^{1-\alpha}} k^\alpha P(|X_1| > s) ds. \end{aligned}$$

Noting that $k^\alpha = (k^{1-\alpha})^{\alpha/(1-\alpha)} \leq s^{\alpha/(1-\alpha)}$ for every $s \geq k^{1-\alpha}$, we get

$$\begin{aligned} \sum_{n=1}^\infty n^{-1+\alpha} \int_{n^{1-\alpha}}^\infty P(|X_1| > s) ds &\leq c'(\alpha) \sum_{k=1}^\infty \int_{k^{1-\alpha}}^{(k+1)^{1-\alpha}} s^{\alpha/(1-\alpha)} P(|X_1| > s) ds \\ &\leq c'(\alpha) \int_0^\infty s^{\alpha/(1-\alpha)} P(|X_1| > s) ds \\ &= \frac{c'(\alpha)(1-\alpha)}{\alpha} \mathbf{E} |X_1|^{1/(1-\alpha)} < \infty. \end{aligned}$$

This achieves the verification of the convergence of $\sum_{n=1}^\infty n^{-1} P_1''(n, \varepsilon)$.

Now consider the series $\sum_{n=1}^{\infty} n^{-1} P'_1(n, \varepsilon)$. We claim that one can find some real $q > 1/(1 - \alpha)$ and some constant $c(\alpha, q)$ such that

$$P'_1(n, \varepsilon) \leq \frac{c(\alpha, q)}{\varepsilon^q} n^{1-q(1-\alpha)} \mathbf{E} |X'_1|^q. \quad (4.4)$$

Postponing the verification of (4.4), let us see how it gives the convergence of $\sum_{n=1}^{\infty} n^{-1} P'_1(n, \varepsilon)$.

First we note that $\mathbf{E} |X'_1|^q \leq 2^q \mathbf{E} |X_1|^q \mathbf{1}_{\{|X_i| \leq n^{1-\alpha}\}}$ and

$$\begin{aligned} \mathbf{E} |X_1|^q \mathbf{1}_{\{|X_i| \leq n^{1-\alpha}\}} &= \int_0^{\infty} P(|X_1|^q \mathbf{1}_{\{|X_i| \leq n^{1-\alpha}\}} > t) dt \\ &= \int_0^{n^{q(1-\alpha)}} P(|X_1|^q \mathbf{1}_{\{|X_i| \leq n^{1-\alpha}\}} > t) dt \\ &= \int_0^{n^{q(1-\alpha)}} P(n^{q(1-\alpha)} \geq |X_1|^q > t) dt \\ &\leq \int_0^{n^{q(1-\alpha)}} P(|X_1|^q > t) dt = \int_0^{n^{1-\alpha}} q s^{q-1} P(|X_1| > s) ds. \end{aligned}$$

Combining this estimate with (4.4) reduces the problem to the convergence of the series

$$\Sigma(q, \alpha) := \sum_{n=1}^{\infty} n^{-q(1-\alpha)} \int_0^{n^{1-\alpha}} s^{q-1} P(|X_1| > s) ds. \quad (4.5)$$

For this aim, it is convenient to exchange the summations as follows:

$$\begin{aligned} \Sigma(q, \alpha) &= \sum_{n=1}^{\infty} n^{-q(1-\alpha)} \sum_{k=1}^n \int_{(k-1)^{(1-\alpha)}}^{k^{(1-\alpha)}} s^{q-1} P(|X_1| > s) ds \\ &= \sum_{k=1}^{\infty} \int_{(k-1)^{(1-\alpha)}}^{k^{(1-\alpha)}} s^{q-1} P(|X_1| > s) ds \sum_{n=k}^{\infty} n^{-q(1-\alpha)} \\ &\leq c'(q, \alpha) \sum_{k=1}^{\infty} \int_{(k-1)^{(1-\alpha)}}^{k^{(1-\alpha)}} k^{1-q(1-\alpha)} s^{q-1} P(|X_1| > s) ds. \end{aligned}$$

Now we note that in the last integral above, $s \leq k^{(1-\alpha)}$, so $k \geq s^{1/(1-\alpha)}$ and as $1 - q(1 - \alpha) < 0$, $k^{1-q(1-\alpha)} \leq (s^{1/(1-\alpha)})^{1-q(1-\alpha)} = s^{1/(1-\alpha)-q}$ and finally $k^{1-q(1-\alpha)} s^{q-1} \leq s^{1/(1-\alpha)-1}$. This leads to

$$\Sigma(q, \alpha) \leq c'(q, \alpha) \int_0^{\infty} s^{1/(1-\alpha)-1} P(|X_1| > s) ds = c''(q, \alpha) \mathbf{E} |X_1|^{1/(1-\alpha)} < \infty.$$

To complete the proof, it only remains to check (4.4). In the case where $0 \leq \alpha < 1/2$, we can choose $q = 2$ and then (4.4) is obtained by Chebyshev inequality:

$$\begin{aligned}
P'_1(n, \varepsilon) &\leq \sum_{j=0}^{\log n} \sum_{r \in \mathbb{D}_j} P \left(\left| \sum_{nr < k \leq n(r+2^{-j})} X'_k \right| > \varepsilon n 2^{-j\alpha} \right) \\
&\leq \frac{1}{n^2 \varepsilon^2} \sum_{j=0}^{\log n} 2^{2j\alpha} \sum_{r \in \mathbb{D}_j} \mathbf{E} \left| \sum_{nr < k \leq n(r+2^{-j})} X'_k \right|^2 \\
&= \frac{1}{n^2 \varepsilon^2} \sum_{j=0}^{\log n} 2^{2j\alpha} 2^j \mathbf{E} X_1'^2 \\
&\leq \frac{2^{1+2\alpha}}{\varepsilon^2 (2^{1+2\alpha} - 1)} n^{-1+2\alpha} \mathbf{E} X_1'^2.
\end{aligned}$$

If $\alpha \geq 1/2$, then by Markov and Rosenthal (1970) inequalities we obtain for any $q > 1/(1 - \alpha) \geq 2$,

$$\begin{aligned}
P'_1(n, \varepsilon) &\leq \frac{1}{n^q \varepsilon^q} \sum_{j=0}^{\log n} 2^{jq\alpha} \sum_{r \in \mathbb{D}_j} \mathbf{E} \left| \sum_{nr < k \leq n(r+2^{-j})} X'_k \right|^q \\
&\leq \frac{C_q}{n^q \varepsilon^q} \sum_{j=0}^{\log n} 2^{jq\alpha} 2^j \left((n 2^{-j} \mathbf{E} (X_1')^2)^{q/2} + n 2^{-j} \mathbf{E} |X_1'|^q \right),
\end{aligned}$$

denoting by C_q the universal constant in Rosenthal inequality. Since $q > 2$, $(\mathbf{E} (X_1')^2)^{q/2} = ((\mathbf{E} |X_1'|^2)^{1/2})^q \leq ((\mathbf{E} |X_1'|^q)^{1/q})^q = \mathbf{E} |X_1'|^q$. Moreover, in the range of summation, $n 2^{-j} \geq 1$ and as $q/2 > 1$, $(n 2^{-j})^{q/2} \geq n 2^{-j}$. This gives

$$P'_1(n, \varepsilon) \leq \frac{2C_q \mathbf{E} |X_1'|^q}{\varepsilon^q n^{q/2}} \sum_{j=0}^{\log n} 2^{j(q(\alpha-1/2)+1)} \leq \frac{4C_q}{(2^{q(\alpha-1/2)+1} - 1) \varepsilon^q} n^{1-q(1-\alpha)} \mathbf{E} |X_1'|^q,$$

so (4.4) is verified. \square

Proof of (c) \Rightarrow (b). Since $\zeta_n(0) = 0$, we have

$$\|\zeta_n - n\mu I\|_\alpha = \omega_\alpha(\zeta_n - n\mu I, 1) = \sup_{0 \leq s < t \leq 1} \frac{|\zeta_n(t) - \zeta_n(s) - n\mu(t-s)|}{|t-s|^\alpha}.$$

Applying Lemma 3.1 to the polygonal line $\zeta_n - \mu I$ gives

$$\|\zeta_n - \mu I\|_\alpha = n^\alpha \max_{1 \leq i < j \leq n} \frac{|S_j - S_i - (j-i)\mu|}{|j-i|^\alpha}. \quad (4.6)$$

From (4.6), it is clear that the sequence $(Z_n)_{n \geq 1}$ defined by

$$Z_n = n^{-\alpha} \|\zeta_n - n\mu I\|_\alpha, \quad n \geq 1,$$

is non-decreasing. Now by (c) we have for every positive ε ,

$$\sum_{n=1}^{\infty} n^{-1} P(Z_n > \varepsilon n^{1-\alpha}) < \infty.$$

Exploiting the monotonicity of $(Z_n)_{n \geq 1}$, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1} P(Z_n > \varepsilon n^{1-\alpha}) &= \sum_{i=0}^{\infty} \sum_{2^i \leq n < 2^{i+1}} n^{-1} P(Z_n > \varepsilon n^{1-\alpha}) \\ &\geq \sum_{i=0}^{\infty} \sum_{2^i \leq n < 2^{i+1}} 2^{-i-1} P(Z_{2^i} > \varepsilon 2^{(i+1)(1-\alpha)}) \\ &= \sum_{i=0}^{\infty} 2^{-1} P(Z_{2^i} > \varepsilon 2^{(i+1)(1-\alpha)}). \end{aligned}$$

It follows that for every positive ε ,

$$\sum_{i=0}^{\infty} P(Z_{2^i} > \varepsilon 2^{i(1-\alpha)}) < \infty. \quad (4.7)$$

For $N > 1$ let $J \geq 0$ be such that $2^{J-1} \leq N < 2^J$. Then

$$\begin{aligned} P\left(\sup_{n \geq N} \|n^{-1}\zeta_n - \mu I\|_{\alpha} > \varepsilon\right) &\leq P\left(\sup_{n \geq 2^{J-1}} n^{-1+\alpha} Z_n > \varepsilon\right) \\ &= P\left(\sup_{i \geq J} \max_{2^{i-1} \leq n < 2^i} n^{-1+\alpha} Z_n > \varepsilon\right) \\ &\leq P\left(\sup_{i \geq J} 2^{(-1+\alpha)(i-1)} Z_{2^i} > \varepsilon\right) \\ &\leq \sum_{i=J}^{\infty} P(Z_{2^i} > \varepsilon 2^{(1-\alpha)(i-1)}). \end{aligned}$$

This upper bound goes to zero when $J \rightarrow \infty$ by (4.7). Hence $\sup_{n \geq N} \|n^{-1}\zeta_n - \mu I\|_{\alpha}$ converges in probability to zero as $N \rightarrow \infty$. From this it is easily deduced that $\|n^{-1}\zeta_n - \mu I\|_{\alpha}$ converges almost surely to zero as $n \rightarrow \infty$, which gives (b). \square

Proof of (b) \Rightarrow (a). Putting $\zeta'_n := \zeta_n - n\mu I$, we note that

$$n^{\alpha} |X_n - \mu| = \frac{|\zeta'_n(1) - \zeta'_n(1 - 1/n)|}{(\frac{1}{n})^{\alpha}} \leq \|\zeta'_n\|_{\alpha},$$

so (b) implies that

$$n^{-1+\alpha} |X_n - \mu| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0.$$

Then, by independence of the X_n 's, the second Borel-Cantelli lemma yields

$$\sum_{n=1}^{\infty} P(|X_n - \mu| \geq n^{1-\alpha}) < \infty,$$

which by identical distribution of the X_n 's can be recast as

$$\sum_{n=1}^{\infty} P(|X_1 - \mu| \geq n^{1-\alpha}) < \infty,$$

what gives $\mathbf{E}|X_1 - \mu|^{1/(1-\alpha)} < \infty$. It follows that $\mathbf{E}|X_1|^{1/(1-\alpha)} < \infty$ and as $(1-\alpha)^{-1} \geq 1$, $\mathbf{E}|X_1| < \infty$. Finally $n^{-1}S_n = n^{-1}\zeta_n(1)$ converges almost surely to μ by (b) and to $\mathbf{E}X_1$ by the classical strong law of large numbers. Hence $\mu = \mathbf{E}X_1$ and the proof of Theorem 1.7 is complete. \square

5. PROOF OF THEOREM 1.10

As a preliminary remark, it seems worth noticing that the proof of Theorem 1.10 cannot be easily reduced to the case where $\mu = 0$. Indeed the centering which substitutes X_i by $X'_i = X_i - \mu$ changes also the random partition of $[0, 1]$ in another one, built on the X'_i , and it seems difficult to find a simple relationship between the two corresponding polygonal lines.

In what follows, we put

$$\nu := \mathbf{E} |X_1|$$

and discard the trivial case where $\nu = 0$ since then all the X_i 's would be almost surely null.

We shall prove Theorem 1.10 following the scheme:

$$(a) \Rightarrow (b) \Rightarrow (a) \Rightarrow (c) \Rightarrow (a).$$

Proof of (a) \Rightarrow (b). The Hölder norm of the polygonal line ζ_n^{ad} is reached at two vertices. Accounting the possibility of several consecutive equal τ_i 's, this property can be translated here by

$$\begin{aligned} \Delta_n &:= \|n^{-1}\zeta_n^{\text{ad}} - \mu I\|_\alpha \\ &= \frac{1}{n} \max_{0 \leq \tau_j < \tau_k \leq 1} \frac{|S_k - S_j - n\mu(\tau_k - \tau_j)|}{(\tau_k - \tau_j)^\alpha} \mathbf{1}_{\{T_n > 0\}} + \mu \mathbf{1}_{\{T_n = 0\}} \\ &= \frac{T_n^\alpha}{n} \max_{0 \leq \tau_j < \tau_k \leq 1} \frac{|S_k - S_j - \frac{n\mu}{T_n}(T_k - T_j)|}{(T_k - T_j)^\alpha} \mathbf{1}_{\{T_n > 0\}} + \mu \mathbf{1}_{\{T_n = 0\}}. \end{aligned} \quad (5.1)$$

By triangle inequality, on the event $\{T_n > 0\}$,

$$\left| \sum_{j < i \leq k} \left(X_i - \frac{n\mu}{T_n} |X_i| \right) \right| \leq \left(1 + \frac{n|\mu|}{T_n} \right) \sum_{j < i \leq k} |X_i|$$

whence

$$(T_k - T_j)^\alpha \geq \left(1 + \frac{n|\mu|}{T_n} \right)^{-\alpha} \left| S_k - S_j - \frac{n\mu}{T_n}(T_k - T_j) \right|^\alpha.$$

With this lower bound for the denominator in (5.1), we obtain

$$\begin{aligned} \Delta_n &\leq \left(\frac{T_n}{n} + |\mu| \right)^\alpha \left(\frac{1}{n} \max_{0 \leq \tau_j < \tau_k \leq 1} \left| S_k - S_j - \frac{n\mu}{T_n}(T_k - T_j) \right| \right)^{1-\alpha} \mathbf{1}_{\{T_n > 0\}} \\ &\quad + \mu \mathbf{1}_{\{T_n = 0\}} \\ &\leq \left(\frac{T_n}{n} + |\mu| \right)^\alpha \left(\frac{2}{n} \max_{1 \leq k \leq n} \left| S_k - \frac{n\mu}{T_n} T_k \right| \right)^{1-\alpha} \mathbf{1}_{\{T_n > 0\}} + \mu \mathbf{1}_{\{T_n = 0\}}. \end{aligned}$$

We introduce some centering by writing, on the event $\{T_n > 0\}$,

$$S_k - \frac{n\mu}{T_n} T_k = S_k - k\mu - \frac{n\mu}{T_n}(T_k - k\nu) + \frac{\mu k}{T_n}(T_n - n\nu),$$

whence

$$\frac{1}{n} \max_{1 \leq k \leq n} \left| S_k - \frac{n\mu}{T_n} T_k \right| \leq \frac{1}{n} \max_{1 \leq k \leq n} |S_k - k\mu| + \frac{2|\mu|}{T_n} \max_{1 \leq k \leq n} |T_k - k\nu|.$$

Finally

$$\begin{aligned} \Delta_n \leq & \left(\frac{T_n}{n} + |\mu| \right)^\alpha \left(\frac{2}{n} \max_{1 \leq k \leq n} |S_k - k\mu| + \frac{4|\mu|}{T_n} \max_{1 \leq k \leq n} |T_k - k\nu| \right)^{1-\alpha} \mathbf{1}_{\{T_n > 0\}} \\ & + \mu \mathbf{1}_{\{T_n = 0\}} \end{aligned} \quad (5.2)$$

Now we recall that $P(T_n = 0) = P(X_1 = 0)^n$ which goes to zero, since we discarded the trivial case where $X_1 = 0$ almost surely. As the sequence T_n is non-decreasing, this implies that

$$\mathbf{1}_{\{T_n > 0\}} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 1. \quad (5.3)$$

As $\mathbf{E}|X_1| < \infty$, we have the following convergences

$$\frac{T_n}{n} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \nu > 0, \quad (5.4)$$

by classical SLLN applied to the $|X_i|$'s,

$$\frac{1}{n} \max_{1 \leq k \leq n} |S_k - k\mu| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0, \quad (5.5)$$

by Theorem 1.2,

$$\frac{1}{T_n} \max_{1 \leq k \leq n} |T_k - k\nu| \mathbf{1}_{\{T_n > 0\}} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0, \quad (5.6)$$

by (5.3), (5.4) and Theorem 1.2 applied to the $|X_i|$'s.

Then the almost sure convergence of $\|n^{-1}\zeta_n^{\text{ad}} - \mu I\|_\alpha$ to zero results from (5.2) to (5.6). \square

Proof of (b) \Rightarrow (a). We note that

$$\|n^{-1}\zeta_n^{\text{ad}} - \mu I\|_\alpha = \omega_\alpha(n^{-1}\zeta_n^{\text{ad}} - \mu I, 1) \geq |n^{-1}(\zeta^{\text{ad}}(1) - \zeta^{\text{ad}}(0)) - \mu| = \left| \frac{S_n}{n} - \mu \right|.$$

Then (b) implies that $n^{-1}S_n$ converges almost surely to μ , which gives (a) by the converse part in the classical Kolmogorov-Khintchine strong law of large numbers for i.i.d. random variables. \square

Proof of (a) \Rightarrow (c). Writing $E_n := \{\Delta_n > \varepsilon\}$, we have to prove that for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P(E_n) < \infty. \quad (5.7)$$

Recalling that $\nu = \mathbf{E}|X_1|$ is assumed to be positive, we split

$$P(E_n) = P(E'_n) + P(E''_n),$$

where

$$E'_n := E_n \cap \left\{ \left| \frac{T_n}{n} - \nu \right| \leq \frac{\nu}{2} \right\}, \quad E''_n := E_n \cap \left\{ \left| \frac{T_n}{n} - \nu \right| > \frac{\nu}{2} \right\}.$$

On the event E'_n , as $0 < n\nu/2 \leq T_n \leq 3n\nu/2$, we deduce from (5.2) the following upper bound.

$$\begin{aligned} \Delta_n &\leq \left(\frac{5\nu}{2}\right)^\alpha \left(\frac{2}{n} \max_{1 \leq k \leq n} |S_k - k\mu| + \frac{8}{n} \max_{1 \leq k \leq n} |T_k - k\nu|\right)^{1-\alpha} \\ &\leq a \left(\frac{1}{n} \max_{1 \leq k \leq n} |S_k - k\mu|\right)^{1-\alpha} + b \left(\frac{1}{n} \max_{1 \leq k \leq n} |T_k - k\nu|\right)^{1-\alpha}, \end{aligned}$$

with $a = 2(5\nu)^\alpha$ and $b = 2^{3-4\alpha}(5\nu)^\alpha$.

It follows that

$$\begin{aligned} P(E'_n) &\leq P\left(\left(\frac{1}{n} \max_{1 \leq k \leq n} |S_k - k\mu|\right)^{1-\alpha} > \frac{\varepsilon}{2a}\right) + P\left(\left(\frac{1}{n} \max_{1 \leq k \leq n} |T_k - k\nu|\right)^{1-\alpha} > \frac{\varepsilon}{2b}\right) \\ &= P\left(\max_{1 \leq k \leq n} |S_k - k\mu| > n\varepsilon_a\right) + P\left(\max_{1 \leq k \leq n} |T_k - k\nu| > n\varepsilon_b\right) \end{aligned}$$

where $\varepsilon_a = (\varepsilon/2a)^{1/(1-\alpha)}$ and $\varepsilon_b = (\varepsilon/2b)^{1/(1-\alpha)}$. Therefore

$$\sum_{n=1}^{\infty} n^{-1} P(E'_n) < \infty, \quad (5.8)$$

by Theorem 1.8 (c'') applied to the sequences $(X_i)_{i \geq 1}$ and $(|X_i|)_{i \geq 1}$.

Next we note that

$$\sum_{n=1}^{\infty} n^{-1} P(E''_n) \leq \sum_{n=1}^{\infty} n^{-1} P\left(|T_n - n\nu| > \frac{\nu}{2}n\right) < \infty, \quad (5.9)$$

by Theorem 1.8 (c'') applied to the random variables $|X_i|$.

Gathering (5.8) and (5.9) gives (5.7), establishing (c). \square

Proof of (c) \Rightarrow (a). As already observed above,

$$\|n^{-1}\zeta_n^{\text{ad}} - \mu I\|_\alpha \geq \left|\frac{S_n}{n} - \mu\right|.$$

Then (c) implies that for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P(|S_n - n\nu| > n\varepsilon) < \infty,$$

whence (a) follows by the part (c'') \Rightarrow (a) in Theorem 1.8. \square

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